Self-Discharge, Memory Phenomena of Fractional Capacitor explained by using formula q = c * v in RC circuit charging/discharging with voltage excitation-with useful detailed derivations

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Abstract

In this deliberation note we apply the newly developed charge storage expression as a function of time i.e. via convolution operation of time varying capacity function and applied voltage function to a capacitor i.e. q = c*v, and see charging discharging of fractional and ideal capacitor in RC circuit. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor i.e. q = cv. We use this new formula to derive expression for self-discharging decay of open circuited voltage of a charged fractional capacitor, along with very useful step-wise derivations; also with this we prove the case of ideal less less capacitor. We also explain the self-discharge phenomena where open circuit voltage of a fractional capacitor decays is rather a 'misnomer'. With this new formula we explain the memory phenomenon observed in super capacitor, which remembers its charging history. We also use this formula for cases with ideal loss less classical capacitor. This note gives validity of usage of this new formula in RC circuits-for fractional capacitors as well as ideal loss less classical capacitors. The detailed derivations and step wise execution of each are very useful for researchers tackling fractional order circuits.

Keywords
Mittag-Leffler function, Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform, Fractional derivative, Super capacitor, Self-Discharging, Memory Effect

1. Introduction

This is continuation of our earlier deliberations regarding verification of the new formula $q(t)=c(t)*v(t)$; [1], [39]. The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality [1] we write $q(t)=c(t)*v(t)$ where $v(t)$ is the input impressed voltage. This is different to usual formula $q(t)=c(t)v(t)$. This new formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t)\propto t^{-\alpha}$; $0<\alpha<1$ and acts only at the time of application of sudden voltage change (mathematically where the differentiability of input voltage function is lost). For ideal case of loss-less capacitor the capacity function is $c(t)\propto\delta(t)$; [1]. In this note we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one, i.e. $0<\alpha<1$. This power-law decay function is in singular at origin and is in tune with singular power law decay relaxation current given by Curie-von

Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t) \propto t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11], [37]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36], for several dynamic systems.

Here we are taking singular function $c(t)$ as 'time varying capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t) \propto t^{-\alpha}$ of Curie-von Schweidler [1], [39]. In this discussion we will take capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula [1], where the voltage excitation $v(t)$ is applied at time $t = 0$ to an uncharged capacitor

$$q(t) = c(t) * v(t) = \int_0^t c(t-\tau)v(\tau)d\tau = \int_0^t c(\tau)v(t-\tau)d\tau$$

With this new formula $q(t) = c(t) * v(t)$ applied we discuss various cases of $q(t)$ i.e. charge stored in capacitor and $i(t)$, the circuital current etc. for RC charging/discharging circuit with ideal capacitor and fractional capacitor.

We note a priori that the constant C_α is proportionality constant of the relation of time varying capacity function i.e. $c(t) \propto t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\mathrm{Farad}/\mathrm{sec}^{1-\alpha}$, and we will use $C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha)$ to relate the two [1], [39]. The voltage, $v(t)$ across a capacitor or dielectric changes at a rate in proportion to the current: $i(t) = D_t^1\left(c(t) * v(t)\right)$, with $c(t) = C_\alpha t^{-\alpha}$ we get $i(t) = \left(c(t)\right)\left(v(0)\right) + c(t) * D_t^1 v(t)$; [1]. The equation of current and voltage, and impedance for fractional capacitor is following, given by fractional derivative $D_t^\alpha \equiv d^\alpha / dt^\alpha$ [6], [7] [8], [12], [13]; comes from $q(t) = c(t) * v(t)$, [1]. The fractional derivative operator is Riemann-Liouville type (Refer Appendix) as derived in [1]; and in [6], [7].

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad Z(s) = \frac{1}{s^\alpha C_{F-\alpha}}; \quad 0 < \alpha < 1$$

With limit $\alpha \to 1$ we get classical ideal loss less capacitor that is following

$$i(t) = C \frac{d\ v(t)}{dt}; \quad Z(s) = \frac{1}{s\ C}$$

The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (that has zero output impedance), in the RC charging circuits. We will use Laplace Transform technique in all our analysis. In all the cases in subsequent sections, we will apply this new formula $q(t) = c(t) * v(t)$ and give the validity justification. Recently this formula $q(t) = c(t) * v(t)$ is getting experimentally validated [38], for super-capacitors.

Therefore charge in a capacitor is $q(t)=c(t)*v(t)$, is given via convolution operation and not with the usual way that we write as $q(t)=c(t)v(t)$. Let us have a capacitor with capacity function in time as power-law $c(t)=C_{\alpha}t^{-\alpha}$ $(0<\alpha<1)$, that is fractional capacitor, is charged via resistance R. Let a voltage $v_{in}(t)$ be applied to an uncharged capacitor in the RC circuit at time $t=0$. Then charge function in time is given as convolution (*) operation as $q(t)=c(t)*v_0(t)$, with $v_0(t)$ is the voltage profile on the capacitor, in the RC circuit of Figure-1. This charge $q(t)$ is also $q(t)=\int_0^t i(\tau)d\tau$, where $i(t)$ is current flowing through the capacitor in the RC circuit. This comes from normal circuit theory application, and we will show that this $q(t)=c(t)*v_0(t)$ is same that we get from normal circuit theory. For each case we also study the ideal loss less capacitor given by capacity function as $c(t)=C\delta(t)$, [1], [39] and apply $q(t)=c(t)*v_0(t)$.

We will also validate self-discharge mechanism of fractional capacitor (super-capacitor) exhibiting memory effect, by using this new formula $q(t)=c(t)*v(t)$. Self-discharge mechanism, is where the fractional capacitor is charged to a voltage V_m for a time T_c, then kept in open circuited condition, we observe the open circuited voltage $v_{oc}(t)$ decays with time; and the decay curves depend on T_c, that is as though the fractional capacitor is memorizing its history of charging pattern! We will use the formula $q(t)=c(t)*v_0(t)$ to derive this self-discharge decay $v_{oc}(t)$ that depends on T_c. We will show that for ideal loss less capacitor V_m is held for infinite time, and $v_{oc}(t)=V_m$.

2. Charge storage $q(t)$ by step input voltage $v_{in}(t)=V_m u(t)$ excitation to RC circuit with ideal loss less capacitor

In classical circuit theory, if we charge an ideal capacitor, C that is initially uncharged ($v_0(0)=0$) through a resistor R, via a step input voltage $v_{in}(t)=V_m u(t)$ (Figure-1) we get voltage across capacitor as exponential rise as $v_0(t)=V_m(1-e^{-t/RC})$; $t\geq 0$. In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is ideal capacitor with capacity function as $c(t)=C\delta(t)$, [1], [39]. Therefore we have following impedance function

$$Z_2(s)=\frac{1}{s\mathcal{L}\{c(t)\}}=\frac{1}{s\mathcal{L}\{C\delta(t)\}}=\frac{1}{sC} \tag{1}$$

The above Eq. (1) is new way of writing $Z(s)$ for capacitor (ideal or fractional) that we got from application of formula $q(t)=c(t)*v(t)$ in our earlier discussion [39]. Eq. (1) we got by differentiating this convolution expression to get $i(t)$ and then taking Laplace transform to arrive at $Z(s)=V(s)/I(s)=\left(s\mathcal{L}\{c(t)\}\right)^{-1}$.

We have from circuit theory and from Figure-1 the following expressions, where $\Delta v_0(t)$ represents change in voltage across Z_2 , and say $v_0(0)$ is the initial voltage at Z_2. With $\Delta V_0(s)=\mathcal{L}\{\Delta v_0(t)\}$, we write the following

$$\begin{aligned} \Delta V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\{v_{in}(t)-v_0(0)\}, \quad v_{in}(t)=V_m u(t), \qquad \mathcal{L}\{v_{in}(t)\}=\frac{V_m}{s} \\ \Delta V_0(s) &= \frac{V_m-v_0(0)}{RCs\left(s+\frac{1}{RC}\right)}=(V_m-v_0(0))\left(\frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\right) \end{aligned} \tag{2}$$

The inverse Laplace Transform of Eq. (2) gives following voltage charging equation for capacitor

$$\begin{aligned} \Delta v_0(t) &= (V_m-v_0(0))(1-e^{-t/RC}); \quad t\ge 0, \quad v_0(0)=0 \\ \Delta v_0(t) &= V_m(1-e^{-t/RC}) \\ v_0(t) &= v_0(0)+\Delta v_0(t)=v_0(0)+(V_m-v_0(0))(1-e^{-t/RC}) \end{aligned} \tag{3}$$

We have $\lim_{t\uparrow\infty} v_0(t)=V_m$. The change in current flowing in the RC circuit at $t\ge 0$ is the following

$$\begin{aligned} \Delta i(t) &= \mathcal{L}^{-1}\left\{\frac{(V_m/s)\text{-}(v_0(0)/s)}{R+\frac{1}{Cs}}\right\}=\mathcal{L}^{-1}\left\{\frac{(V_m-v_0(0))}{R}\left(\frac{1}{s+\frac{1}{RC}}\right)\right\}=\frac{V_m-v_0(0)}{R}e^{-t/RC} \\ &= \frac{V_m}{R}e^{-t/RC}, \quad v_0(0)=0 \end{aligned} \tag{4}$$

This change in current will be manifested as change in coulomb charge $\Delta q(t)$ in capacitor. Therefore the change in charge function $q(t)$ is $\Delta q(t)$

$$\begin{aligned} \Delta q(t) &= \int_0^t \Delta i(\tau)d\tau=\int_0^t \frac{V_m-v_0(0)}{R}e^{-\tau/RC}d\tau \\ &= (V_m - v_0(0))C(1-e^{-t/RC})=V_mC(1-e^{-t/RC}); \quad t\ge 0, \quad v_0(0)=0 \\ q(t) &= q(0)+(V_m - v_0(0))C(1-e^{-t/RC}) \end{aligned} \tag{5}$$

We apply the formula $q(t)=c(t)*v(t)$ to ideal capacitor given by $c(t)=C\delta(t)$ across which we are having a voltage profile as $v_0(t)=(V_m-v_0(0))(1-e^{-t/RC})+v_0(0)$, to write following

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right) \\ &= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{(V_m-v_0(0))(1-e^{-t/RC})+v_0(0)\}\right) \\ &= (C)\left((V_m - v_0(0))\left(\tfrac{1}{s}-\tfrac{1}{(s+1/RC)}\right)+v_0(0)\left(\tfrac{1}{s}\right)\right) \\ &= C(V_m-v_0(0))\left(\frac{1}{s}-\frac{1}{\left(s+\frac{1}{RC}\right)}\right)+Cv_0(0)\left(\frac{1}{s}\right) \end{aligned} \tag{6}$$

The inverse Laplace transform of Eq. (6) above gives

$$\begin{aligned} q(t) &= C(V_m-v_0(0))(1-e^{-t/RC})+Cv_0(0); \quad t\ge 0 \\ &= CV_m(1-e^{-t/RC}); \quad v_0(0)=0 \end{aligned} \tag{7}$$

Eq. (7) is same as Eq. (5) that we got via circuit theory applying $q(t)=q(0)+\int_0^t i(\tau)d\tau$, with identifying $q(0)=Cv_0(0)$.

We differentiate Eq. (7) to write $i(t)=\frac{dq}{dt}=Cv_0(0)\delta(t)+\frac{V_m-v_0(0)}{R}e^{-t/RC}$. The first part is Dirac delta impulse current $i(t)\big|_{t=0}$ what is true for uncharged capacitor excited by constant step voltage, in this case $v_0(0)u(t)$, and the second part is $\Delta i(t)$ Eq.(4), that is via RC circuit theory. This gives validation of formula $q(t)=c(t)*v(t)$ for classical ideal loss less capacitor case.

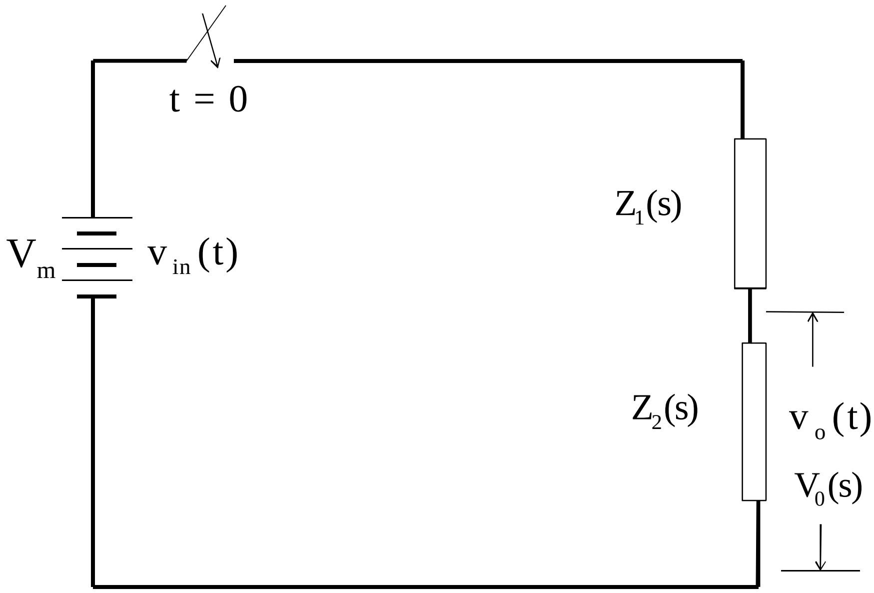

Figure- 1: The constant voltage charging RC circuit

3. Charge storage $q(t)$ by step input voltage $v_{in}(t) = V_m u(t)$ excitation to RC circuit with fractional capacitor

In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is fractional capacitor with capacity function as $c(t)=C_\alpha t^{-\alpha}$; with $0<\alpha<1$. Therefore we have following impedance function [39]

$$\begin{aligned} Z_2(s) &= \frac{1}{s\mathcal{L}\{c(t)\}}=\frac{1}{s\mathcal{L}\{C_\alpha t^{-\alpha}\}}=\frac{1}{s\left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)} \\ &= \frac{1}{s^\alpha C_\alpha\Gamma(1-\alpha)}=\frac{1}{s^\alpha C_{F-\alpha}};\quad C_{F-\alpha}=C_\alpha\Gamma(1-\alpha) \end{aligned} \tag{8}$$

Here we will use a constant voltage excitation of V_m from time $t=0$, to time $t=T_c$ (as charging phase, through a known resistor R) and thereafter we will switch to discharging phase i.e. voltage source will be made zero (Figure-3). By this we record the charging and discharging profile $v_0(t)$, and then apply $q(t)=c(t)*v_0(t)$ to get charge, and then current.

3a. Charging phase equations

From the circuit diagram of Figure-1, we write the following [36]

$$\Delta V_0(s)=\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\left\{v_{in}(t)-v_0(0)\right\},\quad v_{in}(t)=V_m u(t),\qquad \mathcal{L}\left\{v_{in}(t)\right\}=\frac{V_m}{s}$$
$$=\frac{V_m-v_0(0)}{RC_{F-\alpha}s\left(s^\alpha+\frac{1}{RC_{F-\alpha}}\right)}=\frac{(V_m-v_0(0))ks^{-1}}{(s^\alpha+k)};\quad k=\frac{1}{RC_{F-\alpha}} \tag{9}$$

Now use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\}=\frac{p!s^{\alpha-\beta}}{s^\alpha-a}$ [10], [12], [13] to get $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^\alpha-a}\right\}=t^\alpha E_{\alpha,\alpha+1}(at^\alpha)$, by putting $p=0$, $\alpha=\alpha,\beta=\alpha+1$, where the $E_{\alpha,\beta}(at^\alpha)$ is two parameter Mittag-Leffler function (Refer Appendix); as defined in infinite series in following expression

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)},\qquad E_{\alpha,(\alpha+1)}(-kt^\alpha)=\sum_{m=0}^{\infty}\frac{(-kt^\alpha)^m}{\Gamma(m\alpha+\alpha+1)}$$
$$E_{\alpha,1}(x)=E_\alpha(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+1)} \tag{10}$$

With this we obtain the following from inverse Laplace transform of Eq. (9)

$$\Delta v_0(t)=\mathcal{L}^{-1}\left\{\frac{(V_m-v_0(0))k}{s(s^\alpha+k)}\right\}=(V_m-v_0(0))kt^\alpha E_{\alpha,\alpha+1}(-kt^\alpha)=\frac{(V_m-v_0(0))}{RC_{F-\alpha}}t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{RC_{F-\alpha}}\right) \tag{11}$$

We have alternate derivation via series expansion [13], [36] as follows

$$\Delta V_0(s)=\frac{(V_m-v_0(0))k}{s(s^\alpha+k)}=\frac{(V_m-v_0(0))k}{s^{\alpha+1}}\left(1+\frac{k}{s^\alpha}\right)^{-1};\quad (1+x)^{-1}=1-x+x^2-x^3+\dots$$
$$=\frac{(V_m-v_0(0))k}{s^{\alpha+1}}\left(1-\frac{k}{s^\alpha}+\frac{k^2}{s^{2\alpha}}-\frac{k^3}{s^{3\alpha}}+\dots\right)=V_m\left(\frac{k}{s^{\alpha+1}}-\frac{k^2}{s^{2\alpha+1}}+\frac{k^3}{s^{3\alpha+1}}-..\right) \tag{12}$$

Use Laplace pair $\frac{\Gamma(n+1)}{s^{n+1}}=\mathcal{L}\left\{t^n\right\}$ to invert term by term the above Eq. (12) to get following

$$\Delta v_0(t)=(V_m-v_0(0))\left(\frac{kt^\alpha}{\Gamma(\alpha+1)}-\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}-\dots\right)$$
$$=(V_m-v_0(0))\left(1-\left[1-\frac{kt^\alpha}{\Gamma(\alpha+1)}+\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\dots\right]\right)$$
$$=(V_m-v_0(0))\left(1-\sum_{n=0}^{\infty}\frac{(-kt^\alpha)^n}{\Gamma(n\alpha+1)}\right)$$
$$=(V_m-v_0(0))\left[1-E_\alpha(-kt^\alpha)\right]=(V_m-v_0(0))\left[1-E_\alpha\left(-\frac{t^\alpha}{RC_{F-\alpha}}\right)\right] \tag{13}$$

Where, $E_\alpha(x)$ is one parameter Mittag-Leffler function (Refer Appendix) used in Eq. (13), with $E_1(x)=e^x$. Therefore for classical ideal capacitor with limit $\alpha\to 1$, we have normal exponential charging $\Delta v_0(t)=(V_m-v_0(0))(1-e^{-t/RC})$; writing $C_{F-\alpha}\big|_{\alpha\to 1}\equiv C$.

For voltage charging expression for fractional order impedance $Z_2(s)=s^{-\alpha}C^{-1}_{F-\alpha}$, Eq. (8) we have from Eq. (11) and Eq. (13) the following

$$\Delta v_0(t) = (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right) = \frac{(V_m - v_0(0))}{RC_{F\text{-}\alpha}} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)$$
$$v_0(t) = v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right) = v_0(0) + \frac{(V_m - v_0(0))}{RC_{F\text{-}\alpha}} t^\alpha E_{\alpha,\alpha+1}\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right) \quad (14)$$

We have $\lim_{t\uparrow\infty} v_0(t) = V_m$. For charging current of circuit of Figure-1 with $Z_1 = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$, we have $Z(s) = Z_1(s) + Z_2(s)$ and write the following

$$\Delta I(s) = \frac{1}{Z(s)}\left(\frac{V_m}{s} - \frac{v_0(0)}{s}\right) = \frac{V_m - v_0(0)}{s\left(R + \frac{1}{s^\alpha C_{F\text{-}\alpha}}\right)} = \frac{V_m - v_0(0)}{R}\left(\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}}\right) \quad (15)$$

Using $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n-1}}{s^n - a}$, [10], [12], [13] we get inverse Laplace transform of above Eq. (15) as

$$\Delta i(t) = \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right) \quad (16)$$

Clearly for ideal case i.e. in limit $\alpha \to 1$ case we get $\Delta i(t) = \frac{V_m - v_0(0)}{R} e^{-t/RC}$. Therefore the change in charge $\Delta q(t)$ is from Eq. (16) the following with $q(t) = q(0) + \Delta q(t)$

$$\Delta q(t) = \int_0^t \Delta i(\tau) d\tau = \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau$$
$$q(t) = q(0) + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau \quad (17)$$

We apply the formula $q(t) = c(t) * v(t)$, i.e. $Q(s) = \left(\mathcal{L}\left\{c(t)\right\}\right)\left(\mathcal{L}\left\{v_0(t)\right\}\right)$ to fractional capacitor given by $c(t) = C_\alpha t^{-\alpha}$ across which we are having a voltage profile as $v_0(t) = v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)$, to write following steps

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v_0(0) + (V_m - v_0(0))\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)\right\}\right) \\
&= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\left(\tfrac{v_0(0)}{s} + \tfrac{(V_m - v_0(0))k}{s(s^\alpha + k)}\right) = \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\frac{v_0(0)}{s} \\
&\quad + \frac{(V_m - v_0(0))C_{F\text{-}\alpha}\left(\frac{1}{RC_{F\text{-}\alpha}}\right)}{s^{2\text{-}\alpha}\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)} \\
&= \left(C_\alpha \Gamma(1-\alpha) s^{\alpha-1}\right)\frac{v_0(0)}{s} + \left(\frac{V_m - v_0(0)}{R}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)} \\
&= \left(C_\alpha \Gamma(1\text{-}\alpha)\right) v_0(0) s^{-1} s^{\alpha\text{-}1} + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\left(\frac{s^{\alpha\text{-}1}}{\left(s^\alpha + \frac{1}{RC_{F\text{-}\alpha}}\right)}\right)\right) \\
&= \left(C_\alpha \Gamma(1\text{-}\alpha) v_0(0)\right) s^{-1}\left(\frac{\mathcal{L}\{t^{-\alpha}\}}{\Gamma(1\text{-}\alpha)}\right) + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right\}\right) \\
&= \left(C_\alpha v_0(0)\right) s^{-1}\left(\mathcal{L}\{t^{-\alpha}\}\right) + \left(\frac{V_m - v_0(0)}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right\}\right)
\end{aligned} \tag{18}
$$

We used $k = \frac{1}{RC_{F\text{-}\alpha}}$, $\frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)} = C_\alpha$, $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha\text{-}1}}{s^\alpha + k}$ and $s^{\alpha\text{-}1} = \frac{1}{\Gamma(1\text{-}\alpha)}\mathcal{L}\{t^{-\alpha}\}$ in above steps in Eq. (18). Taking inverse Laplace transform of Eq. (18) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$
\begin{aligned}
q(t) &= C_\alpha v_0(0)\int_0^t \tau^{-\alpha} d\tau + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau \\
&= \frac{C_\alpha v_0(0)}{1\text{-}\alpha} t^{1\text{-}\alpha} + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau \\
&= \frac{C_{F\text{-}\alpha}}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)} v_0(0) t^{1\text{-}\alpha} + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau
\end{aligned} \tag{19}
$$

The same result as in Eq. (17) we got by using $\Delta q(t) = \int_0^t i(\tau)d\tau$ validates the verification of formula $q(t) = c(t) * v(t)$; where $q(0) = \frac{C_{F\text{-}\alpha} v_0(0)}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)} t^{1\text{-}\alpha}$. Note here that $q(0)$ is function of time.

Put $\alpha = 1$ in Eq. (19) and we get ideal loss-less capacitor with $C_{F\text{-}\alpha} \equiv C$, and $E_1(x) = e^x$ to write the following case

$$
\begin{aligned}
q(t) &= Cv_0(0) + \int_0^t \frac{V_m - v_0(0)}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right) d\tau \Big|_{\alpha=1} \\
&= Cv_0(0) + \int_0^t \frac{V_m - v_0(0)}{R} e^{-\tau/RC} d\tau = Cv_0(0) + C(V_m - v_0(0))\left(1 - e^{-t/RC}\right)
\end{aligned} \tag{20}
$$

The above Eq. (20) is charge build up relation for ideal-loss less capacitor, same as Eq. (5) and Eq. (7). Interestingly as $t\uparrow\infty$ for a fractional capacitor Eq. (19) $\lim_{t\uparrow\infty} q(t)=\infty$ while for ideal loss less capacitor $\lim_{t\uparrow\infty} q(t)=CV_m$ Eq. (20). We note that $q(0)=\frac{C_\alpha v_0(0)}{1-\alpha}t^{1-\alpha}$ is growing function of time. Differentiating Eq. (19) we write $i(t)=\frac{dq(t)}{dt}=C_\alpha v_0(0)t^{-\alpha}+\frac{V_m-v_0(0)}{R}E_\alpha\left(-\frac{t^\alpha}{RC_{F\text{-}\alpha}}\right)$. This current component has first part as power law decay current $C_\alpha v_0(0)t^{-\alpha}$, as per Curie-von Schwiedler (UDR) law. This component is always flowing in a fractional capacitor when impressed by a constant voltage in this case $v_0(0)u(t)$ appearing across fractional capacitor directly (that is without resistance), the second part is $\Delta i(t)$, given by RC circuit theory Eq. (16).

We take the integration of Mittag-Leffler function as $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau=t\left(E_{\alpha,2}(-kt^\alpha)\right)$ with $E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)}$ (Refer Appendix for proof). So we have charge build up function on a fractional capacitor in RC charging circuit as follows from Eq. (19)

$$\begin{aligned}q(t)&=\frac{C_{F\text{-}\alpha}}{(1-\alpha)\Gamma(1-\alpha)}v_0(0)t^{1-\alpha}+\int_0^t\frac{V_m-v_0(0)}{R}E_\alpha\left(-\frac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau\\&=\frac{C_{F\text{-}\alpha}}{(1-\alpha)\Gamma(1-\alpha)}v_0(0)t^{1-\alpha}+\frac{V_m-v_0(0)}{R}t\left(E_{\alpha,2}(-t^\alpha/RC_{F\text{-}\alpha})\right);\quad t\geq 0 \qquad (21)\\&=\frac{V_m}{R}t\left(E_{\alpha,2}(-t^\alpha/RC_{F\text{-}\alpha})\right),\quad v_0(0)=0\end{aligned}$$

Let us verify this for $\alpha=1$, taking from Eq. (21), with $v_0(0)=0$ where we get $q(t)=\frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha/RC_{F\text{-}\alpha})\right)\Big|_{\alpha=1;C_{F\text{-}\alpha}=C}$. Using $E_{\alpha,2}(-a\,x^\alpha)=\sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)}$ we get the charge profile as $q(t)=V_mC\left(1-e^{-t/RC}\right)$, for $v_0(0)=0$ by simple algebraic manipulations and tricks, as describe below

$$q(t)=\frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha/RC_{F\text{-}\alpha})\right)\Big|_{\alpha=1;C_{F\text{-}\alpha}=C};\qquad E_{\alpha,2}(-ax^\alpha)=\sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)}$$

$$\begin{aligned}&=\frac{V_m}{R}t\left(1-\frac{t}{(RC)\Gamma(3)}+\frac{t^2}{(RC)^2\Gamma(4)}-\frac{t^3}{(RC)^3\Gamma(5)}+\dots\right)\\&=\frac{V_mC}{RC}\left(t-\frac{t^2}{(RC)(2)!}+\frac{t^3}{(RC)^2(3!)}-\frac{t^4}{(RC)^3(4)!}+\dots\right)\\&=V_mC\left(1-1+\frac{\left(\frac{t}{RC}\right)}{1!}-\frac{\left(\frac{t}{RC}\right)^2}{2!}+\frac{\left(\frac{t}{RC}\right)^3}{3!}-\frac{\left(\frac{t}{RC}\right)^4}{4!}+\dots\right)\\&=V_mC\left(1-\left(1-\frac{\left(\frac{t}{RC}\right)}{1!}+\frac{\left(\frac{t}{RC}\right)^2}{2!}-\frac{\left(\frac{t}{RC}\right)^3}{3!}+\frac{\left(\frac{t}{RC}\right)^4}{4!}-\dots\right)\right)\\&=V_mC\left(1-e^{-t/RC}\right)\end{aligned}$$

Thus we have verified the validity of formula $q(t)=c(t)*v(t)$ in RC charging circuit with fractional capacitor.

3b. Charge holding at large times for fractional capacitor

We have from Eq. (21) at $t=T_c$ the charge stored is $q(T_c)=\left(\frac{V_m}{R}\right)T_c\left(E_{\alpha,2}(-T_c^{\alpha}/RC_{F\text{-}\alpha})\right)$, for uncharged capacitor i.e. $v_0(0)=0$. Now we see if we keep the unit step voltage $v_{in}(t)=V_m u(t)$ for large time say $T_c\uparrow\infty$ for a fractional capacitor, what is $\lim_{T_c\uparrow\infty} q(T_c)=\lim_{T_c\uparrow\infty}\left(\frac{V_m}{R}\right)T_c\left(E_{\alpha,2}(-T_c^{\alpha}/RC_{F\text{-}\alpha})\right)$, that we analyze. Whereas for classical ideal capacitor $\lim_{T_c\uparrow\infty} q(T_c)=\lim_{T_c\uparrow\infty} CV_m(1-e^{-T_c/RC})=V_mC$, is a constant independent of $t=T_c$.

This we study from recurring property of $E_{\alpha,\beta}(x)$ which is $E_{\alpha,\beta}(x)=\frac{1}{x}E_{\alpha,\beta\text{-}\alpha}(x)-\frac{1}{x\Gamma(\beta\text{-}\alpha)}$ from which Poincare asymptotic expansion is $E_{\alpha,\beta}(x)\sim-\sum_{n=1}^{\infty}\frac{1}{x^n\Gamma(\beta\text{-}n\alpha)}$ valid for $x\to-\infty$ (Refer Appendix). In the expression asymptotic expansion of $E_{\alpha,2}(-T_c^{\alpha}/RC_{F\text{-}\alpha})$ taking $x=-kT_c^{\alpha}$, where $k=\frac{1}{RC_{F\text{-}\alpha}}$ we write for $T_c\uparrow\infty$ as following

$$\lim_{T_c\uparrow\infty}E_{\alpha,2}(-k\,T_c^{\alpha})=\frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)}-\frac{T_c^{-2\alpha}}{k^2\Gamma(2-2\alpha)}-\frac{T_c^{-3\alpha}}{k^3\Gamma(2-3\alpha)}-\ldots\sim\frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)}\qquad(22)$$

We approximate above infinite series Eq. (22) by neglecting higher powers exponents of power law, as the higher terms will be decaying much faster than the first term. Therefore we write the following

$$\begin{aligned}\lim_{T_c\uparrow\infty}q(T_c)&=\lim_{T_c\uparrow\infty}\left(\frac{V_m}{R}\right)T_c\left(E_{\alpha,2}(-T_c^{\alpha}/RC_{F\text{-}\alpha})\right);\quad 0<\alpha<1\\&\sim\frac{V_m}{R}T_c\left(\frac{T_c^{-\alpha}}{k\Gamma(2-\alpha)}\right)=\frac{V_mC_{F\text{-}\alpha}}{\Gamma(2-\alpha)}T_c^{1-\alpha};\quad\Gamma(m+1)=m\Gamma(m)\qquad(23)\\&=\frac{V_mC_{F\text{-}\alpha}}{(1-\alpha)\Gamma(1-\alpha)}T_c^{1-\alpha}=\infty\end{aligned}$$

In [1] we got $q(t)=\frac{C_\alpha V_m t^{1-\alpha}}{1-\alpha}$ for a fractional capacitor with capacity function $c(t)=C_\alpha t^{-\alpha}$ as a charge build up formula for a fractional capacitor. In [1] we showed $\lim_{t\uparrow\infty}q(t)=\infty$ by use of formula $q(t)=c(t)*v(t)$ for an uncharged fractional capacitor, charged directly from ideal voltage source (i.e. in RC of circuit Figure-1 with $R=0\Omega$).

Here in RC circuit case we see that steady state of charge holding will be never obtained (as we get for an ideal loss less capacitor). For the fractional capacitor case, the charge will keep growing to infinity, leading to electro-static break down of capacitors [1], [6], [7]. Using $C_{F\text{-}\alpha}=C_\alpha\Gamma(1-\alpha)$ in the derived formula for large times in RC charging in asymptotic approximation is $q(t)\sim\frac{V_mC_{F\text{-}\alpha}}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)}t^{1-\alpha}=\frac{V_mC_\alpha}{(1\text{-}\alpha)}t^{1-\alpha}$ that is same that we got in [1]. Here if we put $\alpha\to1$, we have classical ideal capacitor $C_{F\text{-}\alpha}=C_\alpha\Gamma(1-\alpha)\equiv C$ and thus $q(t)=V_mC$ for any $t\ge0$; that is true for classical ideal capacitor case.

In case of classical capacitors, we have $q(t)=CV_m(1-e^{-t/RC})$ and here we get steady-state at $\lim_{t\uparrow\infty}q(t)=V_mC$. This is fundamental to memory effect as observed in a fractional capacitor case [39]. There is no memory effect in the classical capacitor cases the charge store is steady constant $q(t)=CV_m$ for any holding time for $v_{in}(t)=V_mu(t)$. While the charge storage in a fractional capacitor depends on holding time for step voltage, more the holding time more the charge stored in fractional capacitor [1], [39].

4. Self-Discharging a fractional capacitor after holding a step input voltage for a long time--The memory effect, explained by the formula q = c * v

A fractional capacitor (that is uncharged) is charged from time $t=-T_c$ to time t with a constant step input $v_{in}(t)=V_mu(t-(-T_c))$. That is step voltage applied at time $t=-T_c$. The charging current is from general charge equation by following convolution expression $q_{CH}(t)=\left(c(t)*v(t)\right)=\int_{-\infty}^{t}c(t-x)v(x)dx$; [1]. For a fractional capacitor with capacity function $c(t)=C_\alpha t^{-\alpha}$ we write the convolution expression with lower limit of integration as $-T_c$ that is the time where the voltage change is applied, [1].

$$q_{CH}(t)=\left(c(t)*v(t)\right)\Big|_{-T_c}^{t}=\int_{-T_c}^{t}C_\alpha(t-x)^{-\alpha}v(x)dx \tag{24}$$

Where $v(t)$ we say voltage across the capacitor assumed to be at V_m in $t=-T_c$, and $v(t)=0$, for $t<-T_c$. This assumption is valid when we say $t>>-T_c$, that is neglecting the rise part of the charging equation $v(t+T_c)=V_m\left(1-E_\alpha\left(-\frac{(t+T_c)^\alpha}{RC_{F-\alpha}}\right)\right)$ is $v(t+T_c)\cong V_m$ for $t\gg T_c$. The charging current is following

$$\begin{aligned} i_{CH}(t)&=\frac{dq_{CH}(t)}{dt}=\frac{d}{dt}\left(c(t)*v(t)\right)\Big|_{-T_c}^{t},\quad c(t)=C_\alpha t^{-\alpha}\\ &=\frac{d}{dt}\int_{x=-T_c}^{x=t}C_\alpha(t-x)^{-\alpha}v(x)dx=C_\alpha\frac{d}{dt}\int_{-T_c}^{t}\frac{v(x)dx}{(t-x)^\alpha}\end{aligned} \tag{25}$$

The integration by parts for term $\int_{-T_c}^{t}(t-x)^{-\alpha}v(x)dx$ in Eq. (25) gives following result in following steps

$$\begin{aligned}\int_{-T_c}^{t}\frac{v(x)dx}{(t-x)^\alpha}&=\left[v(x)\int\frac{dx}{(t-x)^\alpha}\right]_{x=-T_c}^{x=t}-\int_{-T_c}^{t}\left(v^{(1)}(x)\int\frac{dx}{(t-x)^\alpha}\right)dx\\ &=v(x)\left(-\frac{(t-x)^{1-\alpha}}{1-\alpha}\right)\Bigg|_{x=-T_c}^{x=t}-\int_{-T_c}^{t}v^{(1)}(x)\left(\frac{(-1)(t-x)^{1-\alpha}}{1-\alpha}\right)dx\\ &=\frac{v(-T_c)}{1-\alpha}(t+T_c)^{1-\alpha}+\int_{-T_c}^{t}\frac{v^{(1)}(x)}{1-\alpha}(t-x)^{1-\alpha}dx\end{aligned} \tag{26}$$

Using the derivation of Eq. (26) and using the definition of fractional derivative for $0<\alpha<1$ is Riemann –Liouville (RL) ${}_aD_t^\alpha$ and Caputo ${}_a^CD_t^\alpha$ (Refer Appendix) we write the following steps

$$
\begin{aligned}
i_{CH}(t) &= C_\alpha \frac{d}{dt}\int_{-T_c}^{t}\frac{v(x)dx}{(t-x)^\alpha} = C_\alpha \frac{d}{dt}\left(\frac{v(\text{-}T_c)}{1-\alpha}(t+T_c)^{1-\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)}{1-\alpha}(t-x)^{1-\alpha}dx\right) \\
&= C_\alpha v(\text{-}T_c)\frac{d}{dt}\left(\frac{(t+T_c)^{1-\alpha}}{1-\alpha}\right) + C_\alpha \int_{-T_c}^{t}\left(\frac{v^{(1)}(x)}{1-\alpha}\right)\frac{d}{dt}(t-x)^{1-\alpha}dx \\
&= C_\alpha \frac{v(\text{-}T_c)}{(t+T_c)^\alpha} + C_\alpha \int_{-T_c}^{t}\frac{v^{(1)}(x)}{(t-x)^\alpha}dx \\
&= C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(\text{-}T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right), \quad C_\alpha\left(\Gamma(1-\alpha)\right) = C_{F\text{-}\alpha} \\
&= \frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)}\left(\frac{v(\text{-}T_c)}{(t+T_c)^\alpha}\right) + C_{F\text{-}\alpha}\left({}^{C}_{-T_c}D_t^\alpha\left[v(t)\right]\right) \\
&= C_{F\text{-}\alpha}\left({}_{-T_c}D_t^\alpha\left[v(t)\right]\right), \quad 0<\alpha<1
\end{aligned}
\tag{27}
$$

We set $v(\text{-}T_c) \approx V_m$ and for $t \gg T_c$ we write $v^{(1)}(t) = 0$ for a constant voltage $v(t) = V_m$ for $t \gg T_c$ and get the following

$$
\begin{aligned}
i_{CH}(t) &= C_\alpha\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\left(\frac{v(\text{-}T_c)}{(t+T_c)^\alpha} + \int_{-T_c}^{t}\frac{v^{(1)}(x)dx}{(t-x)^\alpha}\right)\right) \\
&= \frac{C_\alpha V_m}{(t+T_c)^\alpha}
\end{aligned}
\tag{28}
$$

The above $i_{CH}(t)$ in Eq. (28) is Curie-Von Schewdler relaxation current power law for dielectric relaxation when the dielectric is stressed by a constant voltage at time (in this case) $t \approx -T_c$. This we get by other method too as depicted below by using $i_{CH}(t) = C_{F-\alpha}\left({}_{-T_c}D_t^\alpha v(t)\right)$

$$
\begin{aligned}
i_{CH}(t) &= C_{F-\alpha}\left({}_{-T_c}D_t^\alpha v(t)\right); \quad 0<\alpha<1 \\
&= C_{F\text{-}\alpha}\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{t=-T_c}^{t=t}, \quad C_{F\text{-}\alpha} = C_\alpha\Gamma(1-\alpha) \\
&= C_\alpha\Gamma(1-\alpha)\left.\frac{d^\alpha V_m}{dt^\alpha}\right|_{-T_c}^{t} = C_\alpha\Gamma(1-\alpha)\left(\frac{V_m}{\Gamma(1-\alpha)}(t-(\text{-}T_c))^{-\alpha}\right) \\
&= C_\alpha\frac{V_m}{(t+T_c)^\alpha} \qquad 0<\alpha<1 \qquad (t+T_c)>0
\end{aligned}
\tag{29}
$$

In above steps of Eq. (29) we used formula for RL fractional derivative of a constant K as ${}_aD_x^\alpha K = K\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$, with $a = \text{-}T_c$ that is start point of fractional differentiation process, and $x = t$, and $K = V_m$ (Refer Appendix). We note that ${}^{C}_{a}D_x^\alpha K = 0$, that appears in Eq. (28).

At $t = 0$ the voltage source $v_{in}(t) = V_m u(t)$ is disconnected, or we keep the fractional capacitor at open-circuited condition, after keeping this for a long-long time from $t = \text{-}T_c$. There will be a self-discharging of the charged fractional capacitor, and the self discharge current will be proportional to decaying open circuited voltage $v_{oc}(t)$, given as follows from time $t=0$ the time the fractional capacitor was kept open circuited, to time $t \geq 0$. The self-discharging current (the notional current) we write as follows $i_{DIS}(t) = C_{F-\alpha}\left({}_0D_t^{\alpha}v_{oc}(t)\right)$, that is

$$i_{DIS}(t) = C_{F-\alpha}\frac{d^{\alpha}v_{oc}(t)}{dt^{\alpha}}\Bigg|_{t=0}^{t=t} = C_{\alpha}\Gamma(1-\alpha)\frac{d^{\alpha}v_{oc}(t)}{dt^{\alpha}}\Bigg|_{t=0}^{t=t} \tag{30}$$

We will see in subsequent section that $i_{DIS}(t)$ of Eq. (30) is not the conventional current of discharge that flows out to a shunt resistance put for discharging the stored charge, but gives a notion due to special re-distribution of charges inside a spatially distributed system infinite RC circuit-we call it notional discharge current (we will discuss later).

The coulomb of charge $q_{CH}(t)$ pumped into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives the following. We write the following

$$\begin{aligned} & i_{CH}(t) + i_{DIS}(t) = 0 \\ & C_{F-\alpha}\left({}_{-T_c}D_t^{\alpha}v(t)\right) + C_{F-\alpha}\left({}_0D_t^{\alpha}v_{oc}(t)\right) = 0 \end{aligned} \tag{31}$$

That is the following we get using Eq. (28) or Eq. (29)

$$C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}} + C_{\alpha}\Gamma(1-\alpha)\frac{d^{\alpha}v_{oc}(t)}{dt^{\alpha}} = 0 \tag{32}$$

Our interest is finding $v_{oc}(t)$, from $t \geq 0$ that is in self-discharge phase. We do the fractional integration ${}_0I_t^{\alpha}$ (from time 0 to time t) of the above Eq. (32) and write the following

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left({}_0I_t^{\alpha}\left[\frac{d^{\alpha}v_{oc}(t)}{dt^{\alpha}}\right]\right) = 0 \tag{33}$$

For the second term we write $\left({}_0I_t^{\alpha}\left[{}_0D_t^{\alpha}v_{oc}(t)\right]\right) = v_{oc}(t)\big|_0^t = v_{oc}(t) - v_{oc}(0)$ with $v_{oc}(0) = V_m$ and then write the following from Eq. (33)

$${}_0I_t^{\alpha}\left[C_{\alpha}\frac{V_m}{(t+T_c)^{\alpha}}\right] + C_{\alpha}\Gamma(1-\alpha)\left[v_{oc}(t) - V_m\right] = 0 \tag{34}$$

To the first term of Eq. (34) we apply Riemann formula of Fractional Integration (Refer Appendix) that is ${}_0I_t^{\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x)dx}{(t-x)^{1-\alpha}}$ and get

$$C_{\alpha}V_m\frac{1}{\Gamma(\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} + \left[C_{\alpha}\Gamma(1-\alpha)v_{oc}(t) - C_{\alpha}\Gamma(1-\alpha)V_m\right] = 0 \tag{35}$$

Rearranging the Eq. (35) we write the following expression

$$v_{oc}(t) = V_m - \frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{dx}{(T_c+x)^{\alpha}(t-x)^{1-\alpha}} \tag{36}$$

In Eq. (36) put $T_c + x = \tau$, $dx = d\tau$, therefore for $x=0$, $\tau = T_c$ and $x = t$, $\tau = T_c + t$ we have following simplified representation

$$v_{oc}(t)=V_m-\frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=V_m-\frac{V_m}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{T_c}^{T_c+t}F(\tau)d\tau$$
$$F(\tau)=\frac{1}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}} \quad (37)$$

Now we break $\int_{T_c}^{T_c+t}F(\tau)d\tau$ as $\int_{T_c}^{T_c+t}F(\tau)d\tau=\int_{T_c}^{0}F(\tau)d\tau+\int_{0}^{T_c+t}F(\tau)d\tau$ and call the second term as $I_N(t)$. We write $I_N(t)=\int_0^{T_c+t}F(\tau)d\tau$ in terms of convolution of two functions, as demonstrated in steps of Eq. (38). With substitution $T_c+t=\overline{t}$ we write as follows

$$I_N(t)=\int_0^{T_c+t}F(\tau)d\tau=\int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=\int_0^{\overline{t}}\frac{d\tau}{\tau^{\alpha}(\overline{t}-\tau)^{1-\alpha}}=\left(\frac{1}{t^{\alpha}}\right)*\left(\frac{1}{t^{1-\alpha}}\right) \quad (38)$$

Now we use Laplace pair $\mathcal{L}\left\{t^m\right\}=\frac{\Gamma(m+1)}{s^{m+1}}$ to write $\mathcal{L}\left\{I_N(t)\right\}=I_N(s)=\left(\mathcal{L}\left\{t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{t^{-(1-\alpha)}\right\}\right)$ as following

$$I_N(s)=\left(\frac{\Gamma(-\alpha+1)}{s^{-\alpha+1}}\right)\left(\frac{\Gamma\left(-(1-\alpha)+1\right)}{s^{(-(1-\alpha)+1)}}\right)=\frac{\Gamma(1-\alpha)\Gamma(\alpha)}{s} \quad (39)$$

Recognizing $\mathcal{L}\left\{u(t)\right\}=s^{-1}$, we write $\mathcal{L}\left\{I_N(s)\right\}=I_N(t)$, and write

$$I_N(t)=\begin{cases}\Gamma(1-\alpha)\Gamma(\alpha) & ;t\geq 0\\ 0 & ;t<0\end{cases} \quad (40)$$

Therefore we have $\int_0^{T_c+t}F(\tau)d\tau=\int_0^{T_c+t}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}=\Gamma(1-\alpha)\Gamma(\alpha)$. Thus we write the expression for open circuit voltage $v_{oc}(t)$ for a charged fractional capacitor that is charged for a long time T_c to voltage V_m and at $t=0$ kept at self-discharge mode, we get the following

$$\begin{aligned}v_{oc}(t)&=V_m-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int_0^{T_c+t}F(\tau)d\tau+\int_{T_c}^{0}F(\tau)d\tau\right]\\&=V_m-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\Gamma(1-\alpha)\Gamma(\alpha)\right)-\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau\\&=\frac{-V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_{T_c}^{0}F(\tau)d\tau=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}F(\tau)d\tau\\&=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}\end{aligned} \quad (41)$$

In Eq. (41) $v_{oc}(t)$ is the voltage over open capacitor at self discharge mode (oc). This $v_{oc}(t)$ function of time depends on the total time T_c the capacitor has been on the voltage source of constant voltage V_m. More the T_c more $q_{CH}(T_c)$ and more time $v_{oc}(t)$ will take to self-discharge, from charged voltage V_m. This formula for self discharge voltage i.e. $v_{oc}(t)=\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ is noted in [6]; here we derived the same by using the concept $q(t)=\left(c(t)*v(t)\right)$.

We mention here the formula for self discharge as described above is only valid for a constant voltage excitation or a step input case. For a triangular voltage impressed at $t=-T_c$ reaching voltage V_m at time T_{cm} described as $\left(V_m/T_{cm}\right)(t+T_c)$ will be having different $v_{oc}(t)$ self-discharge profile, as the charge storage will be in both the cases will be different [39].

The Figure-2 shows self discharge of a super-capacitor when charged with different times, showing memory effect. Here T_c is 4hr, 8hr and 16hr, charged to $V_m=2.2$ (Courtesy: BRNS Funded joint Project CMET Thrissur-BARC Development of CAG Super-capacitors and application in electronics circuits); [41], [42]. The Figure-2 shows that self discharging curves $v_{oc}(t)$ for each T_c is different, indicating memory effect.

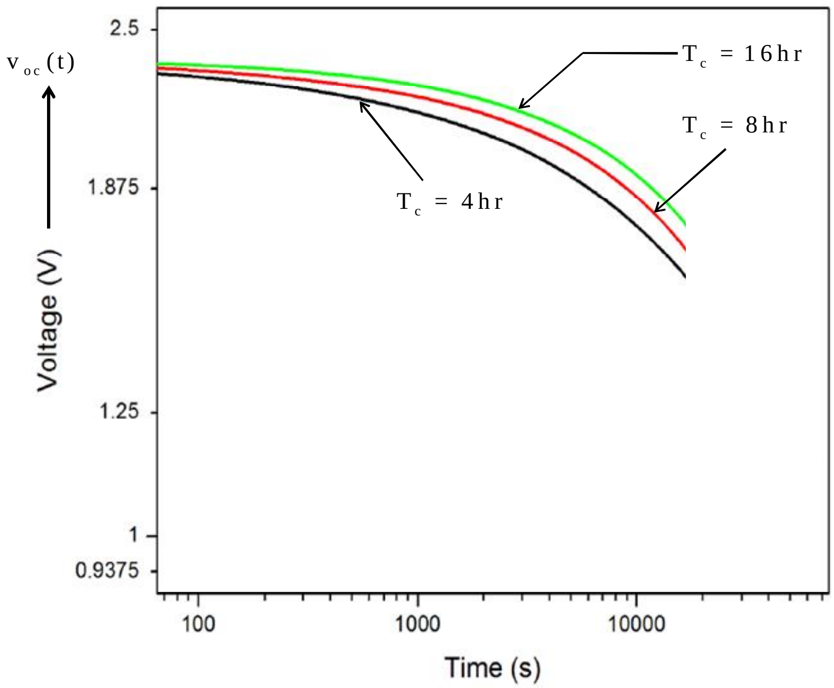

Figure-2: Self-discharge of fractional capacitor, more time we place fractional capacitor on a constant voltage more time it takes decay: Memorizing the charging history.

5. Self discharging of a classical ideal capacitor

We have a constant voltage source applied at $t=-T_c$ for a constant capacitor case with capacity function as $c(t)=C\delta(t)$, [1]. For this case we have the relation Eq. (42) i.e. $i_{CH}(t)=C\delta(t+T_c)\left(v(-T_c)\right)+C\left(v^{(1)}(t)\right)$; that we derive from formula $q_{CH}(t)=c(t)*v(t)$. Compare what we got for a fractional capacitor with $c(t)=C_\alpha t^{-\alpha}$ i.e. $i_{CH}(t)=C_\alpha\frac{v(-T_c)}{(t+T_c)^\alpha}+C_\alpha\int_{-T_c}^{t}\frac{v^{(1)}(x)}{(t-x)^\alpha}dx$, Eq. (28). We follow following steps

$$\begin{aligned}
i_{CH}(t) &= \frac{dq_{CH}(t)}{dt} = \frac{d}{dt}\left(c(t) * v(t)\right)\Big|_{-T_c}^{t}, \quad c(t) = C\,\delta(t) \\
&= \frac{d}{dt}\int_{x=-T_c}^{x=t} C\delta(t-x)v(x)dx = \frac{d}{dt}\left(C\left(v(t)\right)\right), \qquad t \geq -T_c \\
&= v(t)\frac{dC}{dt}\Big|_{t\geq -T_c} + C\frac{dv(t)}{dt}\Big|_{t\geq -T_c} \\
&= \left(v(t)\right)\left(C\left(\delta(t+T_c)\right)\right) + C\frac{dv(t)}{dt} = C\left(v(-T_c)\delta(t+T_c)\right) + C\frac{dv(t)}{dt} \\
&= i(-T_c) + i(t), \qquad t \geq -T_c
\end{aligned} \tag{42}$$

The first term at RHS of above Eq. (42) i.e. $i(-T_c)$ indicate the value of current at $t = -T_c$. The constant function starting at $t = -T_c$ i.e. C when differentiated gives $C\delta(t+T_c)$. This unit delta functions at $t = -T_c$, i.e. $\delta(t+T_c)$ when multiplied by $v(t)$ gives $v(-T_c)\delta(t+T_c)$. This comes from property $\int\left(\delta(x_0 - x)\right)\left(f(x)\right)dx = f(x_0)$, differentiation of this gives $\left(\delta(x_0 - x)f(x)\right) = \frac{d}{dx}f(x_0) = f(x_0)\delta(x)$. Thus at $t = -T_c$ we have $i(-T_c) = Cv(-T_c)$ and $i(-T_c) = 0$ for $t > -T_c$. Compositely we write $i(-T_c) = C_1 v(-T_c)\left(\delta(t+T_c)\right)$, i.e. specifying its value at only $t = -T_c$. The second term is $i(t)$ for $t \neq -T_c$, that is $i(t) = C\left(v^{(1)}(t)\right)$.

The obtained expression $i_{CH}(t) = C\delta(t+T_c)\left(v(-T_c)\right) + C\left(v^{(1)}(t)\right)$ is by the formulation $q(t) = c(t) * v(t)$. As an example, we take $v(t) = V_m u(t+T_c)$ a step input at time $t = -T_c$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > -T_c$; and at $t = -T_c$ we have $v(-T_c) = V_m$. Using this we get $i(-T_c) = CV_m\left(\delta(t+T_c)\right)$; this makes $i_{CH}(t) = CV_m\left(\delta(t+T_c)\right), \quad t \geq -T_c$.

At any time t the coulomb $q_{CH}(t)$ pumped charge into the capacitor plus self-discharged coulombs of charge say $q_{DIS}(t)$ is zero that is $q_{CH}(t) = -q_{DIS}(t)$. Differentiating this we get $i_{CH}(t) + i_{DIS}(t) = 0$ which gives $i_{CH}(t) + i_{DIS}(t) = 0$. That is the following

$$CV_m\left(\delta(t+T_c)\right) + C\frac{dv_{oc}(t)}{dt} = 0 \tag{43}$$

Our interest is finding $v_{oc}(t)$, from $t \geq 0$ self-discharge phase. We do the integration ${}_0I_t^1$ (from time 0 to time t) of the above Eq. (43) and write the following

$$\int_0^t d\tau\left(CV_m\left(\delta(\tau+T_c)\right)\right) + \int_0^t d\tau C\frac{dv_{oc}(\tau)}{d\tau} = 0; \quad t \geq 0 \tag{44}$$

The first integration term is zero since the delta function is outside of the region of integration, thus $C\int_0^t d\tau\left(V_m\left(\delta(\tau+T_c)\right)\right) = 0$. For the second term in Eq. (44) we have $C\int_0^t v_{oc}^{(1)}(\tau)d\tau = C\left[v_{oc}(t)\right]_{t=0}^{t} = C\left(v_{oc}(t) - v(0)\right)$. The value $v(0) = V_m$ that is ideal capacitor is charged to full value of voltage. Using these results we have for ideal classical capacitor $v_{oc}(t) = V_m$, from Eq. (44). This is very true observation. That an ideal loss less classical capacitor, once charged to V_m Volts would retain its charge that is finite and equilibrium value CV_m coulombs; and the terminal voltage $v_{oc}(t)$ will be held constant indefinitely.

Now if a resistance is shunted across the charged capacitor, say R, this voltage $v_{oc}(t) = V_m$ will decay as $v_{DIS}(t) = \left(v_{oc}(t)\right)e^{-t/RC}$ or $v_{DIS}(t) = V_m e^{-t/RC}$, for $t \geq 0$ from the time the resistance was shunted. Similarly for a case of fractional capacitor the self discharge voltage say $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$, Eq. (41) will additionally discharge if the fractional capacitor is shunted by R, and we will record for a fractional capacitor $v_{DIS}(t)$ as following expression

$$v_{DIS}(t) = \left(v_{oc}(t)\right)E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right) = \left(\frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}\right)E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right) \qquad (45)$$

The term $E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right)$ is the discharge decay function of Mittag-Leffler, for a fractional capacitor (that we will derive in subsequent section), is similar to decay function $e^{-t/RC}$ as for the case for a classical loss less capacitor.

6. Self-discharge in fractional capacitor is a misnomer

While we keep the charged fractional capacitor in ideal open circuit condition, (assume ideal infinite open circuit resistance or the ideal case this fractional capacitor having no leakage resistance), then we question why shall the terminal voltage $v_{oc}(t)$ once charged to V_m Volts, decay. We say in ideal case while shunt resistances are infinite there is no discharge current flowing out of fractional capacitor. Yet we observe decay as $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ for different T_c pumping various amounts of charge $q(T_c)$.

A fractional capacitor is like lossy semi-infinite transmission line-that is electrode structure being porous [8], [9], [40]. This infinite transmission line is composed of per unit series resistance r_u and shunt capacitance c_u, giving terminal relation of current and voltage as, [10]

$$i(t) = \sqrt{\frac{c_u}{r_u}}\frac{d^{\alpha}v(t)}{dt^{\alpha}}; \quad \alpha = \tfrac{1}{2}, \quad C_{F-\alpha} = \sqrt{\frac{c_u}{r_u}} \qquad (46)$$

Therefore the fractional capacitor we say is spatially distributed system too, having infinite elements. When we connect a voltage source V_m to this semi infinite transmission line, though the first capacitor (say c_{u-1} gets charged to V_m, yet, the charging current keeps flowing to charge infinite number of c_{u-2}, c_{u-3}…, $c_{u-\infty}$,(charges diffuse spatially). Therefore at time $T_c = \infty$, we have $q_{CH}(\infty) = \infty$, with all the voltages at each distributed capacitors of infinite numbers at V_m.

This system with $q_{CH}(\infty) = \infty$ when kept in open ideal circuit condition will maintain $v_{oc}(t) = V_m$. But see the actual case, we have a limited $T_c < \infty$, but large enough that gives the terminal voltage, say to capacitor c_{u-1} almost $\sim V_m$ with other capacitors c_{u-2}, c_{u-3}….., which are spatially farther away, with lesser terminal voltage as compared to the first capacitor c_{u-1}. While in ideal open circuited condition-this unequally charged semi-infinite transmission line, will have internal spatial charge distribution, to have voltage balancing to equal voltage to all the unit capacitors that are spatially distributed. This gives the notion as if $v_{oc}(t)$ is self-discharging or decaying, though there is no real discharge current flowing out of the fractional capacitor. Since this semi-infinite lossy transmission line has infinite elements, thus this process goes on infinitely for a long time, to have infinite capacitors have infinitesimal small charges and adding up to zero-and while the charge balancing is at play. At open circuited condition the current that flows in all the section will dissipate the stored electrostatic energy. Therefore, a fractional capacitor is a truly lossy capacitor, unlike an ideal loss-less capacitor which holds the stored charge and thus the open circuit voltage) indefinitely. This analysis is assuming that ideal capacitor or fractional capacitor doesn't to have any leakage resistance. Therefore, self-discharging term is misnomer; actually it is voltage redistribution taking place spatially-via diffusion process.

7. Charging/discharging a super-capacitor in RC circuit

A super-capacitor is modeled as Equivalent Series Resistance (ESR) i.e. R_s series with impedance of a Fractional Capacitor of order α i.e. $\frac{1}{s^\alpha C_{F\text{-}\alpha}}$ [15]-[22].

7a. Charging Phase

The differential equation corresponding to Figure-1 for $\alpha = 1$, is ordinary differential equation (ODE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{sC}$ is following

$$RC\frac{dv_0(t)}{dt} + v_0(t) = v_{in}(t) \tag{47}$$

Eq. (3) is solution to this Eq. (47). For $\alpha \neq 1$ we get fractional differential equation (FDE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ is following

$$RC_{F\text{-}\alpha}\frac{d^\alpha v_0(t)}{dt^\alpha} + v_0(t) = v_{in}(t); \quad 0 < \alpha < 1 \tag{48}$$

We now consider a lumped ESR (R_s) for super-capacitor, thus for Figure-1 we have $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}} = \frac{s^\alpha R_s C_{F\text{-}\alpha} + 1}{s^\alpha C_{F\text{-}\alpha}}$ while charging impedance remains at $Z_1(s) = R$. Therefore for any input voltage $V_{in}(s) = \mathcal{L}\{v_{in}(t)\}$, we write the charging current (in Laplace domain) as following considering initial voltage across $C_{F\text{-}\alpha}$ as zero, i.e. $v_c(0) = 0$

$$I_{CH}(s) = \frac{V_{in}(s)}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = \frac{s^\alpha C_{F\text{-}\alpha} V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R + R_s) + 1} \tag{49}$$

Output voltage across $Z_2(s)$ in Laplace domain is therefore is therefore as follows

$$V_0(s) = \left(I_{CH}(s)\right)\left(Z_2(s)\right) = \left(\frac{V_{in}(s)s^{\alpha}C_{F\text{-}\alpha}}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{s^{\alpha}R_sC_{F\text{-}\alpha}+1}{s^{\alpha}C_{F\text{-}\alpha}}\right)$$

$$= \frac{V_{in}(s)+V_{in}s^{\alpha}R_sC_{F\text{-}\alpha}}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1} = \frac{\frac{V_{in}(s)}{C_{F\text{-}\alpha}(R+R_s)}+\frac{V_{in}(s)s^{\alpha}R_s}{(R+R_s)}}{s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}} \quad \text{put} \quad V_{in}(s) = \frac{V_m}{s} \tag{50}$$

$$= \left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) + \left(\frac{V_mR_s}{R+R_s}\right)\left(\frac{s^{\alpha\text{-}1}}{s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\right)$$

To get $v_0(t)$ we do inverse Laplace transform of Eq. (50) as following

$$v_0(t) = \mathcal{L}^{-1}\left\{V_0(s)\right\} = \mathcal{L}^{-1}\left\{\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} + \mathcal{L}^{-1}\left\{\frac{V_mR_ss^{\alpha\text{-}1}}{(R+R_s)\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} \tag{51}$$

Use formula $\mathcal{L}\left\{t^{\alpha p+\beta\text{-}1}E_{\alpha,\beta}^{(p)}(at^{\alpha})\right\} = p!\frac{s^{\alpha\text{-}\beta}}{s^{\alpha}\text{-}a}$. [10], [12], [13] with $p=1$, $\alpha=\alpha$, $\beta=\alpha+1$ and $p=0$, $\alpha=\alpha$, $\beta=1$, to write from Eq. (51) the inverse Laplace as

$$v_0(t) = \frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^{\alpha}E_{\alpha,\alpha+1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) + \frac{V_RR_s}{R+R_s}E_{\alpha,1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{52}$$

Let us keep the step input from time $t=0$ to $t=T_c$, and then at time $t=T_c$, the output voltage is

$$v_0(T_c) = \frac{V_RT_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) + \frac{V_RR_s}{R+R_s}E_{\alpha,1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{53}$$

The charge $q(t)$ will be held only in the element $C_{F\text{-}\alpha}$. We calculate now the voltage profile $v_c(t)$ and then voltage at $t=T_c$, i.e. $v_c(T_c)$ for only fractional impedance part i.e. $\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$ of the impedance $Z_2(s)$ comprising of R_s plus this fractional impedance $\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$. The voltage across $C_{F\text{-}\alpha}$ is thus, with $v_c(0)=0$ no initial voltage at $C_{F\text{-}\alpha}$

$$V_c(s) = I_{CH}\left(\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}\right) = \left(\frac{s^{\alpha}C_{F\text{-}\alpha}V_{in}(s)}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}\right) \quad \text{put} \quad V_{in}(s) = \frac{V_m}{s}$$

$$= \left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) \tag{54}$$

Using the Laplace identity of Mittag-Leffler function $\mathcal{L}\left\{E_n(at^n)\right\} = \frac{s^{n\text{-}1}}{s^n\text{-}a}$, [10], [12], [13] we write

$$v_c(t) = \frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^{\alpha}E_{\alpha,\alpha+1}\left(-\frac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)$$

$$v_c(t) = V_m\left(1-E_{\alpha}\left(-\frac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right), \quad 0 \le t \le T_c \tag{55}$$

At $t=T_c$ we thus have the voltage at the fractional impedance $C_{F\text{-}\alpha}$ as

$$v_c(T_c) = \frac{V_mT_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) = V_m\left(1-E_{\alpha}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{56}$$

The charge $q(t)$ is $q(t) = c(t) * v_c(t)$ with fractional capacitor with capacity function as $c(t) = C_\alpha t^{-\alpha}$ having voltage profile and that is $v_c(t) = V_m\left(1 - E_\alpha\left(-\frac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$ as following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right)\\
&= \left(\mathcal{L}\left\{C_\alpha\, t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right\}\right)\\
&= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{V_m\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s\left(s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right) = \frac{V_m C_{F\text{-}\alpha}\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s^{2\text{-}\alpha}\left(s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}; \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}}, \quad \frac{C_{F\text{-}\alpha}}{\Gamma(1-\alpha)} = C_\alpha\\
&= \left(\frac{V_m}{R+R_s}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha\text{-}1}}{s^\alpha + k} \qquad (57)\\
&= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)\right)\\
&= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right)
\end{aligned}
$$

Taking inverse Laplace transform of Eq. (57) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$q(t) = \int_0^t \frac{V_m}{R+R_s} E_\alpha\left(-\tfrac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau = \frac{V_m t}{R+R_s}\left(E_{\alpha,2}(-t^\alpha/(R+R_s)C_{F\text{-}\alpha})\right) \qquad (58)$$

We used $\int_0^t E_\alpha\left(-\frac{\tau^\alpha}{k}\right)d\tau = t\left(E_{\alpha,2}(-t^\alpha/k)\right)$ in Eq. (58), refer Appendix. Therefore at $t = T_c$ we have charge as

$$q(T_c) = \frac{V_m T_c}{R+R_s} E_{\alpha,2}\left(-\tfrac{T_c{}^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right) \qquad (59)$$

For $Z_2(s) = R_s + \frac{1}{sC}$ i.e. with an ideal capacitor with ESR, we have the following expression

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right), \quad v_c(0) = 0\\
&= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{V_m\left(1 - e^{-\frac{t}{(R+R_s)C}}\right)\right\}\right)\\
&= C\left(\frac{V_m\left(\frac{1}{(R+R_s)C}\right)}{s\left(s + \frac{1}{(R+R_s)C}\right)}\right) = \frac{V_m C\left(\frac{1}{(R+R_s)C}\right)}{s\left(s + \frac{1}{(R+R_s)C}\right)} = V_m C\left(\frac{1}{s} - \frac{1}{s + \frac{1}{(R+R_s)C}}\right) \qquad (60)
\end{aligned}
$$

$$q(t) = CV_m\left(1 - e^{-\frac{t}{(R+R_s)C}}\right)$$

Charge at the end of $t = T_c$ is

$$q(T_c) = CV_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) \qquad (61)$$

The charging current is following from Eq. (60)

$$i_{CH}(t) = \frac{dq(t)}{dt} = \frac{V_m}{(R+R_s)} e^{-\frac{t}{(R+R_s)C}}, \quad 0 \le t \le T_c \qquad (62)$$

The voltage at the end of $t = T_c$ is $v_c(T_c) = V_m(1 - e^{-\frac{T_c}{(R+R_s)C}})$.

7b. Discharging Phase

After $t = T_c$ we make the voltage $v_{in}(t) = 0$ i.e. we are draining out the stored charge i.e. $q(T_c) = CV_m(1 - e^{-T_c/((R+R_s)C)})$ during the discharge phase ($t \ge T_c$); Figure-3. In the discharge phase for ideal loss less capacitor the voltage $v_c(T_c)$ will decay as $v_c(t') = \left(v_c(T_c)\right)e^{-t'/(R+R_s)C}$, for $t \ge T_c$, writing $t' = t - T_c$. At this point the capacity function $c(t') = C\delta(t')$ will again appear, as there is sudden change (differentiability is lost) in voltage from V_m to 0 at $t' = 0$ (i.e. $t = T_c$). Therefore the charge profile while discharging i.e. $q(t')$ we write as $q(t') = c(t') * v_c(t')$ is as follows in Eq. (63) with initial charge as $q(t' = 0) = q(T_c) = Cv_c(T_c)$.

We apply general equation, with changing of $t \equiv t'$ derived Eq. (7), i.e. $q(t') = C(V_m - v_0(0))(1 - e^{-t'/RC}) + Cv_0(0)$, with $R \equiv R + R_s$, $v_0 \equiv v_c$. Here we put $V_m = 0$, that is making $v_{in}(t) = 0$ at $t = T_c; t' = 0$ where we have $t' = t - T_c$. So we have from Eq. (7), the derived expression $q(t') = -Cv_0(0)(1 - e^{-t'/(R+R_s)C}) + Cv_0(0)$, with $v_0(0) = v_c(T_c)$ .we get $q(t') = Cv_c(T_c)e^{-t'/(R+R_s)C}$. We get the same in the following steps Eq. (63), by using $q(t') = c(t') * v_c(t')$ or $\mathcal{L}\{q(t')\} = \mathcal{L}\{c(t') * v_c(t')\}$

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right), \quad t > T_c \\
&= \left(\mathcal{L}\{C\delta(t')\}\right)\left(\mathcal{L}\left\{\left(v_c(T_c)\right)e^{-t'/(R+R_s)C}\right\}\right) \\
&= (C)\left(\frac{\left(v_c(T_c)\right)}{s + \frac{1}{(R+R_s)C}}\right) \\
q(t') &= Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) \\
&= CV_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right)e^{-\frac{t'}{(R+R_s)C}}; \quad t' > 0; \quad t > T_c
\end{aligned}
\tag{63}
$$

At initial time $t' = 0$, we get $q(t') = Cv_c(T_c)$. For $t' \uparrow \infty$ after the ideal capacitor charged to $Cv_c(T_c)$ coulombs, the discharge amount of coulomb from Eq. (63) $\lim_{t' \uparrow \infty} q(t') = \lim_{t' \uparrow \infty} Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}} = 0$. Obvious that all charge is drained out, from ideal loss less capacitor. The discharging current $t \ge T_c$ or $t' \ge 0$ is as follows, by differentiation

$$
\begin{aligned}
i_{DIS}(t') &= \frac{dq(t')}{dt'} = \frac{d}{dt'}\left(C(V_m - v_c(T_c))(1 - e^{-t'/(R+R_s)C}) + Cv_c(T_c)\right), \quad V_m = 0 \\
&= -\frac{v_c(T_c)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}} + Cv_c(T_c)\delta(t') \\
&= i_{DIS}(t')\big|_{t'>0} + i_{DIS}(t')\big|_{t'=0}
\end{aligned}
\tag{64}
$$

$$
i_{DIS}(t')\big|_{t'=0} = Cv_c(T_c)\delta(t') \qquad i_{DIS}(t')\big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}}
$$

In Eq. (64) we have $i_{DIS}(0)$ is the remnant charging current that is given by Eq. (62) i.e. $i_{CH}(T_c) = \frac{V_m}{(R+R_s)} e^{-\frac{T_c}{(R+R_s)C}} = i(0)$. The negative sign in Eq. (64), for $i_{DIS}(t')\big|_{t'>0}$ indicates that the discharge current is opposite to that of charging current. This $i_{DIS}(t')$ current will be flowing through R the discharge resistor, thus discharge voltage across the impedance $Z_2(s) = R_s + \frac{1}{sC}$ is the voltage appearing across $Z_1(s) = R$ is $v_{DIS}(t') = R\left(i_{DIS}(t')\right) = \frac{Rv_c(T_c)}{R+R_s} e^{-t'/(R+R_s)C}$. While we have decay of $v_c(T_c)$ i.e. through $R + R_s$ as $v_c(t') = (R + R_s)i_{DIS}(t') = v_c(T_c)e^{-t'/(R+R_s)C}$; i.e. voltage measured across $R + R_s$.

The Eq. (64) can also be from writing $I_{DIS}(s) = -\frac{1}{R+R_s+(1/sC)}\left(\frac{v_c(T_c)}{s}\right)$ for $t' > 0$, where the initial voltage $v_c(t' = 0) = v_c(T_c)$ appears as step input at $t' = 0$ i.e. $v_c(t') = v_c(T_c)u(t')$, with Laplace transform as $V_c(s) = v_c(T_c)/s$. By inverse Laplace transform we obtain $i_{DIS}(t')\big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)} e^{-\frac{t'}{(R+R_s)C}}$, that is the first term of Eq. (64). Well, look at Eq. (42) which says if applied the voltage at $t' = 0$ we have current as $i(t') = Cv_c(0)\delta(t') + C\frac{dv(t')}{dt'}$ for ideal capacitor; which is $i(t') = i(t')\big|_{t'=0} + i(t')\big|_{t'>0}$. At $t' = 0$, we have $v_c(0) = v_c(T_c)$, that is we are shorting the voltage source, therefore we are in a way applying a $v_c(t') = -v_c(T_c)u(t')$ to a capacitor charged to a voltage $v_c(T_c)$. Thus $i(t')\big|_{t'=0} = Cv_c(T_c)\delta(t')$ .The second term of (42) gives differentiation of voltage as the current, we thus have for a decaying voltage $v_c(t') = v_c(T_c)e^{-t'/(R+R_s)C}$, $i(t')\big|_{t'>0} = C\frac{d}{dt'}v_c(T_c)e^{-t'/(R+R_s)C}$ or $i(t')\big|_{t'>0} = -\frac{v_c(T_c)}{(R+R_s)} e^{-t'/(R+R_s)C}$. The components of (64) are recovered for the case of ideal capacitor. Here we have $\lim_{t'\uparrow\infty} i(t')\big|_{t'>0} = 0^-$.

From Eq. (19) we write $q(t') = \frac{C_\alpha v_0(0)}{(1-\alpha)}(t')^{1-\alpha} + \int_0^{t'} \frac{V_m - v_0(0)}{R+R_s} E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F-\alpha}}\right)d\tau$, by changing $t \equiv t'$, $R \equiv R + R_s$, $C_{F-\alpha} = C_\alpha\Gamma(1-\alpha)$, here we put $V_m = 0$, $v_0(0) \equiv v_c(T_c)$, to write $q(t') = \frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha} - \frac{v_c(T_c)}{R+R_s}\int_0^{t'} E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F-\alpha}}\right)d\tau$, , which we also write as $q(t') = \frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha} - \frac{v_c(T_c)}{R+R_s}t'\left(E_{\alpha,2}\left(-\frac{t'^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)$ is the discharging profile. By using Asymptotic expansion for Mittag-Leffler function we can see that $\lim_{t'\uparrow\infty}\frac{v_c(T_c)}{R+R_s}t'\left(E_{\alpha,2}\left(-\frac{t'^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right) = \lim_{t'\uparrow\infty}\frac{v_c(T_c)C_\alpha(t')^{1-\alpha}}{1-\alpha} = \infty$. This limit is same as first term in $q(t')$, which is $\frac{C_\alpha v_c(T_c)}{(1-\alpha)}(t')^{1-\alpha}$ and the limit is ∞. Thus $\lim_{t'\uparrow\infty} q(t') = 0$.

Now we carry on with the above logic for a fractional capacitor with impedance as $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F-\alpha}}$ .The value $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F-\alpha}}\right)\right)$; Eq. (56) becomes the initial voltage while we discharge the super-capacitor with time defined as $t' = t - T_c$, for discharge phase where $v_{in}(t') = 0$. Now we see the discharge profile, as the charged fractional capacitor $C_{F\text{-}\alpha}$ with above value $v_c(T_c)$ Eq. (56) discharges through R. The discharge current is now for $t'>0$, negative to the charging current is following

$$I_{DIS}(s)\big|_{t'>0} = -\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}} = -\frac{v_c(T_c)s^{\alpha-1}}{(R+R_s)\left(s^{\alpha}+\frac{1}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)}\right)} \quad (65)$$

The inverse Laplace transform of Eq. (65) gives discharge current for $t>T_c$ as following

$$\begin{aligned} i_{DIS}(t')\big|_{t'>0} &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}}\right\} = \mathcal{L}^{-1}\left\{-v_c(T_c)C_{F\text{-}\alpha}\frac{s^{\alpha\text{-}1}}{s^{\alpha}C_{F\text{-}\alpha}(R+R_s)+1}\right\} \\ &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)}{(R+R_s)}\frac{s^{\alpha\text{-}1}}{s^{\alpha}+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\right\} \\ &= -\frac{v_c(T_c)}{R+R_s}E_{\alpha}\left(-\frac{(t')^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right);\quad t>T_c,\quad v_c(T_c)=V_m\left(1-E_{\alpha}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \end{aligned} \quad (66)$$

For $t'\uparrow\infty$, we have $\lim_{t'\uparrow\infty} i_{DIS}(t')\big|_{t'>0}=0^{-}$ .This $i_{DIS}(t')$ for $t'>0$ is real discharge current flowing out of the capacitor, unlike notional discharge current that we used in explaining the self discharge phenomena for $v_{oc}(t)$.

The negative sign in Eq. (66) indicates that discharge current is opposite to that of charging current. This $i_{DIS}(t')$ current will be flowing through R the discharge resistor, thus discharge voltage across the impedance $Z_2(s)=R_s+\frac{1}{s^{\alpha}C_{F\text{-}\alpha}}$ is the voltage appearing across $Z_1(s)=R$ is $v_{DIS}(t')=R\left(i_{DIS}(t')\right)=\frac{Rv_c(T_c)}{R+R_s}E_{\alpha}\left(-\frac{(t')^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)$. While we have decay of $v_c(T_c)$ i.e. through $R+R_s$ as $v_c(t')=(R+R_s)i_{DIS}(t')=v_c(T_c)E_{\alpha}\left(-\frac{(t')^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)$; i.e. measured across $R+R_s$

For $\alpha=1$ we have for ideal loss less capacitor $C_{F\text{-}\alpha}=C$ from Eq. (66)

$$i_{DIS}(t')=\mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{sC}}\right\}=-\frac{v_c(T_c)}{R+R_s}e^{-\frac{t'}{(R+R_s)C}};\quad t>T_c,\quad v_c(T_c)=V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \quad (67)$$

The discharging profile of $q(t')$ with initial charge $q(0)=q(T_c)$ for ideal capacitor is

$$\begin{aligned} \Delta q(t') &= \int_0^{t'} -\frac{v_c(T_c)}{R+R_s}e^{-\frac{\tau}{(R+R_s)C}}d\tau = \left[Cv_c(T_c)e^{-\frac{\tau}{(R+R_s)C}}\right]_{\tau=0}^{\tau=t'};\quad t>T_c \\ &= Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}}-Cv_c(T_c),\quad q(0)=Cv_c(T_c) \\ q(t) &= q(0)+\Delta q(t') = Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}} \end{aligned} \quad (68)$$

Same that we obtained in Eq. (63). Thus we get $q(t')$ for $t\geq T_c$ with $t'=t-T_c$ as following

$$q(t')=Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}};\quad v_c(T_c)=V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right);\quad t\geq T_c \quad (69)$$

The voltage profile across the fractional capacitor, the discharge voltage across $R+R_s$ is $v_c(t')$ while discharge voltage $v_{DIS}(t')$ measured across R is following

$$\begin{aligned} & v_c(t') = v_c(T_c)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \geq T_c, \quad v_c(T_c) = V_m\left(1 - E_\alpha\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \\ & v_{DIS}(t') = \frac{Rv_c(T_c)}{R+R_s}E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \geq T_c; \quad t' = t - T_c \end{aligned} \tag{70}$$

We mention here that Eq. (70) is only having discharge though shunt resistor $R + R_s$ while neglecting the self discharge phenomena. If we consider the self-discharge phenomena of the fractional capacitors, then we have from earlier derivation

$$\begin{aligned} & v_c(t') = V_m\left(\frac{\left(1-E_\alpha\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{\Gamma(1\text{-}\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^\alpha(T_c+t'\text{-}\tau)^{1\text{-}\alpha}}\right)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t' \geq 0 \\ & v_{DIS}(t') = V_m\left(\frac{R\left(1\text{-}E_\alpha\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{(R+R_s)\Gamma(1\text{-}\alpha)\Gamma(\alpha)}\int_0^{T_c}\frac{d\tau}{\tau^\alpha(T_c+t'\text{-}\tau)^{1\text{-}\alpha}}\right)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t' \geq 0 \end{aligned} \tag{71}$$

The self discharge part due to spatial charge diffusion into distributed structure, is a very-very slow process, thus we generally avoid that while calculating the discharge profiles through external shunt resistance. The Eq. (71) is nominal discharge phenomena through resistance are getting modulated by this self-discharge phenomenon.

The charge $q(t')$ profile during the discharge phase is $q(t') = c(t') * v_c(t')$ for $t \geq T_c$ is by utilizing the steps of Eq. (63), we write the following

$$\begin{aligned} q(t') &= c(t') * v_c(t'), \quad \mathcal{L}\{q(t')\} = \mathcal{L}\{c(t') * v_c(t')\} \\ Q(s) &= \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right) \\ &= \left(\mathcal{L}\left\{C_\alpha(t')^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v_c(T_c)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right); \quad C_{F\text{-}\alpha} = C_\alpha\Gamma(1\text{-}\alpha) \\ &= \left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{v_c(T_c)s^{\alpha\text{-}1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\right) = C_{F\text{-}\alpha}v_c(T_c)s^{\alpha-1}\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}} \\ &= C_{F\text{-}\alpha}v_c(T_c)\left(s^{\alpha-1}\mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\}\right) \end{aligned} \tag{72}$$

We used $k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}}$ and $\mathcal{L}\left\{E_\alpha(-kt'^\alpha)\right\} = \frac{s^{\alpha\text{-}1}}{s^\alpha+k}$ in the above steps of Eq. (72). In Eq. (72) placing $\alpha = 1$,and $C_{F\text{-}\alpha} \equiv C$, we write $Q(s) = Cv_c(T_c)\left(\mathcal{L}\left\{e^{\text{-}kt'}\right\}\right)$. Inverse Laplace transform yields $q(t') = Cv_c(T_c)e^{\text{-}kt'}$; where $k = \frac{1}{(R+R_s)C}$. This is same as that of Eq. (63), obtained for ideal loss less capacitor.

Consider the Caputo fractional derivative operator ${}^C D_t^\alpha$. We have the Caputo fractional derivative of Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ as ${}_0^C D_x^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$; [13] (refer Appendix). Using this and relation $\mathcal{L}\left\{{}_0^C D_t^\alpha f(t)\right\} = s^\alpha\left(\mathcal{L}\{f(t)\}\right) \text{-} s^{\alpha\text{-}1}f(0), \quad 0 < \alpha < 1$ i.e. Laplace transform of Caputo Fractional Derivative (refer Appendix) we write the following from Eq. (72)

$$
\begin{aligned}
Q(s) &= C_{F\text{-}\alpha} v_c(T_c)\left(s^{\alpha-1}\mathcal{L}\left\{E_\alpha(-kt'^{\alpha})\right\}\right); \quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}} \\
&= C_{F\text{-}\alpha} v_c(T_c)s^{-1}\left(s^{\alpha}\mathcal{L}\left\{E_\alpha(-kt'^{\alpha})\right\}\right); \quad s^{\alpha}\left(\mathcal{L}\left\{f(t)\right\}\right)=\mathcal{L}\left\{{}_0^C D_t^{\alpha} f(t)\right\}+s^{\alpha\text{-}1}f(0) \\
&= C_{F\text{-}\alpha} v_c(T_c)s^{-1}\left(\mathcal{L}\left\{{}_0^C D_{t'}^{\alpha}\left(E_\alpha(\text{-}kt'^{\alpha})\right)\right\}+s^{\alpha-1}E_\alpha(\text{-}kt'^{\alpha})\Big|_{t'=0}\right); \quad E_\alpha(\text{-}kt'^{\alpha})\Big|_{t'=0}=1 \qquad (73) \\
&= C_{F\text{-}\alpha} v_c(T_c)s^{-1}\mathcal{L}\left\{{}_0^C D_{t'}^{\alpha}\left(E_\alpha(\text{-}kt'^{\alpha})\right)\right\}+C_{F\text{-}\alpha} v_c(T_c)s^{-1}s^{\alpha\text{-}1}, \quad s^{\alpha\text{-}1}=\tfrac{1}{\Gamma(1\text{-}\alpha)}\mathcal{L}\left\{t^{\text{-}\alpha}\right\} \\
&= C_{F\text{-}\alpha} v_c(T_c)s^{-1}\left(\mathcal{L}\left\{\text{-}kE_\alpha(\text{-}kt'^{\alpha})\right\}\right)+\frac{C_{F\text{-}\alpha} v_c(T_c)}{\Gamma(1\text{-}\alpha)}s^{-1}\mathcal{L}\left\{t'^{\text{-}\alpha}\right\}
\end{aligned}
$$

We justify the use of ${}_0^C D_t^{\alpha} f(t)$ the Caputo derivative operator on function $f(t)$, in Eq. (73). That is because it is easy to be using Caputo derivative, rather using Riemann-Liouville (RL)L fractional derivative, where initial states are of fractional order which presently hard to realize [10], [12]. The point that Caputo derivative works for a differentiable function $f(t)$, and $f(t)=E_\alpha(-kt^{\alpha})$ is differentiable for $t>0$.

Recognizing in Eq. (73) $s^{-1}\mathcal{L}\left\{f(t)\right\}=\int_0^t f(\tau)d\tau$ and using inverse Laplace Transform of Eq. (73) we have

$$
\begin{aligned}
q(t') &= \left(C_{F\text{-}\alpha} v_c(T_c)\int_0^{t'} -kE_\alpha(-k\tau^{\alpha})d\tau\right)+\left(\frac{C_{F\text{-}\alpha} v_c(T_c)}{\Gamma(1\text{-}\alpha)}\right)\int_0^{t'}\tau^{\text{-}\alpha}d\tau; \quad k=\tfrac{1}{(R+R_s)C_{F\text{-}\alpha}} \\
&= -\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'}E_\alpha(-k\tau^{\alpha})d\tau+\left(\frac{C_{F\text{-}\alpha} v_c(T_c)}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)}\right)(t')^{1\text{-}\alpha}; \quad t\ge T_c \qquad (74) \\
&= \int_0^{t'} i_{DIS}(\tau)d\tau+\left(\frac{C_{F\text{-}\alpha} v_c(T_c)}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)}\right)(t')^{1\text{-}\alpha}; \quad i_{DIS}(t')\Big|_{t'>0}=\text{-}\frac{v_c(T_c)}{(R+R_s)}E_\alpha(\text{-}kt'^{\alpha})
\end{aligned}
$$

The same result of Eq. (74) we will get by applying Eq. (18) and Eq. (19) with $t\equiv t'$, $R\equiv R+R_s$, $v_0(0)\equiv v_c(T_c)$, setting $V_m=0$, and using $C_{F\text{-}\alpha}=C_\alpha\Gamma(1\text{-}\alpha)$.

Where we have $q(0)=q(T_c)=\frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)$ and $v_c(T_c)=V_m\left(1-E_\alpha\left(-\frac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$.

Now while differentiation Eq. (74) we get $i(t')=\frac{dq(t')}{dt'}=i_{DIS}(t')\Big|_{t'>0}+\frac{C_{F\text{-}\alpha} v_c(T_c)}{\Gamma(1\text{-}\alpha)}(t')^{\text{-}\alpha}$, with $i_{DIS}(t')\Big|_{t'>0}=\text{-}\frac{v_c(T_c)}{(R+R_s)}E_\alpha(\text{-}kt'^{\alpha})$.

From Eq. (27) we have current in a fractional capacitor as $i(t')=\frac{C_{F\text{-}\alpha} v_c(0)}{\Gamma(1\text{-}\alpha)}(t')^{\text{-}\alpha}+C_{F\text{-}\alpha}\left({}_0^C D_{t'}^{\alpha}\left[v_c(t')\right]\right)$, when a voltage $v_c(t')$ is applied at $t'=0$. With $v_c(t')=v_c(T_c)E_\alpha\left(-k(t')^{\alpha}\right)$, and ${}_0^C D_{t'}^{\alpha}\left[E_\alpha\left(-k(t')^{\alpha}\right)\right]=-kE_\alpha\left(-k(t')^{\alpha}\right)$ and also $v_c(0)=v_c(T_c)$, we write $i(t')=\frac{C_{F\text{-}\alpha} v_c(T_c)}{\Gamma(1\text{-}\alpha)}(t')^{\text{-}\alpha}\text{ - }\frac{v_c(T_c)}{R+R_s}\left(E_\alpha(\text{-}kt'^{\alpha}\right)$ for, same that we got by differentiating Eq. (74).

We use $\int_0^t E_\alpha\left(\text{-}k\tau^{\alpha}\right)d\tau=t\left(E_{\alpha,2}(-kt^{\alpha})\right)$ (Refer Appendix) and write the following

$$q(t') = \left(\frac{C_{F\text{-}\alpha} v_c(T_c)}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)}\right)(t')^{1\text{-}\alpha} + \left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'} E_\alpha\left(-\tfrac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right) + ; \quad t \ge T_c$$
$$= \left(\frac{C_{F\text{-}\alpha} v_c(T_c)}{(1\text{-}\alpha)\Gamma(1\text{-}\alpha)}\right)(t')^{1\text{-}\alpha} - \frac{v_c(T_c)}{(R+R_s)}\left[t'\left(E_{\alpha,2}\left(-\tfrac{t'^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right] \quad (75)$$

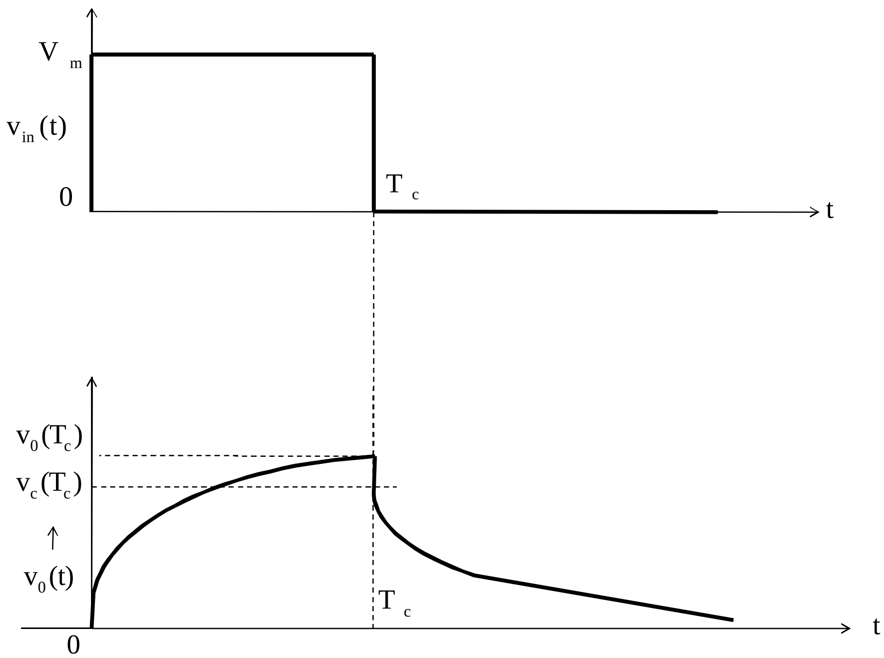

Figure-3: Constant voltage charging and discharging voltage profile at super-capacitor

The Figure-3 displays the curve of voltage profile for a constant voltage charge and discharge case. Here we point out that the charging curve though similar to exponential charging of a text book capacitor $v_0(t) \propto (1 - e^{-t/RC})$, but it is not so, for fractional capacitor that is described via Mittag-Leffler function. Similarly the discharge profile though similar to exponential decay $v_0(t) \propto e^{-t/RC}$, but is not so for fractional capacitor; here too described by Mittag-Leffler function. All the relations we obtained and also verified our formula $q(t) = c(t) * v(t)$.

8. Conclusions

This formula $q(t) = c(t) * v(t)$ is a new development. In this deliberation we have applied this new formula of charge storage i.e. via convolution operation $q(t) = c(t) * v(t)$, of time varying capacity function and voltage stress for a fractional capacitor and ideal loss-less capacitor; for verification in RC charging/discharging circuit; with dc voltage excitation. We have stressed on detailed derivations, to have transparency in each and every steps. We have also shown the effect of memory in self-discharging cases for a fractional capacitor, by this formula. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. The circuit analysis that we described for each cases verifies this formula. Thus this new formulation of stored charge via convolution operation is applicable, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor.

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APPENDIX

A. Preliminaries of fractional calculus

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration of order $\upsilon \in \mathbb{R}^{+}$ is defined as

$$ {}_0I_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)d\tau \qquad \text{A1}$$

Where $\Gamma(\upsilon)$ is Euler's Gamma function, is generalization of factorial function we have $\Gamma(\upsilon) = (\upsilon-1)!$. The formula Eq. (A1) is ${}_0I_t^{\upsilon}\left[f(t)\right] = \left(\frac{t^{\upsilon-1}}{\Gamma(\upsilon)}\right) * f(t)$ is convolution operation, with power-law memory kernel. This is $k_{\upsilon}(t) = \frac{t^{\upsilon-1}}{\Gamma(\upsilon)}$ and is singular function for case $0 < \upsilon < 1$. We have $\lim_{\upsilon\to 0} k_{\upsilon}(t) = \lim_{\upsilon\to 0}\frac{t^{\upsilon-1}}{\Gamma(\upsilon)} = \delta(t)$, which gives ${}_0I_t^{0}\left[f(t)\right] = f(t)$ .The formula Eq. (A1) is appearing as generalization of Cauchy's multiple integration formula of m fold integration where $m \in \mathbb{N}$ given as follows

$$ {}_0I_t^{m}\left[f(t)\right] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)d\tau; \quad m = 1,2,3,.... \qquad \text{A2}$$

The fractional derivative of order β for $0 < \beta < 1$ by Riemann-Liouville (RL) formula is

$$ {}_0D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)d\tau; \quad 0 < \beta < 1 \qquad \text{A3}$$

The Eq. (A3) is fractionally integrating the function by order $(1-\beta)$ by formula Eq. (A1) and then followed by one-whole differentiation. We note that Eq. (A7) is also having convolution operation and with singular kernel as $k_{\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$ .We have thus $\lim_{\beta\to 1} k_{\beta}(t) = \lim_{\beta\to 1}\frac{t^{-\beta}}{\Gamma(1-\beta)} = \delta(t)$ and $\lim_{\beta\to 1}\left({}_0D_t^{\beta}\left[f(t)\right]\right) = \frac{d\ f(t)}{dt}$.

There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative for fractional order $0 < \beta < 1$ is given as

$$ {}_0^{C}D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta} f^{(1)}(\tau)d\tau; \quad 0 < \beta < 1 \qquad \text{A4}$$

Thus for Eq. (A4) we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order $1-\beta$, by formula Eq. (A1). The formula Eq. (A4) also employs singular kernel as $k_{\beta}(t) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, and we have $\lim_{\beta\to 1}\left({}_0^{C}D_t^{\beta}\left[f(t)\right]\right) = f^{(1)}(t)$. The Caputo and Riemann-Liouville (RL) fractional derivative are related by

$$ {}_0D_t^{\beta}\left[f(t)\right] = {}_0^{C}D_t^{\beta}\left[f(t)\right] + \frac{f(0)}{\Gamma(1-\beta)}t^{-\beta}; \quad 0 < \beta < 1 \qquad \text{A5}$$

We write (A5) as following, with non-zero as start point of fractional differentiation process

$$
\begin{aligned}
{}_aD_t^{\beta}\left[f(t)\right] &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_a^t \frac{f(x)}{(t-x)^{\beta}}dx, \quad 0<\beta<1 \\
&= \frac{1}{\Gamma(1-\beta)}\left(\frac{f(a)}{(t-a)^{\beta}} + \int_a^t \frac{f^{(1)}(x)}{(t-x)^{\beta}}dx\right); \quad t>a \\
&= \frac{f(a)}{(t-a)^{\beta}\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f^{(1)}(x)}{(t-x)^{\beta}}dx \\
&= \frac{f(a)}{\Gamma(1-\beta)}(t-a)^{-\beta} + {}_a^CD_t^{\beta}\left[f(t)\right]
\end{aligned}
\qquad A6
$$

We mention that both the fractional derivatives are equal when initial value is zero i.e. $f(0)=0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^{\beta}\left[K\right] = \frac{K}{\Gamma(1-\beta)}t^{-\beta}$. Whereas the Caputo's fractional derivative of a constant is zero, i.e. ${}_0^CD_t^{\beta}\left[K\right]=0$.

The fractional integration and fractional differentiation of delta function is as follows

$$
{}_0I_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}; \quad {}_0D_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(-\upsilon)}t^{-\upsilon-1}, \qquad 0<\upsilon<1 \qquad A7
$$

Fractional derivative and fractional integration of power function $f(t)=Kt^{p}$ is

$$
{}_0I_t^{\upsilon}Kt^{p} = K\frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}t^{p+\upsilon}, \qquad {}_0D_t^{\upsilon}Kt^{p} = K\frac{\Gamma(p+1)}{\Gamma(p-\upsilon+1)}t^{p-\upsilon}, \quad p>-1 \qquad A8
$$

The Laplace transform of fractional integral operation is

$$
\mathcal{L}\left\{{}_0I_t^{\upsilon}f(t)\right\} = s^{-\upsilon}F(s) \qquad A9
$$

Laplace transform of Caputo fractional derivative for fractional order $0<\upsilon<1$ is

$$
\mathcal{L}\left\{{}_0^CD_t^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - s^{\upsilon-1}f(0) \qquad A10
$$

Laplace transform of Riemann-Liouville fractional derivative of order $0<\upsilon<1$ is

$$
\mathcal{L}\left\{{}_0D_t^{\upsilon}f(t)\right\} = s^{\upsilon}F(s) - f^{(\upsilon-1)}(0) \qquad A11
$$

In (A11) $f^{(\upsilon-1)}(0) = \lim_{t\to 0}\left({}_0I_t^{1-\upsilon}f(t)\right)$; that initial states required in (A11) for RL fractional derivative is of fractional order, types $f^{(\upsilon-1)}(0)$ whereas initial states required (A10) for Caputo type fractional derivative is integer order (classical) type $f(0)$.

B. Mittag-Leffler Function

Like in classical calculus, we have exponential function e^{z}; similarly, in fractional calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(z)^{n}}{\Gamma(\alpha n+\beta)}, \quad z\in\mathbb{C}; \quad \mathrm{Re}\left[\alpha,\beta\right]>0 \qquad B1
$$

For $\beta=1$ we have $E_{\alpha,1}(z)=E_{\alpha}(z)$; is called One-Parameter Mittag-Leffler function. The Laplace transformation of Mittag-Leffler function is following

$$
\mathcal{L}\left\{E_{\alpha}(\lambda t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \qquad B2
$$

We observe that for $E_\alpha(-bt^\alpha)\big|_{\alpha=1} = e^{-bt}$, and $E_\alpha(-at^\alpha)\big|_{\alpha=2} = \cos\sqrt{a}t$.

We point here that $f(t) = E_\alpha(\lambda t^\alpha)$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^\alpha f(t) = \lambda f(t)$; and $f(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ is eigen-function for fractional differential equation with RL fractional derivative i.e. ${}_0D_t^\alpha f(t) = \lambda f(t)$.

Recurring property of $E_{\alpha,\beta}(x)$ is

$$E_{\alpha,\beta}(x) = \frac{1}{x}E_{\alpha,\beta-\alpha}(x) - \frac{1}{x\Gamma(\beta-\alpha)} \qquad \text{B3}$$

For one parameter Mittag-Leffler function

$$E_\alpha(x) = E_{\alpha,1}(x) = \frac{1}{x}E_{\alpha,1-\alpha}(x) - \frac{1}{x\Gamma(1-\alpha)} \qquad \text{B4}$$

We use (B3) and write following steps

$$\begin{aligned} E_{\alpha,\beta}(x) &= -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}E_{\alpha,\beta-\alpha}(x) = -\frac{1}{x\Gamma(\beta-\alpha)} + \frac{1}{x}\left(-\frac{1}{x\Gamma(\beta-2\alpha)} + \frac{1}{x}E_{\alpha,\beta-2\alpha}(x)\right) \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} + \frac{1}{x^2}E_{\alpha,\beta-2\alpha}(x) \\ &= -\frac{1}{x\Gamma(\beta-\alpha)} - \frac{1}{x^2\Gamma(\beta-2\alpha)} - \frac{1}{x^3\Gamma(\beta-3\alpha)} + \frac{1}{x^3}E_{\alpha,\beta-3\alpha}(x) \end{aligned} \qquad \text{B5}$$

From (B5) we get Poincare asymptotic expansion of $E_{\alpha,\beta}(x)$ as

$$E_{\alpha,\beta}(x) \sim -\sum_{n=1}^{\infty}\frac{1}{x^n\Gamma(\beta-n\alpha)} \qquad \text{B6}$$

valid for $x \to -\infty$.

C. Proof of formula $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ used

We verify the formula used $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ as in following steps

$$\begin{aligned} \int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau &= \int_0^t\left(1 - \frac{k\tau^\alpha}{\Gamma(\alpha+1)} + \frac{k^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3\tau^{3\alpha}}{\Gamma(3\alpha+1)} + \dots\right)d\tau \\ &= t - \frac{kt^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)} + \frac{k^2t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)} - \frac{k^3t^{3\alpha+1}}{(3\alpha+1)\Gamma(3\alpha+1)} + \dots \\ &= t\left(1 - \frac{kt^\alpha}{\Gamma(\alpha+2)} + \frac{k^2t^{2\alpha}}{\Gamma(2\alpha+2)} - \frac{k^3t^{3\alpha}}{\Gamma(3\alpha+2)} + \dots\right), \quad \Gamma(m+1) = m\Gamma(m) \\ &= t\left(E_{\alpha,2}(-kt^\alpha)\right) \quad ; \quad E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)} \end{aligned} \qquad \text{C1}$$

****