Differential equations with derivatives with orders that is a continuous function and its solutions

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Abstract

The fractional order derivatives are generalization of classical derivatives as we all know. A further generalization may be made as per having a derivative order that is distributed in some functional form between two extreme orders. We call that as continuously distributed order. The differential equation thus formed with continuously distributed order we solve by Laplace transformation technique. We use Laplace transformation techniques by contour integration and residue calculus, and the other by performing direct integration on Bromwich path without going for contour integration. We give various characteristic functions for some continuously distributed orders, and we will show how we can obtain this characteristic equation via use or *r*-Laplace transforms. We thus give time domain and frequency domain responses for the differential equations formed with continuously distributed order.

Keywords

Residue Calculus, Bromwich Integration, Berberan-Santo method, *r*-Laplace transform, Continuously Distributed Order, Characteristic Function

Introduction

As the concept of order (of derivative, integration) is central to the understanding of fractional (or integer) order systems, some discussion of this concept now follows. In this discussion, single-input-output systems are considered. Recalling the characteristic equations or transfer function definition we call a system first order, second order third order etc. similarly the system can be of fractional order too i.e. the characteristic equation having powers of *s*-variable of non integers numbers (real numbers at present). We also consider that system representation is generally of 'minimal phase' and they are linear. For non-minimal phase the system behavior is for a positive step demand the output initially goes in reverse and then changes direction to follow the demand. This is peculiar of gas turbines where for non-minimal phase system the velocity first will marginally decrease and then eventually increase to follow increment in the demanded velocity.
Mathematical order conventionally is defined as highest derivative occurring in a given differential equation. The concept of mathematical order is applicable to both ordinary and fractional differential equations. Normally, when the order is used without qualifier, it implies the meaning of mathematical order.
For linear dynamic systems that are described by ordinary differential equations the system mathematical order implies or is equivalent to the following:
(1). The highest derivative in ordinary differential equation. (2). The highest power of Laplace variable-*s*, in the characteristic equation-or indicial polynomial (3). The number of initializing constants required for the differential equation. (4). The length of the state vector. (5) The number

of singularities in the characteristic equation. (6). The number of energy storage elements. (7). The number of independent spatial directions in which a trajectory can move. (8). The number of devices that can add 90^0 sinusoidal steady state phase lag and (9). The number of devises that retain some memory of the past.

The utility of the definition of mathematical order is that infers all the system characteristics for system, with integer order components. Thus the benefit of having a definition for order for linear ordinary differential equations is that it allows a direct understanding of the behavior of given dynamic system. Unfortunately, for fractional differential equations, the order of the highest derivative does not infer all of the previously mentioned properties. Indeed, the most important characteristics of order in integer order ordinary differential equation are probably item (3) i.e. it indicates number of initializing constants, which together with the differential equations allow prediction of the future behavior. In system terminology this information provides initial 'states', of the system being analyzed. Clearly the order of highest derivative in a fractional differential equations does not have this property nor does it predict the associated number of energy/memory elements associated with fractional differential equation, nor does it infer the number of integrations (even fractional), required to solve simulate the given fractional differential equation. Thus the issue of order and the information required together with the fractional differential equation to predict the future is fundamental and should be treated differently. This is explained as seeming looking first order characteristic polynomial with fractional order components may go into resonance-due to presence of half-order elements.

We generalize the concept of derivative say from classical integer order derivative

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t)$$

to fractional derivative system say

$$\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q} = f(t); \quad q \in \mathbb{R}$$

the q is fractional order. Now if we say the fractional order is defined as a continuous function, call it $k(q)$ between the numbers $a \le q \le b$ that is one more step in generalization. Well we will form a transfer function of above system in Laplace domain, and then we see how we can use techniques of inverse Laplace transform to get solution to the differential equation formed by continuously distributed order differential equation.

Concept of continuous-distribution of order in a system of differential equation

A very basic of mass spring damper system of force balance is taken here to study the concept of continuous order distribution. The familiar system is represented (with un-initialized derivative) as:

$$m\left(\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}\right) + b\left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right) + k\left(x(t)\right) = f(t)$$

where $x(t)$ is position of the mass m, $f(t)$ is the forcing function on the mass, b is the damping and k is the spring (restoring) force. In the Laplace domain this takes following form (assuming initial conditions $x(0) = 0$ and $x^{(1)}(t)\big|_{t=0} = 0$ are at rest).

$$(ms^2 + bs + k)X(s) = F(s)$$

Where $\mathcal{L}\left\{x(t)\right\} = X(s)$, $\mathcal{L}\left\{f(t)\right\} = F(s)$.

It is well known that some element intermediate between spring and dashpot behaves and balances the forces called viscoelastic element-described via fractional derivative. Such element is described as:

$$k_q\left(\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q}\right)=f(t) \qquad 0\le q\le 1$$

The Laplace representation of above is:

$$k_q s^q X(s)=F(s), \quad \mathcal{L}\{x(t)\}=X(s); \quad \mathcal{L}\{f(t)\}=F(s)$$

In above fractional derivative of Caputo type is considered with condition $x(0)=0$, and thus used is the relation

$$\mathcal{L}\left\{\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q}\right\}=s^q X(s)-x(0) \qquad 0\le q\le 1$$

Adding this viscoelastic element to the original assumed (lumped) system i.e. $(ms^2+bs+k)X(s)=F(s)$ we get:

$$\left(ms^2+bs+k_q s^q+k\right)X(s)=F(s)$$

It is known that viscoelastic elements will posses any order q between 0 and 1, so we can add another viscoelastic element and then keep on adding several others too, like the following:

$$\left(ms^2+bs+k_{q2}s^{q2}+k_{q1}s^{q1}+k\right)X(s)=F(s)$$

$$\left(ms^2+k_{q4}s^{q4}+k_{q3}s^{q3}+bs+k_{q2}s^{q2}+k_{q1}s^{q1}+k\right)X(s)=F(s)$$

This process could be continued so that the system can therefore be expressed as power series of s^{q_n} with $0\le q_n\le 2$, with N as integer as following:

$$\left(\sum_{n=0}^{N}k_n s^{q_n}\right)X(s)=F(s); \quad 0\le q_n\le 2$$

In the limit of infinitesimally small elements the above $\left(\sum_{n=0}^{N}k_n s^{q_n}\right)X(s)=F(s)$ will tend to continuum, and the summation be replaced then by integral. This gives fundamental motivating procedure for concept with continuous order distribution. This system with continuous order distribution is expressed as following

$$\left(\int_0^2 k(q)s^q \mathrm{d}q\right)X(s)=F(s)$$

This is very general representation of a dynamic system of any type taken for system identification studies.

From continuously distributed order distribution we recover classical differential equations of integer and fractional orders and fractional orders

For demonstration the familiar integer order dynamic spring damper mass element equation $(ms^2+bs+k)X(s)=F(s)$ can be re-written with form expressed in $\left(\int_0^2 k(q)s^q\mathrm{d}q\right)X(s)=F(s)$ as:

$$\left(\int_0^\infty\left(m\delta(q-2)+b\delta(q-1)+k\delta(q)\right)s^q\mathrm{d}q\right)X(s)=F(s)$$

$$k(q)=m\delta(q-2)+b\delta(q-1)+k\delta(q)$$

Figure-3 show the plot of $k(q)$ and q, for the classical mass –spring –damper, i.e. represented as $(ms^2+bs+k)X(s)=F(s)$ and also by above equation order distribution. Here the order is discrete, Dirac-delta functions at 0, 1, 2. With weights as k , b and m respectively.
The Figure-3b has few fractional order elements and its order distribution function is

$$k(q)=m\delta(q-2)+b_1\delta(q-\tfrac{3}{2})+b\delta(q-1)+b_0\delta(q-\tfrac{1}{2})+k\delta(q)$$

and corresponding fractional differential equation is

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+b_1\frac{\mathrm{d}^{3/2}x(t)}{\mathrm{d}t^{3/2}}+b\frac{\mathrm{d}x(t)}{\mathrm{d}t}+b_0\frac{\mathrm{d}^{1/2}x(t)}{\mathrm{d}t^{1/2}}+kx(t)=f(t)$$

The Figure-3c gives the limiting case in case the distributions are in continuum limit. This is called continuously distributed order. In Figure-1c $k(q)$ is distribution function, which is a continuous function in a particular interval of order $0\le q\le q_{\max}$, elsewhere $k(q)=0$. For convenience we have shown $k(q)$ as positive, in general it can be positive or negative.

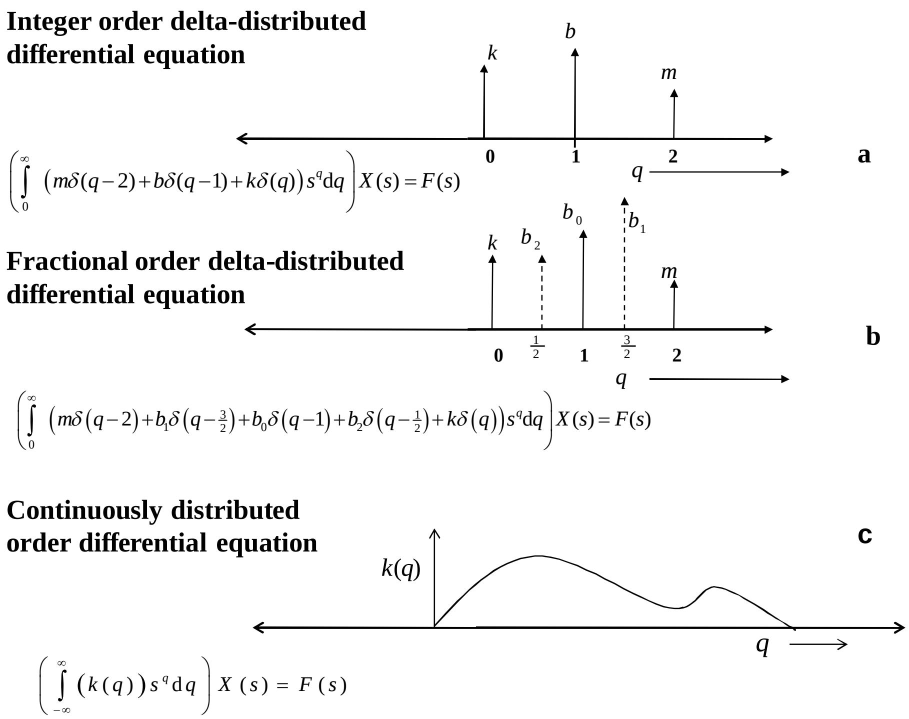


Figure-3: Order distribution of mass-spring- damper integer order system.

Further generalization of continuously distributed order differential equation system

Allowed the restriction on the maximum possible order (i.e. second order in the present case) the equation $\left(\int_0^2 k(q)s^q\mathrm{d}q\right)X(s)=F(s)$ can be still generalized as continuum power series as following.

$$\left(\int_0^{\infty} k(q)s^q \mathrm{d}q\right) X(s) = F(s)$$

The time domain representation of the above is

$$\left(\int_0^{\infty} k(q)\left(s^q \left(X(s)\right)\right)\mathrm{d}q\right) = F(s)$$

$$\int_0^{\infty} k(q)\left(\frac{\mathrm{d}^q x(t)}{\mathrm{d}t}\right)\mathrm{d}q = f(t)$$

Mathematically system can be also described by continuum asymptotic series instead of power series as:

$$\left(\int_0^{\infty} k(q)s^{-q} \mathrm{d}q\right) X(s) = F(s)$$

Combining $\left(\int_0^{\infty} k(q)s^q \mathrm{d}q\right) X(s) = F(s)$ and $\left(\int_0^{\infty} k(q)s^{-q} \mathrm{d}q\right) X(s) = F(s)$ we obtain a very general system descriptive method as:

$$\left(\int_{-\infty}^{+\infty} k(q)s^q \mathrm{d}q\right) X(s) = F(s)$$

The time domain symbolic representation of above is

$$\int_{-\infty}^{+\infty} k(q)\left(\frac{\mathrm{d}^q x(t)}{\mathrm{d}t}\right)\mathrm{d}q = f(t)$$

Transfer function of continuous distributed order by *r*-Laplace transform

By rewriting the integral in system equation i.e. $\int_0^{\infty} k(q)s^q \mathrm{d}q$, with the exponent $s^q = \mathrm{e}^{q\ln(s)}$, we obtain:

$$\left(\int_0^{\infty} k(q)\mathrm{e}^{q\ln(s)} \mathrm{d}q\right) X(s) = P(s)X(s) = F(s)$$

The expression $\int_0^{\infty}\left(\mathrm{e}^{q\ln s}k(q)\right)\mathrm{d}q, \quad \ln s = -r, \quad \int_0^{\infty}\left(\mathrm{e}^{-qr}k(q)\right)\mathrm{d}q$ gives, is effectively a Laplace transform (Laplace integral) of the function $k(q)$ with the new Laplace variable r i.e. $r = -\ln(s)$. We call $\mathcal{L}_r\left\{k(q)\right\} = K(r)$, that is transforming vial Laplace integral, the function $k(q)$ in q domain to r domain. As long as the order distribution $k(q)$ is of exponential order then the resulting $P(s)$, is easy to evaluate using this *r*-Laplace transform.

Table 1 presents the transfer function $P(s)$, for various systems with continuous order distribution.

The order distribution $k(q)$ is taken for all $q \geq 0$. The system descriptions with the characteristic equations are expressed in differential equations with differentiation order greater than zero. So the integration terms are also converted to differentiation and the characteristic equations are in polynomial of powers of s^q with $q \geq 0$. In the following derivations thus, the q is always taken as greater than zero, and $k(q) = 0$ for all $q < 0$.

To evaluate r-Laplace transform from given order distribution (continuous spiked or mixed), the Laplace identities are used. So $K(r)=\int_0^\infty k(q)\mathrm{e}^{-rq}\mathrm{d}q$, obtained is r-Laplace transform, and then in the obtained expression of $K(r)$ substitution for $r=-\ln(s)$ is carried out to get $P(s)$. Meaning obtain r-Laplace as

$$\mathcal{L}_r\{k(q)\}=K(r)$$

$$\mathcal{L}_r\{k(q)\}=\int_0^\infty k(q)\mathrm{e}^{-rq}\mathrm{d}q=K(r)$$

and then get $P(s)$.

$$K(r)\big|_{r=-\ln s}=P(s)$$

Following examples demonstrates derivation of r-Laplace $K(r)$ and then Laplace transform $P(s)$ of continuous order, spiked ordered and mixed order distributions $k(q)$

Order distribution as exponential distribution its Transfer Function

For exponential order distribution $k(q)=\mathrm{e}^{-q}$ for all $q\geq 0$. r-Laplace transform is following

$$K(r)=\int_0^\infty \mathrm{e}^{-q}\mathrm{e}^{-rq}\mathrm{d}q$$

$$=\int_0^\infty \mathrm{e}^{-q(r+1)}\mathrm{d}q=\frac{1}{r+1}$$

$$\mathcal{L}_r\left\{e^{-q}\right\}=\frac{1}{1+r}$$

in this expression putting $r=-\ln(s)$, gives

$$P(s)=\frac{1}{1-\ln(s)}$$

Order distribution as hyperbolic function and its Transfer Function

For $k(q)=(q+a)^{-1}$ for all $q\geq 0$,

$$K(r)=\int_0^\infty \frac{1}{q+a}\mathrm{e}^{-qr}\mathrm{d}q\,.$$

This integral we solve by using definition of Exponential Integral as: $Ei(x)\overset{\text{def}}{=}-\int_{-x}^{\infty}\frac{\mathrm{e}^{-y}}{y}\mathrm{d}y$

Put $u=q+a$, then

$$K(r)=\int_a^\infty \frac{1}{u}\mathrm{e}^{-r(u-a)}\mathrm{d}u$$

$$=\mathrm{e}^{ar}\int_a^\infty \frac{1}{u}\mathrm{e}^{-ru}\mathrm{d}u$$

In this take $ru=v$, then

$$K(r)=\mathrm{e}^{ar}\int_{ar}^{\infty}\frac{r}{v}\mathrm{e}^{-v}\frac{\mathrm{d}v}{r}$$
$$=\mathrm{e}^{ar}\int_{-(-ar)}^{\infty}\frac{1}{v}\mathrm{e}^{-v}\mathrm{d}v$$
$$=-\mathrm{e}^{ar}\left\{-\int_{-(-ar)}^{\infty}\frac{\mathrm{e}^{-v}}{v}\mathrm{d}v\right\}$$
$$=-\mathrm{e}^{ar}Ei(-ar)$$

Therefore we have

$$\mathcal{L}_r\left\{\tfrac{1}{q+a}\right\}=-\mathrm{e}^{ar}Ei(-ar);\quad Ei(x)\stackrel{\text{def}}{=}-\int_{-x}^{\infty}\tfrac{\mathrm{e}^{-y}}{y}\mathrm{d}y$$

Substitute $r=-\ln(s)$ to get

$$P(s)=-\mathrm{e}^{a[-\ln(s)]}Ei\left(-a\left(-\ln(s)\right)\right)$$
$$=-\frac{1}{s^a}Ei\left[\ln(s^a)\right];\qquad Ei(x)\stackrel{\text{def}}{=}-\int_{-x}^{\infty}\tfrac{\mathrm{e}^{-y}}{y}\mathrm{d}y$$

Order distribution as series of alternating delta functions and Transfer function

For a train of spikes alternating at $q=1,2,3,4\ldots$, we have

$$k(q)=\delta(q-1)-\delta(q-2)+\delta(q-3)-\delta(q-4)+\ldots$$
$$=\sum_{n=1}^{\infty}(-1)^{n+1}\left(\delta(q-n)\right)$$

By using the shifted Dirac delta and its transform as $\delta(t-t_0)\leftrightarrow e^{-st_0}$, we get:

$$K(r)=\mathrm{e}^{-r}-\mathrm{e}^{-2r}+\mathrm{e}^{-3r}-\mathrm{e}^{-4r}+\ldots$$
$$=(\mathrm{e}^{-r}+\mathrm{e}^{-3r}+\mathrm{e}^{-5r}+\ldots)-(\mathrm{e}^{-2r}+\mathrm{e}^{-4r}+\mathrm{e}^{-6r}+\ldots)$$
$$=\frac{\mathrm{e}^{-r}}{1-\mathrm{e}^{-2r}}-\frac{\mathrm{e}^{-2r}}{1-\mathrm{e}^{-2r}}$$
$$=\frac{1-\mathrm{e}^{-r}}{\mathrm{e}^{r}-\mathrm{e}^{-r}}$$
$$=\frac{1}{2\sinh(r)}(1-\mathrm{e}^{-r})$$

Therefore we have

$$\mathcal{L}_r\left\{\sum_{n=1}^{\infty}(-1)^{n+1}\left(\delta(q-n)\right)\right\}=\frac{(1-\mathrm{e}^{-r})}{2\sinh(r)}$$

Substituting $r=-\ln(s)$ we obtain

$$P(s)=\frac{1}{2\sinh\left(-\ln(s)\right)}\left(1-\mathrm{e}^{-(-\ln(s))}\right)$$
$$=\frac{1}{\sinh\left(\ln(s)\right)}\left(\frac{s-1}{2}\right)$$

Uniformly distributed order between 0 and 1 and its Transfer Function

For

$$k_{\mathrm{W}(0,1)}(q)=\begin{cases}1 & q\geq 0\\ 0 & q>1\end{cases}$$

call this as $k_{\mathrm{W}(0,1)}(q)$

Then we have

$$K(r)=\int_0^\infty (1)\mathrm{e}^{-rq}\mathrm{d}q$$
$$=\int_0^1 \mathrm{e}^{-rq}\mathrm{d}q$$
$$=\frac{1-\mathrm{e}^{-r}}{r}$$

Thus we get

$$\mathcal{L}_r\left\{k_{\mathrm{W}(0,1)}(q)\right\}=\frac{1-\mathrm{e}^{-r}}{r}$$

and substituting $r=-\ln(s)$ we have:

$$P(s)=\frac{1-\mathrm{e}^{-(-\ln(s))}}{-\ln(s)}=\frac{s-1}{\ln(s)}$$

Uniformly distributed order between 0 and 2 and its Transfer Function

For

$$k_{\mathrm{W}(0,2)}(q)=\begin{cases}1 & q\geq 0\\ 0 & q>2\end{cases}$$

We have

$$K(r)=\int_0^\infty (1)\mathrm{e}^{-rq}\mathrm{d}q$$
$$=\int_0^2 \mathrm{e}^{-rq}\mathrm{d}q$$
$$=\frac{1-\mathrm{e}^{-2r}}{r}$$

Therefore we have

$$\mathcal{L}_r\left\{k_{\mathrm{W}(0,2)}\right\}=\frac{1-\mathrm{e}^{-r}}{r}$$

and substituting $r=-\ln(s)$ we have

$$P(s)=\frac{s^2-1}{\ln(s)}$$

Uniform distributed order between 0 and 1 and with delta distribution at 2 and its Transfer Function

For order distribution with continuous from 0-1 as $k_{\mathrm{W}(0,1)}(q)$ and with delta distribution function at $q=2$ is $k(q)=k_{\mathrm{W}(0,1)}(q)+\delta(q-2)$,

Then

$$K(r)=\mathcal{L}_r\left\{k_{\mathrm{W}(0,1)}(q)\right\}+\mathcal{L}_r\left\{\delta(q-2)\right\}$$

gives:

$$\begin{aligned}K(r)&=\mathcal{L}_r\left\{k_{\mathrm{W}(0,1)}(q)\right\}+\mathcal{L}_r\left\{\delta(q-2)\right\}\\&=\frac{1-\mathrm{e}^{-r}}{r}+\mathrm{e}^{-2r}\\&=\frac{1-\mathrm{e}^{-r}+r\mathrm{e}^{-2r}}{r}\end{aligned}$$

and substituting $r=-\ln(s)$ we have

$$\begin{aligned}P(s)&=\frac{1-\mathrm{e}^{-(-\ln(s))}+\left(-\ln(s)\right)\mathrm{e}^{-2(-\ln(s))}}{-\ln(s)}\\&=\frac{s^2\ln(s)+s-1}{\ln(s)}\end{aligned}$$

Order distribution as linear ramp between 0 and 2 and its Transfer Function

For

$$k_{+\mathrm{R}(0,2)}(q)=\begin{cases}q & q\geq 0\\0 & q>2\end{cases}$$

In this derivation we use Laplace identity:

$$\mathcal{L}\left\{t^n f(t)\right\}=(-1)^n\frac{\mathrm{d}^n}{ds^n}F(s);\quad \mathcal{L}\left\{f(t)\right\}=F(s)$$

This observation makes construction of $k_{+\mathrm{R}(0,2)}(q)$ from $k_{\mathrm{W}(0,2)}(q)$ as:

$$k_{+\mathrm{R}(0,2)}(q)=q\left(k_{\mathrm{W}(0,2)}(q)\right)$$

So Laplace will be

$$\mathcal{L}_r\left\{q\left(k_{\mathrm{W}(0,2)}(q)\right)\right\}=(-1)^1\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{L}_r\left\{k_{\mathrm{W}(0,2)}(q)\right\}$$

We derive the following

$$
\begin{aligned}
K(r) &= \mathcal{L}_r\left\{k_{+\mathrm{R}(0,2)}(q)\right\} \\
&= (-1)^1 \frac{\mathrm{d}}{\mathrm{d}r}\mathcal{L}_r\left\{k_{\mathrm{W}(0,2)}(q)\right\}; \qquad \mathcal{L}_r\left\{k_{\mathrm{W}(0,2)}(q)\right\} = \frac{1-\mathrm{e}^{-2r}}{r} \\
&= (-1)\frac{\mathrm{d}}{\mathrm{d}r}\frac{1-\mathrm{e}^{-2r}}{r} \\
&= \frac{1-2r\mathrm{e}^{-2r}-\mathrm{e}^{-2r}}{r^2} \\
&= \frac{1-(2r+1)\mathrm{e}^{-2r}}{r^2}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\mathcal{L}_r\left\{k_{+\mathrm{R}(0,2)}(q)\right\} &= K(r) \\
&= \frac{1-(2r+1)\mathrm{e}^{-2r}}{r^2}
\end{aligned}
$$

and substituting $r = -\ln(s)$ we have

$$
\begin{aligned}
P(s) &= \frac{1-\left(2\left(-\ln(s)\right)+1\right)\mathrm{e}^{-2(-\ln(s))}}{\left(\ln(s)\right)^2} \\
&= \frac{1+\left(2\ln(s)-1\right)s^2}{\left(\ln(s)\right)^2}
\end{aligned}
$$

Order distribution as linear ramp between 0 and 1 and its Transfer Function

For

$$
k_{+\mathrm{R}(0,1)}(q) = \begin{cases} q & q \geq 0 \\ 0 & q > 1 \end{cases}
$$

using similar procedure as above we get:

$$
\begin{aligned}
\mathcal{L}_r\left\{k_{+\mathrm{R}(0,1)}(q)\right\} &= K(r) \\
&= \frac{1-(1+r)\mathrm{e}^{-r}}{r^2}
\end{aligned}
$$

and substituting $r = -\ln(s)$ we get

$$
P(s) = \frac{1+\left(\ln(s)-1\right)s}{\left(\ln(s)\right)^2}
$$

Order distribution as negative ramp between 0 and 2 and its Transfer Function

For

$$
k_{-\mathrm{R}(0,2)}(q) = \begin{cases} -q+2 & q \geq 0 \\ 0 & q > 2 \end{cases},
$$

can be composed as:

$$
k_{-\mathrm{R}(0,2)}(q) = -\left(k_{+\mathrm{R}(0,2)}(q)\right) + 2\left(k_{\mathrm{W}(0,2)}(q)\right),
$$

By using derived Laplace of these constituents we get:

$$\begin{aligned}K(r) &= \mathcal{L}\left\{k_{-\mathrm{R}(0,2)}(q)\right\} \\ &= -\mathcal{L}\left\{\left(k_{+\mathrm{R}(0,2)}(q)\right)\right\} + 2\mathcal{L}\left\{\left(k_{\mathrm{W}(0,2)}(q)\right)\right\} \\ &= -\left(\frac{1-\mathrm{e}^{-2r} - 2r\mathrm{e}^{-2r}}{r^2}\right) + 2\left(\frac{1-\mathrm{e}^{-2r}}{r}\right) \\ &= \frac{2r - 1 + \mathrm{e}^{-2r}}{r^2}\end{aligned}$$

and substituting $r = -\ln(s)$ we get:

$$P(s) = \frac{s^2 - 1 - 2\ln(s)}{\left(\ln(s)\right)^2}$$

Order distribution as negative ramp between 0 and 1 and its Transfer Function

For

$$k_{-R(0,1)}(q) = \begin{cases} -q+1 & q \geq 0 \\ 0 & q > 1 \end{cases}$$

and by the above procedure we compose this and write:

$$k_{-\mathrm{R}(0,1)}(q) = -\left(k_{+\mathrm{R}(0,1)}(q)\right) + 1\left(k_{\mathrm{W}(0,1)}(q)\right)$$

then taking Laplace of the constituents we get:

$$\begin{aligned}K(r) &= \mathcal{L}\left\{k_{-R(0,1)}(q)\right\} \\ &= -\left(\frac{1-(1+r)\mathrm{e}^{-r}}{r^2}\right) + 1\left(\frac{1-\mathrm{e}^{-r}}{r}\right) \\ &= \frac{r - 1 + \mathrm{e}^{-r}}{r^2}\end{aligned}$$

and substituting $r = -\ln(s)$ we get:

$$P(s) = \frac{s - 1 - \ln(s)}{\left(\ln(s)\right)^2}$$

Order distribution as negative ramp between 1 and 2 and its Transfer Function

For

$$k_{(-R(0,1))-1}(q) = \begin{cases} 0 & q \geq 0 \\ -q+2 & q \geq 1 \\ 0 & q > 2 \end{cases},$$

is $k_{-R(0,1)}(q)$ shifted from $q = 0$ to $q = 1$, that is $k_{(-R(0,1))-1}(q) = k_{-R(0,1)}(q-1)$. Here Laplace shift identity $\mathcal{L}\left\{f(t-t_0)\right\} = \mathrm{e}^{-st_0}\left(\mathcal{L}\left\{f(t)\right\}\right)$ is used to get

$$\mathcal{L}_r\left\{k_{-R(0,1)-1}(q)\right\} = K(r)$$

$$= \mathcal{L}_r\left\{k_{-R(0,1)}(q-1)\right\} = \mathrm{e}^{-r}\mathcal{L}_r\left\{k_{-R(0,1)}(q)\right\}$$

$$= \mathrm{e}^{-r}\left(\frac{r-1+\mathrm{e}^{-r}}{r^2}\right) = \frac{r\mathrm{e}^{-r}-\mathrm{e}^{-r}+\mathrm{e}^{-2r}}{r^2}$$

$$= \frac{(r-1)\mathrm{e}^{-r}+\mathrm{e}^{-2r}}{r^2}$$

and substituting $r = -\ln(s)$ we get:

$$P(s) = \frac{\left(\left(-\ln(s)\right)-1\right)\mathrm{e}^{-(-\ln(s))}+\mathrm{e}^{-2(-\ln(s))}}{\left(\ln(s)\right)^2}$$

$$= \frac{s^2+s\left(\ln(s)+1\right)}{\left(\ln(s)\right)^2}$$

Order distribution triangular between 0 and 2 and its Transfer Function

For

$$k_{\mathrm{TR}}(q) = \begin{cases} q & q \geq 0 \\ -q+2 & q \geq 1 \\ 0 & q > 2 \end{cases}$$

can be composed by

$$k_{\mathrm{TR}}(q) = k_{+\mathrm{R}(0,1)}(q) + k_{-\mathrm{R}(0,1)}(q-1)$$

From above obtained results for shifted we get:

$$\mathcal{L}_r\left\{k_{\mathrm{TR}}(q)\right\} = K(r) = \mathcal{L}_r\left\{k_{+\mathrm{R}(0,1)}(q)\right\} + \mathcal{L}_r\left\{k_{-\mathrm{R}(0,1)}(q-1)\right\}$$

$$= \frac{1-\mathrm{e}^{-r}-r\mathrm{e}^{-r}}{r^2} + \frac{r\mathrm{e}^{-r}-\mathrm{e}^{-r}-\mathrm{e}^{-2r}}{r^2}$$

$$= \frac{1-2\mathrm{e}^{-r}+\mathrm{e}^{-2r}}{r^2}$$

and substituting $r = -\ln(s)$ we get:

$$P(s) = \frac{1-2s+s^2}{\left(\ln(s)\right)^2}$$

Order distribution as off-set cosine function and its Transfer Function

For

$$k(q) = \frac{1}{2} - \frac{1}{2}\cos(2\pi q)$$

we use standard Laplace transform for Heaviside step and cosine expressions to obtain

$$
\begin{aligned}
K(r) &= \mathcal{L}_r\left\{\tfrac{1}{2}-\tfrac{1}{2}\cos(2\pi q)\right\} \\
&= \frac{1}{2}\times\frac{1}{r}-\frac{1}{2}\left(\frac{r}{r^2+4\pi^2}\right) \\
&= \frac{2\pi^2}{r(r^2+4\pi^2)}
\end{aligned}
$$

and substituting $r=-\ln(s)$ we get:

$$
P(s)=\frac{-2\pi^2}{\left(\ln(s)\right)\left(\left(\ln(s)\right)^2+4\pi^2\right)}
$$

Order distribution as offset cosine function between 0 and 2 and its Transfer Function

We now use this derived expression to get truncated

$$
k(q)=\begin{cases} 0.5-0.5\cos(2\pi q) & q\geq 0 \\ 0 & q>2 \end{cases},
$$

this can be composed by continuous function $\tfrac{1}{2}-\tfrac{1}{2}\cos(2\pi q)$ and from this subtracting shifted function at $q=2$, that is $\tfrac{1}{2}-\tfrac{1}{2}\cos\left(2\pi(q-2)\right)$. Using shift identity of Laplace operation we get:

$$
\begin{aligned}
K(r) &= \frac{2\pi^2}{r(r^2+4\pi^2)}-\mathrm{e}^{-2r}\left(\frac{2\pi^2}{r(r^2+4\pi^2)}\right) \\
&= \frac{2\pi^2(1-\mathrm{e}^{-2r})}{r(r^2+4\pi^2)}
\end{aligned}
$$

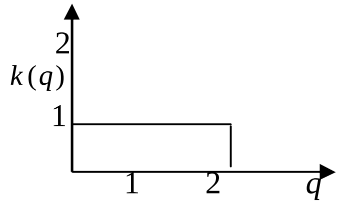
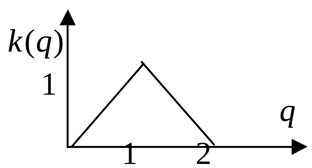
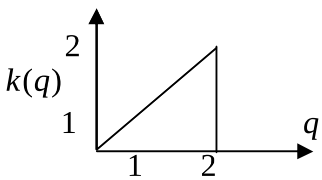
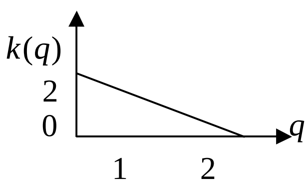
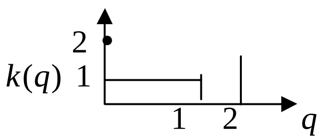
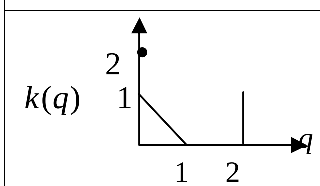
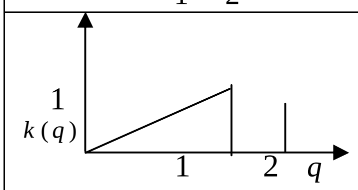
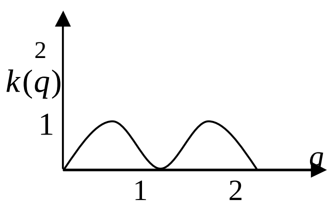
and substituting $r=-\ln(s)$ we get:

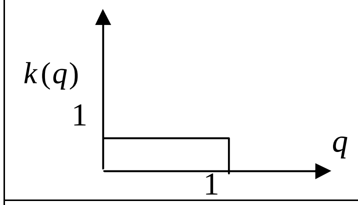
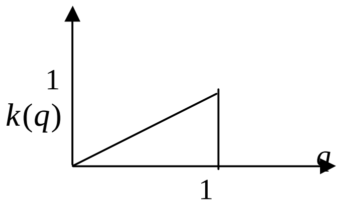
$$
P(s)=\frac{2\pi^2(s^2-1)}{\left(\ln(s)\right)\left(\left(\ln(s)\right)^2+4\pi^2\right)}
$$

If $k(q)$ is composed of one continuous function $f(q)$ multiplied by $a\left(k_{W(0,q_n)}(q)\right)$ then to get *r*-Laplace transform, convolution identity is used. Meaning

$$
K(r)=\mathcal{L}_r\left\{f(q)\right\}*\mathcal{L}_r\left\{a\left(k_{W(0,q_n)}(q)\right)\right\}
$$

Table-1

Order-distribution $k(q)$ vs. q	***r*-Laplace-Transform** $K(r)=\mathcal{L}_r\left\{k(q)\right\}$	**Laplace transfer function** $P(s)=K(r)\big\|_{r=-\ln s}$
	$\dfrac{1-\mathrm{e}^{-2r}}{r}$	$\dfrac{s^2-1}{\ln(s)}$
	$\dfrac{1-2\mathrm{e}^{-r}+\mathrm{e}^{-2r}}{r^2}$	$\dfrac{1-2s+s^2}{\left(\ln(s)\right)^2}$
	$\dfrac{1-(1+2r)\mathrm{e}^{-2r}}{r^2}$	$\dfrac{1-s^2\left(1-2\ln(s)\right)s^2}{\left(\ln(s)\right)^2}$
	$\dfrac{2r-1+\mathrm{e}^{-2r}}{r^2}$	$\dfrac{s^2-1-2\ln(s)}{\left(\ln(s)\right)^2}$
	$\dfrac{r\mathrm{e}^{-2r}+1-\mathrm{e}^{-r}}{r}$	$\dfrac{s-1+s^2\ln(s)}{\ln(s)}$
	$\dfrac{r-1+\mathrm{e}^{-r}+r^2\mathrm{e}^{-2r}}{r^2}$	$\dfrac{s^2\left(\ln(s)\right)^2-\ln(s)+s-1}{\left(\ln(s)\right)^2}$
	$\dfrac{1-(1+r)\mathrm{e}^{-r}+r^2\mathrm{e}^{-2r}}{r^2}$	$\dfrac{1-s+s\ln(s)+s^2\left(\ln(s)\right)^2}{\left(\ln(s)\right)^2}$
	$\dfrac{4\pi^2(1-\mathrm{e}^{-2r})}{2r(r^2+4\pi^2)}$	$\dfrac{4\pi^2(s^2-1)}{\left(2\ln(s)\right)\left(\left(\ln(s)\right)^2+4\pi^2\right)}$

	$\dfrac{1-\mathrm{e}^{-r}}{r}$	$\dfrac{s-1}{\ln(s)}$
	$\dfrac{1-(1+r)\mathrm{e}^{-r}}{r^2}$	$\dfrac{1-s\left(1-\ln(s)\right)}{\left(\ln(s)\right)^2}$

Order distribution as Gaussian distribution with peak at zero

Let us take Gaussian distribution of $q \geq 0$ as

$$k(q) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-q^2/4}$$

The Laplace transform of the above we write as

$$K(r) = \mathcal{L}_r\left\{k(q)\right\} = \mathcal{L}_r\left\{\tfrac{1}{\sqrt{\pi}} \mathrm{e}^{-q^2/4}\right\}$$
$$= E_{1/2}(-r)$$

Where $E_{1/2}(z)$ is Mittag-Leffler function $E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ and in our case $\alpha = 1/2$. This we write from the Laplace transform relation for M-Wright function ($M_\alpha(t)$) i.e. $\mathcal{L}\left\{M_\alpha(t)\right\} = E_\alpha(-s)$ and while $\alpha = 1/2$, $M_{1/2}(t) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-t^2/4}$. Thus transfer function $P(s) = K(r)\big|_{r=-\ln s}$ is

$$P(s) = E_{1/2}(\ln s)$$

Getting response in time domain for a continuously distributed ordered system via inverse Laplace technique

Let us try and get system response from the equation $\left(\int_{-\infty}^{\infty} k(q) s^q \mathrm{d}q\right) X(s) = F(s)$. In inverted form we can write the equation in Laplace domain for obtaining $X(s)$ as

$$X(s) = \frac{F(s)}{\int_{-\infty}^{+\infty} k(q) s^q \mathrm{d}q} = \frac{F(s)}{P(s)}$$

With

$$P(s) = \int_{-\infty}^{+\infty} k(q) s^q \mathrm{d}q$$

Consider a simple case of a second order system with uniformly distributed order of value $k(q) = \mathrm{K}$ for the order interval $0 \le q \le 2$. Then calculation of $P(s)$ is:

$$
\begin{aligned}
P(s) &= \int_{-\infty}^{\infty} k(q) s^q \mathrm{d}q \\
&= \int_0^2 \mathrm{K} s^q \mathrm{d}q; \quad s^q = \mathrm{e}^{\ln s^q} = \mathrm{e}^{q \ln s} \\
&= \mathrm{K}\int_0^2 \left(\mathrm{e}^{(\ln s)q}\right)\mathrm{d}q = \mathrm{K}\left[\frac{\mathrm{e}^{q\ln s}}{\ln s}\right]_{q=0}^{q=2} \\
&= \mathrm{K}\left(\frac{\mathrm{e}^{2\ln s} - 1}{\ln s}\right) = \mathrm{K}\left(\frac{\mathrm{e}^{\ln s^2} - 1}{\ln s}\right) \\
&= \mathrm{K}\left(\frac{s^2 - 1}{\ln s}\right)
\end{aligned}
$$

For forcing function $f(t) = \delta(t)$ as delta-function $F(s) = 1$ gives the response function in Laplace domain as:

$$
X(s) = \frac{\ln(s)}{\mathrm{K}\left(s^2 - 1\right)}
$$

Therefore we have to perform inverse Laplace transform to get

$$
x(t) = \mathcal{L}^{-1}\left\{\frac{\ln(s)}{\mathrm{K}\left(s^2-1\right)}\right\}
$$

Consider another simple case that order is uniformly distributed equally at $q = \overline{q}$ with spread of $\pm\Delta\overline{q}$, and strength as $k(q) = \mathrm{K}$, a constant. The response for $f(t) = \delta(t)$, the unit delta will consider $F(s) = 1$, and the integral expression for the response is thus, as demonstrated above is

$$
P(s) = \int_{-\infty}^{+\infty} k(q) s^q \mathrm{d}q = \int_{\overline{q}-\Delta\overline{q}}^{\overline{q}+\Delta\overline{q}} \mathrm{K} s^q \mathrm{d}q
$$

With change of variable as $p = q - \overline{q}$, $q = p + \overline{q}$ we have $\mathrm{d}q = \mathrm{d}p$ with new limits:

$$
\begin{aligned}
P(s) &= \int_{q=\overline{q}-\Delta\overline{q}}^{q=\overline{q}+\Delta\overline{q}} \mathrm{K} s^q \mathrm{d}q; \quad q = p + \overline{q}; \quad p = q - \overline{q} \\
&= \int_{-\Delta\overline{q}}^{+\Delta\overline{q}} \mathrm{K} s^{(p+\overline{q})} \mathrm{d}p = \int_{-\Delta\overline{q}}^{+\Delta\overline{q}} \mathrm{K} s^p s^{\overline{q}} \mathrm{d}p; \quad s^p = \mathrm{e}^{p\ln s} \\
&= \mathrm{K}s^{\overline{q}} \int_{-\Delta\overline{q}}^{+\Delta\overline{q}} \mathrm{e}^{(p)\ln s} \mathrm{d}p = \mathrm{K}s^{\overline{q}}\left(\frac{\mathrm{e}^{(p)\ln s}}{\ln s}\right)_{p=-\Delta\overline{q}}^{p=+\Delta\overline{q}} = \mathrm{K}s^{\overline{q}}\,\frac{s^{\Delta\overline{q}} - s^{-\Delta\overline{q}}}{\ln s} \\
&= 2\mathrm{K}s^{\overline{q}}\,\frac{\sinh\left(\Delta\overline{q}\ln s\right)}{\ln s}
\end{aligned}
$$

Therefore the response to the impulse input is (in Laplace variables):

$$
X(s) = s^{-(\Delta\overline{q})}\,\frac{\ln(s)}{2\mathrm{K}\sinh(\Delta\overline{q}\ln s)}
$$

This is basic response from which other responses are formed by convolution.

Obtaining the inverse Laplace transform for $\frac{\ln(s)}{\mathrm{K}\left(s^b - s^a\right)}$ to get $x(t)$ the solution

Let us find

$$x(t) = \mathcal{L}^{-1}\left\{\frac{\ln(s)}{\mathrm{K}\left(s^b - s^a\right)}\right\}; \quad b > a > 0$$

The function $X(s) = \frac{\ln(s)}{\mathrm{K}\left(s^b - s^a\right)}$ behavior we see, that at $s \uparrow \infty$ is by L'Hospital rule is following

$$\lim_{s\uparrow\infty} X(s) = \lim_{s\uparrow\infty} \frac{\frac{1}{s}}{\mathrm{K}(bs^{b-1} - as^{a-1})}$$
$$= 0$$

We may feel $s = 1$ is singular point of $X(s)$, but by L'Hospital rule, we see it is not

$$\lim_{s\to 1} X(s) = \lim_{s\uparrow\infty} \frac{\frac{1}{s}}{\mathrm{K}(bs^{b-1} - as^{a-1})}$$
$$= \frac{1}{\mathrm{K}(b-a)}$$

So we find $s = 1$ is not a singular point. The point $s = 0$ is singularity in the $X(s)$. Due to multi-valued nature of $\ln s$ we call the singular point as branch point and not pole. The contour shown in Figure-2 excludes the branch point, and the branch-cut is taken as standard branch-cut – the negative real axis. The inverse Laplace transform integral is following.

$$x(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \mathrm{e}^{st} X(s)\mathrm{d}s$$
$$= \frac{1}{2\pi i}\int_{\text{Bromwich}-\text{Path}} \mathrm{e}^{st} X(s)\mathrm{d}s$$

The above integration is Bromwich path integration on the line *AB* as in Figure-2. The $\mathrm{Re}[s] = \sigma$ is the line, to right of which we do not have any singularity, or the singularity is contained in the left side of the line *AB* (Figure-2). So we select $\sigma = 0^+$ close to imaginary axis as shown in Figure-2. By use of Residue Calculus, we get

$$\int_{A\to B\to C\to D\to E\to F\to G\to A} \mathrm{e}^{st} X(s)\mathrm{d}s = 2\pi i \sum \text{Residues of}\left[\mathrm{e}^{st} X(s)\right]$$

The discussion about $X(s) = \frac{\ln(s)}{\mathrm{K}\left(s^b - s^a\right)}$ indicate that there are no singularities in the path of integration A, B, C, D, E, F, G, A; thus Residues are zero, thus we write the following

$$\int_{A\to B\to C\to D\to E\to F\to G\to A} \mathrm{e}^{st} X(s)\mathrm{d}s = 0$$

$$\int_{A\to B} \mathrm{e}^{st} X(s)\mathrm{d}s = -\int_C \mathrm{e}^{st} X(s)\mathrm{d}s - \int_D \mathrm{e}^{st} X(s)\mathrm{d}s - \int_E \mathrm{e}^{st} X(s)\mathrm{d}s - \int_F \mathrm{e}^{st} X(s)\mathrm{d}s - \int_G \mathrm{e}^{st} X(s)\mathrm{d}s$$

That is we need to evaluate integral on big arcs *C* and *G*, lines just above and below negative real axis *D* and *F,* and on small circle at origin i.e. *E*.

Examples of uniformly distributed order and obtain inverse Laplace transform for $\frac{\ln s}{s^2-1}$, to get solution

We take $X(s)=\frac{\ln s}{s^2-1}$, that is $b=2, \quad a=0, \quad \mathrm{K}=1$, in our function $X(s)=\frac{\ln(s)}{\mathrm{K}\left(s^b-s^a\right)}$ and do the calculations.

Let the big arcs be represented by $C: R\mathrm{e}^{i\theta}; \quad \frac{\pi}{2}\le\theta\le-\pi$ and $G: R\mathrm{e}^{i\theta}; \quad -\pi\le\theta\le-\frac{\pi}{2}$, then the integrals is

$$\int_C \mathrm{e}^{st}X(s)\mathrm{d}s=\lim_{R\uparrow\infty}\int \mathrm{e}^{R\mathrm{e}^{i\theta}t}X(R\mathrm{e}^{i\theta})\mathrm{e}^{i\theta}\,\mathrm{d}R; \quad s=R\mathrm{e}^{i\theta}$$

$$=\lim_{R\uparrow\infty}\int \mathrm{e}^{R\mathrm{e}^{i\theta}t}\left(\frac{\ln(R\mathrm{e}^{i\theta})}{R^2\mathrm{e}^{i2\theta}-1}\right)\mathrm{e}^{i\theta}\,\mathrm{d}R$$

$$=\lim_{R\uparrow\infty}\int \mathrm{e}^{R\mathrm{e}^{i\theta}t}\left(\frac{\ln(R)+i\theta}{R^2\mathrm{e}^{i2\theta}-1}\right)\mathrm{e}^{i\theta}\,\mathrm{d}R$$

It may be noted that the function $X(s)=\frac{\ln s}{s^2-1}$ vanishes at large values; hence the integral on the large arcs C and G are zero.

$$\int_C \mathrm{e}^{st}X(s)\mathrm{d}s=\int_G \mathrm{e}^{st}X(s)\mathrm{d}s=0$$

Now we do integration on the small circle E, with $s=\in\mathrm{e}^{i\theta}$ encircling the origin where $\pi\le\theta\le-\pi$ that is integration in positive sense (anti-clockwise). So we have the following

$$\int_E \mathrm{e}^{st}X(s)\mathrm{d}s=\lim_{\in\downarrow 0}\int_{\pi}^{-\pi}\mathrm{e}^{\in\mathrm{e}^{i\theta}t}\left(\frac{\ln\in+i\theta}{\in^2 e^{i2\theta}-1}\right)i\in\mathrm{e}^{i\theta}\mathrm{d}\theta$$

$$\sim\lim_{\in\downarrow 0}\int_{\pi}^{-\pi}(1)\left(\frac{\ln\in+i\theta}{-1}\right)i\in\mathrm{e}^{i\theta}\mathrm{d}\theta$$

$$=\lim_{\in\downarrow 0, M\uparrow\infty}\int_{\pi}^{-\pi}\left(M-i\theta\right)\left(i\in\mathrm{e}^{i\theta}\right)\mathrm{d}\theta=0$$

In above we took for $\in^2\mathrm{e}^{i\theta}-1\sim-1; \quad \mathrm{e}^{\in t\mathrm{e}^{i\theta}}\sim 1$ and $\ln\in\sim -M$ with $M\uparrow\infty$ as $\in\downarrow 0$, the entire expression when gets multiplied by $\in\downarrow 0$ gives zero. Thus we have integration on the small circle is zero. We note that $\lim_{s\downarrow 0}X(s)=\frac{\ln s}{s^2-1}=\infty$, but the integration on a small circle encircling $s=0$ with $\lim_{\in\downarrow 0}\int_{s=\in\mathrm{e}^{i\theta}}\mathrm{e}^{st}X(s)\mathrm{d}s=0$.

Thus we are left with integration on D where we take $s=z\mathrm{e}^{i\pi}; \quad \mathrm{d}s=\mathrm{e}^{i\pi}\mathrm{d}z$ with $\infty<z<0$ and on line F with $s=z\mathrm{e}^{-i\pi}, \quad \mathrm{d}s=\mathrm{e}^{-i\pi}\,\mathrm{d}z$ with $0<z<\infty$. We note that we are on negative real axis in both cases where z is positive and $\mathrm{e}^{i\pi}=\mathrm{e}^{-i\pi}=-1$. Thus the integration on line D is

$$\int_D \mathrm{e}^{st}X(s)\mathrm{d}s=\int_{\infty}^{0}\mathrm{e}^{z\mathrm{e}^{i\pi}t}\left(\frac{\ln\left(z\mathrm{e}^{i\pi}\right)}{z^2\mathrm{e}^{i(2\pi)}-1}\right)\mathrm{e}^{i\pi}\mathrm{d}z$$

$$=\int_{\infty}^{0}\mathrm{e}^{-zt}\left(\frac{\ln z+i\pi}{z^2-1}\right)(-\mathrm{d}z)=\int_{0}^{\infty}\mathrm{e}^{-zt}\left(\frac{\ln z+i\pi}{z^2-1}\right)\mathrm{d}z$$

$$\int_F \mathrm{e}^{st} X(s)\mathrm{d}s = \int_0^{\infty} \mathrm{e}^{z\mathrm{e}^{-i\pi}t} \left(\frac{\ln\left(z\mathrm{e}^{-i\pi}\right)}{z^2\mathrm{e}^{i(-2\pi)} - 1} \right) \mathrm{e}^{-i\pi}\mathrm{d}z$$

$$= \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{\ln z - i\pi}{z^2 - 1} \right)(-\mathrm{d}z)$$

Adding the two we get

$$\int_{D+F} \mathrm{e}^{st} X(s)\mathrm{d}s = \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{\ln z + i\pi}{z^2 - 1} \right)\mathrm{d}z - \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{\ln z - i\pi}{z^2 - 1} \right)\mathrm{d}z$$

$$= \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{2i\pi}{z^2 - 1} \right)\mathrm{d}z$$

The inverse Laplace transform integral on Bromwich path is thus

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\} = \frac{1}{2\pi i} \int_{\text{Bromwich}-\text{Path}} \mathrm{e}^{st} X(s)\mathrm{d}s$$

$$= -\frac{1}{2\pi i} \int_{D+F} \mathrm{e}^{st} X(s)\mathrm{d}s$$

$$= -\frac{1}{2\pi i} \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{2\pi i}{z^2 - 1} \right)\mathrm{d}z$$

$$= -\int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{1}{z^2 - 1} \right)\mathrm{d}z$$

$$= \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{1}{1 - z^2} \right)\mathrm{d}z$$

Therefore we have obtained the following

$$x(t) = \mathcal{L}^{-1}\left\{\tfrac{\ln s}{s^2 - 1}\right\} = \int_0^{\infty} \mathrm{e}^{-zt} \left(\frac{1}{1 - z^2} \right)\mathrm{d}z$$

The observation is now for $X(s) = \frac{\ln s}{s^b - s^a}; \quad b > a \geq 0$. We may note that if $a \neq 0$ then the integration on the small circle E blows up, and the method fails.

Using Berberan Santo Method to get inverse Laplace transform of $X(s) = \frac{\ln s}{s^b - s^a}$

The $g(t)$ inverse Laplace Transform of $G(s)$ is obtained by following formula

$$g(t) = \frac{e^{\sigma_0 t}}{\pi} \int_0^{\infty} \left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t) \right)\mathrm{d}\omega$$

Here $s = \sigma_0 + i\omega$ where $\mathrm{Re}\left[s\right] = \sigma_0$ is the line, where right of that line there is no singularity. This above formula is Berberan-Santo formula.

The condition for Laplace inverse to exist is $G(s)$ tends zero at large s (look at the Tables of Laplace Transforms, all the listed functions follow this). However it is not always the case, for example $G(s) = \sqrt{s - a}$ does not vanish at large s yet this function has inverse Laplace transform i.e. $g(t) = -\frac{\mathrm{e}^{ax}}{2\sqrt{\pi t^3}}$. This can be found by the contour integration method of Figure-2.

The function $X(s)=\frac{\ln s}{s^b-s^a}; \quad b>a\geq 0$ vanishes at infinity (apply L'Hospital rule to see that $\lim_{s\uparrow\infty} X(s)=\frac{\frac{1}{s}}{bs^{b-1}-as^{a-1}}=\frac{1}{bs^b-as^a}=\frac{1}{\infty}=0$); and thus is well suited for inverse Laplace transform. We also observe that to the right side of line $\mathrm{Re}[s]=0$ we do not have singularity (though one may feel $s=1$ is singular, but we ruled that out earlier). So we choose $\sigma_0=0$. Then we go about to use Berberan-Santos method.

$$X(s)=\frac{\ln s}{s^b-s^a}; \quad b>a\geq 0; \quad s=\sigma_0+i\omega; \quad \sigma_0=0$$

$$X(i\omega)=\frac{\ln(i\omega)}{(i\omega)^b-(i\omega)^a}=\frac{\ln\left(\omega \mathrm{e}^{i\pi/2}\right)}{\omega^b\mathrm{e}^{ib\pi/2}-\omega^a\mathrm{e}^{ia\pi/2}}$$

$$=\frac{\ln\omega+i\left(\frac{\pi}{2}\right)}{\left(\omega^b\cos\frac{b\pi}{2}-\omega^a\cos\frac{a\pi}{2}\right)+i\left(\omega^b\sin\frac{b\pi}{2}-\omega^a\sin\frac{a\pi}{2}\right)}$$

$$=\frac{\left(\ln\omega+i\frac{\pi}{2}\right)\left(\left(\omega^b\cos\frac{b\pi}{2}-\omega^a\cos\frac{a\pi}{2}\right)-i\left(\omega^b\sin\frac{b\pi}{2}-\omega^a\sin\frac{a\pi}{2}\right)\right)}{\left(\left(\omega^b\cos\frac{b\pi}{2}-\omega^a\cos\frac{a\pi}{2}\right)^2+\left(\omega^b\sin\frac{b\pi}{2}-\omega^a\sin\frac{a\pi}{2}\right)^2\right)}$$

From above we write

$$\mathrm{Re}\left[X(i\omega)\right]=\frac{\omega^b\left(\ln\omega\cos\frac{b\pi}{2}+\frac{\pi}{2}\sin\frac{b\pi}{2}\right)-\omega^a\left(\ln\omega\cos\frac{a\pi}{2}+\frac{\pi}{2}\sin\frac{a\pi}{2}\right)}{\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\frac{(b-a)\pi}{2}}$$

$$\mathrm{Im}\left[X(i\omega)\right]=-\frac{\omega^b\left(\ln\omega\sin\frac{b\pi}{2}-\frac{\pi}{2}\cos\frac{b\pi}{2}\right)-\omega^a\left(\ln\omega\sin\frac{a\pi}{2}-\frac{\pi}{2}\cos\frac{a\pi}{2}\right)}{\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\frac{(b-a)\pi}{2}}$$

Use Berberan-Santos formula to get the following

$$x(t)=\frac{1}{\pi}\int_0^\infty\left(\mathrm{Re}\left[X(i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[X(i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega$$

$$=\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\frac{\omega^b\left(\ln\omega\cos\frac{b\pi}{2}+\frac{\pi}{2}\sin\frac{b\pi}{2}\right)-\omega^a\left(\ln\omega\cos\frac{a\pi}{2}+\frac{\pi}{2}\sin\frac{a\pi}{2}\right)}{\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\frac{(b-a)\pi}{2}}\cos\omega t$$

$$+\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\frac{\omega^b\left(\ln\omega\sin\frac{b\pi}{2}-\frac{\pi}{2}\cos\frac{b\pi}{2}\right)-\omega^a\left(\ln\omega\sin\frac{a\pi}{2}-\frac{\pi}{2}\cos\frac{a\pi}{2}\right)}{\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\frac{(b-a)\pi}{2}}\sin\omega t$$

With $b=2, \quad a=0, \quad \omega\equiv z$, the following is Laplace inverse

$$x(t)=\mathcal{L}^{-1}\left\{\frac{\ln s}{s^2-1}\right\}=\frac{1}{\pi}\int_0^\infty\frac{\left(\sin zt\right)\left(\frac{\pi}{2}z^2-\ln z\right)-\left(\cos zt\right)\left(z^2\ln z+\frac{\pi}{2}\right)}{(z^2+1)^2}\mathrm{dz}$$

Solution to differential equations with continuously distributed order

For the differential equation defined above in Laplace domain is $F(s)=P(s)X(s)$. We need to find the solution $x(t)$ for function $f(t)$ that is given as

$$x(t)=\left(\mathcal{L}^{-1}\left\{\frac{1}{P(s)}\right\}\right)*\left(f(t)\right)$$

For $f(t)=\delta(t)$ we have homogeneous solution as

$$x(t)=\mathcal{L}^{-1}\left\{\tfrac{1}{P(s)}\right\}$$

For any other input say $f(t)=A\sin\omega t$ we will get solution as

$$x(t)=\left(\mathcal{L}^{-1}\left\{\tfrac{1}{P(s)}\right\}\right)*\left(f(t)\right)$$

$$=\frac{A}{\pi}\left(\int_0^\infty\frac{(\sin zt)\left(\frac{\pi}{2}z^2-\ln z\right)-(\cos zt)\left(z^2\ln z+\frac{\pi}{2}\right)}{(z^2+1)^2}\mathrm{d}z\right)*(\sin\omega t)$$

We have seen $P(s)X(s)=F(s)$ where $P(s)=\int_{-\infty}^{\infty}k(q)s^q\mathrm{d}q$. For a simple case of a system with uniformly distributed order of value $k(q)=1$ for the order interval $a\le q\le b;\quad q\ge 0$; then $P(s)$ is, $P(s)=\frac{s^b-s^a}{\ln s}$. We want to now solve for $F(s)$, the differential equation $P(s)X(s)=F(s)$ for some function $X(s)$. Obviously by Laplace transform technique we can write the solution $f(t)=\mathcal{L}^{-1}\{F(s)\}$, by following convolution.

$$f(t)=\left(\mathcal{L}^{-1}\{P(s)\}\right)*\left(\mathcal{L}^{-1}\{X(s)\}\right)$$

Here we need to do inverse Laplace transform of $P(s)=\frac{s^b-s^a}{\ln s},\quad b>a$.

Conditioning the transfer function to get inverse Laplace transform

We note here that $\lim_{s\uparrow\infty}P(s)=\infty$, thus we say that if we modify $P(s)$ as $\bar{P}(s)=\frac{s^b-s^a}{s\ln s}$, then we get $\lim_{s\uparrow\infty}\bar{P}(s)=0$ and thus we can comfortably apply the formulas for Laplace inverse transforms. Let us take $b=2$ and $a=0$ then via contour integration method as we did earlier we see-integral on the big arcs C and G are zero, as $\bar{P}(s)=\frac{s^2-1}{s\ln s}$ goes to zero for large R with $s=R\,\mathrm{e}^{i\theta}$. For the small circle $s=\in\mathrm{e}^{i\theta};\quad \pi\le\theta\le\pi$, i.e. E enclosing the origin we have following

$$\int_E \mathrm{e}^{st}\bar{P}(s)\mathrm{d}s=\lim_{\in\downarrow 0}\int_{\pi}^{-\pi}\mathrm{e}^{st}\,\frac{s^2-1}{s\ln s}\mathrm{d}s\Bigg|_{s=\in\mathrm{e}^{i\theta}}$$

$$=\lim_{\in\downarrow 0}\int_{\pi}^{-\pi}\mathrm{e}^{\in\mathrm{e}^{i\theta}}\,\frac{\in^2\mathrm{e}^{i2\theta}-1}{\in\mathrm{e}^{i\theta}\left(\ln\in+i\theta\right)}\in i\mathrm{e}^{i\theta}\mathrm{d}\theta$$

$$=\int_{\pi}^{-\pi}(1)\lim_{M\uparrow\infty}\frac{(-1)}{(-M+i\theta)}i\mathrm{d}\theta$$

$$=\int_{\pi}^{-\pi}\lim_{M\uparrow\infty}\frac{1}{\sqrt{M^2+\theta^2}\,\tan^{-1}\left(-\frac{\theta}{M}\right)}i\mathrm{d}\theta=0$$

Thus we are left with integrals on the lines D with $s=z\mathrm{e}^{i\pi}$, $\infty>z>0$ and F with $s=z\mathrm{e}^{-i\pi}$, $0<z<\infty$ and $\mathrm{d}s=-\mathrm{d}z$. So we have following

$$\int_{D+F} e^{st}\bar{P}(s)ds = \int_{\infty}^{0} e^{-zt}\frac{z^2-1}{(-z)(\ln z+i\pi)}(-dz)+\int_{0}^{\infty} e^{-zt}\frac{z^2-1}{(-z)(\ln z-i\pi)}(-dz)$$

$$=\int_0^{\infty} e^{-zt}\left(\frac{z^2-1}{(-z)}\right)\left(\frac{1}{\ln z+i\pi}-\frac{1}{\ln z-i\pi}\right)dz$$

$$=\int_0^{\infty} e^{-zt}\left(\frac{z^2-1}{z}\right)\left(\frac{2i\pi}{(\ln z)^2+\pi^2}\right)dz$$

As we have seen inverse Laplace transform is integration on Bromwich path we have from the residue calculus, and contour integration the following, noting that there are no poles enclosed in the contour of Figure-2;

$$\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}=\frac{1}{2\pi i}\int_{A\to B} e^{st}\bar{P}(s)ds=-\frac{1}{2\pi i}\int_{D+F} e^{st}\bar{P}(s)ds$$

$$=\int_0^{\infty} e^{-zt}\left(\frac{1-z^2}{z\left((\ln z)^2+\pi^2\right)}\right)dz$$

Now we see another representation by Berberan-Santos method for $\bar{P}(s)=\frac{s^b-s^a}{s\ln s}=\frac{s^{b-1}-s^{a-1}}{\ln s}$ to get inverse Laplace transformation. Since there is no singularity for $\mathrm{Re}[s]>0$ we select $s=\sigma_0+i\omega$, with $\sigma_0=0$ and do the following

$$\bar{P}(i\omega)=\frac{(i\omega)^{b-1}-(i\omega)^{a-1}}{\ln(i\omega)};\quad \ln(i\omega)=\ln\omega+\ln i;\quad i=e^{i\pi/2};\quad \ln(i\omega)=\ln\omega+i\left(\tfrac{\pi}{2}\right)$$

$$=\frac{\omega^{b-1}\left(\cos\left(\frac{(b-1)\pi}{2}\right)+i\sin\left(\frac{(b-1)\pi}{2}\right)\right)-\omega^{a-1}\left(\cos\left(\frac{(a-1)\pi}{2}\right)+i\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\ln\omega+i\left(\frac{\pi}{2}\right)}$$

$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)+i\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\ln\omega+i\left(\frac{\pi}{2}\right)}$$

$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)+i\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\left(\ln\omega-i\left(\frac{\pi}{2}\right)\right)$$

$$=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)(\ln\omega)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$

$$-i\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$

This above steps gives real and imaginary parts as

$$\mathrm{Re}\left[\bar{P}(i\omega)\right]=\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)(\ln\omega)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$

$$\mathrm{Im}\left[\bar{P}(i\omega)\right]==\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}$$

We apply Berberan Santo formula to get the following

$$\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}=\mathcal{L}^{-1}\left\{\frac{s^b-s^a}{s\ln s}\right\}=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left\{\bar{P}(\sigma_0+i\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{\bar{P}(\sigma_0+i\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega,\quad \sigma_0=0$$

$$=\frac{1}{\pi}\int_0^\infty\left(\begin{array}{l}\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)(\ln\omega)+\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\cos(\omega t)\\ +\frac{\left(\omega^{b-1}\cos\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\cos\left(\frac{(a-1)\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)-\left(\omega^{b-1}\sin\left(\frac{(b-1)\pi}{2}\right)-\omega^{a-1}\sin\left(\frac{(a-1)\pi}{2}\right)\right)\ln\omega}{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}\sin(\omega t)\end{array}\right)\mathrm{d}\omega$$

For $b=2$, $a=0$ and with $\omega\equiv z$ we get

$$\mathcal{L}^{-1}\left\{\frac{s^2-1}{s\ln s}\right\}=\frac{1}{\pi}\int_0^\infty\left(\frac{\pi}{2}\cos zt-\ln z\sin zt\right)\left(\frac{z^2+1}{z\left((\ln z)^2+\frac{\pi^2}{4}\right)}\right)\mathrm{dz}$$

Our main equation is following where the symbol $*$is convolution integration i.e. defined as. $g_1(t)*g_2(t)=\int_{-\infty}^{t}\left(g_1(t-x)\right)\left(g_2(x)\right)\mathrm{d}x$

$$f(t)=\left(\mathcal{L}^{-1}\left\{P(s)\right\}\right)*\left(\mathcal{L}^{-1}\left\{X(s)\right\}\right)$$
$$=\left(\mathcal{L}^{-1}\left\{P(s)\right\}\right)*\left(x(t)\right)$$

But we found inverse Laplace transform of $\bar{P}(s)=\frac{P(s)}{s}$ as demonstrated above. Therefore we get in Laplace domain the following

$$F(s)=P(s)X(s)=\left(\frac{P(s)}{s}\right)\left(sX(s)\right)$$
$$=\bar{P}(s)\left(sX(s)\right)$$
$$f(t)=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}*\left(x^{(1)}(t)\right)$$

By doing so as demonstrated we need derivative of the function $x(t)$ i.e. $x^{(1)}(t)$ and with obtained $\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}$ we get the solution. For $P(s)=\frac{s^2-1}{\ln s}$ and say $x(t)=u(t)$ a unit step input at $t=0$ we get for $x^{(1)}(t)=\delta(t)$ (unit Delta function at origin)

$$f(t)=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}*x^{(1)}(t)=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}*\left(\delta(t)\right)$$

$$=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}=\frac{1}{\pi}\int_0^{\infty}\left(\tfrac{\pi}{2}\cos zt-\ln z\sin zt\right)\left(\frac{z^2+1}{z\left((\ln z)^2+\frac{\pi^2}{4}\right)}\right)\mathrm{dz}$$

For say $x(t)=A\sin\omega t$ we have $x^{(1)}(t)=A\omega\cos\omega t$, then

$$f(t)=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}*x^{(1)}(t)=\mathcal{L}^{-1}\left\{\bar{P}(s)\right\}*\left(A\omega\cos\omega t\right)$$

$$=\frac{A\omega}{\pi}\left(\int_0^{\infty}\left(\tfrac{\pi}{2}\cos zt-\ln z\sin zt\right)\left(\frac{z^2+1}{z\left((\ln z)^2+\frac{\pi^2}{4}\right)}\right)\mathrm{dz}\right)*\left(\cos\omega t\right)$$

Frequency Plot (Bode Magnitude and Phase Plot) of Transfer Function for continuous order distribution system

The Bode plots are important for magnitude and phase angle of the Transfer Function for analysis purposes. Let us take the Transfer Function

$$P(s)=\frac{s^2-1}{\ln s}$$

Put $s=i\omega$ to write the following

$$P(\omega)=\frac{(i\omega)^2-1}{\ln(i\omega)},\quad i=\mathrm{e}^{i\pi/2}$$

$$=\frac{-\omega^2-1}{\ln\omega+i\left(\frac{\pi}{2}\right)}$$

$$=\frac{-(\omega^2+1)\left(\ln\omega-i\left(\frac{\pi}{2}\right)\right)}{\left(\ln\omega\right)^2+\frac{\pi^2}{4}}$$

$$=\frac{-(\omega^2+1)\ln\omega}{\left(\ln\omega\right)^2+\frac{\pi^2}{4}}+i\frac{(\omega^2+1)\left(\frac{\pi}{2}\right)}{\left(\ln\omega\right)^2+\frac{\pi^2}{4}}$$

The magnitude is

$$\left|P(\omega)\right|=\frac{\omega^2+1}{\sqrt{(\ln\omega)^2+\frac{\pi^2}{4}}}$$

$$\left|P(\omega)\right|_{\mathrm{dB}}=20\log\left(\frac{\omega^2+1}{\sqrt{(\ln\omega)^2+\frac{\pi^2}{4}}}\right)=20\log\left(\omega^2+1\right)-10\log\left((\ln\omega)^2+\tfrac{\pi^2}{4}\right)$$

The Bode magnitude plot is $\left|P(\omega)\right|_{\mathrm{dB}}$ with ω in log scale.

The phase angle of transfer function is

$$\angle P(\omega)=\tan^{-1}\left(-\frac{\pi}{2\ln\omega}\right)$$

The phase angle of Bode plot is $\angle P(\omega)$ in degree with ω in log scale.

Conclusion

Here we discussed the further generalization of fractional differential equation, where the order of differentiation is not a fixed number, instead is a function or we call order is distributed as continuous function. We have seen how we can use the Laplace transformation techniques to get a compact transfer function of the system. With this transfer function (or characteristic equation), we can use the inverse Laplace transformation to obtain time responses, and we can plot the Bode magnitude and phase plots for analysis of transfer function. We see that the classical method of application of residue calculus by using contour integration in complex plane or modern method of Berberan-Santo, (i.e. using direct Bromwich path integration) gives time domain solution, in different integral representations-however can be numerically plotted. We have shown few examples of continuously distributed functions and their respective transfer functions. This can be interesting project in mathematical science to get to know the responses of continuously distributed dynamic systems.

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Appendix

Revising basics of Laplace Transforms

For a function $x(t)$ which is zero for $t<0$, the Laplace transform is following Laplace integral

$$X(s)=\int_0^{\infty}\mathrm{e}^{-st}x(t)\mathrm{d}t$$

Here we have complex variable $s=\sigma+i\omega$, so we have

$$X(s)=\int_0^{\infty}\mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t$$

The inverse problem is given $X(s)$ how to get $x(t)$. Whereas the Fourier transform of $x(t)$ is $\hat{x}(\omega)$ defined as $\hat{x}(\omega)=\int_{-\infty}^{\infty}\mathrm{e}^{-i\omega t}x(t)\mathrm{d}t$ and inverse Fourier transform is the following integral

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\omega t}\hat{x}(\omega)\mathrm{d}\omega$$

In the Laplace transform expression $X(s)=\int_0^{\infty}\mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t$ as obtained above, let us take

$$\begin{aligned}\phi(t)&=\mathrm{e}^{-\sigma t}x(t); \quad && t\geq 0\\ &=0 && t<0\end{aligned}$$

Where σ is a constant. Taking the Fourier transform of $\phi(t)$ we write

$$\begin{aligned}\hat{\phi}(\omega)&=\int_0^{\infty}\mathrm{e}^{-i\omega t}\phi(t)\mathrm{d}t\\ &=\int_0^{\infty}\mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t\\ &=\int_0^{\infty}\mathrm{e}^{-(\sigma+i\omega)t}x(t)\mathrm{d}t=X(\sigma+i\omega)\end{aligned}$$

Now we take inverse Fourier transform of $X(\sigma+i\omega)$ as following

$$\phi(t)=\mathrm{e}^{-\sigma t}x(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{i\omega t}X(\sigma+i\omega)\mathrm{d}\omega$$

Therefore

$$\begin{aligned}x(t)&=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{\sigma t}\left(\mathrm{e}^{i\omega t}X(\sigma+i\omega)\right)\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{(\sigma+i\omega)t}X(\sigma+i\omega)\mathrm{d}\omega\end{aligned}$$

Letting $s=\sigma+i\omega$ we have $\mathrm{d}s=i\mathrm{d}\omega$, we get the following from above

$$x(t) = \frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega} \mathrm{e}^{st} X(s)\mathrm{d}s; \quad t \geq 0$$
$$= 0; \quad t < 0$$

The inverse Laplace transform via contour integration

The formula that we derived that is

$$x(t) = \frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega} \mathrm{e}^{st} X(s)\mathrm{d}s$$

is the integral that gives inverse Laplace transform. Now we ask the following

1). How we choose the real part of s i.e. σ

2). How do we calculate/evaluate the above integral in complex domain.

We already know that $x(t) = 0$ for $t < 0$; that will help to answer (1).

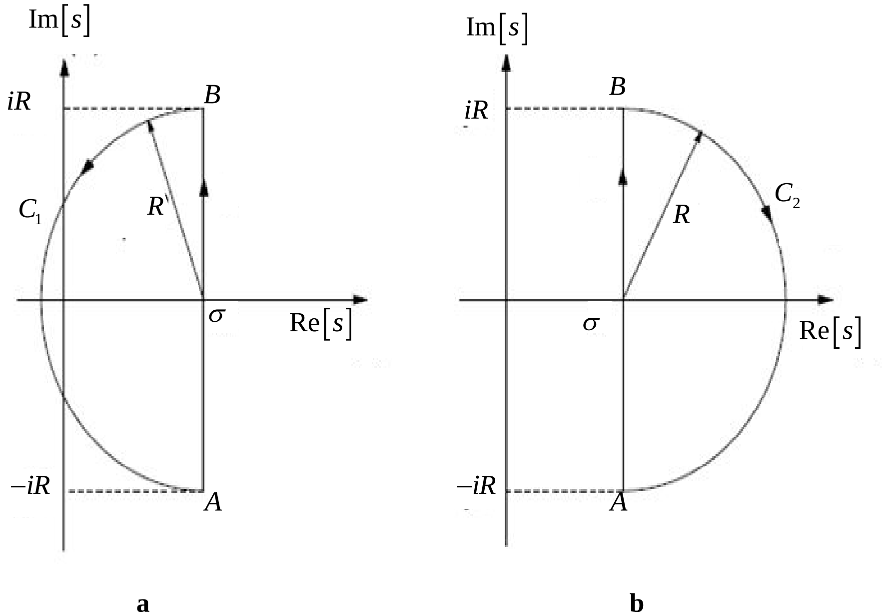

Figure-1 Contour in complex-plane to evaluate inverse Laplace transforms

Consider Figure-1a the closed contour is $A \to B \to C_1$, we write the contour integration as following

$$\int_{A\to B\to C_1} \mathrm{e}^{st} X(s)\mathrm{d}s = \int_{A\to B} \mathrm{e}^{st} X(s)\mathrm{d}s + \int_{C_1} \mathrm{e}^{st} X(s)\mathrm{d}s$$
$$= \int_{\sigma-iR}^{\sigma+iR} \mathrm{e}^{st} X(s)\mathrm{d}s + \int_{C_1} \mathrm{e}^{st} X(s)\mathrm{d}s$$
$$= 2\pi i \sum_{\text{poles}} \text{Residues}\left[\mathrm{e}^{st} X(s)\right]$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_1$ of Figure-1a. Now as $R \uparrow \infty$ the integral $\int_{\sigma-iR}^{\sigma+iR} \mathrm{e}^{st} X(s)\mathrm{d}s$ is the integral of interest that we require to evaluate inverse Laplace transform. We note that the integral on the line $A \to B$ is the Bromwich integral, and is for finding the inverse Laplace transform. We will use Jordan lemma (described shortly) which says for $t > 0$, $\lim_{R\uparrow\infty} \int_{C_1} \mathrm{e}^{st} X(s) \mathrm{d}s = 0$.

Therefore for $t > 0$ we write the following (as $R \uparrow \infty$)

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathrm{e}^{st} X(s) \mathrm{d}s = \sum_{\text{poles left of } \sigma} \text{Residues}\left[\mathrm{e}^{st} X(s)\right]$$

Consider Figure-1b the closed contour is $A \to B \to C_2$, we write the contour integration as following

$$\int_{A\to B\to C_2} \mathrm{e}^{st} X(s) \mathrm{d}s = \int_{A\to B} \mathrm{e}^{st} X(s) \mathrm{d}s + \int_{C_2} \mathrm{e}^{st} X(s) \mathrm{d}s$$

$$= \int_{\sigma - iR}^{\sigma + iR} \mathrm{e}^{st} X(s) \mathrm{d}s + \int_{C_2} \mathrm{e}^{st} X(s) \mathrm{d}s$$

$$= -2\pi i \sum_{\text{poles}} \text{Residues}\left[\mathrm{e}^{st} X(s)\right]$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_2$ of Figure-1b; and the negative sign indicate that contour is taken in clock-wise direction. We will use Jordan lemma (described shortly) which says for $t < 0$, $\lim_{R\uparrow\infty} \int_{C_2} \mathrm{e}^{st} X(s) \mathrm{d}s = 0$, thus we write for $t < 0$ the following

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathrm{e}^{st} X(s) \mathrm{d}s = -\sum_{\text{poles right of } \sigma} \text{Residues}\left[\mathrm{e}^{st} X(s)\right]$$

We know that this above integral must be zero, since for $t < 0$, we have $x(t) = 0$. Therefore the $\text{Re}\left[s\right] = \sigma$, the line AB (Figure-1), or Bromwich line, must be chosen such that contour of Figure-1b i.e. $A \to B \to C_2$ does not contain any poles of $\mathrm{e}^{st} X(s)$ as $R \uparrow \infty$; and thus the contour of Figure-1a, i.e. $A \to B \to C_1$ must have all poles of $\mathrm{e}^{st} X(s)$. This gives answer to point (1) above. Also we note that since e^{st} is analytic everywhere (i.e. it has no poles in entire complex-s plane), the poles of $\mathrm{e}^{st} X(s)$ are same as of $X(s)$. This gives reply to point (2) as posed above, thus we apply residue calculus of complex analysis and write

$$x(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathrm{e}^{st} X(s) \mathrm{d}s = \sum_{\text{all poles of } X(s)} \text{Residues}\left[\mathrm{e}^{st} X(s)\right]$$

This is how we need evaluate the integral for obtaining inverse Laplace transform.

The Jordan Lemma

While we discussed the inverse Laplace transform using contour integration and Residue calculus, we very well stated that we need a condition that is (arc in Figure-1a)

$$\lim_{R\uparrow\infty} \int_{C_1} \mathrm{e}^{st} X(s) \mathrm{d}s = 0 \qquad t > 0$$

While for $t > 0$ the points on this arc C_1 is given as

$$s = \sigma + R\,\mathrm{e}^{i\theta}; \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

Examining standard Laplace transform tables we observe most functions satisfy $\lim_{|s|\uparrow\infty} X(s)=0$; for example $X(s)=\frac{1}{s}$, $X(s)=\frac{1}{s+a}$, $X(s)=\frac{a}{s^2+a^2}$ etc. Therefore in those cases as $R\uparrow\infty$, $X(s)\downarrow 0$. This means that for any $M_R>0$ a radius R can be found such that $|X(s)|=|X(\sigma+R\mathrm{e}^{i\theta})|<M_R$. By using the inequality i.e. $\left|\int_C f(s)\mathrm{d}s\right|\le\int_C|f(s)|\mathrm{d}s$ for this R we have the following,

$$\left|\int_{C_1}\mathrm{e}^{st}X(s)\mathrm{d}s\right|\le\int_{C_1}\left|\mathrm{e}^{st}X(s)\right|\mathrm{d}s$$

$$\left|\int_{C_1}\mathrm{e}^{st}X(s)\mathrm{d}s\right|\le M_R\int_{C_1}\left|\mathrm{e}^{st}\right|\mathrm{d}s$$

On the arc C_1 for $t>0$, $s=\sigma+R\mathrm{e}^{i\theta}$; $\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$, $\mathrm{d}s=iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta$, we write the following

$$\begin{aligned}\left|\mathrm{e}^{st}\right|=\left|\mathrm{e}^{(\sigma+R\mathrm{e}^{i\theta})t}\right|&=\left|\mathrm{e}^{(\sigma+R\cos\theta+iR\sin\theta)t}\right|\\&=\left|\mathrm{e}^{(\sigma+R\cos\theta)t}\mathrm{e}^{iRt\sin\theta}\right|,\quad\left|\mathrm{e}^{i(Rt\sin\theta)}\right|=1\\&=\left|\mathrm{e}^{(\sigma+R\cos\theta)t}\right|,\quad\mathrm{e}^{(\sigma t+Rt\cos\theta)}>0\\&=\mathrm{e}^{\sigma t}\mathrm{e}^{Rt\cos\theta}\end{aligned}$$

Further we write the following steps

$$\begin{aligned}M_R\int_{C_1}\left|\mathrm{e}^{st}\mathrm{d}s\right|&=M_R\int_{\pi/2}^{3\pi/2}\left|\mathrm{e}^{\sigma t}\mathrm{e}^{Rt\cos\theta}iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta\right|\\&\le M_RR\mathrm{e}^{\sigma t}\int_{\pi/2}^{3\pi/2}\mathrm{e}^{Rt\cos\theta}\mathrm{d}\theta=2M_RR\mathrm{e}^{\sigma t}\int_{\pi/2}^{\pi}\mathrm{e}^{Rt\cos\theta}\mathrm{d}\theta\end{aligned}$$

Since the function i.e. $\mathrm{e}^{Rt\cos\theta}$ is even function we write the last step as above. By changing variable $\theta=\phi+\frac{\pi}{2}$ we obtain

$$\begin{aligned}M_R\int_{C_1}\left|\mathrm{e}^{st}\mathrm{d}s\right|&\le 2M_RR\mathrm{e}^{\sigma t}\int_0^{\pi/2}\mathrm{e}^{Rt\cos\left(\phi+\frac{\pi}{2}\right)}\mathrm{d}\phi\\&=2M_RR\mathrm{e}^{\sigma t}\int_0^{\pi/2}\mathrm{e}^{-Rt\sin\phi}\mathrm{d}\phi\end{aligned}$$

In the above equation, plotting the graphs of $y=\sin\phi$ and a straight line $y=\frac{2}{\pi}\phi$ says that in the region $0\le\phi\le\frac{\pi}{2}$, we see $\sin\phi\ge\frac{2}{\pi}\phi$; with this we write the following

$$\begin{aligned}M_R\int_{C_1}\left|\mathrm{e}^{st}\mathrm{d}s\right|&\le 2M_RR\mathrm{e}^{\sigma t}\int_0^{\pi/2}\mathrm{e}^{-Rt\sin\phi}\mathrm{d}\phi\\&\le 2M_RR\mathrm{e}^{\sigma t}\int_0^{\pi/2}\mathrm{e}^{-Rt\left(\frac{2}{\pi}\phi\right)}\mathrm{d}\phi=\frac{M_R\pi\mathrm{e}^{\sigma t}}{t}\left(1-\mathrm{e}^{-Rt}\right)\end{aligned}$$

Therefore, for any $t>0$ as $R\uparrow\infty$, we have $M_R\downarrow 0$, the above quantity tends to zero. This proves our case $\lim_{R\uparrow\infty}\int_{C_1}\mathrm{e}^{st}X(s)\mathrm{d}s=0$.

Application of Residue Calculus in obtaining inverse Laplace transform via contour integration

We would like to evaluate inverse Laplace transform of $X(s)=\frac{2\mathrm{e}^{-2s}}{s^2+4}$, the Bromwich path integral for inverse Laplace transform is

$$x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}\,\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{d}s$$
$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}\mathrm{d}s$$

We observe that the given function has two simple poles at $s=2i$ and $s=-2i$, both have $\mathrm{Re}\left[s\right]=0$. In Figure-1a, thus $\sigma=0$. We thus take an arbitrary (positive) small σ. We can distinguish the two cases (i) $t<2$ and (ii) $t>2$.

For $t<2$ the exponent $s(t-2)$ has negative real part if $\mathrm{Re}\left[s\right]>0$. We note that $\mathrm{e}^{s(t-2)}=\mathrm{e}^{(\mathrm{Re}[s]+i\,\mathrm{Im}[s])(t-2)}=\mathrm{e}^{(t-2)\mathrm{Re}[s]}\mathrm{e}^{i(t-2)\mathrm{Im}[s]}$, therefore the part $\mathrm{e}^{(t-2)\mathrm{Re}[s]}$ determines the function at infinity, since $\left|\mathrm{e}^{i(t-2)\mathrm{Im}[s]}\right|=1$. Therefore as $\mathrm{Re}\left[s\right]\uparrow\infty$ the function $\mathrm{e}^{s(t-2)}$ goes to zero. At the same time the denominator s^2+4 diverges as $\mathrm{Re}\left[s\right]\uparrow\infty$; and means the term $\frac{1}{s^2+4}$ that multiplies $\mathrm{e}^{s(t-2)}$ along the path C_2 (Figure-1b), for $R\uparrow\infty$. Therefore $\lim_{R\uparrow\infty}\int_{C_2}\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\mathrm{d}s=0$, i.e. integral on curve C_2 as $R\uparrow\infty$. We can calculate Bromwich path integral by considering $A\to B\to C_2$, but since this closed contour does not have any poles we say Residues are zero, resulting $x(t)$ as zero.

For $t>2$ the function $\mathrm{e}^{s(t-2)}$ goes to zero as $\mathrm{Re}\left[s\right]\downarrow-\infty$. That means the $\lim_{R\uparrow\infty}\int_{C_1}\frac{2\mathrm{e}^{(s-2)}}{s^2+4}\mathrm{d}s=0$ along the curve C_1 (Figure 1a) for $R\uparrow\infty$. For the residue theorem, this integral on the closed path $A\to B\to C_1$ is given by sum of residues of the function $\mathrm{e}^{st}X(s)=\frac{2\mathrm{e}^{(s-2)}}{s^2+4}$ at all the poles namely;

$$x(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\mathrm{d}s=\mathrm{Residue}_{\text{at } s=2i}\left(\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\right)+\mathrm{Residue}_{\text{at } s=-2i}\left(\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\right)$$
$$=\lim_{s=2i}(s-2i)\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}+\lim_{s=-2i}(s+2i)\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}$$
$$=\lim_{s=2i}\frac{2\mathrm{e}^{s(t-2)}}{s+2i}+\lim_{s=-2i}\frac{2\mathrm{e}^{s(t-2)}}{s-2i}=\frac{\mathrm{e}^{2i(t-2)}}{2i}-\frac{\mathrm{e}^{-2i(t-2)}}{2i}$$
$$=\sin\left(2(t-2)\right)$$

Thus we write

$$x(t)=\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}=\begin{cases}\sin\left(2(t-2)\right) & t>2\\ 0 & t<2\end{cases}$$

Application of Residue Calculus for Multi-valued function with branch cut on the complex plane

We take a function $X(s)=\sqrt{s-a}$, with $a\in\mathbb{R}$ .The inverse Laplace transform is following

$$x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}\sqrt{s-a}\,\mathrm{d}s$$

The function $e^{st}\sqrt{s-a}$ has no poles but the function $\sqrt{z}$ (taking $z=s-a$ is a multi-valued function in complex plane. Therefore we see a branch point at $z=0$, namely at $s=a$. This is the only singularity of our function $X(s)e^{st}$ and therefore to evaluate Bromwich path integral, we have to take σ larger than a. Thus integral will be

$$x(t)=\mathcal{L}^{-1}\left\{\sqrt{s-a}\right\}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{st}\sqrt{s-a}\,ds; \quad s-a=z$$

$$=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{(a+z)t}\sqrt{z}\,dz$$

$$=\frac{e^{at}}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{zt}\sqrt{z}\,dz$$

In this case the branch point is zero, therefore $\mathrm{Re}[z]=\lambda$ can be arbitrarily small (but always larger than zero). Since $z=0$ is a branch point of the function to integrate we have to introduce a branch cut to evaluate the integral. Although we have taken so far the positive real axis as branch cut, we have said that this choice is arbitrary and to make the function $\sqrt{z}$ single value it is enough that closed curves are not allowed enclosing the origin. We can therefore take a branch cut as the negative real axis. Figure-2 we indicate the contour used to integrate the given function. Since the closed contour $ABCDEFGA$ does not enclose any singularity, its integral is zero. To evaluate Bromwich path integral, (namely along AB) we have to calculate the integral along the arcs C,G along the straight lines D,F and along the circumference on the small circle E (Figure-2).

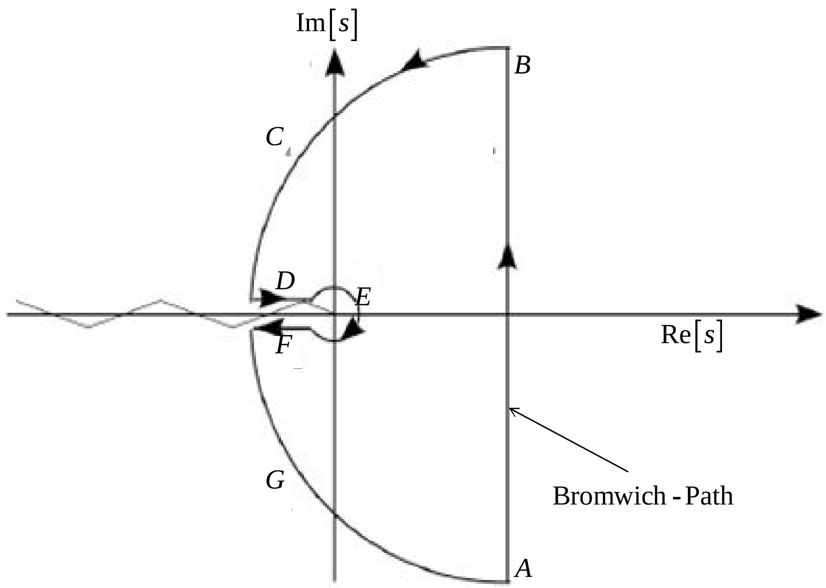

Figure-2: Contour of integration and Bromwich path on a branch cut complex plane

The function $e^{zt}\sqrt{z}$ goes to zero, for $\mathrm{Re}[z]\downarrow-\infty$ (the term $\sqrt{z}$ cannot match the exponential decay of e^{zt} for $t>0$ as always); thus the integral along the arcs C,G is zero that disappears as the radius of these arcs grows. To evaluate the integral along small circle E, we take $z = \in \mathrm{e}^{i\theta}$; $\pi \le \theta \le \pi$ and we take the limit $\in \downarrow 0$. With this we have

$$\int_E \mathrm{e}^{zt}\sqrt{z}\mathrm{d}z = \int_{\pi}^{-\pi} \mathrm{e}^{\in t\mathrm{e}^{i\theta}} \sqrt{\in}\mathrm{e}^{i\left(\frac{\theta}{2}\right)} i \in \mathrm{e}^{i\theta}\mathrm{d}\theta$$

The above integrating function clearly tends to zero for $\in \downarrow 0$, thus there is no contribution from the integration over the circumference of E.

Along the straight lines D, F we can assume that the arguments of the complex variables lying on them are π (along D) and $-\pi$ (along F) and that their imaginary parts are close to zero. Therefore we have $z = r\mathrm{e}^{i\pi}$ (for D) and $z = r\mathrm{e}^{-i\pi}$ (for F). Consequently $\mathrm{d}z = \mathrm{e}^{i\pi}\mathrm{d}r$ (for D) and $\mathrm{d}z = \mathrm{e}^{-i\pi}\mathrm{d}r$ (for F). We note $\mathrm{e}^{i\pi} = \mathrm{e}^{-i\pi} = -1$.

The parameter r runs between $+\infty$ and 0 for line D and between 0 and $+\infty$ for line F. The integrals are given as following

$$\int_D \sqrt{z}\mathrm{e}^{zt}\mathrm{d}z = \int_{\infty}^{0} \sqrt{r}\mathrm{e}^{i\left(\frac{\pi}{2}\right)}\mathrm{e}^{tr\mathrm{e}^{i\pi}}\mathrm{e}^{i\pi}\mathrm{d}r$$
$$= \int_{\infty}^{0} \sqrt{r}(i)(\mathrm{e}^{-rt})(-1)\mathrm{d}r = i\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$
$$\int_F \sqrt{z}\mathrm{e}^{zt}\mathrm{d}z = \int_0^{\infty} \sqrt{r}\mathrm{e}^{i\left(-\frac{\pi}{2}\right)}\mathrm{e}^{tr\mathrm{e}^{-i\pi}}\mathrm{e}^{-i\pi}\mathrm{d}r$$
$$= \int_0^{\infty} \sqrt{r}(-i)(\mathrm{e}^{-rt})(-1)\mathrm{d}r = i\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$

From the above calculations of contour integrations on C, D, E, F, G with residue theorem, we get the following (the closed contour Figure-2, A, B, C, D, E, F, G, A does not enclose any poles, so residue is zero)

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$$
$$= \mathcal{L}^{-1}\left\{\sqrt{s-a}\right\}$$
$$= \frac{1}{2\pi i}\int_{A\to B} \mathrm{e}^{st}\sqrt{s-a}\mathrm{d}s = -\frac{\mathrm{e}^{at}}{2\pi i}\int_{D+E} \sqrt{z}\mathrm{e}^{zt}\mathrm{d}z$$
$$= -\frac{\mathrm{e}^{at}}{\pi}\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$

In order to evaluate $\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$ put $rt = \tau^2$ and $\mathrm{d}r = \frac{2\tau\mathrm{d}\tau}{t}$, and write

$$\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r = \frac{1}{t^{3/2}}\int_0^{\infty} \tau\mathrm{e}^{-\tau^2}\, 2\tau\mathrm{d}\tau$$

We observe that $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{e}^{-\tau^2} = -2\tau\mathrm{e}^{-\tau^2}$, and thus we do integration by parts for above expression and write

$$\int_0^{\infty} \sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r = -\frac{1}{t^{3/2}}\left(\left[\tau e^{-\tau^2}\right]_0^{\infty} - \int_0^{\infty} \mathrm{e}^{-\tau^2}\mathrm{d}\tau\right)$$
$$= \frac{\sqrt{\pi}}{2t^{3/2}}$$

We used known result $\int_0^{\infty} \mathrm{e}^{-\tau^2}\mathrm{d}\tau = \frac{\sqrt{\pi}}{2}$ in above derivation. Now compactly we write the result

$$\mathcal{L}^{-1}\left\{\sqrt{s-a}\right\} = -\frac{e^{at}}{2\sqrt{\pi t^3}}$$

We can write form above derivation $\mathcal{L}^{-1}\left\{-\sqrt{s-a}\right\} = \frac{e^{at}}{2\sqrt{\pi t^3}}$, and thus we have following useful Laplace transform identity

$$\mathcal{L}^{-1}\left\{\sqrt{s-a}-\sqrt{s-b}\right\} = \frac{e^{bt}-e^{at}}{2\sqrt{\pi t^3}}$$

Inverse Laplace transformation without contour integration-Berberan Santo Method

The $x(t)$ inverse Laplace Transform of $X(s)$ is obtained by following formula

$$x(t) = \frac{e^{\sigma t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[X(\sigma+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[X(\sigma+i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega$$

Here $s=\sigma+i\omega$ where $\mathrm{Re}\left[s\right]=\sigma$ is the line, where right of that line there is no singularity. This above formula is Berberan-Santo formula. Here there is no requirement of contour integration, but comes directly from Bromwich integral i.e. integration on the line *AB* (Figure-1a);

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)\mathrm{d}s, \quad s=\sigma+i\omega, \quad \mathrm{d}s = i\mathrm{d}\omega$$

$$= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t + i\sin\omega t\right)\left(X(\sigma+i\omega)\right)\mathrm{d}\omega; \quad X = \mathrm{Re}\left[X\right]+i\,\mathrm{Im}\left[X\right]$$

$$x(t) = \mathrm{Re}\left[\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t + i\sin\omega t\right)\left(\mathrm{Re}\left[X\right]+i\,\mathrm{Im}\left[X\right]\right)\mathrm{d}\omega\right]$$

$$= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t - \mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega$$

Since $x(t)$ is real function we have only extracted the real part and say

$$\mathrm{Im}\left[\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t + i\sin\omega t\right)\left(\mathrm{Re}\left[X\right]+i\,\mathrm{Im}\left[X\right]\right)\mathrm{d}\omega\right] = 0$$

$$\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\sin\omega t + \mathrm{Im}\left[X\right]\cos\omega t\right)\mathrm{d}\omega = 0$$

But we have from definition of Laplace transform i.e. $X(s) = \int_0^{\infty}\left(x(t)\right)e^{-st}\mathrm{d}t$ and by putting $s=\sigma+i\omega$ we get following

$$X(\sigma+i\omega) = \int_0^{\infty}\left(x(t)\right)e^{-t(\sigma_0+i\omega)}\mathrm{d}t$$

$$= \int_0^{\infty}e^{-\sigma_0 t}\left(x(t)\right)\cos(\omega t)\mathrm{d}t - i\int_0^{\infty}e^{-\sigma_0 t}\left(x(t)\right)\sin(\omega t)\mathrm{d}t$$

This gives following

$$\mathrm{Re}\left[X\right]=\int_0^\infty \mathrm{e}^{-\sigma_0 t}\left(x(t)\right)\cos(\omega t)\mathrm{d}t$$

$$\mathrm{Im}\left[X\right]=-\int_0^\infty \mathrm{e}^{-\sigma_0 t}\left(x(t)\right)\sin(\omega t)\mathrm{d}t$$

We find that function $\mathrm{Re}\left[X\right]$ is even function, call it $e(\omega)$ in variable ω and the function $\mathrm{Im}\left[X\right]$ is odd function in variable ω, call it $o(\omega)$ .We get

$$\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)=\left(e(\omega)\right)\cos\omega t-\left(o(\omega)\right)\sin\omega t$$

as ‘even function’. That is even function $e(\omega)$ multiplied by even function i.e. $\cos\omega t$ gives even function, and odd function $o(\omega)$ multiplied by odd function i.e. $\sin\omega t$ gives even function. Therefore for overall even function integrand we have the integral as

$$x(t)=\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega$$

$$=\frac{\mathrm{e}^{\sigma t}}{\pi}\int_{0}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega$$