Tutorial Note on Kramers-Kronig Transformation its Application and Utility in determining Inverse Laplace Transformation for Causal Susceptibility Functions

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Abstract

A simple principle of nature that effect can only happen after cause, i.e. called causality has great mathematical treatment and development we term that as Kramers-Kronigs transformation, analyticity, Titchmarsh principle. Though in complex-analysis terms these principles looks very complicated here in this note we simplify the derivation of Kramers-Kronigs relations and obtain these expressions in time and frequency domains. Then as practical utility we utilize the same to have inverse Laplace transformation without the usual contour integration and residue calculus method-to get response function from given frequency domain data. We start from the basics of Impulse Response Function or Green's Function and then define generalized susceptibility. From there we elaborate by use of Fourier transformation techniques the Kramers-Kronig relations-in time domain and then in frequency domain. We develop the inverse Transform Method of Berberan-Santos technique, and apply the Kramers-Kronigs relation to simplify the same, for obtaining causal response function. In this note we are not discussing the details of causality vis-à-vis analyticity, we are keeping the treatment of Kramer-Kronig relation thus simple and practical. We use this derived Berberan-Santos method and obtain the simplified formulations to determine inverse Laplace transform for a causal response function, where the frequency dispersion data of either real or imaginary part of susceptibility function is known via experiments for positive frequencies. We give application of this technique of Berberan-Santos method to extract normalize electric field relaxation function, from frequency dispersion data for permittivity. This method gets applied to various fields, i.e. in impedance studies, in dielectric relaxation studies, in refractive index studies, in electric polarization studies, in magnetic systems studies, in stress-strain relaxation studies etc.

Keywords:
Generalized Susceptibility, Impulse Response Function, Hilbert-transform, Fourier Transform, Berberan-Santos method of inverse Laplace transform, Convolution

Introduction

Many observable quantities obey the Kramers-Kroning relations. For instance the electric susceptibility describes the electric polarization of a material responds to an applied electric field. This response must be causal so the real and imaginary parts of the electric susceptibility must be related by the Kramers-Kronig relations. This is also true for the magnetic susceptibility, the electrical conductivity, the thermal conductivity, and the dielectric constant, strain and stress compliance functions. Sometimes it is experimentally easier to measure the real part (or the imaginary part) of the susceptibility. The Kramer-Kronig relations can then be used to calculate the part that is difficult to measure. If both real and imaginary parts can be measured, it is possible to check for experimental errors using the Kramers-Kronig relations. If susceptibility is calculated theoretically, it is a good idea to check and see if it satisfies the Kramers-Kronig relations. It is considered a serious error to present a result that violates causality. The Kramers-Kronig relations describe how the real and imaginary parts of the susceptibility are related to each other. If either the real part or the imaginary part of the susceptibility is known for positive frequencies the entire susceptibility can be calculated at all frequencies (negative as well as positive). In this note we start with basics of greens function, i.e. impulse response function, and then develop the concept of generalized susceptibility. We then take the idea of susceptibility and compose the causal green's function in time domain, by non-causal even and odd functions, and then derive the relation of these parts. We see that the non-causal components are the Hilbert Transform of each other in time domain, and doing the Fourier transformation of these relations we arrive at Kramer-Kronigs relation in frequency domain. This frequency domain representation of Kramer-Kronigs relation are usual integral representations relating real part of susceptibility to the imaginary part of susceptibility function. After that we derive the technique of Berberan-Santos method of performing the inverse Laplace transform without performing usual contour integration and residue calculus. We apply the Kramer-Kronigs relations to this analytical method of Berberan-Santos and obtain the simplified formulations to determine inverse Laplace transform for a causal response function, where the frequency dispersion data of either real or imaginary part of susceptibility function is known for positive frequencies. We give application of this technique of Berberan-Santos method to extract normalize electric field relaxation function, from frequency dispersion data for permittivity.

Revising Impulse Response Function (Green's Function) of a system

Consider a particle of mass m moving in a viscous fluid. The linear first order differential equation that describes this system is following

$$m\frac{\mathrm{d}v(t)}{\mathrm{d}t}+bv(t)=F(t) \tag{1}$$

Here b is the damping constant, v is the velocity, and $F(t)$ is a driving force. A special case for the driving force is a δ-function force which strikes the system at $t=0$. Only after the force is struck, the system starts responding at $t>0$, given by function $v(t)$. We assume $v(t)=0$ for $t<0$. The solution to the differential equation for a δ-function drive force is called the 'impulse response function $g(t)$. The symbol g is used because the impulse response function is also called the Green's function. So we write the above dynamic equation as follows, in terms of greens function $v(t)\leftarrow g(t)$ i.e. response to delta function as input $F(t)\leftarrow\delta(t)$.

$$m\frac{\mathrm{d}g(t)}{\mathrm{d}t}+bg(t)=\delta(t) \tag{2}$$

The solution to this equation (or the Green's function) is following

$$g(t)=\frac{1}{m}\mathrm{e}^{-t/\tau}; \qquad \tau=m/b; \qquad t\geq 0 \tag{3}$$

Where τ is called the decay time-is also called 'Time-Constant'.

The velocity equation i.e. equation of motion as given in (1) has counter parts in Electric field relaxation phenomena described as

$$\tau\varepsilon \frac{\mathrm{d}\mathbf{E}(t)}{\mathrm{d}t} + \varepsilon\mathbf{E}(t) = \mathbf{D}(t) \tag{4}$$

Where $\mathbf{E}$ is Electric Field, which is Potential per unit distance $\mathbf{E} = \mathrm{V}/\mathrm{d}$, $\mathbf{D}$ is Dielectric Displacement of charges described as $\mathbf{D} = \mathrm{Q}/\mathrm{A}$ .Where Q is the charge, and A is electrode area, the V is the potential across the electrode plates separated by dielectric of distance d We have for a dielectric system $\mathbf{D} \equiv (\mathrm{Q}/\mathrm{A}) \propto (\mathrm{V}/\mathrm{d}) \equiv \mathbf{E}$, $\mathbf{D} = \varepsilon\mathbf{E}$. This Electric Field Relaxation Equation (4) also has a relaxing electric $\mathbf{E}(t) = \frac{1}{\varepsilon\tau}\mathrm{e}^{-t/\tau}$, for $t \geq 0$ with impulse forced Dielectric Displacement, $\mathbf{D}(t) = \delta(t)$ as applied as forcing function at $t = 0$; with condition $\mathbf{E}(t) = 0$ for $t < 0$.

Thus system Green's function gives the characteristic response or impulse response function. This $g(t)$ is always 'decaying type of function'-may be exponentially decaying, stretched exponential decaying function, may be decaying like power law, may be decaying as damped oscillatory function, or even may have sustained oscillation, may be decaying as per Mittag-Leffler function. The equation (1) and (4) describes single-time constant simple relaxation systems, and are called Debye relaxation. There could be complex relaxation phenomena too.

This Green's function or impulse response function will however never be growing type, and will be effective only at $t \geq 0$ and will be zero for $t < 0$. This is causality, i.e. cause will always preceded the effect. The cause is forcing function and effect is 'relaxation' of $v(t)$ or $\mathbf{E}(t)$, in our two described system-by first ordered differential equations. We can also have second order, or fractional order systems, and those will have different Green's function; however for causal system it will be decaying type. Generally thus, $\lim_{t\uparrow\infty} g(t) = 0$.

Utility of Impulse Response or Green's Function

The utility of the 'impulse response function' is that any driving force can be thought of as being built up of many δ-function forces.

$$F(t) = \int_{-\infty}^{\infty} \delta(t - t')F(t')\mathrm{d}t' \tag{5}$$

The above (5) integral is property of delta function. In convolution expression, the above (5) integral is expressed as $f(\mathrm{x}) = \delta(\mathrm{x}) * f(\mathrm{x})$ i.e.

$$\begin{aligned} f(\mathrm{x}_0) &= \int_{-\infty}^{\infty} \delta(\mathrm{x}_0 - \mathrm{x}) f(\mathrm{x})\mathrm{dx} = (\delta(\mathrm{x})) * (f(\mathrm{x})) \\ f(\mathrm{x}_0) &= \int_{-\infty}^{\infty} f(\mathrm{x}_0 - \mathrm{x})\delta(\mathrm{x})\mathrm{dx} = (\delta(\mathrm{x})) * (f(\mathrm{x})) \end{aligned} \tag{6}$$

By superposition, the response to a driving force $F(t)$ is a sum of the impulse response functions.

$$v(t) = \int_{-\infty}^{\infty} g(t - t')F(t')\mathrm{d}t' \tag{7}$$

For a differential equation if $g(t)$ is Green's function, i.e. solution to the homogeneous equation, or solution to driving force as impulse excitation, that we call impulse response function, then to any other forcing function driving the differential equation, the solution is convolution integral with Green's function, i.e. (7) $v(t) = g(t) * F(t) = \int_{-\infty}^{\infty} g(t - t')F(t')\mathrm{d}t$.

Driving Force as Harmonic Function

A special driving force is a harmonic driving force, $F(t) = F(\omega)\mathrm{e}^{i\omega t}$. The response will occur at the same frequency as the driving force, $v(t) = v(\omega)\mathrm{e}^{i\omega t}$. This is by the definition of linearity of systems. We could even choose $F(t)=F(\omega)\mathrm{e}^{-i\omega t}$; that we will discuss later. , but let us proceed with this $F(t)=F(\omega)\mathrm{e}^{i\omega t}$. To show this, insert a harmonic force into the equation above.

$$\begin{aligned} v(t) &= g(t)*F(t) \qquad F(t)=F(\omega)\mathrm{e}^{i\omega t} \\ &= \int_{-\infty}^{\infty} g(t-t')F(\omega)\mathrm{e}^{i\omega t'}\mathrm{d}t' \\ &= \int_{-\infty}^{\infty} g(t-t')F(\omega)\,\mathrm{e}^{i\omega t-i\omega t+i\omega t'}\,\mathrm{d}t' \\ &= \mathrm{e}^{i\omega t}\int_{-\infty}^{\infty} g(t-t')F(\omega)\mathrm{e}^{-i\omega(t-t')}\mathrm{d}t' \end{aligned} \tag{8}$$

Make a change of variables: $t''=t-t'$, $\mathrm{d}t''=-\mathrm{d}t'$ and reverse the limits of integration, to get $v(t)=\mathrm{e}^{i\omega t}F(\omega)\int_{-\infty}^{\infty} g(t'')\mathrm{e}^{-i\omega t''}\mathrm{d}t''$. The only time dependence of $v(t)$ is the factor of $\mathrm{e}^{i\omega t}$ because the t'' variable gets integrated out. Thus a harmonic driving force $F(\omega)\mathrm{e}^{i\omega t}$ produces a harmonic response $v(\omega)\mathrm{e}^{i\omega t}$ where $v(\omega)=F(\omega)\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-i\omega t}\mathrm{d}t$.

Defining Generalized Susceptibility

The 'generalized susceptibility' χ is the ratio of response to driving force, in our example is

$$\chi(\omega)=\frac{v(\omega)}{F(\omega)}=\int_{-\infty}^{\infty} g(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \tag{9}$$

The above expression suggests that the generalized susceptibility $\chi(\omega)$ is the Fourier transform of the impulse response function or Green's function $g(t)$; we write $\chi(\omega)=\mathcal{F}\{g(t)\}$.

For the case of a particle moving in a viscous (1) fluid, $g(t)=\frac{1}{m}\mathrm{e}^{-t/\tau}$, the generalized susceptibility is $\chi(\omega)=\mathcal{F}\left\{\frac{1}{m}\mathrm{e}^{-t/\tau}\right\}$.

$$\chi(\omega)=\frac{\tau}{m}\frac{(1-i\omega\tau)}{1+\omega^2\tau^2} \tag{10}$$

We used the Fourier Transform pair is $\mathcal{F}\left\{\mathrm{e}^{-at}u(t)\right\}=\frac{1}{a+i\omega}$ where $u(t)$ is Heaviside unit step function; this we have used in getting generalized susceptibility.

Another way to calculate the generalized susceptibility is to assume that the driving force and the response both have harmonic time dependence $v(t)=v(\omega)\,\mathrm{e}^{i\omega t}$ and $F(t)=F(\omega)\mathrm{e}^{i\omega t}$ substituting this form into the differential equation $m\left(\mathrm{d}v/\mathrm{d}t\right)+bv=F(t)$ yields,

$$\begin{aligned} & i\omega m v(\omega)+bv(\omega)=F(\omega) \\ & \chi(\omega)=\frac{v(\omega)}{F(\omega)}=\frac{1}{b+i\omega m}=\frac{1}{m}\left(\frac{1}{\frac{b}{m}+i\omega}\right), \quad \tau=\frac{m}{b} \\ & \quad =\frac{1}{m}\left(\frac{1}{\frac{1}{\tau}+i\omega}\right)=\frac{\tau}{m}\left(\frac{1}{1+i\omega\tau}\right)=\frac{\tau}{m}\left(\frac{1-i\omega\tau}{1+\omega^2\tau^2}\right) \end{aligned} \tag{11}$$

As pointed out earlier it is equally valid to assume that the harmonic dependencies of the drive and the response have the form $v(t) = v(\omega)\,\mathrm{e}^{-i\omega t}$ and $F(t) = F(\omega)\mathrm{e}^{-i\omega t}$. Notice here that the minus sign that has appeared in the exponent. With this choice, the imaginary part of the susceptibility changes sign and we have $\chi(\omega) = \frac{1}{-i\omega m + b} = \frac{\tau}{m}\left(\frac{1+i\omega\tau}{1+\omega^2\tau^2}\right)$.

In (11) we see $\mathrm{Re}\left[\chi(\omega)\right] = \frac{\tau}{m}\left(\frac{1}{1+\omega^2\tau^2}\right)$ is even function in ω. Also $\lim_{\omega\downarrow 0}\mathrm{Re}\left[\chi(\omega)\right] = (\tau/m)$, and $\lim_{\omega\uparrow\infty}\mathrm{Re}\left[\chi(\omega)\right] = 0$. We see $\mathrm{Im}\left[\chi(\omega)\right] = -\frac{\tau}{m}\left(\frac{\omega\tau}{1+\omega^2\tau^2}\right)$ is odd function in variable ω. We also note that $\lim_{\omega\downarrow 0}\mathrm{Im}\left[\chi(\omega)\right] = 0$ and $\lim_{\omega\uparrow\infty}\mathrm{Im}\left[\chi(\omega)\right] = 0$. These are characteristics of susceptibility function $\chi(\omega) = \mathrm{Re}\left[\chi(\omega)\right] + i\,\mathrm{Im}\left[\chi(\omega)\right]$.

In the examples so far for relaxation dynamics (1), (4) we have seen first order differential equations, and we described from them the concept of susceptibility $\chi(\omega)$. The relaxation system can be second order or even fractional order differential equation. Say we take for example $mx^{(2)} + mx^{(1)}/\tau + m\omega_0^2 x = F(t)$, where $x \equiv x(t)$ position variable in time. The $x^{(2)} \equiv \mathrm{d}^2 x(t)/\mathrm{d}t^2$ i.e. double derivative w.r.t. time and $x^{(1)} \equiv \mathrm{d}\;x(t)/\mathrm{d}t$ is first derivative. This is a damped oscillator system where Forcing function is $F(t)$. Exciting this system with a harmonic force of a single frequency ω i.e. $F(t) = F_0\,\mathrm{e}^{-i\omega t}$, we will get $x(t) = \chi(\omega)F_0\mathrm{e}^{-i\omega t}$. Where $\chi(\omega) = -\frac{1}{m}\left(\frac{1}{\omega^2 + i\omega/\tau - \omega_0^2}\right) = \frac{(-1/m)}{(\omega-\omega_1)(\omega-\omega_2)}$. We have $\omega_{1,2} = (-i/2\tau) \pm \tilde{\omega}$ with $\tilde{\omega} = \sqrt{\omega_0^2 - (1/4\tau^2)}$. Here the response function $\chi(\omega)$ has two simple poles at ω_1 and ω_2. Fourier inverse of $\chi(\omega)$ gives $x(t) = \left(\frac{1}{m\tilde{\omega}}\mathrm{e}^{-t/\tau}\sin\tilde{\omega}t\right)u(t)$; where $u(t)$ is unit step function. This response is response to force $F(t) = \delta(t)$ unit impulse at $t = 0$ gives green's function $g(t) = \left(\frac{1}{m\tilde{\omega}}\mathrm{e}^{-t/\tau}\sin\tilde{\omega}t\right)$ for $t \geq 0$.

Composing the Causal Impulse Response Function as non-Causal Even and Odd Functions

To be of the causal nature of the impulse response function $g(t)$ (it has to be zero for $t < 0$). This causality has consequences for the form of the susceptibility $\chi(\omega)$. Any function say $g(t)$ can be written in terms of an even component $g_e(t)$ and an odd component $g_o(t)$. We write our impulse response function $g(t)$ as following composition

$$g(t) = g_e(t) + g_o(t) \tag{12}$$

Since the impulse response function must be zero for $t < 0$ the even and the odd components must add to zero for $t < 0$. We have in our example (1) $g(t) = \frac{1}{m}\mathrm{e}^{-t/\tau}$, for $t \geq 0$ and $g(t) = 0$ for $t < 0$. Considering $m = 1$, we have even and odd components for $g(t)$ as

$$g_e(t) = \tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0} + \tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0} \qquad g_o(t) = -\tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0} + \tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0} \tag{13}$$

We note that in though the Green's function $g(t)$ is causal, i.e. it appears as response at $t \geq 0$, and not for $t < 0$, the $g_e(t)$ and $g_o(t)$ are non-causal (13), i.e. they are defined for $t < 0$ as well as $t \geq 0$. By this construct (12) we get $g_e(t) + g_0(t) = 0$ for $t < 0$ and $g_e(t) + g_0(t) = \mathrm{e}^{-t/\tau} = g(t)$ for $t \geq 0$. We note that $g_e(t)$ is continuous at $t = 0$, and we have $g_e(0^+) = \mathrm{g}_e(0^-) = \frac{1}{2}$, whereas $g_o(t)$ is discontinuous at $t = 0$, with $g_o(0^-) = -\frac{1}{2}$ and $g_o(0^+) = \frac{1}{2}$. Therefore we have composed causal function in terms of anti-causal functions (12), (13).

We apply $\mathrm{sgn}(t)$ function to $g_o(t)$ and $g_e(t)$ to get the following, noting that $\mathrm{sgn}(t) = -1$ for $t < 0$ and $\mathrm{sgn}(t) = +1$ for $t \geq 0$.

$$\begin{aligned}(\mathrm{sgn}(t))(g_o(t)) &= (\mathrm{sgn}(t))\left(-\tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0}\right) + (\mathrm{sgn}(t))\left(\tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0}\right) \\ &= \tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0} + \tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0} = g_e(t)\end{aligned} \tag{14}$$

$$\begin{aligned}(\mathrm{sgn}(t))(g_e(t)) &= (\mathrm{sgn}(t))\left(\tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0}\right) + (\mathrm{sgn}(t))\left(\tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0}\right) \\ &= -\tfrac{1}{2}\mathrm{e}^{t/\tau}\Big|_{t<0} + \tfrac{1}{2}\mathrm{e}^{-t/\tau}\Big|_{t\geq 0} = g_o(t)\end{aligned} \tag{15}$$

Note that if we know the either the even component or the odd component of $g(t)$ we construct the other, by following rule

$$\begin{aligned}g_e(t) &= (\mathrm{sgn}(t))(g_0(t)) = \frac{1}{2}\left(g(-t)|_{t<0} + g(t)|_{t\geq 0}\right) \\ g_o(t) &= (\mathrm{sgn}(t))(g_e(t)) = \frac{1}{2}\left(-g(-t)|_{t<0} + g(t)|_{t\geq 0}\right)\end{aligned} \tag{16}$$

Taking Fourier Transform of Even and Odd Components

We take Fourier Transform and write the following for $\chi(\omega)$ the generalized susceptibility function, defined as real and imaginary parts as follows

$$\chi(\omega) = \mathrm{Re}\left[\chi(\omega)\right] + i\,\mathrm{Im}\left[\chi(\omega)\right] \tag{17}$$

From (9) we write $\chi(\omega) = \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-i\omega t}\mathrm{d}t$ and using (12) we have following steps

$$\begin{aligned}\chi(\omega) &= \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\left(g_e(t)+g_o(t)\right)\mathrm{e}^{-i\omega t}\mathrm{d}t = \int_{-\infty}^{\infty}\left(g_e(t)+g_o(t)\right)\left(\cos\omega t - i\sin\omega t\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g_e(t)\cos\omega t\mathrm{d}t - i\int_{-\infty}^{\infty} g_o(t)\sin\omega t\mathrm{d}t = \mathrm{Re}\left[\chi(\omega)\right] + i\,\mathrm{Im}\left[\chi(\omega)\right]\end{aligned} \tag{18}$$

Because $g_e(t)\sin\omega t$ is odd function and $\int_{-\infty}^{\infty} g_e(t)\sin\omega t = 0$ also $g_o(t)\cos\omega t$ is odd function and $\int_{-\infty}^{\infty} g_o(t)\cos\omega t\mathrm{d}t = 0$ these terms do not appear in above (18) steps. Therefore we write

$$\mathrm{Re}\left[\chi(\omega)\right] = \int_{-\infty}^{\infty} g_e(t)\cos\omega t\mathrm{d}t \qquad \mathrm{Im}\left[\chi(\omega)\right] = -\int_{-\infty}^{\infty} g_o(t)\sin\omega t\mathrm{d}t \tag{19}$$

Moreover $\mathrm{Re}\left[\chi(\omega)\right]$ is an even function $\mathrm{Re}\left[\chi(\omega)\right] = \mathrm{Re}\left[\chi(-\omega)\right]$ while $\mathrm{Im}\left[\chi(\omega)\right]$ is an odd function $\mathrm{Im}\left[\chi(\omega)\right] = -\mathrm{Im}\left[\chi(-\omega)\right]$.

The Kramers-Kronig relations in time domain

The Kramers-Kronig relations describe how the real and imaginary parts of the susceptibility are related to each other. If either the real part or the imaginary part of the susceptibility is known for positive frequencies $\omega > 0$ the entire susceptibility can be calculated at all frequencies (negative as well as positive). Suppose we know $\mathrm{Re}\left[\chi(\omega)\right]$ for $\omega > 0$. Then $\mathrm{Re}\left[\chi(\omega)\right]$ for all frequencies can be constructed because $\mathrm{Re}\left[\chi(\omega)\right] = \mathrm{Re}\left[\chi(-\omega)\right]$.The even component of the impulse response function $g_e(t)$ can be found by inverse Fourier 'cosine' transform of $\mathrm{Re}\left[\chi(\omega)\right]$, that is following

$$\mathrm{Re}\left[\chi(\omega)\right] = \int_{-\infty}^{\infty} g_e(t)\cos\omega t\mathrm{d}t \qquad g_e(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[\chi(\omega)\right]\cos\omega t\mathrm{d}\omega \tag{20}$$

The odd component of the impulse response function $g_o(t)$ is related to the even component by $g_o(t) = \left(\mathrm{sgn}(t)\right)\left(g_e(t)\right)$ (14), (15) (16) that we re-write as follows

$$g_o(t) = \left(\mathrm{sgn}(t)\right)\left(g_e(t)\right) \qquad g_e(t) = \left(\mathrm{sgn}(t)\right)\left(g_o(t)\right) \tag{21}$$

This above (21) relation is Kramers-Kronigs relation in time domain. Thus we see odd component is Hilbert Transform of even component and vice-versa (21).

The imaginary part of the susceptibility $\mathrm{Im}\left[\chi(\omega)\right]$ can then be constructed since it is the Fourier 'sine' transform of the odd component.

$$\mathrm{Im}\left[\chi(\omega)\right] = -\int_{-\infty}^{\infty} g_o(t)\sin\omega t\mathrm{d}t \qquad g_o(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[\chi(\omega)\right]\sin\omega t\mathrm{d}\omega \tag{22}$$

Many observable quantities obey the Kramers-Kroning relations. For instance the electric susceptibility describes the electric polarization of a material responds to an applied electric field. This response must be causal so the real and imaginary parts of the electric susceptibility must be related by the Kramers-Kronig relations. This is also true for the electric modulus, magnetic susceptibility, the electrical conductivity, the thermal conductivity, and the dielectric constant, stress compliance etc.

Sometimes it is experimentally easier to measure the real part (or the imaginary part) of the susceptibility. The Kramer-Kronig relations then are utilized to calculate the part that is difficult to measure. If both real and imaginary parts can be measured, it is possible to check for experimental errors using the Kramers-Kronig relations. If a susceptibility is calculated theoretically, it is a good idea to check and see if it satisfies the Kramers-Kronig relations. It should be considered a serious error to present a result that violates causality-and Kramer-Kronig relations.

Kramers-Kronig relation in frequency domain

The Kramers-Kronig relations in the frequency domain-are traditionally expressed. This unfortunately introduces a singularity in the formula. The singularity in the integral makes the form that is given below (23) less suitable for a numerical evaluation of the Kramers-Kronig relation. Nevertheless, it commonly appears in the literature and is given for completeness.

$$\mathrm{Re}\left[\chi(\omega)\right] = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left[\chi(\omega')\right]}{\omega'-\omega}\mathrm{d}\omega', \qquad \mathrm{Im}\left[\chi(\omega)\right] = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left[\chi(\omega')\right]}{\omega'-\omega}\mathrm{d}\omega' \tag{23}$$

We get above (23) expression by using $g_o(t) = \left(\mathrm{sgn}(t)\right)\left(g_e(t)\right)$ and $g_e(t) = \left(\mathrm{sgn}(t)\right)\left(g_o(t)\right)$ (21) by Fourier transforming and using convolution theorem.

$$\begin{aligned} \mathcal{F}\left\{g_o(t)\right\} &= \left(\mathcal{F}\left\{\left(\mathrm{sgn}(t)\right)\right\}\right) * \left(\mathcal{F}\left\{\left(g_e(t)\right)\right\}\right) \\ \mathcal{F}\left\{g_e(t)\right\} &= \left(\mathcal{F}\left\{\left(\mathrm{sgn}(t)\right)\right\}\right) * \left(\mathcal{F}\left\{\left(g_o(t)\right)\right\}\right) \end{aligned} \tag{24}$$

We find Fourier transform of $g_e(t)$ as follows

$$\begin{aligned}\mathcal{F}\{g_e(t)\} &= \int_{-\infty}^{\infty} g_e(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\left(g_e(t)\right)\mathrm{e}^{-i\omega t}\mathrm{d}t = \int_{-\infty}^{\infty}\left(g_e(t)\right)\left(\cos\omega t - i\sin\omega t\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g_e(t)\cos\omega t\mathrm{d}t - i\int_{-\infty}^{\infty} g_e(t)\sin\omega t\,\mathrm{d}\,t \\ &= \mathrm{Re}\left[\chi(\omega)\right]\end{aligned} \quad (25)$$

Since $g_e(t)\sin\omega t$ is odd function, the second term is zero above (25) i.e. $\int_{-\infty}^{\infty} g_e(t)\sin\omega t\,\mathrm{d}\,t = 0$.

We find Fourier Transform of $g_o(t)$ as follows

$$\begin{aligned}\mathcal{F}\{g_o(t)\} &= \int_{-\infty}^{\infty} g_o(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\left(g_o(t)\right)\mathrm{e}^{-i\omega t}\mathrm{d}t = \int_{-\infty}^{\infty}\left(g_o(t)\right)\left(\cos\omega t - i\sin\omega t\right)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} g_o(t)\cos\omega t\mathrm{d}t - i\int_{-\infty}^{\infty} g_o(t)\sin\omega t\,\mathrm{d}\,t \\ &= -i\,\mathrm{Im}\left[\chi(\omega)\right]\end{aligned} \quad (26)$$

Since $g_o(t)\cos\omega t\mathrm{d}t$ is odd function, the first term above (26) is zero i.e. $\int_{-\infty}^{\infty} g_o(t)\cos\omega t\mathrm{d}t = 0$.

We have Fourier transform of $\mathrm{sgn}(t)$ as $\mathcal{F}\{\mathrm{sgn}(t)\} = -\frac{i}{\pi\omega}$, and use above (25), (26) derived $\mathcal{F}\{g_e(t)\}$ and $\mathcal{F}\{g_o(t)\}$ to get the following using convolution integral

$$\begin{aligned}\mathcal{F}\{g_o(t)\} &= \left(\mathcal{F}\{(\mathrm{sgn}(t))\}\right) * \left(\mathcal{F}\{(g_e(t))\}\right) \\ -i\,\mathrm{Im}\left[\chi(\omega)\right] &= \left(-\frac{i}{\pi\omega}\right) * \left(\mathrm{Re}\left[\chi(\omega)\right]\right) \\ \mathrm{Im}\left[\chi(\omega)\right] &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\omega' - \omega}\right)\mathrm{Re}\left[\chi(\omega')\right]\mathrm{d}\omega'\end{aligned} \quad (27)$$

and

$$\begin{aligned}\mathcal{F}\{g_e(t)\} &= \left(\mathcal{F}\{(\mathrm{sgn}(t))\}\right) * \left(\mathcal{F}\{(g_o(t))\}\right) \\ \mathrm{Re}\left[\chi(\omega)\right] &= \left(-\frac{i}{\pi\omega}\right) * \left(-i\,\mathrm{Im}\left[\chi(\omega)\right]\right) \\ &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\omega' - \omega}\right)\mathrm{Im}\left[\chi(\omega')\right]\mathrm{d}\omega'\end{aligned} \quad (28)$$

We have thus obtained frequency in (27) and (28) domain representation of Krammer-Kronig relation (23). This is in integration in both sides of frequency domain $-\infty < \omega < \infty$, can be further simplified to one sided integration for $0 < \omega < \infty$. We are not discussing this in this note.

Basic formula to get inverse Laplace Transform without doing contour integration

The $x(t)$ inverse Laplace Transform of $X(s)$, that is $x(t) = \mathcal{L}^{-1}\{X(s)\}$ is obtained by following formula

$$x(t) = \frac{e^{\sigma t}}{\pi}\int_{0}^{\infty}\Big(\mathrm{Re}\big[X(\sigma + i\omega)\big]\cos(\omega t) - \mathrm{Im}\big[X(\sigma + i\omega)\big]\sin(\omega t)\Big)\mathrm{d}\omega \tag{29}$$

Here $s = \sigma + i\omega$ where $\mathrm{Re}[s] = \sigma$ is the line, where right of that line there is no singularity, in the function $X(s)$.

Considering $x(t)$ as Causal response function like Electric Field Relaxation function, then the above formula (29) gets simplified to following

$$x(t) = \frac{2}{\pi}\int_{0}^{\infty}\Big(\mathrm{Re}\big[X\big]\cos\omega t\Big)\mathrm{d}\omega \tag{30}$$

We consider in above $\mathrm{Re}[s] = \sigma = 0$ with no singularity expected at Right-Half Plane of complex Laplace Domain, i.e. s -plane; and writing $\mathrm{Re}\big[X(\sigma + i\omega)\big] = \mathrm{Re}\big[X\big]$. We will derive this above expression (30).

Derivation of Berberan-Santos Formula

This above formula is Berberan-Santo formula (29), (30). Here there is no requirement of contour integration and use of Residue Calculus, but comes directly from Bromwich integral as follows

$$\begin{aligned}
x(t) &= \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}X(s)\mathrm{d}s, \quad s = \sigma + i\omega, \quad \mathrm{d}s = i\mathrm{d}\omega \\
&= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\big(\cos\omega t + i\sin\omega t\big)\big(X(\sigma + i\omega)\big)\mathrm{d}\omega; \quad X = \mathrm{Re}\big[X\big] + i\,\mathrm{Im}\big[X\big] \\
x(t) &= \mathrm{Re}\left[\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\big(\cos\omega t + i\sin\omega t\big)\big(\mathrm{Re}\big[X\big] + i\,\mathrm{Im}\big[X\big]\big)\mathrm{d}\omega\right] \\
&= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\big[X\big]\cos\omega t - \mathrm{Im}\big[X\big]\sin\omega t\Big)\mathrm{d}\omega
\end{aligned} \tag{31}$$

Since $x(t)$ is real function we have only extracted the real part and say

$$\begin{aligned}
&\mathrm{Im}\left[\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\big(\cos\omega t + i\sin\omega t\big)\big(\mathrm{Re}\big[X\big] + i\,\mathrm{Im}\big[X\big]\big)\mathrm{d}\omega\right] = 0 \\
&\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\Big(\mathrm{Re}\big[X\big]\sin\omega t + \mathrm{Im}\big[X\big]\cos\omega t\Big)\mathrm{d}\omega = 0
\end{aligned} \tag{32}$$

The above (32) also comes from observation that when the susceptibility functions $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\mathrm{d}t$ in this case $X(\omega) = \mathrm{Re}\big[X(\omega)\big] + i\,\mathrm{Im}\big[X(\omega)\big]$, has $\mathrm{Re}\big[X\big]$ as even function and $\mathrm{Im}\left[X\right]$ as odd function, for a causal response function $x(t)$. This we have noted in earlier section, of describing susceptibility in general terms.

Composing the function of time as even and odd components and their Fourier transformation

Consider Fourier Transform of $x(t)$ that we write as following

$$X(\omega)=\int_{-\infty}^{\infty}x(t)\mathrm{e}^{-i\omega t}\mathrm{d}t \tag{33}$$

Any function $x(t)$ can be composed as even component $x_e(t)$ and odd component $x_o(t)$.

$$x(t)=x_e(t)+x_o(t) \tag{34}$$

For Causal function since $x(t)=0$ for $t<0$, the even and odd parts must add to zero for $t<0$. Note that if we know either the even or the odd component we have the following

$$\begin{aligned} x_e(t)&=\left(\mathrm{sgn}(t)\right)\left(x_o(t)\right)=\frac{1}{2}\left(x(-t)\big|_{t<0}+x(t)\big|_{t\geq 0}\right)\\ x_0(t)&=\left(\mathrm{sgn}(t)\right)\left(x_0(t)\right)=\frac{1}{2}\left(-x(-t)\big|_{t<0}+x(t)\big|_{t\geq 0}\right)\end{aligned} \tag{35}$$

With this we write Fourier transform of $x(t)$ as we did in (18)

$$\begin{aligned} X(\omega)&=\int_{-\infty}^{\infty}x(t)\mathrm{e}^{-i\omega t}\mathrm{d}t\\ &=\int_{-\infty}^{\infty}\left(x_e(t)+x_o(t)\right)\left(\cos\omega t-i\sin\omega t\right)\mathrm{d}t\end{aligned} \tag{36}$$

This enables us to write the following cosine and sine Fourier transforms and pairs as following

$$\begin{aligned} \mathrm{Re}\left[X\right]&=\int_{-\infty}^{\infty}x_e(t)\cos\omega t\mathrm{d}t \qquad & x_e(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega\\ \mathrm{Im}\left[X\right]&=-\int_{-\infty}^{\infty}x_o(t)\sin\omega t\mathrm{d}t \qquad & x_o(t)&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega\end{aligned} \tag{37}$$

We see from above (37) $\mathrm{Re}\left[X\right]$ is even function and $\mathrm{Im}\left[X\right]$ as odd function in variable ω .

Also we have from definition of Laplace transform i.e. $X(s)=\int_0^{\infty}\left(x(t)\right)\mathrm{e}^{-st}\mathrm{d}t$ and by putting $s=\sigma+i\omega$ we get following

$$\begin{aligned} X(\sigma+i\omega)&=\int_0^{\infty}\left(x(t)\right)\mathrm{e}^{-t(\sigma+i\omega)}\mathrm{d}t\\ &=\int_0^{\infty}\mathrm{e}^{-\sigma t}\left(x(t)\right)\cos(\omega t)\mathrm{d}t-i\int_0^{\infty}\mathrm{e}^{-\sigma t}\left(x(t)\right)\sin(\omega t)\mathrm{d}t\end{aligned} \tag{38}$$

This gives following

$$\mathrm{Re}\left[X\right]=\int_0^{\infty}\mathrm{e}^{-\sigma t}\left(x(t)\right)\cos(\omega t)\mathrm{d}t \qquad \mathrm{Im}\left[X\right]=-\int_0^{\infty}\mathrm{e}^{-\sigma t}\left(x(t)\right)\sin(\omega t)\mathrm{d}t \tag{39}$$

We find that function $\mathrm{Re}\left[X\right]$ is even function, call it $e(\omega)$ in variable ω and the function $\mathrm{Im}\left[X\right]$ is odd function in variable ω , call it $o(\omega)$ .We get

$$\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)=\left(e(\omega)\right)\cos\omega t-\left(o(\omega)\right)\sin\omega t \tag{40}$$

as 'even function'. That is even function $e(\omega)$ multiplied by even function i.e. $\cos\omega t$ gives even function, and odd function $o(\omega)$ multiplied by odd function i.e. $\sin\omega t$ gives even function. Therefore for overall even function integrand we have the integral as

$$\begin{aligned} x(t) &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega \\ &= \frac{\mathrm{e}^{\sigma t}}{\pi}\int_{0}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega \end{aligned} \tag{41}$$

Since $\mathrm{Re}\left[X\right]$ is even we have $\mathrm{Re}\left[X\right]\sin\omega t$ as odd function, and thus $\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\sin\omega t\mathrm{d}\omega=0$. We have $\mathrm{Im}\left[X\right]$ as odd function, so $\mathrm{Im}\left[X\right]\cos\omega t$ is odd function, and thus $\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\cos\omega t\mathrm{d}\omega=0$. This justifies also $\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\sin\omega t+\mathrm{Im}\left[X\right]\cos\omega t\right)\mathrm{d}\omega=0$ (32) that we have taken earlier, i.e. $\mathrm{Im}\left[\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t+i\sin\omega t\right)\left(\mathrm{Re}\left[X\right]+i\,\mathrm{Im}\left[X\right]\right)\mathrm{d}\omega\right]=0$.

Considering $x(t)=x_e(t)+x_o(t)$ (34) and using $x_o(t)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega$ (37) we write the following steps

$$\begin{aligned} x(t) &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega \\ &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega-\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega;\quad \int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega=-2\pi x_0(t) \\ &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega+\frac{\mathrm{e}^{\sigma t}}{2\pi}2\pi\left(x_0(t)\right);\quad x_0(t)=\left(\mathrm{sgn}(t)\right)\left(x_e(t)\right) \\ &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega+\mathrm{e}^{\sigma t}\left(\mathrm{sgn}(t)\right)\left(x_e(t)\right);\quad x_e(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega \\ &= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega+\frac{\mathrm{e}^{\sigma t}}{2\pi}\left(\mathrm{sgn}(t)\right)\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega \end{aligned} \tag{42}$$

Using $\mathrm{sgn}(t)=1$ for $t>0$ and $\mathrm{sgn}(t)=-1$ for $t<0$ we write following from (42)

$$x(t)=\begin{cases} \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega+\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega & ;\quad t>0 \\ \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega-\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega & ;\quad t<0 \end{cases} \tag{43}$$

$$x(t)=\begin{cases} \frac{\mathrm{e}^{\sigma t}}{\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega & ;\quad t>0 \\ 0 & ;\quad t<0 \end{cases} \tag{44}$$

Using observation that $\mathrm{Re}\left[X\right]$ is even function in ω we have $\mathrm{Re}\left[X\right]\cos\omega$ as even function, and thus $\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega=2\int_{0}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega$ write the following

$$x(t)=\begin{cases} 2\dfrac{e^{\sigma t}}{\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega & ; \quad t>0 \\ 0 & ; \quad t<0 \end{cases} \tag{45}$$

On similar lines (43)-(45) we can derive following steps

$$\begin{aligned}
x(t) &= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X\right]\cos\omega t-\mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega \\
&= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega\mathrm{d}\omega-\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega; \quad \int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega=2\pi x_e(t) \\
&= \frac{e^{\sigma t}}{2\pi}2\pi\left(x_e(t)\right)-\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega; \quad x_e(t)=\left(\mathrm{sgn}(t)\right)\left(x_o(t)\right) \\
&= e^{\sigma t}\left(\mathrm{sgn}(t)\right)\left(x_o(t)\right)-\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega; \quad x_o(t)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega \\
&= \left(\mathrm{sgn}(t)\right)\left(-\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega\right)-\frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega
\end{aligned} \tag{46}$$

Using $\mathrm{sgn}(t)=1$ for $t>0$ and $\mathrm{sgn}(t)=-1$ for $t<0$ we write the following from (46)

$$x(t)=\begin{cases} \left(-\dfrac{e^{\sigma t}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega\right)-\dfrac{e^{\sigma t}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Re}\left[X\right]\cos\omega t\mathrm{d}\omega & ; \quad t>0 \\ \dfrac{e^{\sigma t}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega-\dfrac{e^{\sigma t}}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega & ; \quad t<0 \end{cases} \tag{47}$$

Using the observation that $\mathrm{Im}\left[X\right]$ is odd function then $\mathrm{Im}\left[X\right]\sin\omega t$ is even and we have $\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega=2\int_{0}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega$ we write following

$$x(t)=\begin{cases} -2\dfrac{e^{\sigma t}}{\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{Im}\left[X\right]\sin\omega t\mathrm{d}\omega & ; \quad t>0 \\ 0 & ; \quad t<0 \end{cases} \tag{48}$$

Interpretation

Thus we have following formula derived from Berberan-Santos method if Inverse Laplace Transformation, where $X(s)=\mathcal{L}\left\{x(t)\right\}$, for a causal response function $x(t)$, which obeys Kramer-Kronigs relation. Considering $\sigma=0$, i.e. expecting no singularity in $X(s)$ at the Right Half Plane of complex plane $s=\sigma+i\omega$ from (45) and (48) we write the following.

$$x(t) = \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[X(\omega)\right]\cos\omega t - \mathrm{Im}\left[X(\omega)\right]\sin\omega t\right)\mathrm{d}\omega$$

$$= 2\frac{e^{\sigma t}}{\pi}\int_{0}^{\infty}\left(\mathrm{Re}\left[X(\omega)\right]\cos\omega t\right)\mathrm{d}\omega = -2\frac{e^{\sigma t}}{\pi}\int_{0}^{\infty}\left(\mathrm{Im}\left[X(\omega)\right]\sin\omega t\right)\mathrm{d}\omega \tag{49}$$

$$\sigma = 0 \qquad x(t) = \frac{2}{\pi}\int_{0}^{\infty}\left(\mathrm{Re}\left[X(\omega)\right]\cos\omega t\right)\mathrm{d}\omega$$

$$\sigma = 0 \qquad x(t) = -\frac{2}{\pi}\int_{0}^{\infty}\left(\mathrm{Im}\left[X(\omega)\right]\sin\omega t\right)\mathrm{d}\omega$$

That is for data of frequency dispersion of $\mathrm{Im}\left[X(\omega)\right]$ (or $\mathrm{Re}\left[X(\omega)\right]$) for frequency $\omega > 0$ given we can extract $x(t)$ for $t > 0$ by above formula.

Application in extracting Electric Relaxation Function from Frequency dispersion of dissipation data by using inverse Laplace Transform of Berberan-Santos formula

The expression for relaxation of Electric Modulus in Laplace domain is following

$$\frac{M^*(s)}{M_\infty} = 1 - \int_0^\infty e^{-st}\left(-\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)\mathrm{d}t \tag{50}$$

Where $M^*(s)$ is complex Electric Modulus M_∞ is constant or called steady state value of electric modulus. The $\phi(t)$ is 'normalized' Electric Field Decay function. Using the definition of Laplace transform i.e. $F(s) = \mathcal{L}\{f(t)\}$ that is $F(s) = \int_0^\infty e^{-st}\left(f(t)\right)\mathrm{d}t$ we write from (50) the following

$$\frac{M^*(s)}{M_\infty} = 1 - \mathcal{L}\left\{-\tfrac{\mathrm{d}}{\mathrm{d}t}\phi(t)\right\} = 1 + s\mathcal{L}\left\{\phi(t)\right\} \tag{51}$$

We have used $\mathcal{L}\left\{f^{(1)}(t)\right\} = s\left(\mathcal{L}\left\{f(t)\right\}\right)$ considering $f(0) = 0$ in (51). Rearranging (51) we write the following

$$\mathcal{L}\left\{\phi(t)\right\} = \frac{M^*(s) - M_\infty}{sM_\infty}; \qquad \phi(t) = \mathcal{L}^{-1}\left\{\tfrac{M^*(s)-M_\infty}{sM_\infty}\right\} \tag{52}$$

Thus (52) is the extraction formula to get Electric Field Relaxation function from Electric Modulus Data. We will apply the Berberan-Santos formula to get Inverse Laplace Transform, considering $\phi(t)$ as Causal Function that follows Kramer-Kronigs relation.We take

$$X(s) = \frac{M^*(s) - M_\infty}{sM_\infty}$$

Put $s = i\omega$ to write and $M^*(s) = \mathrm{Re}\left[M(s)\right] + i\,\mathrm{Im}\left[M(s)\right]$

$$X(\omega) = \frac{\mathrm{Re}\left[M(\omega)\right] + i\,\mathrm{Im}\left[M(\omega)\right] - M_\infty}{i\omega M_\infty} = \frac{\mathrm{Im}\left[M(\omega)\right]}{\omega M_\infty} - i\frac{\mathrm{Re}\left[M(\omega)\right] - M_\infty}{\omega M_\infty} \tag{53}$$

We write from (53) the following parts

$$\mathrm{Re}\left[X\right] = \frac{\mathrm{Im}\left[M(\omega)\right]}{\omega M_\infty}, \qquad \mathrm{Im}\left[X\right] = -\frac{\mathrm{Re}\left[M(\omega)\right] - M_\infty}{\omega M_\infty} \tag{54}$$

Choosing $\sigma = 0$ implying that we do not expect any singularity for $\mathrm{Re}\left[s\right] > 0$; we get from Berberan-Santos method of inverse Laplace Transform (41) the following

$$\phi(t) = \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left[X\right]\cos\omega t - \mathrm{Im}\left[X\right]\sin\omega t\right)\mathrm{d}\omega$$
$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\mathrm{Im}\left[M(\omega)\right]}{\omega M_\infty}\cos\omega t\right)\mathrm{d}\omega + \frac{1}{\pi}\int_0^\infty \left(\frac{\mathrm{Re}\left[M(\omega)\right]-M_\infty}{\omega M_\infty}\sin\omega t\right)\mathrm{d}\omega \tag{55}$$

We note that the response functions dielectric $\varepsilon(\omega)$ and Electric Modulus $M^*(\omega)$ with relation $M^*(\omega) = \left(\varepsilon(\omega)\right)^{-1}$. We have

$$\varepsilon(\omega) = \mathrm{Re}\left[\varepsilon(\omega)\right] + i\,\mathrm{Im}\left[\varepsilon(\omega)\right] \qquad M^*(\omega) = \mathrm{Re}\left[M(\omega)\right] + i\,\mathrm{Im}\left[M(\omega)\right] \tag{56}$$

Experimentally, we have data for $\varepsilon(\omega)$ that gives us $M^*(\omega)$ for frequencies $\omega > 0$. The functions $\varepsilon(\omega)$ as well as $M^*(\omega)$ follow this causality principle, and their Real and Imaginary Parts are Hilbert Transforms of each other-given by Kramer-Kronigs relations. This gives the above formula (55) in following form using (45)

$$\phi(t) = \frac{2}{\pi}\int_0^\infty \left(\frac{\mathrm{Im}\left[M(\omega)\right]}{\omega M_\infty}\cos\omega t\right)\mathrm{d}\omega \tag{57}$$

Thus from the experimental measurements of dielectric dispersion $\varepsilon(\omega) = \mathrm{Re}\left[\varepsilon(\omega)\right] + i\,\mathrm{Im}\left[\varepsilon(\omega)\right]$ and loss vis-à-vis frequency $\omega > 0$, we can compute the electric modulus function i.e. $M^*(\omega) = \mathrm{Re}\left[M(\omega)\right] + i\,\mathrm{Im}\left[M(\omega)\right]$, as $M^*(\omega) = 1/\varepsilon(\omega)$ and then estimate the nature of normalized electric field decay function $\phi(t)$, by using (57).

Conclusions

This deliberation gave us the Kramer-Kronigs relations for causal system response that we tried to develop in a simplified way without deliberating analyticity, and other complex theorems. This note gave us practical application of Kramer-Kronig relation to apply for obtaining inverse Laplace Transformation for Causal function, and we saw its application in obtaining normalized electric field relaxation function, from given dispersion data of imaginary part of the electric modulus. This general method can be applied for Debye or non-Debye relaxation systems-even for fractional order systems. The feature of inverse Laplace transformation without going for usual contour integration and residue calculus as described in Berberan-Santos method is useful for various experimental approaches.

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