Formula $q(t) = c(t) * v(t)$ used and described for various cases of $v(t)$ applied to fractional capacitor and classical ideal capacitor

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Abstract
In this presentation note, we apply the newly developed charge storage expression as a function of time i.e. via convolution operation of time varying capacity function and applied voltage function to a capacitor. We apply this new formula to various types of input excitation voltage functions- step, ramp, square pulse, triangular, sinusoidal (cosine/sine) and analyze the charge stored expressions. We deliberate this for case of fractional capacitor as well as classical ideal capacitor. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor. This new deliberation with convolution operation works well for classical ideal loss less capacitors, where one says that is a constant capacity, and also for a time varying capacity function given by power-law: that gives the formation of fractional capacitor. This presentation note gives validity of usage of this new formula.

Keywords
Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform

1. Introduction
The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality we write $q(t)=c(t)*v(t)$ where $v(t)$ is the input impressed voltage. This is contrary to usual usage of $q(t)=c(t)v(t)$ i.e. the product of the two. This formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t)\sim t^{-\alpha}$; $0<\alpha<1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t)\sim\delta(t)$; [1]. In this presentation note we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one. This power-law decay function is in singular at origin and in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t)\sim t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36]. Here we are taking singular function $c(t)$ as 'time varying

capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t) \sim t^{-\alpha}$ of Curie-von Schweidler.

In this note we will take capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula

$$q(t) = c(t) * v(t) = \int_0^t c(t-\tau)v(\tau)\mathrm{d}\tau = \int_0^t c(\tau)v(t-\tau)\mathrm{d}\tau$$

and discuss various cases for $v(t)$ as sinusoidal voltage excitation, step voltage excitation, ramp voltage excitation, square pulse excitation and triangular voltage excitation.

We note a priori that the constant C_α is proportionality constant of the relation of time varying capacity function i.e. $c(t) \sim t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad}/\text{sec}^{1-\alpha}$. The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (having output impedance as zero). We will use Laplace Transform technique in all analysis. In all the cases in subsequent sections, we will apply this new formula and give the validity justification.

Charge in a Capacitor is $q(t) = c(t) * v(t)$, is given via convolution operation and not with the usual way that we write as $q(t) = c(t)v(t)$. Let us have a Capacitor with capacity function in time as power-law $c(t) = C_\alpha t^{-\alpha}$ $(0 < \alpha < 1)$, that is fractional capacitor. Let a voltage be applied to an uncharged capacitor $v(t)$, at time $t = 0$ .Then charge function in time is given as convolution $(*)$ operation as $q(t) = c(t) * v(t)$. For each case we also study the ideal loss less capacitor given by capacity function as $c(t) = C_1\delta(t)$, and apply $q(t) = c(t) * v(t)$. For each case we will write the expression of current $i(t)$ from the obtained charge function $q(t) = c(t) * v(t)$.

2. Charge storage $q(t)$ by Step Input Voltage $v(t) = V_m u(t)$ excitation

The unit step is $u(t)$, applied at $t = 0$. Let at $t = 0$ $v(t) = V_m$ a step input is given to uncharged capacitor with time varying capacity function as $c(t) = C_\alpha t^{-\alpha}$ that is a fractional capacitor following the relation [6]-[10]

$$i(t) = C_{F-\alpha} \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} v(t)$$

Where $C_{F-\alpha}$ is Fractional Capacity in units of $\text{Farad}/\text{sec}^{1-\alpha}$. The DC-voltage V_m is maximum voltage of circuit, or maximum voltage that is rated for a capacitor. Thus we have following from the relation $q(t) = c(t) * v(t)$

$$\begin{aligned} q(t) &= c(t) * v(t) \\ Q(s) &= \mathcal{L}\{q(t)\} = \mathcal{L}\{c(t)\} \times \mathcal{L}\{v(t)\} \\ &= \frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}} \times \frac{V_m}{s} = \frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}} \end{aligned}$$

Doing inverse Laplace transform of above, we get the following

$$\begin{aligned} q(t) &= \frac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} \\ &= \frac{V_m C_{F-\alpha}}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha) \end{aligned}$$

The current is following

$$i(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = V_m C_\alpha t^{-\alpha}$$

This above is Curie-von Schweidler relaxation law for dielectric stressed with constant voltage or Electric field. If the case we take of loss less ideal capacitor given by capacity function $c(t) = C_1 \delta(t)$, then $q(t) = c(t) * v(t)$, with $v(t) = V_m u(t)$, where $u(t)$ is unit step function at $t = 0$, is

$$\begin{aligned} q(t) &= c(t) * v(t) \\ Q(s) &= \mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{C_1 \delta(t)\right\} \times \mathcal{L}\left\{V_m u(t)\right\} \\ &= C_1 \times \frac{V_m}{s} \end{aligned}$$

Doing Laplace inverse we write the following

$$\begin{aligned} q(t) &= \mathcal{L}^{-1}\left\{C(s)V(s)\right\} \\ &= \mathcal{L}^{-1}\left\{C_1 V_m / s\right\}. \\ &= C_1 V_m u(t) \end{aligned}$$

The current in this ideal case is

$$\begin{aligned} i(t) &= \frac{\mathrm{d}}{\mathrm{d}t} q(t) = \frac{\mathrm{d}}{\mathrm{d}t} V_m C_1 u(t) \\ &= V_m C_1 \delta(t) \end{aligned}$$

Both the cases are depicted in Figure-2[1].

Capacity, charge, current for ideal capacitor vis-a-vis time varying capacitor to a step voltage excitation

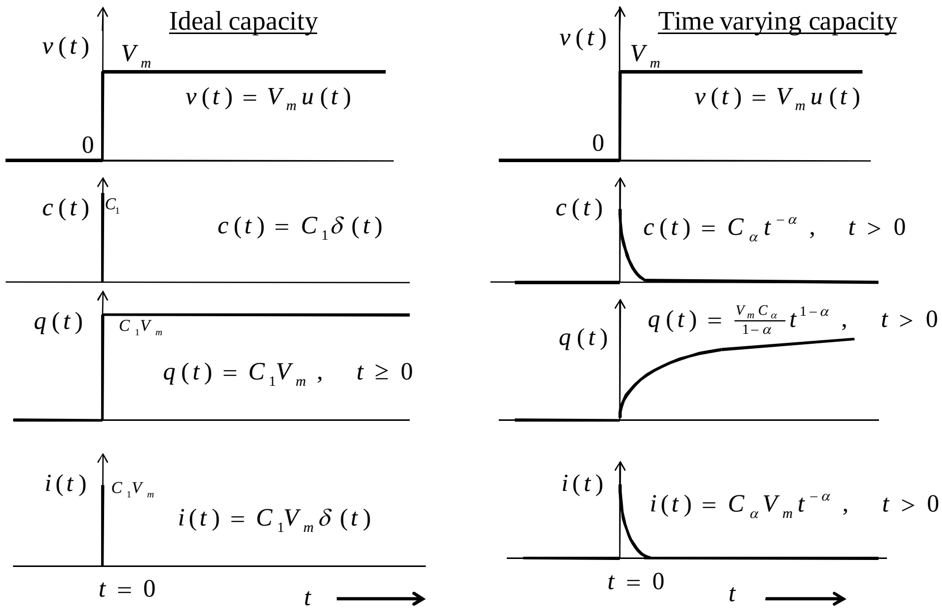

Figure-1: Charge in a ideal capacitor vis-à-vis time varying capacity function

We see when the step input $v(t)$ is kept ON at V_m for time T, $2T$, $3T$ and so on, we get $q(T)=\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$, $q(2T)=2^{1-\alpha}\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$, $q(3T)=3^{1-\alpha}\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$ respectively. If the $T\uparrow\infty$, we get $q(T)\big|_{T\uparrow\infty}=\infty$. Refer Figure-1. This process leads to a new breakdown mechanism of capacitors noted in [1], [6], [7], called electrostatic breakdown of capacitors.

3. Charge storage $q(t)$ in a square wave voltage on for time T and thereafter zero

$$v(t)=\begin{cases}0 & , \quad t<0\\ V_m & , \quad 0\le t\le T\\ 0 & , \quad t>T\end{cases}$$

We construct the above excitation as follows, with $u(t-\tau)=1$ for $t\ge 0$ and $u(t-\tau)=0$ for $t<\tau$;

$$v(t)=V_mu(t)-V_mu(t-T)$$

The Laplace transform is

$$V(s)=\frac{V_m}{s}-\frac{V_m}{s}\mathrm{e}^{-sT}$$

We used $\mathcal{L}\left\{f(t-t_d)\right\}=\mathrm{e}^{-st_d}\mathcal{L}\left\{f(t)\right\}=\mathrm{e}^{-st_d}F(s)$ in above

When this voltage is applied to a time varying capacity function $c(t)=C_1\delta(t)$ i.e. ideal loss less capacitor we write from $q(t)=c(t)*v(t)$ the following

$$
\begin{aligned}
Q(s) &= \mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{c(t)\right\} \times \mathcal{L}\left\{v(t)\right\} \\
&= \left(C_1\right) \times \left(\frac{V_m}{s} - \frac{V_m}{s} \mathrm{e}^{-sT}\right) \\
&= \frac{V_m C_1}{s} - \mathrm{e}^{-sT} \frac{V_m C_1}{s}
\end{aligned}
$$

Taking inverse Laplace transform we get

$$
q(t) = V_m C_1 u(t) - V_m C_1 u(t-T) = \begin{cases} 0 & , \quad t < 0 \\ V_m C_1 & , \quad 0 \le t \le T \\ 0 & , \quad t > T \end{cases}
$$

Now when this square-wave is applied for a time varying capacity function as $c(t) = C_\alpha t^{-\alpha}$ i.e. for fractional capacitor we write from $q(t) = c(t) * v(t)$ the following

$$
\begin{aligned}
Q(s) &= \mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{c(t)\right\} \times \mathcal{L}\left\{v(t)\right\} \\
&= \frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}} \times \left(\frac{V_m}{s} - \frac{V_m}{s} \mathrm{e}^{-sT}\right) \\
&= \frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}} - \mathrm{e}^{-sT} \frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}}
\end{aligned}
$$

Taking inverse Laplace Transform of above we obtain

$$
q(t) = \frac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} - \frac{V_m C_\alpha}{1-\alpha} (t-T)^{1-\alpha} = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} - \dfrac{V_m C_\alpha}{1-\alpha} (t-T)^{1-\alpha} & , \quad t > T \end{cases}
$$

The charge at $t = T$ is $q(T) = \frac{V_m C_\alpha T^{1-\alpha}}{1-\alpha}$, charge at $t = 2T$ $q(2T) = \frac{V_m C_\alpha T^{1-\alpha}}{1-\alpha}(2^{1-\alpha} - 1)$, charge at $t = 3T$ is $q(t) = \frac{V_m C_\alpha T^{1-\alpha}}{(1-\alpha)}(3^{1-\alpha} - 2^{1-\alpha})$.

We observe that for a fractional capacitor while the voltage is zero, after $t = T$, there still is charge holding, as compared with ideal capacitor. The current wave form is

$$
i(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = V_m C_\alpha (t^{-\alpha} - (t-T)^{-\alpha}) = \begin{cases} 0 & , \quad t < 0 \\ V_m C_\alpha t^{-\alpha} & , \quad 0 \le t \le T \\ V_m C_\alpha (t^{-\alpha} - (t-T)^{-\alpha}) & , \quad t > T \end{cases}
$$

4. Charge storage by Ramp Input Voltage $v(t) = (V_m / T) r(t)$ excitation

The function $r(t)$ is unit ramp of slope one applied at $t = 0$. We define $r(t-\tau) = (t-\tau)$ for $t \ge \tau$ and $r(t-\tau) = 0$ for $t < \tau$. Say we have a Ramp voltage input as $v(t) = \left(V_m / T\right) t$, i.e. applied at $t = 0$ and it linearly rises from zero volts to V_m volts, in time $t = T$ .We write this as $v(t) = \left(\frac{V_m}{T}\right) r(t)$. We have $V(s) = \mathcal{L}\left\{\frac{V_m}{T} r(t)\right\} = \left(V_m / Ts^2\right)$.

We apply $q(t) = c(t) * v(t)$; with $c(t) = C_\alpha t^{-\alpha}$, as time varying capacity function. To get the following charge function in Laplace domain

$$
\begin{aligned}
Q(s) &= \mathcal{L}\{q(t)\} = \mathcal{L}\{c(t)\} \times \mathcal{L}\{v(t)\} \\
&= \left(\frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2}\right) \\
&= \frac{V_m C_\alpha \Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}} \\
&= \frac{V_m C_{F-\alpha}}{s^{1+(2-\alpha)} T}
\end{aligned}
$$

Doing inverse Laplace transform we obtain the $q(t)$ as follows

$$
\begin{aligned}
q(t) &= \frac{V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)} t^{2-\alpha} = \frac{V_m C_\alpha}{T(1-\alpha)(2-\alpha)} t^{2-\alpha}, \quad 0 \le t \le T \\
&= \frac{V_m C_{F-\alpha}}{T(2-\alpha)\Gamma(2-\alpha)} t^{2-\alpha} \\
&= \frac{V_m}{T} \frac{C_{F-\alpha}}{\Gamma(3-\alpha)} t^{2-\alpha}
\end{aligned}
$$

The current is the following

$$
i(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = \frac{V_m C_\alpha}{T(1-\alpha)} t^{1-\alpha}, \quad 0 \le t \le T
$$

We get the charge at the end of $t = T$ as

$$
q(T) = \frac{V_m C_\alpha}{(1-\alpha)(2-\alpha)} T^{1-\alpha}
$$

For ideal capacitor we have $c(t) = C_1 \delta(t)$, we write charge as $q(t) = c(t) * v(t)$ as

$$
\begin{aligned}
Q(s) &= \mathcal{L}\{c(t)\} \times \mathcal{L}\{v(t)\} = \mathcal{L}\{C_1 \delta(t)\} \times \mathcal{L}\{(V_m / T) t\} \\
&= (C_1)\left(\frac{V_m}{Ts^2}\right) = \frac{V_m C_1}{Ts^2}
\end{aligned}
$$

With inverse Laplace of above we write

$$
q(t) = \frac{V_m C_1}{T} t, \quad t > 0
$$

The current in the ideal capacitor is

$$
i(t) \frac{\mathrm{d}}{\mathrm{d}t} q(t) = \frac{V_m C_1}{T}
$$

This we verify from ideal capacitor equation that is following

$$
\begin{aligned}
i(t) &= C_1 \frac{\mathrm{d}v(t)}{\mathrm{d}t} \\
&= C_1 \frac{\mathrm{d}}{\mathrm{d}t}(V_m t / T) = \frac{V_m C_1}{T}
\end{aligned}
$$

The charge is

$$q(t)=\int_0^t i(\tau)\mathrm{d}\tau=\int_0^t\left(\frac{V_mC_1}{T}\right)\mathrm{d}\tau$$
$$=\frac{V_mC_1}{T}t$$

The above derivation for ideal-loss-less capacitor is verification and justifies that we apply $c(t)=C_\alpha t^{-\alpha}$ as we did for step input case in previous section when voltage is changed at $t=0$, (in ramp case too), for a fractional capacitor. This comes from the above observation that for ideal loss less capacitor case, the $c(t)=C_1\delta(t)$ gets applied at $t=0$ (for ramp case too).

5. Comparison of charge storage by step $v(t)=V_mu(t)$ and ramp input $v(t)=\left(V_m/T\right)r(t)$ excitations

For the step input with voltage, held for time $t=T$ we have the charge as $q(T)=\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$, and we have charge at the end of $t=T$ for a ramp input as $q(T)=\frac{V_mC_\alpha}{(1-\alpha)(2-\alpha)}T^{1-\alpha}$. We write the ratio as follows

$$\frac{q(T)\big|_{STEP}}{q(T)\big|_{RAMP}}=\frac{\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}}{\frac{V_mC_\alpha}{(1-\alpha)(2-\alpha)}T^{1-\alpha}}=2-\alpha$$

We observe that for time T if we hold the voltage to V_m and charge a capacitor, then we will be holding $(2-\alpha)$ times the charge if we ramp the voltage at rate V_m/T from zero to V_m. Now after this process if we keep the capacitors in self discharge mode, for both the cases the voltage decay will start from V_m, but for step-charging case, since amount of charges held is more, it will take longer time to self discharge as compared to case with ramp-charging. This we expect from the memory effect. The comparison between step input voltage charging and ramp input voltage charging is depicted in Figure-2. This study on similar lines about memory effect from step and ramp charging is also shown in [15].

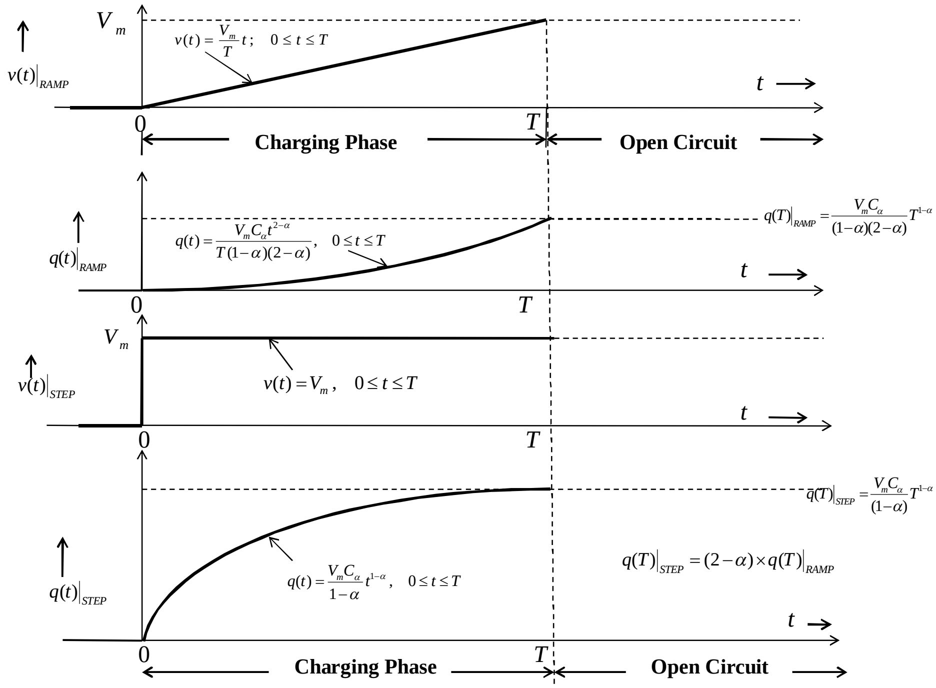

Figure-2: Step input voltage charging and Ramp input voltage charging

6. Charge storage by voltage as triangular input

$$v(t)=\begin{cases} 0 & , \quad t<0 \\ \dfrac{V_m}{T}t & , \quad 0\le t\le T \\ \dfrac{V_m}{T}t-\dfrac{2V_m}{T}(t-T) & , \quad T\le t\le 2T \\ 0 & , \quad t\ge 2T \end{cases}$$

We can write the above excitation as

$$v(t)=\frac{V_m}{T}r(t)-\frac{2V_m}{T}r(t-T), \quad 0\le t\le 2T$$

With $r(t)$ unit ramp at $t=0$ and is zero for $t<0$ and $r(t-T)$ as unit ramp at $t=T$ and zero at $t<T$. With this applied to a ideal capacitor, with $c(t)=C_1\delta(t)$, we get

$$\begin{aligned} Q(s) &= \mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{c(t)\right\} \times \mathcal{L}\left\{v(t)\right\} \\ &= \left(C_1\right) \times \left(\frac{V_m}{Ts^2} - \frac{2V_m}{Ts^2}\mathrm{e}^{-sT}\right) \\ &= \frac{V_m C_1}{Ts^2} - \mathrm{e}^{-sT}\,\frac{2V_m C_1}{Ts^2} \end{aligned}$$

Doing inverse Laplace transform we get

$$q(t) = \frac{V_m C_1}{T} r(t) - \frac{2V_m C_1}{T} r(t-T) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_1}{T} t & , \quad 0 \le t \le T \\ \dfrac{V_m C_1}{T} t - \dfrac{2V_m C_1}{T}(t-T) & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases}$$

Current is

$$i(t) = \frac{V_m C_1}{T} u(t) - \frac{2V_m C_1}{T} u(t-T) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_1}{T} & , \quad 0 \le t \le T \\ -\dfrac{V_m C_1}{T} & , \quad T \le t \le 2T \\ 0 & , \quad t \ge 2T \end{cases}$$

We take a fractional capacitor and do the following as done above

$$\begin{aligned} Q(s) &= \mathcal{L}\left\{q(t)\right\} = \mathcal{L}\left\{c(t)\right\} \times \mathcal{L}\left\{v(t)\right\} \\ &= \left(\frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2} - \frac{2V_m}{Ts^2}\mathrm{e}^{-sT}\right) \\ &= \frac{V_m C_\alpha \Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}} - \mathrm{e}^{-sT}\,\frac{2V_m C_\alpha \Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}} \end{aligned}$$

Taking inverse Laplace transform of above, we write, with notation as follows

$$p(t-\tau) = \begin{cases} (t-\tau)^{2-\alpha}, & t \ge \tau \\ 0, & t < \tau \end{cases}$$

Thus the charge function is following

$$
\begin{aligned}
q(t) &= \frac{V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)} p(t) - \frac{2V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)} p(t-T) \\
&= \frac{V_m C_\alpha}{T(1-\alpha)(2-\alpha)} p(t) - \frac{2V_m C_\alpha}{T(1-\alpha)(2-\alpha)} p(t-T)
\end{aligned}
$$

We re-write above

$$
q(t) = \frac{V_m C_\alpha}{T(1-\alpha)(2-\alpha)} p(t) - \frac{2V_m C_\alpha}{T(1-\alpha)(2-\alpha)} p(t-T)
$$

$$
q(t) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha t^{2-\alpha}}{T(1-\alpha)(2-\alpha)} - \dfrac{2V_m C_\alpha (t-T)^{2-\alpha}}{T(1-\alpha)(2-\alpha)} & , \quad T \le t \le 2T \end{cases}
$$

We have at $t = T$, $q(T) = \frac{V_m C_\alpha T^{1-\alpha}}{(1-\alpha)(2-\alpha)}$ at $t = 2T$, $q(2T) = \frac{V_m C_\alpha T^{1-\alpha}(2^{2-\alpha}-2)}{(1-\alpha)(2-\alpha)}$ .We observe that at $t = 2T$, the voltage is zero, but we have charge as non-zero. With $\alpha \approx 1$, we get $q(2T) \approx 0$, that we have analyzed for an ideal loss less capacitor earlier in this section. Differentiating the above we write current as

$$
i(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = \begin{cases} 0 & , \quad t < 0 \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} & , \quad 0 \le t \le T \\ \dfrac{V_m C_\alpha t^{1-\alpha}}{T(1-\alpha)} - \dfrac{2V_m C_\alpha (t-T)^{1-\alpha}}{T(1-\alpha)} & , \quad T \le t \le 2T \end{cases}
$$

7. Charge storage by Sinusoidal voltage $v(t) = V_m \cos \omega_0 t$

Let a sinusoidal voltage be applied to an uncharged capacitor $v(t) = V_m \cos \omega_0 t$, at time $t = 0$ for a fractional capacitor given by capacity function $c(t) = C_\alpha t^{-\alpha}$. We write charge $q(t)$ as follows

$$
q(t) = c(t) * v(t) = \left(C_\alpha t^{-\alpha}\right) * \left(V_m \cos \omega_0 t\right)
$$

We apply Laplace Transform to the above and write the following

$$\mathcal{L}\left\{q(t)\right\}=\mathcal{L}\left\{c(t)*v(t)\right\}; \qquad Q(s)=C(s)V(s)$$

We have $C(s)=\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}=C_\alpha\Gamma(1-\alpha)s^{-(1-\alpha)}$ and $V(s)=\mathcal{L}\left\{V_m\cos\omega_0 t\right\}=\frac{V_m s}{s^2+\omega_0^2}$. This gives $Q(s)$ as follows

$$\begin{aligned}Q(s)&=\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\times\frac{V_m s}{s^2+\omega_0^2}\\&=V_mC_\alpha\Gamma(1-\alpha)\left(s^\alpha\frac{1}{s^2+\omega_0^2}\right)=\frac{V_mC_\alpha\Gamma(1-\alpha)}{\omega_0}\left(s^\alpha\frac{\omega_0}{s^2+\omega_0^2}\right)\\&=\frac{V_mC_\alpha\Gamma(1-\alpha)}{\omega_0}\left(\mathcal{L}\left\{D_t^\alpha\sin\omega_0 t\right\}\right)\end{aligned}$$

In above we used Laplace Transform of Fractional Derivative as $\mathcal{L}\left\{D_t^\alpha f(t)\right\}=s^\alpha F(s)$ with $f(0)=0$, [10], [12], [13] and $\mathcal{L}\left\{\sin at\right\}=\frac{a}{s^2+a^2}$. Taking the inverse Laplace transform of the above, we get

$$q(t)=\frac{V_mC_\alpha\Gamma(1-\alpha)}{\omega_0}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sin\omega_0 t=\frac{V_mC_{F-\alpha}}{\omega_0}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sin\omega_0 t$$

Here we introduce a constant $C_{F-\alpha}=C_\alpha\Gamma(1-\alpha)$ as fractional capacity in units of $\mathrm{Farad}/\mathrm{sec}^{1-\alpha}$ [1], [6], [7], [8], [9], [11]. We have fractional derivative of $\sin x$ as following [12], [13]

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\sin x=\sin\left(x+\tfrac{\alpha\pi}{2}\right)+\frac{x^{-1-\alpha}}{\Gamma(-\alpha)}-\frac{x^{-3-\alpha}}{\Gamma(-\alpha-2)}+\ldots$$

We write $x=\omega_0 t$ thus we have $\mathrm{d}x=\omega_0\mathrm{d}t$, gives $\mathrm{d}t^\alpha=\omega_0^{-\alpha}\mathrm{d}x$, with this we write the following

$$\begin{aligned}q(t)&=\frac{V_mC_{F-\alpha}}{\omega_0}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sin\omega_0 t\\&=\frac{V_mC_{F-\alpha}}{\omega_0^{1-\alpha}}\left(\sin\left(\omega_0 t+\tfrac{\alpha\pi}{2}\right)+\frac{(\omega_0 t)^{-1-\alpha}}{\Gamma(-\alpha)}-\frac{(\omega_0 t)^{-3-\alpha}}{\Gamma(-\alpha-2)}+\ldots\right)\\&=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0 t+\tfrac{\alpha\pi}{2}\right)+\frac{t^{-1-\alpha}}{\omega_0^2\Gamma(-\alpha)}-\frac{t^{-3-\alpha}}{\omega_0^4\Gamma(-\alpha-2)}+\ldots\end{aligned}$$

The transient terms i.e. $t^{-1-\alpha}$, $t^{-3-\alpha}$ … in the above expression decays to zero for large times, thus we write the steady state charge function from above as following

$$q(t)=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)$$

From above steady state current is

$$\begin{aligned}i(t)=\frac{\mathrm{d}}{\mathrm{d}t}q(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\left(V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)\right)\\&=V_mC_{F-\alpha}\omega_0^{\alpha}\cos\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)\end{aligned}$$

This shows at steady state the current in fractional capacitor leads the voltage by angle $\alpha\times 90^0$. This is true as the way to validate experimentally a fractional integrator or differentiator circuit, by sinusoidal input. The leading current is 90^0 to voltage excitation for ideal loss less capacitor where ($\alpha=1$).

$$i(t)=V_mC_{F-1}\omega_0\cos\left(\omega_0t+\tfrac{\pi}{2}\right);\quad C_{F-1}\equiv C_1\text{Farads}$$

We do the following steps to re-write above for $q(t)=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)$ as following

$$\begin{aligned}q(t)&=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)\\&=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\tfrac{\pi}{2}+\left(\omega_0t-\tfrac{\pi}{2}+\tfrac{\alpha\pi}{2}\right)\right)=V_mC_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\tfrac{\pi}{2}+\left(\omega_0t-\tfrac{(1-\alpha)\pi}{2}\right)\right)\\&=V_mC_{F-\alpha}\omega_0^{\alpha-1}\cos\left(\omega_0t-\tfrac{(1-\alpha)\pi}{2}\right)\\&=Q_p\cos\left(\omega_0t-\tfrac{(1-\alpha)\pi}{2}\right)\\Q_p&=V_mC_\alpha\Gamma(1-\alpha)\omega_0^{\alpha-1}=V_mC_{F-\alpha}\omega_0^{\alpha-1}\end{aligned}$$

We observe that charge function $q(t)$ lags voltage function at steady state by angle $\frac{(1-\alpha)\pi}{2}$. For $\alpha=1$ these is no phase difference (lag) between charge function and voltage function [1]. This is for ideal loss less capacitor where capacity function is $c(t)=C_1\delta(t)$. With this we get

$$\begin{aligned}q(t)&=c(t)*v(t)=\left(C_1\delta(t)\right)*\left(V_m\cos\omega_0t\right)\\Q(s)&=\mathcal{L}\left\{C_1\delta(t)\right\}\times\mathcal{L}\left\{V_m\cos\omega_0t\right\}\\&=V_mC_1\frac{s}{s^2+\omega_0^2}\end{aligned}$$

By using inverse Laplace Transform we get with writing $Q_m=V_mC_1$ the following

$$q(t)=V_mC_1\cos\omega_0t=Q_m\cos\omega_0t;\quad t\geq 0$$

That is $q(t)$ in same phase with voltage function $v(t)=V_m\cos\omega_0 t$.

We see that for a time varying capacity function $c(t)=C_\alpha t^{-\alpha}$; i.e. fractional capacitor, the charge function $q(t)=Q_p\cos\left(\omega_0 t-\frac{(1-\alpha)\pi}{2}\right)$, the peak value of charge $Q_p=V_m C_{F-\alpha}\omega_0^{\alpha-1}$ varies with operating frequency ω_0 of the excitation voltage $v(t)=V_m\cos\omega_0 t$. With parameters $C_{F-\alpha}$ and α assuming to be constant with varying ω_0 (which is not actual case though), and V_m as maximum rated value of the operational circuit; the peak charge Q_p decreases as ω_0 the input frequency is increased. While for ideal loss less capacitors the peak charge $Q_m=V_m C_1$ remains invariant with frequency of input voltage. Therefore with various input excitation voltages we will be getting different peak charge values, though the excitation voltage is within the capacitor maximum rating V_m. This means square wave, triangular wave, trapezoidal wave of voltages will be giving different peak charge stored, as they will be having different fundamental and harmonic frequencies. A square wave voltage of positive and negative cycles, a symmetric triangular wave voltage, and a pure sinusoidal with same period will have different peak charge stored.

8. Charge storage by Sinusoidal voltage $v(t)=V_m\sin\omega_0 t$ excitation

Let a sinusoidal voltage be applied to an uncharged capacitor $v(t)=V_m\sin\omega_0 t$, at time $t=0$ for a fractional capacitor given by capacity function $c(t)=C_\alpha t^{-\alpha}$. We write charge $q(t)$ as follows

$$q(t)=c(t)*v(t)=\left(C_\alpha t^{-\alpha}\right)*\left(V_m\sin\omega_0 t\right)$$

We apply Laplace Transform to the above and write the following

$$\mathcal{L}\left\{q(t)\right\}=\mathcal{L}\left\{c(t)*v(t)\right\};\qquad Q(s)=C(s)V(s)$$

We have $C(s)=\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}=C_\alpha\Gamma(1-\alpha)s^{-(1-\alpha)}$ and $V(s)=\mathcal{L}\left\{V_m\sin\omega_0 t\right\}=\frac{V_m\omega_0}{s^2+\omega_0^2}$. This gives $Q(s)$ as follows

$$\begin{aligned}Q(s)&=\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\times\frac{V_m\omega_0}{s^2+\omega_0^2}\\&=V_m C_\alpha\Gamma(1-\alpha)\left(s^{\alpha-1}\frac{\omega_0}{s^2+\omega_0^2}\right)\\&=V_m C_\alpha\Gamma(1-\alpha)\left(\mathcal{L}\left\{D_t^{\alpha-1}\sin\omega_0 t\right\}\right)\end{aligned}$$

In above we used Laplace Transform of Fractional Derivative as $\mathcal{L}\left\{D_t^\alpha f(t)\right\} = s^\alpha F(s)$ with $f(0)=0$, [10], [12], [13] and $\mathcal{L}\left\{\sin at\right\} = \frac{a}{s^2+a^2}$. Taking the inverse Laplace transform of the above, we get

$$q(t) = V_m C_\alpha \Gamma(1-\alpha)\left(D_t^{\alpha-1}\sin\omega_0 t\right)$$

From the formula as noted above for $D^\alpha \sin x$ we get by replacing α with $\alpha-1$ the following.

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\sin x = \sin\left(x+\tfrac{\alpha\pi}{2}\right) + \frac{x^{-1-\alpha}}{\Gamma(-\alpha)} - \frac{x^{-3-\alpha}}{\Gamma(-\alpha-2)} + \ldots \qquad \alpha \to (\alpha-1)$$
$$\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}x^{\alpha-1}}\sin x = \sin\left(x+\tfrac{(\alpha-1)\pi}{2}\right) + \frac{x^{-1-(\alpha-1)}}{\Gamma(-(\alpha-1))} - \frac{x^{-3-(\alpha-1)}}{\Gamma(-(\alpha-1)-2)} + \ldots$$

Since $0<\alpha<1$ and for $x\uparrow\infty$ the terms $x^{-1-(\alpha-1)}$, $x^{-3-(\alpha-1)}$ …goes to zero, and we write

$$\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}x^{\alpha-1}}\sin x = \sin\left(x+\tfrac{(\alpha-1)\pi}{2}\right)$$

With $x=\omega_0 t$, we have $(\mathrm{d}x)^{\alpha-1} = \omega_0^{\alpha-1}(\mathrm{d}t)^{\alpha-1}$, we write the following

$$\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}x^{\alpha-1}}\sin x = \frac{1}{\omega_0^{\alpha-1}}\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}\sin\omega_0 t = \sin\left(\omega_0 t+\tfrac{(\alpha-1)\pi}{2}\right)$$
$$\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}\sin\omega_0 t = \omega_0^{\alpha-1}\sin\left(\omega_0 t+\tfrac{(\alpha-1)\pi}{2}\right)$$

Using above, we write steady state charge as follows

$$\begin{aligned} q(t) &= V_m C_\alpha \Gamma(1-\alpha)\left(\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}\sin\omega_0 t\right) \\ &= V_m C_\alpha \Gamma(1-\alpha)\omega_0^{\alpha-1}\sin\left(\omega_0 t+\tfrac{(\alpha-1)\pi}{2}\right) \\ &= V_m C_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0 t-\tfrac{(1-\alpha)\pi}{2}\right) \end{aligned}$$

The charge $q(t)$ lags the voltage function $v(t)$ by an angle $\frac{(1-\alpha)\pi}{2}$. We differentiate and write the current as follows

$$i(t)=\frac{\mathrm{d}}{\mathrm{d}t}q(t)=V_mC_{F-\alpha}\omega_0^{\alpha-1}\omega_0\cos\left(\omega_0t-\tfrac{(1-\alpha)\pi}{2}\right)$$
$$=V_mC_{F-\alpha}\omega_0^{\alpha}\cos\left(\omega_0t-\tfrac{\pi}{2}+\tfrac{\alpha\pi}{2}\right)$$
$$=V_mC_{F-\alpha}\omega_0^{\alpha}\sin\left(\omega_0t+\tfrac{\alpha\pi}{2}\right)$$

The current leads voltage by angle $\frac{\alpha\pi}{2}$ for fractional capacitor.

9. Conclusions

We have applied the new formula of charge storage i.e. via convolution operation, of time varying capacity function and voltage stress for a fractional capacitor. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. We have discussed various results obtained for different excitation voltages- sinusoidal, step and ramp, square pulse, triangular voltage excitation. We have given interpretations of the various analytical results that were obtained by this new formulation-for classical ideal loss-less capacitor as well as for fractional capacitor. This verifies that this new formulation of stored charge via convolution operation is correct, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor.

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