Inserting various Memory Kernels in basic Evolution Equation in Process Dynamics-and formation of new constituent equations for various Physical Laws- (different from Classical formulas)

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Abstract

This presentation is taken from earlier notes on a Project Proposal -to have projects on derivation on Memory Based Physical Laws.-for different types of memory kernel function; and study reality. We recall that the system responds with its 'characteristic function' call it $\mathrm{h}(t)$; by universal law given as $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$; i.e. convolution operation (*). This is 'causality principle' and we term as general evolution equation, relating cause-function $\mathrm{x}(t)$ and effect-function $\mathrm{y}(t)$. The cause-function $\mathrm{x}(t)$ is as pinching action acts on body with $\mathrm{y}(t)$ the effect-function is uncomfortable pain. The relaxation of pain may have memory and decaying with time after the pinch action is over, or the pain may vanish as soon as pinch action is over i.e. with no memory! The relaxation characteristic of pain is by property of body defined by function $\mathrm{h}(t)$. The convolution operation symbol as (*) is integral given by $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)=\int_{-\infty}^{t}\mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau$. The characteristics function (is also termed as response function) $\mathrm{h}(t)$ we term here as 'memory-kernel' in this evolution equation given as convolution integral. This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with 'memory-kernel', that is $\mathrm{h}(t)$. Implying that the value of the process $\mathrm{y}(t)$ at present instant is being influenced by all the states of $\mathrm{x}(t)$ the system had been through in past.

We will discuss formation of basic constitutive equations with zero-memory case where the memory-kernel is delta-function (which is singular function) i.e. $\mathrm{h}(t)\propto\delta(t)$, and with $\mathrm{h}(t)$ singular and non-singular memory kernels that decays with time, like $\mathrm{h}(t)\propto t^{-\alpha};\quad 0<\alpha<1$ or $\mathrm{h}(t)\propto \mathrm{e}^{-kt};\quad k>0$ etc. These decaying functions used as memory kernel gives a reality in which memory fades as time grows. After all 'Dead Matter' does have Memory, like relaxation in dielectric, visco-elastic material responses-etc.

We note that for the zero-memory case, that is memory kernel as delta function $\mathrm{h}(t)\propto\delta(t)$ (singular in nature) returns classical constitutive equations (like classical Newton's law of motion i.e. force law $f(t)\propto\upsilon^{(1)}(t),\quad f(t)\propto{}_0I_t^0\left[\upsilon^{(1)}(t)\right]$ with classical Newtonian derivative, with $f(t)$ as force function and $\upsilon(t)$ as velocity function). Here $\upsilon^{(1)}(t)=D_t^1\left[\upsilon(t)\right]$ is classical integer order one-whole derivative, and ${}_0I_t^0$ is 'zero order' integration operation. However, with the case with memory kernel as other than delta function (say $\mathrm{h}(t)\propto t^{-\alpha},\quad 0<\alpha<1$- a singular decaying power-law memory kernel) returns constitutive equations with fractional derivatives and fractional integrations, as $f(t)\propto\upsilon^{(\alpha)}(t),\quad f(t)\propto{}_0I_t^{1-\alpha}\left[\upsilon^{(1)}(t)\right]$ i.e. different from classical laws. Here $\upsilon^{(\alpha)}(t)={}_0D_t^{\alpha}\left[\upsilon(t)\right]$ is fractional derivative operation of non-integer order α, and ${}_0I_t^{1-\alpha}$ is fractional integration operation of order, non-integer i.e. $1-\alpha$.

We will observe that singular function that we use for time-decaying memory kernel gives rise to conjugation to classical constitutive equations where its fractional counterpart (fractional derivative/integrals) replaces integer-order (classical) derivative or integral operation. We will see that non-singular memory kernel gives rise to more complicated constitutive equations as weighted infinite series sum of repeated integrations or weighted series sum of fractional integrations, like for force law we will get $f(t)\propto\sum_{n=1}^{\infty}\mathrm{a}_n\left({}_0I_t^n\upsilon^{(1)}(t)\right)$, $f(t)\propto\sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+1}\upsilon^{(1)}(t)\right)$, for decaying memory kernel given by exponential function $\mathrm{h}(t)\propto\mathrm{e}^{-kt};\quad k>0$ and decaying memory kernel given by Mittag-Leffler function $\mathrm{h}(t)\propto E_{\alpha}(-kt^{\alpha});\quad k>0;\quad 0<\alpha<1$. Interesting to note that these infinite sums do converge-that we will discuss.

The physical laws are constitutive equations, relating cause-function and response (effect-function). The cause is input function of time t call it $\mathrm{x}(t)$ and its response function (effect) is $\mathrm{y}(t)$, also a function of time. For example, we have law for force $f(t)$ acting on a body of mass m is; $f(t)=m\upsilon^{(1)}(t)$ that is proportional to time rate of change of velocity $\upsilon(t)$. The cause in this case is $\upsilon^{(1)}(t)\leftarrow\mathrm{x}(t)$; and effect is $f(t)\leftarrow\mathrm{y}(t)$. We have radioactive decay law as $N^{(1)}(t)=-\lambda N(t)$, we relate cause as number of decaying nuclei at a particular time i.e. $N(t)\leftarrow\mathrm{x}(t)$ and effect as $N^{(1)}(t)\leftarrow\mathrm{y}(t)$. The diffusion laws (without advection) in one-dimensional case is, $\frac{\partial}{\partial t}\mathrm{c}(x,t)=\mathbb{D}\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)$ with $\mathrm{c}(x,t)$ denoting concentration or density of diffusing species. Here the cause-function and effect-function are $\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\leftarrow\mathrm{x}(t)$ and $\frac{\partial}{\partial t}\mathrm{c}(x,t)\leftarrow\mathrm{y}(t)$ respectively. The wave mechanics is governed by constitutive equation $\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)=c^2\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)$, with cause-function as $\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)\leftarrow\mathrm{x}(t)$ and effect-function as $\frac{\partial^2}{\partial t^2}\mathrm{w}(x,t)\leftarrow\mathrm{y}(t)$ respectively. All these constituent laws are classical laws (that we noted are from classical physics books), and are for system without any memory, i.e. $\mathrm{h}(t)\propto\delta(t)$. We will modify these constituent laws for system having memory kernel-a time dependent decaying function singular and non-singular-and get new-relations.

We will present two types of system with evolution equation defined as $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$. First considering the cause $\mathrm{x}(t)$ is proportional to rate of change of some other physical quantity, i.e. $\mathrm{x}(t)\propto \mathrm{f}^{(1)}(t)$ and second is a system where response is proportional to rate of change of cause i.e. $\mathrm{y}(t)\propto \mathrm{x}^{(1)}(t)$. We note the first type of system is like 'response current' to a change in 'applied voltage' observed in dielectric relaxations and capacitors or force law of Newton's law of motion. The second one corresponds to population growth or radioactive decay type system. The corollary to second type of system we study where cause x is replaced by $\mathrm{L}[\mathrm{x}]$; where L denotes a 'Linear operator' and $\mathrm{y}(t)\propto \mathrm{x}^{(1)}(t)$. With L as Laplacian operator $\mathrm{L}\equiv\nabla^2$, we see that we will be getting various types of diffusion equations, for different memory-kernels. Extending further making effect as $\mathrm{y}(t)\propto \mathrm{x}^{(2)}(t)$, we will write various types of wave-equations, with different type of memory kernels.

We will present the formation of constituent expressions of physical laws with memory kernel $\mathrm{h}(t)$ as delta function, singular power-law decay function, non-singular power-law decay function, Mittag-Leffler function, pure exponential function and stretched exponential function. The motivation to have this presentation is to discuss, issues about using singular and non-singular functions as memory kernels; in basic evolution equation in process dynamics-and its implications to obtain new type of constituent equations for various systems and processes. This gives a generalization of system studies. We will restrict our analysis to simple constitutive equations of physical laws that we deal in everyday studies.

Keywords

Susceptibility, Convolution, Memory Kernel, Fractional Derivative, Fractional Integration, power-law, Mittag-Leffler function, stretched exponential function

Introduction

In this presentation attempt is made to tackle the basic evolution equation and then derive constituent laws in general, not specific to any system, by inserting various memory kernel functions in the basic system equation. There exist a motivation to have this presentation note is to discuss, issues about using singular and non-singular functions in system studies, especially as memory kernel; and get the corresponding constituent equations as physical laws. We observe complications in the physical laws that we derive, by use of non-singular memory kernel in basic evolution equation, as compared to using singular kernels. In this note, we give simple mathematical treatment to derive the relaxation laws (or constitutive equations) of system with several types of memory kernels to a relaxation law that we formulate as convolution integral $\mathrm{y}(t)\propto \mathrm{h}(t)*\mathrm{x}(t)$. The function $\mathrm{h}(t)$ is the time varying memory kernel, $\mathrm{x}(t)$ is the cause-function, and $\mathrm{y}(t)$ as effect-function or response functions of time (t); for process. This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with 'memory-kernel', that is $\mathrm{h}(t)$. This implies the value of the process $\mathrm{y}(t)$ at present instant is being influenced by all the states of $\mathrm{x}(t)$ the system had been through in past since the application of cause. For all the cases, we take the application of cause at time $t=0$.

Here in this presentation we take the memory kernel $\mathrm{h}(t)$ as singular and non-singular functions. The singular functions considered for memory kernel $\mathrm{h}(t)$ are delta function $\sim\delta(t)$, power-law decay function $\sim t^{-\alpha}$. The non-singular functions for $\mathrm{h}(t)$ are considered as $\sim(1+\upsilon t)^{-\alpha}$; (that is non-singular decaying power-law), Mittag-Leffler function $\sim E_\alpha(-\lambda t^\alpha)$ exponential decay function $\sim \mathrm{e}^{-\kappa t}$, and stretched exponential function $\sim \mathrm{e}^{-(\kappa t)^{\alpha}}$. We arrive at the constitutive laws of $\mathrm{y}(t)$ in relation to $\mathrm{x}(t)$, with all these memory-kernels. All memory kernels we will use are of decaying nature to derive constituent equation from universal evolution equation that is given as convolution, i.e. $\mathrm{y}(t) \propto \mathrm{h}(t) * \mathrm{x}(t)$. This is reality, as we all know memory shall be fading away with time.

First we study $\mathrm{y}(t) \propto \mathrm{h}(t) * \mathrm{x}(t)$, with cause as proportional to derivative of some other physical quantity call it $\mathrm{f}(t)$, i.e. make $\mathrm{x}(t) = \mathrm{f}^{(1)}(t)$. In this type of systems, when the memory kernel is delta function, we note we get constituent law as $\mathrm{y}(t) \propto \mathrm{f}^{(1)}(t)$. This is classical case like for constituent equation for dielectric relaxation current when the dielectric is stressed with a constant voltage stress; that is classical capacitor current equation. This type of classical law is also for case of Newtonian viscous element relating stress to first derivative of strain. We will see by choosing various types of memory kernel this classical law is changed, by use of fractional derivative, and fractional integrals.

The second type of physical laws we will formulate from $\mathrm{y}(t) \propto \mathrm{h}(t) * \mathrm{x}(t)$ by taking $\mathrm{y}(t) \propto \mathrm{x}^{(1)}(t)$. We will show for a zero-memory case with memory kernel as delta function i.e. $\mathrm{h}(t) \propto \delta(t)$ we get classical growth or decay law given by classical constitutive equation i.e. $\mathrm{x}^{(1)}(t) \propto \mathrm{x}(t)$. The growth or decay is given by classical exponential function in this case, i.e. $\mathrm{x}(t) \sim \mathrm{e}^{\mathrm{C}t}$. We will consider several memory kernels as we have done for first case. We will show that a singular power law kernel returns the constitutive equation where first derivative is replaced by fractional derivative i.e. $D^{\upsilon}\mathrm{x}(t) \propto \mathrm{x}(t)$ (or $\mathrm{x}^{(\upsilon)}(t) \propto \mathrm{x}(t)$) with $0 < \upsilon < 1$. We will see that non-singular memory kernels gives rise to more complicated constitutive equations as weighted series sum of repeated integrations or weighted series sum of fractional integrations; for both types of systems considered for evolution equation i.e. $\mathrm{y}(t) \propto \mathrm{h}(t) * \mathrm{x}(t)$.

Corollary to the second case, i.e. $\mathrm{x}^{(1)}(t) \propto \mathrm{h}(t) * \mathrm{x}(t)$ we modify this to have $\mathrm{x}^{(1)} \propto \mathrm{h} * \left(\mathrm{L}\left[\mathrm{x}\right]\right)$. Where L is a linear operator, say Laplacian, i.e. $\mathrm{L} \equiv \frac{\partial^2}{\partial x^2}$ with $\mathrm{x} \equiv \mathrm{c}(x,t)$ and $\mathrm{x}^{(1)} \equiv \frac{\partial}{\partial t}\left[\mathrm{c}(x,t)\right] = \mathrm{c}^{(1)}(x,t)$ i.e. say concentration variable in space-time coordinates. That is we will write various diffusion equations with the considered memory kernels. Extending this case, with $\mathrm{x}^{(2)} \propto \mathrm{h} * \left(\mathrm{L}\left[\mathrm{x}\right]\right)$ as evolution equation, we formulate various wave equations with the considered memory kernels. In recent years several researchers of physics engineering and mathematics are working with non-singular kernel in the constituent equations say like diffusion, heat transfer, electrical circuits, visco-elasticity etc; [42]-[57].-, [62].

Memory observed in real life materials relaxation

Figure-1 shows that a super-capacitor charged with a ramp input to a voltage level $\mathrm{V_m}$, charged for time T, gets the coulombs accumulated as $\mathrm{q(T)} = \frac{\mathrm{V_m C_\alpha}}{(1-\alpha)(2-\alpha)}\mathrm{T}^{1-\alpha}$. The next experiment show that a step input of voltage $\mathrm{V_m}$ is kept for time T and the accumulated coulombs as $\mathrm{q(T)} = \frac{\mathrm{V_m C_\alpha}}{1-\alpha}\mathrm{T}^{1-\alpha}$. We note that $\left(\mathrm{q(T)}\big|_{\mathrm{STEP}} / \mathrm{q(T)}\big|_{\mathrm{RAMP}}\right) = 2-\alpha$; that is the super-capacitor memorizes the shape of the charging profile. If there were no memory and the ideal, capacitor will store the charge as $\mathrm{q(T)}\big|_{\mathrm{RAMP}} = \mathrm{q(T)}\big|_{\mathrm{STEP}} = \mathrm{CV_m}$ i.e. same irrespective of ramp or step, [59], [60], [61], [62].

Figure-2 show that a super-capacitor charged for T hours, $2T$ hours, $4T$ hours and then kept in open circuited condition, the voltage profile on open circuit condition v_{oc} is dependent on time that it was kept on float charge for charge time T_c, i.e. $v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{T_c} \frac{d\tau}{\tau^{\alpha}(T_c+t-\tau)^{1-\alpha}}$ [12], [22], [59]-[61] Therefore, the super-capacitor memorizes the time that it was kept on charge. If it were without memory, then v_{oc} will be same as V_m irrespective, i.e. $v_{oc} = V_m$ of T hours, $2T$ hours, $4T$ hours.

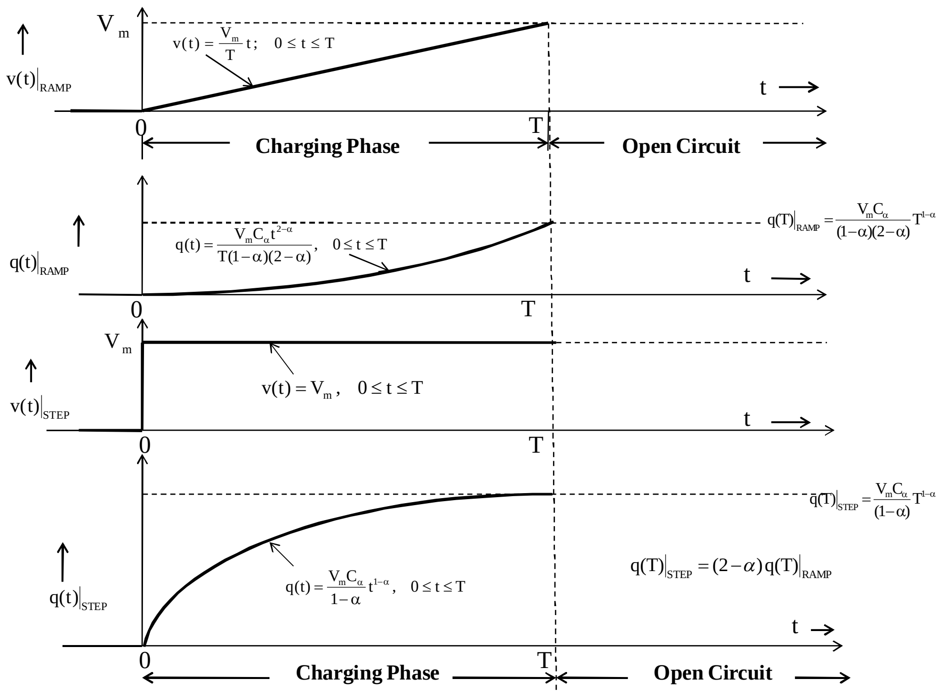

Figure-1: Step input voltage charging and ramp input voltage charging of super-capacitor showing memory

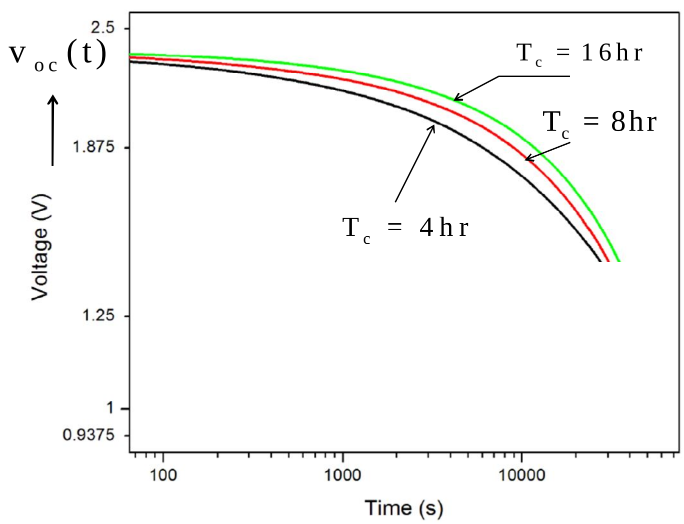

Figure-2: Memorizing the time that a super-capacitor is put afloat a constant voltage

The above experimental evidence shows that indeed, there is memory in relaxation dynamics, but the question is what type of memory-that we listed in the introduction section. We really do not know what is exactly the function that depicts memory kernel in the basic system equation.

Impulse Response of System

Consider a linear device (a linear system), which is supplied with an input field a function i.e. time varying call it $\mathrm{x}(t)$ and as a result the device gives output $\mathrm{y}(t)$. Assume the input field $\mathrm{x}(t')$ is applied at a time t' is sustained for a short infinitesimal period call it $\mathrm{d}t'$ and we say output of the system at some later time $t > t'$ is proportional to input field, i.e. in the form $\mathrm{dy}(t) = \left(\mathrm{h}(t,t')\right)\left(\mathrm{x}(t')\mathrm{d}t'\right)$, with proportionality term as function $\mathrm{h}(t,t')$. Hence, the function $\mathrm{h}(t,t')$ describes the operation of a linear device or linear system.

The application of input field can be anywhere in time axis $t' \in (-\infty,\infty)$. Similarly, the measured response can be anywhere in time axis $t \in (-\infty,\infty)$, but with condition that is $t > t'$. Assuming this operation has no explicit dependence (i.e. no in-built clock that changes its behavior) then the relation between the input and the output will only depend on the time interval i.e. $t - t'$ and not on the absolute time i.e. t. Therefore, we may replace the function $\mathrm{h}(t,t')$ with a function of single variable i.e. $\mathrm{h}(t - t')$; i.e. we have $\mathrm{h}(t-t') \leftarrow \mathrm{h}(t,t')$. This $\mathrm{h}(t-t')$ is called response function of system, or fundamental impulse response; or linear response coefficient; or susceptibility. The general method of describing $\mathrm{h}(t)$ is description in frequency domain, and the complex function that describes susceptibility is $\mathrm{H}(\omega)$ (frequency transform of $\mathrm{h}(t)$). Why $\mathrm{h}(t-t')$ is termed as impulse response is because we will see, that response of the system (or output) is $\mathrm{y}(t) = \mathrm{h}(t)$ when the forcing function $\mathrm{x}(t) = \delta(t)$ i.e. $\mathrm{h}(t)$ is the response of system when excitation is unit delta function. This impulse response $\mathrm{h}(t)$ is also called Green's function.

Experimentally giving excitation as $\mathrm{x}(t) \propto \delta(t)$ is however, many a times difficult (except you strike a bell once that is delta function or you give a small duration pinch to body is delta function). In those difficult cases we give step input excitation, $\mathrm{x}(t) \propto u(t)$ and get output $\mathrm{y}(t)$, then by differentiation of this output response, we get the Green's function $\mathrm{h}(t)$.

The concepts presented above are empirical descriptions of phenomena that are the concept of a primary response function or susceptibility $\mathrm{h}(t)$. It is like a time dependent shear compliance i.e. $\mathrm{J}(t)$, time dependent permittivity $\varepsilon(t)$, time dependent magnetic permeability $\mu(t)$ time dependent shear modulus $\mathrm{G}(t)$, time dependent Electric Modulus $\mathrm{M}(t)$ or a time dependent magnetic susceptibility $\chi_m(t)$ or time dependent electric-susceptibility $\chi_e(t)$ etc. -that do not involve understanding a given material, but are just general functions within which a discussion of the response to perturbation can be considered in a generic sense. All the above measured time dependent response functions $\mathrm{J}(t)$, $\varepsilon(t)$, $\mu(t)$, $\mathrm{G}(t)$, $\mathrm{M}(t)$ $\chi_m(t)$, $\chi_e(t)$ are experimentally obtained in relaxation (or retardation) experiments, by step-input excitation. In order to obtain the Green's function, one has to differentiate the same, and proceed to get dynamic quantities in the frequency domain.

Causality

Strike a bell and then hear its sound and not the other way round, is simply the principle of Causality. This is daily observed natural phenomena. The assumption of 'causality', namely that effect is after the cause, implies that output $\mathrm{y}(t)$ at any time t is obtained only due to input at or before t. Hence, the expression $\mathrm{dy}(t)=\left(\mathrm{h}(t\text{-}t')\right)\left(\mathrm{x}(t')\mathrm{d}t'\right)$ applies only for $t \geq t'$, or equivalently $\mathrm{h}(t,t')=0$ for $t'>t$ or we say $\mathrm{h}(t-t')=0$ for $t'>t$. We get the total output at any time $t\in(-\infty,\infty)$ by integration of $\mathrm{dy}(t)=\left(\mathrm{h}(t,t')\right)\left(\mathrm{x}(t')\mathrm{d}t'\right)$ from application of input field at time $t'=-\infty$ to $t'=\infty$ (since, $\mathrm{h}(t,t')=0$ when $t'>t$: i.e. 'the cause cannot be preceding the effect'). We express as the causality principle as $\mathrm{y}(t)=\int_{-\infty}^{\infty}\mathrm{h}(t-t')\mathrm{x}(t')\mathrm{d}t'$.

This integral is convolution operation of output function and input function $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$. Where in convolution operation is denoted as ($*$) and the convolution of two functions $\mathrm{f}_1(t)$ and $\mathrm{f}_2(t)$ is described as $\mathrm{f}_1(t)*\mathrm{f}_2(t)$ is $\int_{-\infty}^{\infty}\left(\mathrm{f}_1(t-t')\right)\left(\mathrm{f}_2(t')\right)\mathrm{d}t'$ or $\int_{-\infty}^{\infty}\left(\mathrm{f}_1(t')\right)\left(\mathrm{f}_2(t-t')\right)\mathrm{d}t'$. We note that in frequency-transformed (Laplace Transformed) domain we have $\mathcal{L}\left\{\mathrm{f}_1(t)*\mathrm{f}_2(t)\right\}=\mathcal{L}\left\{\mathrm{f}_1(t)\right\}\mathcal{L}\left\{\mathrm{f}_2(t)\right\}$.

If the application of input is at time $t'=0$ then in, $\mathrm{y}(t)=\int_{-\infty}^{\infty}\mathrm{h}(t-t')\mathrm{x}(t')\mathrm{d}t'$ we have output at time after $t>t'$ described as $\mathrm{y}(t)=\int_0^t\mathrm{h}(t-t')\mathrm{x}(t')\mathrm{d}t'$. Note here the upper-limit is $t'=t$. If we change the variable calling it $t-t'=\tau$, then we have following

$$\mathrm{y}(t)=\int_0^t\mathrm{h}(t-t')\mathrm{x}(t')\mathrm{d}t'=\int_t^0\mathrm{h}(\tau)\mathrm{x}(t-\tau)(-\mathrm{d}\tau)$$
$$=\int_0^t\mathrm{h}(\tau)\mathrm{x}(t-\tau)\mathrm{d}\tau$$

While in equilibrium, the system is invariant in time-that we call steady state condition. Therefore, the response at time t can only depend on the interval $t-t'$ between the time of measurement t and the time t' at which the input field acts. Before the input field $\mathrm{x}(t')$ is applied, there can be no response (this is "principle of causality"). Suppose that input field is unit impulse, i.e. $\mathrm{x}(t)=\delta(t)$ then $\mathrm{y}(t)=\mathrm{h}(t)$. This implies $\mathrm{h}(t)$ is the response at time t to a unit impulse applied at time $t=0$. This we have described in the previous section and this $\mathrm{h}(t)$ is also called Green's function. By principle of causality, $\mathrm{h}(t)=0$ for negative times i.e. for $t<0$. The forcing delta function, is like strike bell once, and then you hear the ring, is impulse response, and its ringing is Green's function.

For a case of $\mathrm{x}(t)=u(t)$ the unit-step-input, say we have the response $\mathrm{y}(t)$ as following

$$\mathrm{y}(t)=\int_{-\infty}^{\infty}\mathrm{h}(t-t')u(t')\mathrm{d}t'=\mathrm{h}(t)*u(t)$$
$$\frac{\mathrm{dy}(t)}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{h}(t)*u(t)\right)$$
$$=\mathrm{h}(t)*\left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right)=\mathrm{h}(t)*\delta(t)=\mathrm{h}(t)$$

We have used derivative of convolution as $\left(\mathrm{f}_1*\mathrm{f}_2\right)^{(1)}=\mathrm{f}_1^{(1)}*\mathrm{f}_2=\mathrm{f}_1*\mathrm{f}_2^{(1)}$, in above steps. Therefore if we excite a system with an unit step function, then the derivative of response of that step input gives the Green's function ($\mathrm{h}(t)$).

General Evolution Equation for Response (or relaxation) of a system

The output call it $\mathrm{y}(t)$ a variable in time ($t\in\mathbb{R}$), of a system represented by function of time variable $\mathrm{h}(t)$ to an input variable call it $\mathrm{x}(t)$ acting at time $t=0$, is given by evolution equation [5], [6], [25] as follows

$$\mathrm{y}(t)=\int_0^t\mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau;\quad t\geq 0 \tag{1}$$

In (1) we assume the input variable starts acting at $t=0$ and the response gets recorded at $t\geq 0$.
If we take Laplace transform, with $\mathcal{L}\left\{\mathrm{x}(t)\right\}=\mathrm{X}(s)$, $\mathcal{L}\left\{\mathrm{y}(t)\right\}=\mathrm{Y}(s)$ and $\mathcal{L}\left\{\mathrm{h}(t)\right\}=\mathrm{H}(s)$, we get Laplace Transformed equation for (1) as (2)

$$\mathrm{Y}(s)=\left(\mathrm{H}(s)\right)\left(\mathrm{X}(s)\right) \tag{2}$$

Where variable s is Complex variable, i.e. $s\in\mathbb{C}$.

The input $\mathrm{x}(t)=\delta(t)$ in case is delta-function (an impulse) at $t=0$; we have $\mathrm{y}(t)=\mathrm{h}(t)$. This $\mathrm{h}(t)$ is called 'impulse response' of the system [6], [25]. The evolution equation (1) of $\mathrm{y}(t)$ in time domain is convolution integral [25]. We will call $\mathrm{h}(t)$ as 'memory-kernel', and derive constituent expressions with several types of $\mathrm{h}(t)$.

In (1), if we set effect $\mathrm{y}(t)$ as $\mathrm{q}(t)$ as charge function of a capacitor, which is excited by a voltage $\mathrm{v}(t)$ i.e. the cause function $\mathrm{x}(t)$,acting as time $t=0$ then $\mathrm{h}(t)$ we call as 'capacity-function' call it $\mathrm{c}(t)$. Then we say the charge function $\mathrm{q}(t)=\int_0^t\mathrm{c}(t-\tau)\mathrm{v}(\tau)\mathrm{d}\tau$. Thus, the formula for charge stored is $\mathrm{q}(t)=\mathrm{c}(t)*\mathrm{v}(t)$ [61].

System with cause-function as derivative of action variable

We have several physical systems that are proportional to rate of change of some other physical quantity that is acting as input. Say we have rate of change of a quantity call it $\mathrm{f}(t)$ represented by first time derivative i.e. $\mathrm{f}^{(1)}(t)$, then our input variable in (1) is $\mathrm{x}(t)=\mathrm{f}^{(1)}(t)$; then we have evolution equation in terms of impulse response function of the system, $\mathrm{h}(t)$ as

$$\mathrm{y}(t)=\int_0^t\left(\mathrm{h}(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau \tag{3}$$

Some physical systems can be casted as (1) and (3). For example, current through a capacitor classically related to voltage given as $\mathrm{i}(t)=\mathrm{Cv}^{(1)}(t)$; force as rate of change of momentum as

$f(t)=m\left(\upsilon^{(1)}(t)\right)$ and stress related to rate of change in Newtonian viscous element as $\sigma(t)=\eta\left(\varepsilon^{(1)}(t)\right)$. For these physical systems, as per (3) we have response $\mathrm{y}(t)$ as $\mathrm{i}(t)$, $f(t)$ and $\sigma(t)$ to excitation $\mathrm{f}(t)$ as $\mathrm{v}(t)$, $\upsilon(t)$ and $\varepsilon(t)$ respectively. These described systems only respond to rate of change; i.e. output effect is proportional to rate of change of input cause. For capacitor case from (1) as we wrote $\mathrm{q}(t)=\mathrm{c}(t)*\mathrm{v}(t)$, we have $\mathrm{i}(t)=\mathrm{c}(t)*\mathrm{v}^{(1)}(t)$, from (3) where $\mathrm{i}(t)$ as current through capacitor ($\mathrm{y}(t)$ the effect); to a cause rate of change of applied voltage $\mathrm{v}^{(1)}(t)$, (i.e. cause $\mathrm{f}^{(1)}(t)$).

System Relaxation with Memory

Looking at the time evolution equation (1), if the input variable $\mathrm{x}(t)$ acts only at time $t=0$ thereafter vanishes for $t>0$ and we observe $\mathrm{y}(t)$ even at $t>0$ while ($\mathrm{x}(t)=0$ for $t>0$); we may term that system is remembering its past input or cause. In that case, we say system relaxes with 'memory' [6], [22], [27] [28]. In ideal cases as described by the constitutive equations for capacitor, force function and stress, behave 'without memory'. It can be seen when the 'rate terms' input in the RHS of these constitutive equations vanishes after application at $t=0$, we have no observation of the output at $t>0$. Simply if the rate terms in the RHS of all these constitutive equations is described by delta function, then the output is also delta function at $t=0$.

We have analogy with a real life situation. Say someone gives a pinch to our body. We have its effect linger after the cause. This is memorized relaxation. The pinch is impulse input that vanishes after the application; while our body pain relaxes even after that and gradually, the effect of pinch dies out. The response of lingering pain from pinch definitely depends on the body's memory-kernel.

System with response as derivative of cause variable

In the evolution equation (1), if we call response $\mathrm{y}(t)$ as rate of change of cause i.e. $\mathrm{y}(t)=\mathrm{x}^{(1)}(t)$, then we have following evolution equation

$$\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau ; \quad t\geq 0 \tag{4}$$

The expression (4) denotes physical systems where $\mathrm{x}(t)$ grows or decays from an initial value $\mathrm{x}(0)$. This is like nuclear reaction growth or radioactive decay type systems, depending on memory-kernel $\mathrm{h}(t)$. In

subsequent sections we will form constitutive equation from (3) and (4) types of systems, by having various types of decaying memory kernel (singular as well as non-singular in nature).

The convolution integral (1) can in general have lower terminal of integration as $t=-\infty$ or $a>-\infty$ as the case may be for application of input $\mathrm{x}(t)$; that we depict as follows

$$\mathrm{y}(t)=\int_{-\infty}^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau ; \quad \mathrm{y}(t)=\int_a^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau \tag{5}$$

The (5) is the most general representation of the convolution process. In our discussion of forming the constitutive equation, we will consider that the memory kernel acts only when the cause $\mathrm{x}(t)$ acts; and we take $t=0$, and thus the response is described for $t\geq 0$ with convolution process as $\mathrm{y}(t)=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau$.

Forming Constituent expression for $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$ where $\mathrm{x}(t)=\mathrm{f}^{(1)}(\mathrm{t})$ with various memory kernels

a. Memory Kernel as delta function (zero-memory case)

We study a simple case, the response $\mathrm{y}(t)$ in a system given by (1), when $\mathrm{x}(t)$ the input is rate of change of $\mathrm{f}(t)$ i.e. $\mathrm{x}(t)=\mathrm{f}^{(1)}(t)$. That is classically we have relaxation of $\mathrm{y}(t)$ as $\mathrm{y}(t)\propto \mathrm{f}^{(1)}(t)$. The classical case is therefore is following with C as a constant

$$\mathrm{y}(t)=\mathrm{C}\frac{\mathrm{df}(t)}{\mathrm{d}t}=\mathrm{C}\left(\mathrm{f}^{(1)}(t)\right) \tag{6}$$

We can modify the above expression i.e. $\mathrm{y}(t)=\mathrm{Cf}^{(1)}(t)$ or $\mathrm{y}(t)=\mathrm{C}\left(D_t^1\mathrm{f}(t)\right)$ and write

$$\begin{aligned}\mathrm{y}(t)&=\mathrm{C}\int_{-\infty}^{t}\left(\delta(t-\tau)\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\&=\int_{-\infty}^{t}\left(\mathrm{C}\delta(t-\tau)\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau=\left(\mathrm{C}\delta(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\end{aligned} \tag{7}$$

This comes from property of delta function, i.e. $\int\delta(x-y)f(y)\mathrm{d}y=f(x)$ [23]. In above expression (7) for the convolution integral, we have kernel of integration as delta function call it $\mathrm{h}(t)=\mathrm{C}\delta(t)$. With this we get for $\mathrm{y}(t)=\mathrm{Cf}^{(1)}(t)$ the following

$$\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right),\quad \mathrm{h}(t)=\mathrm{C}\delta(t),\quad \mathrm{y}(t)=\mathrm{Cf}^{(1)}(t) \tag{8}$$

This (8) expression we have casted as (1) and (3). The kernel $\mathrm{h}(t)$ we will now term as 'memory kernel' of the basic evolution equation i.e. $\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)$.

Let us give a unit step input, $\mathrm{f}(t)=u(t)$ applied at $t=0$. This means $\mathrm{f}(t)=1$ for $t\geq 0$ to an initially rest system i.e. $\mathrm{f}(t)=0$ for $t<0$; then we have $\mathrm{f}^{(1)}(t)=\delta(t)$ i.e. differentiation of unit-step input. Placing this value in expression $\mathrm{y}(t)=\mathrm{Cf}^{(1)}(t)$, we get

$$\begin{aligned}\mathrm{y}(t)&=\int_{0}^{t}\left(\mathrm{C}\delta(t-\tau)\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau=\mathrm{C}\int_{0}^{t}\left(\delta(t-\tau)\delta(\tau)\right)\mathrm{d}\tau\\&=\mathrm{C}\delta(t)\end{aligned} \tag{9}$$

This (9) above is direct result of $\mathrm{y}(t)=\mathrm{Cf}^{(1)}(t)$; as the differentiation of unit step function is delta-function. The Laplace transformed relations of $\mathrm{y}(t)=\mathrm{Cf}^{(1)}(t)$ is

$$\mathrm{Y}(s)=\mathrm{C}\left(s\mathrm{F}(s)-\mathrm{f}(0)\right) \tag{10}$$

Doing Laplace transform of $\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)$ we get the following

$$\begin{aligned}&\mathcal{L}\left\{\mathrm{y}(t)\right\}=\mathcal{L}\left\{\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\right\}\\&\mathrm{Y}(s)=\mathcal{L}\left\{\mathrm{h}(t)\right\}\mathcal{L}\left\{\mathrm{f}^{(1)}(t)\right\}\\&\qquad=\mathrm{C}\left(\mathrm{H}(s)\right)\left(s\mathrm{F}(s)-\mathrm{f}(0)\right),\quad \mathrm{H}(s)=\mathcal{L}\left\{\mathrm{h}(t)\right\}=\mathcal{L}\left\{\mathrm{C}\delta(t)\right\}=\mathrm{C}\\&\qquad=\mathrm{C}\left(s\mathrm{F}(s)-\mathrm{f}(0)\right)\end{aligned} \tag{11}$$

We get the same result as we got earlier (10), i.e. $\mathrm{Y}(s)=\mathrm{C}\left(s\mathrm{F}(s)-\mathrm{f}(0)\right)$.

From the classical theory with Newtonian Calculus as the constitutive equation (6), we get a delta impulse output when system is impressed with a constant step input. This is a singular relaxation current function.

From the classical law we have arrived at the equation, which is following

$$\begin{aligned}\mathrm{y}(t)&=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\\&=\int_{-\infty}^{t}\left(\mathrm{h}(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\end{aligned}\tag{12}$$

It so happens that the classical equation i.e. $\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(1)}(t)$ is associated with memory kernel $\mathrm{h}(t)=\mathrm{C}\delta(t)$. This physically implies that the system $\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(1)}(t)$ has zero-memory. That is

just after the instance of application of input i.e. at $t=0^{+}$ the memory kernel vanishes i.e. $\mathrm{h}(t)=0$ for $t>0$. Whereas $\mathrm{h}(t)=\infty$ only at $t=0$; and is singular function. This is a 'singular memory kernel'. Now we will study the formation of constituent equation relaxation of output to unit step input of for various kernels-singular and non-singular, from the evolution equation $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)$.

b. Constitutive equation due to Singular Power Law Memory Kernel

The Nature has many examples of 'power-law' distributions [13]-[16]. Let us have the power law decay memory kernel described as

$$\mathrm{h}(t)=\mathrm{C}t^{-\alpha};\quad 0<\alpha<1\tag{13}$$

In above (13) expression C is a positive constant. The kernel $\mathrm{h}(t)=\mathrm{C}t^{-\alpha}$ is singular at origin with its derivative as minus infinity. This means that we have memory kernel $\mathrm{h}(t)=\infty$ at $t=0$ and monotonically decaying after that i.e. $t>0$, with $\left.\mathrm{h}^{(1)}(t)\right|_{t=0}=-\infty$ this is some way mimicking the actual memory or forgetfulness. That is as the time goes the memory fades away. With this we have following steps

$$\begin{aligned}&\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\&\mathcal{L}\left\{\mathrm{y}(t)\right\}=\mathcal{L}\left\{\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\right\}\\&\mathrm{Y}(s)=\left(\mathcal{L}\left\{\mathrm{h}(t)\right\}\right)\left(\mathcal{L}\left\{\mathrm{f}^{(1)}(t)\right\}\right),\quad \mathcal{L}\left\{\mathrm{h}(t)\right\}=\mathcal{L}\left\{\mathrm{C}t^{-\alpha}\right\}=\mathrm{C}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}};\quad \mathrm{f}(t)=u(t)\\&\qquad=\left(\mathrm{C}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(s\mathrm{F}(s)-\mathrm{f}(0)\right),\quad \mathcal{L}\left\{\mathrm{f}(t)\right\}=\mathrm{F}(s)\\&\qquad=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left(s^{\alpha}\mathrm{F}(s)-s^{\alpha-1}\mathrm{f}(0)\right),\quad \mathrm{f}(0)=0,\qquad \mathrm{F}(s)=\frac{1}{s}=\mathcal{L}\left\{u(t)\right\}\\&\mathrm{Y}(s)=\mathrm{C}\left(\Gamma(1-\alpha)\right)s^{\alpha-1}\end{aligned}\tag{14}$$

From above (14) we get $\mathrm{y}(t)=\mathcal{L}^{-1}\left\{\mathrm{Y}(s)\right\}$ by using $\mathcal{L}\left\{t^{p}\right\}=\Gamma(p+1)s^{-p-1}$

$$\mathrm{y}(t)=\mathrm{C}t^{-\alpha};\quad 0<\alpha<1\tag{15}$$

Therefore, we are getting a power-law decay response (output) i.e. $\mathrm{y}(t)\sim t^{-\alpha}$ for the memory kernel in evolution equation i.e. $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)$ as a power law $\mathrm{h}(t)\sim t^{-\alpha}$; for system unit step input $\mathrm{f}(t)=u(t)$.

Now we obtain constitutive relation with memory kernel $\mathrm{h}(t)=\mathrm{C}t^{-\alpha}$ that is singular, at the application time of the cause i.e. at $t=0$ and has no derivative at start point. We write the following steps

$$\begin{aligned}\mathrm{y}(t)&=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\\&=\int_{-\infty}^{t}\left(\mathrm{h}(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau;\quad \mathrm{h}(t)=\mathrm{C}t^{-\alpha};\quad t\ge 0\\&=\int_{0}^{t}\left(\mathrm{C}(t-\tau)^{-\alpha}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\&=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left((t-\tau)^{-\alpha}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\right)\\&=\mathrm{C}_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{f}(t)\right),\quad \mathrm{C}_{\alpha}=\mathrm{C}\left(\Gamma(1-\alpha)\right)\end{aligned}\tag{16}$$

In above (16) steps we have used the definition of Caputo fractional derivative [6], [26], [29] for fractional order $0<\alpha<1$ i.e. ${}_{0}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left((t-\tau)^{-\alpha}\right)\left(f^{(1)}(\tau)\right)\mathrm{d}\tau$; refer Appendix-1

Thus, our constitutive equation for a system having Singular Power Law Memory kernel is given by fractional differential equation, and is changed from classical with zero-memory case i.e. $\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(1)}(t)$ to following case with fractional derivative

$$\begin{aligned}\mathrm{y}(t)&=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{f}(t)\right);\quad \mathrm{C}_{\alpha}=\mathrm{C}\left(\Gamma(1-\alpha)\right)\\\mathrm{y}(t)&=\mathrm{C}_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{f}(t)\right);\quad 0<\alpha<1\end{aligned}\tag{17}$$

In above (17) putting limit $\alpha\to 1$ we get classical relation i.e. $\mathrm{y}(t)=\mathrm{C}_{1}\mathrm{f}^{(1)}(t)$. The above (17) expression as obtained is used in [5], [7]-[12], [17]-[20], [22], [28]. That is because in (16) $\lim_{\alpha\to 1}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}=\delta(t)$, $\lim_{\alpha\to 1}\int_{0}^{t}\left(\frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau=\int_{0}^{t}\left(\delta(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau=\mathrm{f}^{(1)}(t)$.

The Laplace Transform of Caputo Fractional Derivative [6], [26], [29] for fractional order $0<\alpha<1$ is $\mathcal{L}\left\{{}_{0}^{C}D_{t}^{\alpha}f(t)\right\}=s^{\alpha}F(s)-s^{\alpha-1}f(0)$ (Appendix-1). Using this we write Laplace Transform of $\mathrm{y}(t)=\mathrm{C}_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{f}(t)\right)$ as following

$$\begin{aligned}\mathcal{L}\left\{\mathrm{y}(t)\right\}&=\mathrm{C}_{\alpha}\mathcal{L}\left\{{}_{0}^{C}D_{t}^{\alpha}\mathrm{f}(t)\right\};\quad 0<\alpha<1\\\mathrm{Y}(s)&=\mathrm{C}_{\alpha}\left(s^{\alpha}\mathrm{F}(s)-s^{\alpha-1}\mathrm{f}(0)\right)\end{aligned}\tag{18}$$

We note that in above (18) putting $\alpha=1$ we obtain the classical result of Laplace transform of classical-derivative i.e. $\mathrm{Y}(s)=\mathrm{C}\left(s\mathrm{F}(s)-\mathrm{f}(0)\right)$. We verify the relaxation expression of $\mathrm{y}(t)$ with $\mathrm{F}(s)=s^{-1}$ and $\mathrm{f}(0)=0$ i.e. for unit step input $\mathrm{f}(t)=u(t);\ t\ge 0$, applied to system with $\mathrm{f}(0)=0,\quad t<0$ from above

$$\begin{aligned}
&\mathrm{Y}(s)=\mathrm{C}_\alpha\left(s^\alpha \mathrm{F}(s)-s^{\alpha-1}\mathrm{f}(0)\right); \quad \mathrm{F}(s)=\frac{1}{s}, \quad \mathrm{f}(0)=0\\
&\mathrm{Y}(s)=\mathrm{C}_\alpha s^{\alpha-1}, \quad \mathcal{L}\left\{t^p\right\}=\frac{\Gamma(p+1)}{s^{p+1}}\\
&\mathrm{y}(t)=\mathcal{L}^{-1}\left\{\mathrm{Y}(s)\right\}=\mathcal{L}^{-1}\left\{\mathrm{C}_\alpha s^{\alpha-1}\right\}\\
&\qquad=\frac{\mathrm{C}_\alpha}{\Gamma(1-\alpha)}t^{-\alpha}, \quad 0<\alpha<1, \qquad t\geq 0
\end{aligned} \tag{19}$$

With $\mathrm{C}_\alpha=\mathrm{C}\left(\Gamma(1-\alpha)\right)$ we get $\mathrm{y}(t)=\mathrm{C}t^{-\alpha}$; same as we got earlier (15).

c. Difference between zero-memory and memory based relaxation cases

We observed that for a classical case the relaxation response i.e. $\mathrm{y}(t)$ is delta function at the start of application of input $\mathrm{f}(t)$, which is unit step function, i.e. $\mathrm{f}(t)=u(t)$. Therefore, as soon as the rate of change i.e. $\mathrm{f}^{(1)}(t)$ vanishes at $t>0$ we have relaxing output $\mathrm{y}(t)$ as zero. This is zero memory case with memory kernel as $\mathrm{h}(t)\propto\delta(t)$. Where we observe from above (30) with a power-law memory kernel as $\mathrm{h}(t)\propto t^{-\alpha}$ ($0<\alpha<1$), we have a finite output response $\mathrm{y}(t)$ even the rate of change of cause (input) $\mathrm{f}^{(1)}(t)$ vanished at $t>0$. Therefore, the system is memorizing the excitation that once took place as a rate of change in input (cause) and system is relaxing with memory. Well this above case was the case with singular power law memory kernel. Now in subsequent sections we will discuss non-singular memory kernels and see the constitutive relations that we get for these systems.

d. Constitutive equation due to Non-Singular power law Memory Kernel

We have seen earlier that the kernel of singular power-law i.e. $\mathrm{h}(t)\propto t^{-\alpha}$; $0<\alpha<1$ gives a constitutive equation with fractional derivative i.e. $\mathrm{y}(t)\propto\mathrm{f}^{(\alpha)}(t)$. We modify the power-law to a non-singular type with following type

$$\mathrm{h}(t)=\mathrm{C}(1+\upsilon t)^{-\alpha}; \quad 0<\alpha<1, \quad \upsilon>0, \quad t\geq 0 \tag{20}$$

In above (20) C is a positive constant. In above (20) we have $\mathrm{h}(0)=\mathrm{C}$ and $\mathrm{h}^{(1)}(0)=-\mathrm{C}\alpha$, unlike the singular kernel, i.e. $\mathrm{h}(t)\propto t^{-\alpha}$ (discussed previously). With this we do following calculations for obtaining constitutive equation

$$
\begin{aligned}
\mathrm{y}(t)&=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)\\
&=\int_0^t \mathrm{C}\left((1+\upsilon(t-\tau)^{-\alpha}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(\binom{-\alpha}{0}\left(\upsilon(t-\tau)\right)^0+\binom{-\alpha}{1}\left(\upsilon(t-\tau)\right)+\binom{-\alpha}{2}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(1+(-\alpha)\left(\upsilon(t-\tau)\right)+\tfrac{(-\alpha)(-\alpha-1)}{2!}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\begin{pmatrix}\int_0^t\left(\mathrm{f}^{(1)}(\tau)\right)+\frac{(-\alpha)}{1!}\int_0^t\left(\upsilon(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
+\frac{(-\alpha)(-\alpha-1)}{2!}\int_0^t\left(\upsilon(t-\tau)\right)^2\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
+\frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}\int_0^t\left(\upsilon(t-\tau)\right)^3\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau..............\end{pmatrix}
\end{aligned}
\tag{21}
$$

In above (21) we used binomial expansion [23] for series representation for $(1+z)^{-\alpha}$. We now use repeated integration formula (Appendix) i.e. ${}_0I_t^m g(t)=\frac{1}{(m-1)!}\int_0^t(t-\tau)^{m-1}g(\tau)\mathrm{d}\tau$, [6], [26], [29] and get the following expression

$$
\begin{aligned}
\mathrm{y}(t)&=\mathrm{C}\begin{pmatrix}\int_0^t\left(\mathrm{f}^{(1)}(\tau)\right)+\frac{(-\alpha)}{1!}\int_0^t\left(\upsilon(t-\tau)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\ldots\\
\ldots.+\frac{(-\alpha)(-\alpha-1)}{2!}\int_0^t\left(\upsilon(t-\tau)\right)^2\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau+\frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}\int_0^t\left(\upsilon(t-\tau)\right)^3\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau..\end{pmatrix}\\
&=\mathrm{C}\begin{pmatrix}\int_0^t\left(\mathrm{f}^{(1)}(\tau)\right)+(-\alpha)\upsilon\left(\frac{1}{(2-1)!}\int_0^t\left(t-\tau\right)^{2-1}\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\right)+\\
(-\alpha)(-\alpha-1)\upsilon^2\left(\frac{1}{(3-1)!}\int_0^t\left(t-\tau\right)^{3-1}\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\right)+\ldots\end{pmatrix}\\
&=\mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)+(-\alpha)\upsilon\left({}_0I_t^2\mathrm{f}^{(1)}(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\mathrm{f}^{(1)}(t)\right)+\ldots\ldots....\right)
\end{aligned}
\tag{22}
$$

It so happens that for this memory-kernel $\mathrm{h}(t)=\mathrm{C}(1+\upsilon t)^{-\alpha}$ which is non-singular power-law the constitutive equation is for $\mathrm{y}(t)$ is weighted sum of integrals (one whole, two whole, three whole; and so on) of $\mathrm{f}^{(1)}(t)$; that is rate of change of $\mathrm{f}(t)$ (the input).

With excitation $\mathrm{f}^{(1)}(t)=\delta(t)$ to initially rest system with $\mathrm{y}(0)=0$, that is applying a unit step input $\mathrm{f}(t)=u(t)$ we obtain following from above (22), we write following steps

$$
\begin{aligned}
\mathrm{y}(t)&=\mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)+(-\alpha)\upsilon\left({}_0I_t^2\mathrm{f}^{(1)}(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\mathrm{f}^{(1)}(t)\right)+\ldots\ldots....\right)\\
&=\mathrm{C}\left({}_0I_t^1\delta(t)+(-\alpha)\upsilon\left({}_0I_t^2\delta(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\delta(t)\right)+\ldots\ldots....\right)\\
&=\mathrm{C}\left(1-\alpha\upsilon t^2+\alpha(\alpha+1)\upsilon^2\left(\tfrac{t^2}{2}\right)-\alpha(\alpha+1)(\alpha+2)\left(\tfrac{t^3}{(3)(2)}\right)+\ldots...\right)\\
&=\mathrm{C}(1+\upsilon t)^{-\alpha}
\end{aligned}
\tag{23}
$$

The above (23) says that the response $\mathrm{y}(t)$ lingers in the system while the rate of change of input $\mathrm{f}^{(1)}(t)$ vanishes at $t>0$; giving memorized relaxation response. This was also observed with singular power law memory kernel. However, the constitutive equation for this case is in

$$\mathrm{y}(t)=\mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)+(-\alpha)\upsilon\left({}_0I_t^2\mathrm{f}^{(1)}(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\mathrm{f}^{(1)}(t)\right)+.........\right) \quad (24)$$

This is very different from $\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(1)}(t)$ and $\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(\alpha)}(t)$; the classical case and case with fractional derivative respectively; that we got for singular memory kernels. Here in (24), above we are getting a series sum of weighted repeated integration

$$\mathrm{y}(t)\propto\sum_{n=1}^{\infty}\mathrm{a}_n\left({}_0I_t^n\mathrm{f}^{(1)}(t)\right) \quad (25)$$

With weights in above as $\mathrm{a}_1=1$, $\mathrm{a}_2=-\alpha\upsilon$, $\mathrm{a}_3=(\alpha)(\alpha+1)\upsilon^2\ldots$

e. Constitutive equation due to Mittag-Leffler function as Non-Singular Memory Kernel

Here we take Memory Kernel as following for $t\geq 0$, i.e. "One parameter Mittag-Leffler function"

$$\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha);\quad 0<\alpha<1,\quad t\geq 0 \quad (26)$$

In above C and λ are a positive real constants. Where the "One Parameter Mittag-Leffler function" is defined [6], [23], [26], [29] as following (Appendix-1)

$$E_\alpha(-\lambda t^\alpha)=\sum_{n=0}^{\infty}\frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+1)},\quad t\geq 0;\quad \lambda t^\alpha\in\mathbb{C},\quad \alpha\in\mathbb{C},\quad \mathrm{Re}\left[\alpha\right]>0 \quad (27)$$

The constitutive equation with Memory Kernel $\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha)$ we write the following

$$\begin{aligned}\mathrm{y}(t)&=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\&=\mathrm{C}\int_0^t\left(E_\alpha(-\lambda(t-\tau)^\alpha)\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\&=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n+1)}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\&=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\lambda^n}{\Gamma(\alpha n+1)}\right)\int_0^t(t-\tau)^{\alpha n}\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\\&=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\lambda^n\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)\end{aligned} \quad (28)$$

Where in above (28) we used the operator ${}_0I_t^\upsilon$, $\upsilon=\alpha n+1$, which is Riemann-Liouville fractional integration of order υ defined as ${}_0I_t^\upsilon\left[g(t)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^t(t-\tau)^{\upsilon-1}g(\tau)\mathrm{d}\tau$ (Appendix-1). We write above (28) as series sum of weighted fractional integration

$$\begin{aligned}\mathrm{y}(t)&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)\\&=\mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)-\lambda\left({}_0I_t^{\alpha+1}\mathrm{f}^{(1)}(t)\right)+\lambda^2\left({}_0I_t^{2\alpha+1}\mathrm{f}^{(1)}(t)\right)-\lambda^3\left({}_0I_t^{3\alpha+1}\mathrm{f}^{(1)}(t)\right)+...\right)\\&=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+1}\mathrm{f}^{(1)}(t)\right)\end{aligned} \quad (29)$$

With weights in this case as $\mathrm{b}_0 = 1, \mathrm{b}_1 = -\lambda; \dots \quad \mathrm{b}_n = (-1)^n \lambda^n \dots\dots$ We get similar result with $\mathrm{h}(t) = \mathrm{C}E_\alpha(-\lambda t^\alpha)$ that of with Non-Singular power law memory-kernel i.e. $\mathrm{h}(t) = \mathrm{C}(1+\upsilon t)^{-\alpha}$ and this too is very different from results of singular memory kernels i.e. $\mathrm{h}(t) = \mathrm{C}\delta(t)$ and $\mathrm{h}(t) = \mathrm{C}t^{-\alpha}$ giving $\mathrm{y}(t) = \mathrm{C}\mathrm{f}^{(1)}(t)$ and $\mathrm{y}(t) = \mathrm{C}\mathrm{f}^{(\alpha)}(t)$ respectively.

Thus, a Memory Kernel $\mathrm{h}(t) = \mathrm{C} - \frac{\lambda \mathrm{C} t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 \mathrm{C} t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots; \quad 0 < \alpha < 1$ i.e. series-sum of power laws acting on derivative of voltage function i.e. $\mathrm{f}^{(1)}(t)$, gives a relaxing response $\mathrm{y}(t)$ with series sum of fractional integrations of various orders acting on rate of change of applied excitation i.e. $\mathrm{f}^{(1)}(t)$, i.e. $\mathrm{y}(t) = \mathrm{C}\sum_{n=0}^{\infty} \mathrm{b}_n \left({}_0I_t^{\alpha n+1} \mathrm{f}^{(1)}(t) \right)$.We note that Memory Kernel in this case i.e. $\mathrm{h}(t) = \mathrm{C}E_\alpha(-\lambda t^\alpha)$ is not singular function at $\mathrm{h}(0) = \mathrm{C}$, and its derivative is not defined i.e. $\mathrm{h}^{(1)}(t)\big|_{t=0} = -\infty$.

Now we give a unit step input to this system so we have $\mathrm{f}(t) = 1, \quad t \ge 0$; with $\mathrm{f}^{(1)}(t) = \delta(t)$. Placing this in above $\mathrm{y}(t) = \mathrm{C}\sum_{n=0}^{\infty} \mathrm{b}_n \left({}_0I_t^{\alpha n+1} \mathrm{f}^{(1)}(t) \right)$(30), we write the following

$$\begin{aligned}
\mathrm{y}(t) &= \mathrm{C}\mathrm{f}(t) - \lambda\mathrm{C}\left({}_0I_t^{\alpha+1}\mathrm{f}^{(1)}(t) \right) + \lambda^2\mathrm{C}\left({}_0I_t^{2\alpha+1}\mathrm{f}^{(1)}(t) \right) - \lambda^3\mathrm{C}\left({}_0I_t^{3\alpha+1}\mathrm{f}^{(1)}(t) \right) + \dots \\
&= \mathrm{C} - \lambda\mathrm{C}\left({}_0I_t^{\alpha+1}\delta(t) \right) + \lambda^2\mathrm{C}\left({}_0I_t^{2\alpha+1}\delta(t) \right) - \lambda^3\mathrm{C}\left({}_0I_t^{3\alpha+1}\delta(t) \right) + \dots \\
&= \mathrm{C} - \lambda\mathrm{C}\left(\frac{t^\alpha}{\Gamma(\alpha+1)} \right) + \lambda^2\mathrm{C}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) - \lambda^3\mathrm{C}\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right) + \dots \qquad (30) \\
&= \mathrm{C}\left(1 + \frac{(-\lambda)t^\alpha}{\Gamma(\alpha+1)} + \frac{(-\lambda)^2(t^\alpha)^2}{\Gamma(2\alpha+1)} + \frac{(-\lambda)^3(t^\alpha)^3}{\Gamma(3\alpha+1)} + \dots. \right) \\
&= \mathrm{C}\left(\sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+1)} \right) = \mathrm{C}E_\alpha(-\lambda t^\alpha), \quad t \ge 0
\end{aligned}$$

In above (30) steps we have used formula for fractional integration of delta function i.e. ${}_0I_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(\upsilon)} t^{\upsilon-1}$; also we used ${}_0I_t^1 \mathrm{f}^{(1)}(t) = \mathrm{f}(t)$. What we observe that the relaxation response $\mathrm{y}(t)$ to a system excited by unit-step input i.e. $\mathrm{f}(t) = u(t)$; relaxes in proportional to the memory kernel function i.e. $\mathrm{y}(t) \propto \mathrm{h}(t)$ in this case is $\mathrm{h}(t) \propto E_\alpha(-\lambda t^\alpha)$. Here in above (30) the effect function $\mathrm{y}(t)$ relaxes at $t > 0$ even while the rate of change of voltage has vanished; therefore memorizing the past excitation.We note that by placing $\alpha = 1$ we are not getting classical case i.e. $\mathrm{y}(t) = \mathrm{C}\mathrm{f}^{(1)}(t)$.

Let us do Laplace Transformation for Mittag-Leffler memory kernel, as depicted as follows

$$\begin{aligned}
&\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\
&\mathcal{L}\{\mathrm{y}(t)\}=\mathcal{L}\left\{(\mathrm{h}(t))*\left(\mathrm{f}^{(1)}(t)\right)\right\}\\
&\mathrm{Y}(s)=\left(\mathcal{L}\{\mathrm{h}(t)\}\right)\left(\mathcal{L}\left\{\mathrm{f}^{(1)}(t)\right\}\right),\quad \mathcal{L}\{\mathrm{h}(t)\}=\mathcal{L}\left\{\mathrm{C}E_\alpha(-\lambda t^\alpha)\right\}=\mathrm{C}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\\
&\qquad=\mathrm{C}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\left(s\mathrm{F}(s)-\mathrm{f}(0)\right)\\
&\qquad=\mathrm{C}\left(\left(\tfrac{s^\alpha}{s^\alpha+\lambda}\right)\mathrm{F}(s)-\left(\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\mathrm{f}(0)\right),\quad \mathrm{f}(0)=0,\qquad \mathrm{F}(s)=\frac{1}{s}\\
&\mathrm{Y}(s)=\mathrm{C}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\\
&\mathrm{y}(t)=\mathrm{C}\mathcal{L}^{-1}\left\{\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\right\}\\
&\qquad=\mathrm{C}E_\alpha(-\lambda t^\alpha)
\end{aligned}\tag{31}$$

The same that we got earlier in (30).

f. Constitutive equation due to exponential function as Non-Singular Memory Kernel

Here we take Memory kernel as

$$\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t},\quad t\ge 0,\quad \kappa>0;\quad \mathrm{C}>0 \tag{32}$$

The constitutive equation with Memory Kernel as $\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t}$ gives the following

$$\begin{aligned}
\mathrm{y}(t)&=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\
&=\mathrm{C}\int_0^t\left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t(t-\tau)^n\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\\
&=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t(t-\tau)^n\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\
&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)
\end{aligned}\tag{33}$$

Thus, the memory Kernel which is pure exponential function i.e. $\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t}$ gives a relaxation response $\mathrm{y}(t)$ which is weighted series sum of integer order multiple integration of rate of change of input function $\mathrm{f}^{(1)}(t)$. We describe as follows

$$\begin{aligned}
\mathrm{y}(t) &= \mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)\\
&= \mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)\right)-\kappa\,\mathrm{C}\left({}_0I_t^2\mathrm{f}^{(1)}(t)\right)+\kappa^2\mathrm{C}\left({}_0I_t^3\mathrm{f}^{(1)}(t)\right)-\lambda^3\mathrm{C}\left({}_0I_t^4\mathrm{f}^{(1)}(t)\right)+\ldots\\
&= \mathrm{C}\sum_{n=0}^{\infty}\mathrm{c}_n\left({}_0I_t^{n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)
\end{aligned}\tag{34}$$

$$\mathrm{c}_0=1,\quad \mathrm{c}_1=-\kappa,\quad \mathrm{c}_3=\kappa^2,\quad \ldots\mathrm{c}_n=(-1)^n\kappa^n$$

In above (29) the constitutive equation with memory kernel as non-singular Mittag-Leffler function, i.e. $\mathrm{y}(t)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)$ if we place $\alpha=1$ we get (34), by calling λ as κ .

We give a step input to system having Memory kernel $\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t}$ and observe the following

$$\begin{aligned}
\mathrm{y}(t) &= \mathrm{C}\left({}_0I_t^1\mathrm{f}^{(1)}(t)\right)\mathrm{f}(t)-\kappa\,\mathrm{C}\left({}_0I_t^2\mathrm{f}^{(1)}(t)\right)+\kappa^2\mathrm{C}\left({}_0I_t^3\mathrm{f}^{(1)}(t)\right)-\kappa^3\mathrm{C}\left({}_0I_t^4\mathrm{f}^{(1)}(t)\right)+\ldots\\
&= \mathrm{C}-\kappa\,\mathrm{C}\left({}_0I_t^2\delta(t)\right)+\kappa^2\mathrm{C}\left({}_0I_t^3\delta(t)\right)-\kappa^3\mathrm{C}\left({}_0I_t^4\delta(t)\right)+\ldots\\
&= \mathrm{C}-\kappa\mathrm{C}t+\kappa^2\mathrm{C}\left(\frac{t^2}{2!}\right)-\kappa^3\mathrm{C}\left(\frac{t^3}{3!}\right)+\ldots\\
&= \mathrm{C}\left(1+\frac{(-\kappa)t}{1!}+\frac{(-\kappa)^2(t)^2}{2!}+\frac{(-\kappa)^3(t)^3}{3!}+\ldots.\right)\\
&= \mathrm{C}\left(\sum_{n=0}^{\infty}\frac{(-\kappa t)^n}{n!}\right)=\mathrm{C}\mathrm{e}^{-\kappa t},\quad t\ge 0
\end{aligned}\tag{35}$$

In above (35) we have used ${}_0I_t^m\delta(t)=\frac{1}{(m-1)!}t^{m-1},\quad m=1,2,3,\ldots.$; that is integrations of delta-function [23]. In addition, we assumed $\mathrm{f}(0)=0$, thus we wrote ${}_0I_t^1\,\mathrm{f}^{(1)}(\mathrm{t})=\mathrm{f}(t)$. That is in above (35) the relaxation output $\mathrm{y}(t)$ to unit step input, $\mathrm{f}(t)=u(t)$ to a system having the memory kernel as exponential decay function $\mathrm{h}(t)\sim\mathrm{e}^{-\kappa t}$ has relaxation as $\mathrm{y}(t)\propto\mathrm{h}(t)$.

We note that the Memory Kernel $\mathrm{h}(t)\sim\mathrm{e}^{-\kappa t}$ is non-singular function and has derivative everywhere. Let us apply Laplace Transformation as depicted below

$$\begin{aligned}
&\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\
&\mathcal{L}\{\mathrm{y}(t)\}=\mathcal{L}\left\{(\mathrm{h}(t))*\left(\mathrm{f}^{(1)}(t)\right)\right\}\\
&\mathrm{Y}(s)=\left(\mathcal{L}\{\mathrm{h}(t)\}\right)\left(\mathcal{L}\{\mathrm{f}^{(1)}(t)\}\right),\quad \mathcal{L}\{\mathrm{h}(t)\}=\mathcal{L}\{\mathrm{C}e^{-\kappa t}\}=\mathrm{C}\left(\frac{1}{s+\kappa}\right)\\
&\qquad=\left(\frac{\mathrm{C}}{s+\kappa}\right)(s\mathrm{F}(s)-\mathrm{f}(0))\\
&\qquad=\mathrm{C}\left(\left(\tfrac{s}{s+\kappa}\right)\mathrm{F}(s)-\left(\tfrac{1}{s+\kappa}\right)\mathrm{f}(0)\right),\quad \mathrm{f}(0)=0,\qquad \mathrm{F}(s)=\frac{1}{s}\\
&\mathrm{Y}(s)=\mathrm{C}\left(\frac{1}{s+\kappa}\right)\\
&\mathrm{y}(t)=\mathrm{C}\mathcal{L}^{-1}\left\{\tfrac{1}{s+\kappa}\right\}\\
&\qquad=\mathrm{C}e^{-\kappa t}
\end{aligned}\tag{36}$$

We get same result as earlier in (35).

g. Constitutive equation due to stretched exponential Non-Singular Memory Kernel

Here we take Memory Kernel as stretched exponential function

$$\mathrm{h}(t)=\mathrm{C}e^{-(\kappa t)^{\alpha}}\,,\quad t\geq 0,\quad \kappa>0;\quad 0<\alpha<1;\quad \mathrm{C}>0 \tag{37}$$

With $\alpha=1$ the situation is same as for the case of pure exponential kernel for memory. We now proceed in following steps

$$\begin{aligned}
\mathrm{y}(t)&=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)\\
&=\mathrm{C}\int_0^t\left(e^{-(\kappa(t-\tau))^{\alpha}}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{\left((-\kappa(t-\tau))^{\alpha}\right)^n}{n!}\right)\left(\mathrm{f}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\int_0^t(t-\tau)^{\alpha n}\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\\
&=\mathrm{C}\left(\sum_{n=0}^{\infty}\Gamma(\alpha n+1)\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\mathrm{f}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\
&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)
\end{aligned}\tag{38}$$

This gives the constitutive equation for $\mathrm{y}(t)$ with series weighted sum of fractional integration of various orders of input $\mathrm{f}^{(1)}(t)$; similar to the case with Mittag-Leffler function as Memory kernel, represented as following

$$\mathrm{y}(t)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{d}_n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right);\quad \mathrm{d}_n=(-1)^n\left(\tfrac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right) \tag{39}$$

We note that with $\alpha=1$ we obtain the exact case for memory kernel with pure-exponential function.

Forming Constitutive Equation from evolution equation $\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{x}(t)\right)$ with $\mathrm{y}(t)=\mathrm{x}^{(1)}(t)$ for various memory-kernels $\mathrm{h}(t)$

a. Memory kernel as delta function the zero memory case

We have the following evolution equation

$$\begin{aligned}&\mathrm{y}(t)=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau ;\quad t\ge 0;\quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t)\\&\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\end{aligned}\tag{40}$$

Take memory-kernel as $\mathrm{h}(t)=\mathrm{C}\,\delta(t)$, with C as real constant. Then we have

$$\begin{aligned}\frac{\mathrm{dx}(t)}{\mathrm{d}t}&=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\\&=\int_0^t \mathrm{C}\,\delta(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau=\mathrm{Cx}(t)\end{aligned}\tag{41}$$

From above (41) we have a solution

$$\mathrm{x}(t)=\mathrm{x}(0)\mathrm{e}^{\mathrm{C}t};\quad t\ge 0\tag{42}$$

We get classical expression of exponential growth for $\mathrm{C}>0$ and case of classical exponential decay with $\mathrm{C}<0$ for the above equation, i.e. $\mathrm{x}(t)=\mathrm{x}(0)\,\mathrm{e}^{\mathrm{C}t}$. The case with memory kernel as delta function $\mathrm{h}(t)=\mathrm{C}\,\delta(t)$ gives classical growth or decay equation given as

$$\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\mathrm{Cx}(t)\tag{43}$$

i.e. rate of change of growing (or decaying) quantity is proportional to quantity itself. This is a classical first order linear homogeneous equation i.e. $\mathrm{x}^{(1)}(t)-\mathrm{Cx}(t)=0$; solution of this is $\mathrm{x}(t)=\mathrm{x}(0)\,\mathrm{e}^{\mathrm{C}t}$. The constituent equation for zero-memory case we got (43) as $\mathrm{x}^{(1)}(t)=\mathrm{Cx}(t)$, can be re-casted in following way (by integrating both sides)

$$\begin{aligned}\mathrm{x}(t)-\mathrm{x}(0)&=\mathrm{C}\int_0^t \mathrm{x}(\tau)\mathrm{d}\tau\\\mathrm{x}(t)-\mathrm{x}(0)&=\mathrm{C}\left({}_0I_t^1\mathrm{x}(t)\right)\end{aligned}\tag{44}$$

b. Memory kernel as singular power law function

Take memory kernel as $\mathrm{h}(t)=\mathrm{C}\,t^{-\alpha}$ with $1<\alpha\le 2$, and C as real constant. Then we have following steps

$$\begin{aligned}\frac{\mathrm{dx}(t)}{\mathrm{d}t}&=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\\ &=\int_0^t\left(\mathrm{C}(t-\tau)^{-\alpha}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\ &=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t\left((t-\tau)^{-\alpha}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\right)\\ &=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left({}_0I_t^{(1-\alpha)}\mathrm{x}(t)\right)\end{aligned} \tag{45}$$

We have used fractional integration formula i.e. ${}_0I_t^{\upsilon}\left[g(t)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^t(t-\tau)^{\upsilon-1}g(\tau)\mathrm{d}\tau$, in above (45) steps. Integrate (45) once to write the following

$$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left({}_0I_t^{(2-\alpha)}\mathrm{x}(t)\right) \tag{46}$$

Operate ${}_0D_t^{2-\alpha}$ i.e. fractional derivative operator (RL) on both sides to get the following

$$\begin{aligned}&{}_0D_t^{(2-\alpha)}\left(\mathrm{x}(t)-\mathrm{x}(0)\right)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left({}_0D_t^{(2-\alpha)}\,{}_0I_t^{(2-\alpha)}\mathrm{x}(t)\right)\\ &{}_0D_t^{(2-\alpha)}\mathrm{x}(t)-\frac{\mathrm{x}(0)}{\Gamma(\alpha-1)}t^{\alpha-2}=\mathrm{C}\left(\Gamma(1-\alpha)\right)\mathrm{x}(t)\end{aligned} \tag{47}$$

We have used ${}_0D_t^{\upsilon}\mathrm{K}=\mathrm{K}\frac{t^{-\upsilon}}{\Gamma(1-\upsilon)}$ in above (47) steps. We can always write from above (47) i.e. ${}_0D_t^{(2-\alpha)}\left(\mathrm{x}(t)-\mathrm{x}(0)\right)={}_0^CD_t^{(2-\alpha)}\mathrm{x}(t)$ with condition $0<(2-\alpha)<1$. That is we used the relation of RL and Caputo derivatives for order less than unity, with non-zero initial value. This gives the condition $1<\alpha<2$. Thus for a memory-kernel as $\mathrm{h}(t)=\mathrm{C}t^{-\alpha}$ with $1<\alpha<2$, we have constituent expression given by Caputo fractional derivative.

$${}_0^CD_t^{(2-\alpha)}\mathrm{x}(t)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\mathrm{x}(t) \tag{48}$$

For $0<(2-\alpha)<1$ we have solution to above as

$$\begin{aligned}&{}_0^CD_t^{(2-\alpha)}\mathrm{x}(t)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\mathrm{x}(t)\\ &\mathrm{x}(t)=\mathrm{x}(0)E_{2-\alpha}\left(\mathrm{C}\left(\Gamma(1-\alpha)\right)t^{(2-\alpha)}\right)\end{aligned} \tag{49}$$

In (49) we have used One-parameter Mittag-Leffler function as eigen-function, for Caputo differential operator. With $\mathrm{C}<0$, call it $\mathrm{C}=-\lambda;\quad \lambda>0$, we write the law as

$$\begin{aligned}&{}_0^CD_t^{(2-\alpha)}\mathrm{x}(t)=-\lambda\left(\Gamma(1-\alpha)\right)\mathrm{x}(t)\\ &\mathrm{x}(t)=\mathrm{x}(0)E_{2-\alpha}\left(-\lambda\left(\Gamma(1-\alpha)\right)t^{(2-\alpha)}\right)\end{aligned} \tag{50}$$

With $\alpha=1$ in (50) we get $\mathrm{x}(t)\sim\mathrm{e}^{-\lambda t}$ that is we have classical relaxation law as exponential decay $\mathrm{x}^{(1)}(t)=-\lambda\,\mathrm{x}(t)$ that we described with memory kernel as delta function (44). With $\alpha=0$ we have memory kernel $\mathrm{h}(t)=\mathrm{C}$ which is constant function (i.e. memory kernel is constant) and relaxation function is sustained oscillations, i.e. $\mathrm{x}(t)\sim\cos\sqrt{\lambda}t$, with constitutive equation as $\mathrm{x}^{(2)}(t)=-\lambda\mathrm{x}(t)$. Thus, we have a generalization from a zero memory case to a constant memory case; in $\mathrm{h}(t)=\mathrm{C}t^{-\alpha}$ for $0\le\alpha\le 2$.

$$x(t) = x(0)E_{2-\alpha}\left(C_\alpha t^{(2-\alpha)}\right), \quad 1<\alpha<2$$

$$2-\alpha=\upsilon, \quad C_\alpha=-k, \quad x(0)=1 \qquad x(t)=E_\upsilon(-kt^\upsilon), \quad t>0; \quad k=1; \quad 0\le\upsilon\le1$$

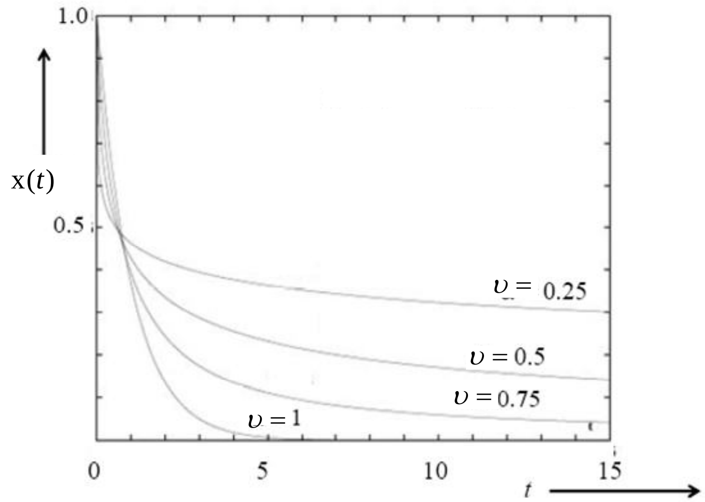

Figure-3: Monotonically decaying Mittag-Leffler function

$$x(t)=E_\upsilon(-kt^\upsilon), \quad t>0; \quad k=1; \quad 1<\upsilon\le2$$

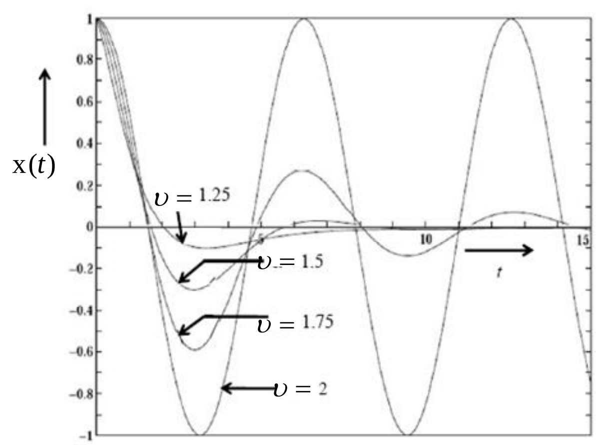

Figure-4: Oscillatory Decaying Mittag-Leffler function

We can write the constituent equation with memory-kernel as $\mathrm{h}(t)=\mathrm{C}t^{-\alpha}$ as in terms of fractional integral as we have for zero-memory case $\mathrm{h}(t)=\mathrm{C}\delta(t)$ i.e. $\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\left({}_0I_t^1\mathrm{x}(t)\right)$ as follows

$$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left({}_0I_t^{(2-\alpha)}\mathrm{x}(t)\right) \tag{51}$$

Let us take $\upsilon=2-\alpha$ and write $\mathrm{C}\left(\Gamma(1-\alpha)\right)=\mathrm{C}_\alpha$. With this, we write Laplace-transform of $\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}_\alpha\left({}_0I_t^{\upsilon}\mathrm{x}(t)\right)$ as $\mathrm{X}(s)-\frac{\mathrm{x}(0)}{s}=\mathrm{C}_\alpha\left(\frac{1}{s^\upsilon}\mathrm{X}(s)\right)$. Rearranging this Laplace transformed equation we get $\mathrm{X}(s)=\frac{s^{\upsilon-1}}{s^\upsilon-\mathrm{C}_\alpha}\mathrm{x}(0)$. Taking inverse Laplace transform, we get $\mathrm{x}(t)=\mathrm{x}(0)E_\upsilon(\mathrm{C}_\alpha t^\alpha)$. That is the same we wrote above as solution above (49).

c. Non singular power law memory-kernel

We take the case as $\mathrm{h}(t)=\mathrm{C}(1+\upsilon t)^{-\alpha}$, with C as a real constant. That is a non-singular memory kernel. With this we write following steps to get constituent equation

$$\begin{aligned}
\frac{\mathrm{dx}(t)}{\mathrm{d}t}&=\mathrm{h}(t)*\mathrm{x}(t)=\int_0^t\mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\\
&=\int_0^t\mathrm{C}\left((1+\upsilon(t-\tau)\right)^{-\alpha}\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(\binom{-\alpha}{0}\left(\upsilon(t-\tau)\right)^0+\binom{-\alpha}{1}\left(\upsilon(t-\tau)\right)+\binom{-\alpha}{2}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(1+(-\alpha)\left(\upsilon(t-\tau)\right)+\tfrac{(-\alpha)(-\alpha-1)}{2!}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{C}\begin{pmatrix}\int_0^t\left(\mathrm{x}(\tau)\right)+\frac{(-\alpha)}{1!}\int_0^t\left(\upsilon(t-\tau)\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
+\frac{(-\alpha)(-\alpha-1)}{2!}\int_0^t\left(\upsilon(t-\tau)\right)^2\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
+\frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}\int_0^t\left(\upsilon(t-\tau)\right)^3\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau..............\end{pmatrix}\\
&=\mathrm{C}\left({}_0I_t^1\,\mathrm{x}(t)+(-\alpha)\upsilon\left({}_0I_t^2\,\mathrm{x}(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\,\mathrm{x}(t)\right)+..........\right)
\end{aligned} \tag{52}$$

We have used ${}_0I_t^m\left[f(t)\right]=\frac{1}{(m-1)!}\int_0^t(t-\tau)^{m-1}f(\tau)\mathrm{d}\tau$ in above derivation. We integrate once to get the following constituent relation

$$\begin{aligned}
\mathrm{x}(t)-\mathrm{x}(0)&=\mathrm{C}\left({}_0I_t^2\,\mathrm{x}(t)+(-\alpha)\upsilon\left({}_0I_t^3\,\mathrm{x}(t)\right)+(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^4\,\mathrm{x}(t)\right)+..........\right)\\
\mathrm{x}(t)-\mathrm{x}(0)&=\mathrm{C}\left(\sum_{n=0}^{\infty}\mathrm{a}_n\left({}_0I_t^{n+2}\,\mathrm{x}(t)\right)\right)\\
\mathrm{a}_0&=1,\quad \mathrm{a}_1=-\alpha\upsilon,\quad \mathrm{a}_2=\alpha(\alpha+1)\upsilon^2...
\end{aligned} \tag{53}$$

In contrary to the cases for constituent equations with memory kernel as singular type (44), (49), here in (53) we are having weighted sum series of multiple integrations of orders, 2, 3, 4… We note the expression (53) is difficult to solve for obtaining $\mathrm{x}(t)$. Since there is no close form standard, Laplace transform formula exists for $\mathrm{h}(t)=\mathrm{C}(1+\upsilon t)^{-\alpha}$ we are unable to proceed with solution using Laplace transform technique to (52). We will use Laplace transform technique for Mittag-Leffler kernel and exponential kernel.

d. Memory Kernel non-singular Mittag-Leffler function

We take $\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha)$ with C and λ as positive constants, and write following steps

$$\begin{aligned}
&\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{x}(t)\right)=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\,, \quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t)\\
&\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\mathrm{C}\int_0^t\left(E_\alpha(-\lambda(t-\tau)^\alpha)\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n+1)}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\lambda^n}{\Gamma(\alpha n+1)}\right)\int_0^t(t-\tau)^{\alpha n}\,\mathrm{x}(\tau)\,\mathrm{d}\tau\\
&\qquad=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\lambda^n\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\,\mathrm{x}(\tau)\,\mathrm{d}\tau\right)\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{x}(t)\right]\right)
\end{aligned} \tag{54}$$

We perform one whole integration on both sides to write the following constituent equation

$$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+2}\left[\mathrm{x}(t)\right]\right) \tag{55}$$

Here we see the constituent equation (55) is weighted infinite series sum of fractional integration of various orders. The (55) is difficult to solve for $\mathrm{x}(t)$. However one can use Laplace transformation to (54) and write following

$$\begin{aligned}
&\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\mathrm{C}E_\alpha(-\lambda t^\alpha)*\mathrm{x}(t)\\
&s\,\mathrm{X}(s)-\mathrm{x}(0)=\mathrm{C}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\mathrm{X}(s)\\
&\mathrm{x}(t)=\mathrm{x}(0)\mathcal{L}^{-1}\left\{\frac{s^\alpha+\lambda}{s^{\alpha+1}+\lambda s-\mathrm{C}s^{\alpha-1}}\right\}
\end{aligned} \tag{56}$$

The inverse Laplace transform of (56) one can obtain via Contour Integration technique or Berberan Santos method [29], [38], [39]. That will give solutions as integral representation of $\mathrm{x}(t)$.

e. Memory kernel as pure Exponential function

Take $\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t}$ with C and κ as positive real constants, and write following steps

$$\begin{aligned}
&\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{x}(t)\right);\quad \mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t};\quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t)\\
&\frac{\mathrm{d}\mathrm{x}(t)}{\mathrm{d}t}=\mathrm{C}\int_0^t\left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t(t-\tau)^n\,\mathrm{x}(\tau)\,\mathrm{d}\tau\\
&\qquad=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t(t-\tau)^n\,\mathrm{x}(\tau)\,\mathrm{d}\tau\right)\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{x}(t)\right]\right)
\end{aligned}\tag{57}$$

In above steps we used ${}_0I_t^m\left[g(t)\right]=\frac{1}{(m-1)!}\int_0^t(t-\tau)^{m-1}g(\tau)\mathrm{d}\tau$. Integrate once both the sides to get constituent equation as

$$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+2}\left[\mathrm{x}(t)\right]\right)\tag{58}$$

The constituent equation (58) is difficult to solve where the formation is infinite series sum of multiple integer order integrals. However, one can apply Laplace transform to (57) and write solution as

$$\begin{aligned}
&\frac{\mathrm{d}\mathrm{x}(t)}{\mathrm{d}t}=\mathrm{C}\mathrm{e}^{-\kappa t}*\mathrm{x}(t)\\
&s\,\mathrm{X}(s)-\mathrm{x}(0)=\mathrm{C}\left(\frac{1}{s+\kappa}\right)\mathrm{X}(s)\\
&\mathrm{x}(t)=\mathrm{x}(0)\mathcal{L}^{-1}\left\{\frac{s+\kappa}{s^2+\kappa s-\mathrm{C}}\right\}
\end{aligned}\tag{59}$$

The (59) can be solved by usual partial fraction methods and with roots of $s^2+\kappa s-\mathrm{C}$. For example taking $\mathrm{C}=-20$, $\kappa=12$, doing partial fractions of (59) and then taking inverse Laplace transformation we get, $\mathrm{x}(t)=\mathrm{x}(0)\left(\frac{5}{4}\mathrm{e}^{-2t}-\frac{1}{4}\mathrm{e}^{-10t}\right)$.

f. Stretched Exponential function as memory-kernel

Take $\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-(\kappa t)^{\alpha}}$, $0<\alpha<1$ and with C , κ as positive constants, and write following steps

$$
\begin{aligned}
&\mathrm{y}(t)=\big(\mathrm{h}(t)\big)*\big(\mathrm{x}(t)\big),\quad \mathrm{h}(t)=\mathrm{C}e^{-(\kappa t)^{\alpha}},\quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t)\\
&\frac{\mathrm{dx}(t)}{\mathrm{d}t}=\mathrm{C}\int_0^t\Big(e^{-(\kappa(t-\tau))^{\alpha}}\Big)\big(\mathrm{x}(\tau)\big)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{\big((-\kappa(t-\tau))^{\alpha}\big)^n}{n!}\right)\big(\mathrm{x}(\tau)\big)\mathrm{d}\tau\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\int_0^t(t-\tau)^{\alpha n}\,\mathrm{x}(\tau)\,\mathrm{d}\tau\\
&\qquad=\mathrm{C}\left(\sum_{n=0}^{\infty}\Gamma(\alpha n+1)\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\,\mathrm{x}(\tau)\,\mathrm{d}\tau\right)\\
&\qquad=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\Big({}_0I_t^{\alpha n+1}\big[\mathrm{x}(t)\big]\Big)
\end{aligned}
\tag{60}
$$

In above (60) we used the formula ${}_0I_t^{\upsilon}\big[g(t)\big]=\frac{1}{\Gamma(\upsilon)}\int_0^t(t-\tau)^{\upsilon-1}g(\tau)\mathrm{d}\tau$, the RL formula of fractional integration. Integrating (60) once, we write the constitutive expression

$$
\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\Big({}_0I_t^{\alpha n+2}\big[\mathrm{x}(t)\big]\Big)
\tag{61}
$$

The (61) is difficult to solve for $\mathrm{x}(t)$, we are getting constitutive equation as infinite series sum of weighted fractional integrations of various orders.

Corollary-1: Application of the devised method to get various diffusion equations for corresponding Memory Kernels

Here we take evolution equation as described in previous section

$$
\begin{aligned}
&\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t);\quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t);\quad \mathrm{x}(t)\leftarrow\mathrm{L}\big[\mathrm{x}(t)\big]\\
&\mathrm{x}^{(1)}(t)=\int_0^t\mathrm{h}(t-\tau)\big(\mathrm{L}\big[\mathrm{x}(\tau)\big]\big)\mathrm{d}\tau,\quad \mathrm{L}\big[\mathrm{x}(t)\big]\equiv\mathrm{L}\big[\mathrm{c}(x,t)\big],\quad \mathrm{x}^{(1)}(t)\equiv\mathrm{c}^{(1)}(x,t)=\frac{\partial\mathrm{c}(x,t)}{\partial t}
\end{aligned}
\tag{62}
$$

Say the quantity $\mathrm{c}(x,t)$ represents concentration at position x and time t. The (62) implies that response $\mathrm{y}(t)$ evolves in time as convolution of memory kernel $\mathrm{h}(t)$ with $\mathrm{L}\big[\mathrm{x}\big]$, where L is a linear operator, operating on x. We take one-dimensional case with $\mathrm{L}\equiv\frac{\partial^2}{\partial x^2}$, to write

$$
\begin{aligned}
&\mathrm{x}^{(1)}(t)=\mathrm{h}(t)*\left(\frac{\partial^2}{\partial x^2}\mathrm{x}(t)\right)\\
&\mathrm{x}^{(1)}(t)=\int_0^t\mathrm{h}(t-\tau)\left(\frac{\partial^2\big[\mathrm{x}(\tau)\big]}{\partial x^2}\right)\mathrm{d}\tau\\
&\frac{\partial\mathrm{c}(x,t)}{\partial t}=\int_0^t\mathrm{h}(t-\tau)\left(\frac{\partial^2\big[\mathrm{c}(x,\tau)\big]}{\partial x^2}\right)\mathrm{d}\tau
\end{aligned}
\tag{63}
$$

The (63) gives a generalized one dimensional diffusion equation. We can have different linear operator say

$\mathrm{L}\equiv\frac{\partial^2}{\partial x^2}-\mathrm{A}\frac{\partial}{\partial x}$ or other types too. Here we restrict to a simple one-dimensional type Laplacian operator described in (63). In this section, we will only form the constitutive equations for diffusion with types of memory kernel that we took for previous discussions. We are not going to solve these diffusion equations.

a. Memory kernel as Delta function-zero memory case

The following steps are derived

$$\mathrm{y}=\int_0^t \mathrm{h}(t-\tau)\big(\mathrm{L}\left[\mathrm{x}(\tau)\right]\big)\mathrm{d}\tau\,;\quad t\geq 0;\quad \mathrm{y}(t)=\mathrm{x}^{(1)}(t)$$
$$\frac{\mathrm{dx}}{\mathrm{d}t}=\int_0^t \mathrm{h}(t-\tau)\big(\mathrm{L}\left[\mathrm{x}(\tau)\right]\big)\mathrm{d}\tau\,,\quad \mathrm{x}(t)\equiv \mathrm{c}(x,t)$$
$$\frac{\partial \mathrm{c}(x,t)}{\partial t}=\int_0^t \mathrm{h}(t-\tau)\big(\mathrm{L}\left[\mathrm{c}(x,\tau)\right]\big)\mathrm{d}\tau\,,\quad \mathrm{h}(t)=\mathrm{C}\delta(t),\quad \mathrm{L}\left[\mathrm{c}(x,t)\right]=\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2} \tag{64}$$
$$=\mathrm{C}\int_0^t \delta(t-\tau)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau$$
$$\frac{\partial \mathrm{c}(x,t)}{\partial t}=\mathrm{C}\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}$$

We recover the classical one dimensional diffusion equation. This is zero-memory case. Doing integration once, we write the following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\int_0^t\frac{\partial^2}{\partial x^2}\mathrm{c}(x,\tau)\mathrm{d}\tau$$
$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\left({}_0I_t^1\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right) \tag{65}$$

b. Memory Kernel as Singular Power law

The following steps are derived

$$\mathrm{x}^{(1)}=\mathrm{h}(t)*\mathrm{L}\left[\mathrm{x}\right];\quad \mathrm{h}(t)=\mathrm{C}t^{-\alpha},\quad 0<\alpha<1$$
$$\mathrm{c}^{(1)}(x,t)=\int_0^t\left(\mathrm{C}(t-\tau)^{-\alpha}\right)\left(\frac{\partial^2}{\partial x^2}\mathrm{c}(x,\tau)\right)\mathrm{d}\tau$$
$$\frac{\partial}{\partial t}\mathrm{c}(x,t)=\mathrm{C}\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t\left((t-\tau)^{-\alpha}\right)\left(\frac{\partial^2}{\partial x^2}\mathrm{c}(x,\tau)\right)\mathrm{d}\tau\right);\quad \mathrm{C}_\alpha=\mathrm{C}\left(\Gamma(1-\alpha)\right) \tag{66}$$
$$=\mathrm{C}_\alpha\left({}_0I_t^{(1-\alpha)}\left[\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\right]\right)$$

We get fractional integral equation in (66). Integrate once to write the following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}_\alpha\left({}_0I_t^{(2-\alpha)}\left[\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\right]\right) \tag{67}$$

Operate $D_t^{(2-\alpha)}$ on both sides of (67) to write the following

$$
\begin{aligned}
&{}_0D_t^{(2-\alpha)}\left(\mathrm{c}(x,t)-\mathrm{c}(x,0)\right)=\mathrm{C}_\alpha\left({}_0D_t^{(2-\alpha)}\,{}_0I_t^{(2-\alpha)}\left[\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\right]\right)\\
&{}_0D_t^{(2-\alpha)}\mathrm{c}(x,t)-\frac{\mathrm{c}(x,0)}{\Gamma(\alpha-1)}t^{\alpha-2}=\mathrm{C}_\alpha\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\\
&{}_0^CD_t^{(2-\alpha)}\mathrm{c}(x,t)=\mathrm{C}_\alpha\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t),\quad 0<(2-\alpha)<1
\end{aligned}
\tag{68}
$$

c. Memory Kernel as Non-Singular Power Law

The following steps are derived for the kernel $\mathrm{h}(t)=\mathrm{C}\left((1+\upsilon t)\right)^{-\alpha}$

$$
\begin{aligned}
\frac{\partial \mathrm{c}(x,t)}{\partial t}&=\left(\mathrm{h}(t)\right)*\left(\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right),\quad \mathrm{h}(t)=\mathrm{C}\left((1+\upsilon(t-\tau)\right)^{-\alpha}\\
&=\int_0^t\mathrm{C}\left((1+\upsilon(t-\tau)\right)^{-\alpha}\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(\binom{-\alpha}{0}\left(\upsilon(t-\tau)\right)^0+\binom{-\alpha}{1}\left(\upsilon(t-\tau)\right)+\binom{-\alpha}{2}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\
&=\mathrm{C}\int_0^t\left(1+(-\alpha)\left(\upsilon(t-\tau)\right)+\tfrac{(-\alpha)(-\alpha-1)}{2!}\left(\upsilon(t-\tau)\right)^2+\ldots\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\
&=\mathrm{C}\left(\begin{aligned}&\int_0^t\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)+\tfrac{(-\alpha)}{1!}\int_0^t\left(\upsilon(t-\tau)\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\
&\quad+\tfrac{(-\alpha)(-\alpha-1)}{2!}\int_0^t\left(\upsilon(t-\tau)\right)^2\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\
&\quad+\tfrac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}\int_0^t\left(\upsilon(t-\tau)\right)^3\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\ldots\ldots\ldots\ldots\end{aligned}\right)\\
&=\mathrm{C}\left(\begin{aligned}&{}_0I_t^1\left(\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)+(-\alpha)\upsilon\left({}_0I_t^2\left(\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)\right)+\\
&(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^3\left(\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)\right)+\ldots\ldots\ldots\end{aligned}\right)
\end{aligned}
\tag{69}
$$

Integrate once, to write following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\left(\begin{array}{l} {}_0I_t^2\left(\dfrac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)+(-\alpha)\upsilon\left({}_0I_t^3\left(\dfrac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)\right) \\ +(-\alpha)(-\alpha-1)\upsilon^2\left({}_0I_t^4\left(\dfrac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)\right)+\dots\dots\dots \end{array}\right)$$

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\left(\sum_{n=0}^{\infty}\mathrm{a}_n\left({}_0I_t^{n+2}\left(\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right)\right)\right) \tag{70}$$

$$\mathrm{a}_0=1,\quad \mathrm{a}_1=-\alpha\upsilon,\quad \mathrm{a}_2=\alpha(\alpha+1)\upsilon^2\dots$$

d. Memory Kernel as Mittag-Leffler Function

The following steps are derived for $\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha)$

$$\mathrm{c}^{(1)}(x,t)=\left(\mathrm{h}(t)\right)*\left(\frac{\partial^2}{\partial x^2}\mathrm{c}(x,t)\right)=\int_0^t\mathrm{h}(t-\tau)\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

$$\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha)$$

$$\begin{aligned}\frac{\partial\mathrm{c}(x,t)}{\partial t}&=\mathrm{C}\int_0^t\left(E_\alpha(-\lambda(t-\tau)^\alpha)\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\ &=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n+1)}\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\ &=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\lambda^n}{\Gamma(\alpha n+1)}\right)\int_0^t(t-\tau)^{\alpha n}\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\ &=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\lambda^n\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\right)\\ &=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right)\end{aligned} \tag{71}$$

Integrate once and write the following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+2}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right) \tag{72}$$

e. Memory Kernel as Pure exponential function

The following steps are derived

$$\mathrm{c}^{(1)}(x,t)=\left(\mathrm{h}(t)\right)*\left(\frac{\partial^2 \mathrm{c}(x,t)}{\partial x^2}\right);\quad \mathrm{h}(t)=\mathrm{Ce}^{-\kappa t}$$

$$\begin{aligned}\frac{\partial \mathrm{c}(x,t)}{\partial t}&=\mathrm{C}\int_0^t\left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\&=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\&=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t(t-\tau)^n\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\&=\mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t(t-\tau)^n\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\right)\\&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right)\end{aligned}\tag{73}$$

Integrating once, we write the following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+2}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right)\tag{74}$$

f. Stretched exponential function

The following steps are derived

$$\mathrm{c}^{(1)}(x,t)=\left(\mathrm{h}(t)\right)*\left(\frac{\partial^2 \mathrm{c}(x,t)}{\partial x^2}\right),\quad \mathrm{h}(t)=\mathrm{Ce}^{-(\kappa t)^{\alpha}}$$

$$\begin{aligned}\frac{\partial \mathrm{c}(x,t)}{\partial t}&=\mathrm{C}\int_0^t\left(\mathrm{e}^{-(\kappa(t-\tau))^{\alpha}}\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\&=\mathrm{C}\int_0^t\left(\sum_{n=0}^{\infty}\frac{\left((-\kappa(t-\tau))^{\alpha}\right)^n}{n!}\right)\left(\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau\\&=\mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\int_0^t(t-\tau)^{\alpha n}\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\mathrm{d}\tau\\&=\mathrm{C}\left(\sum_{n=0}^{\infty}\Gamma(\alpha n+1)\left(\frac{(-1)^n\kappa^{\alpha n}}{n!}\right)\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\frac{\partial^2\mathrm{c}(x,\tau)}{\partial x^2}\mathrm{d}\tau\right)\\&=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right)\end{aligned}\tag{75}$$

Integrating once, we get the following

$$\mathrm{c}(x,t)-\mathrm{c}(x,0)=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\left({}_0I_t^{\alpha n+2}\left[\frac{\partial^2\mathrm{c}(x,t)}{\partial x^2}\right]\right)\tag{76}$$

Corollary-2: Application of this devised Method to get various Wave Equations for various Memory Kernels

The evolution equation in this case is

$$\begin{aligned} &\mathrm{y}=\int_0^t \mathrm{h}(t-\tau)\left(\mathrm{L}\left[\mathrm{x}(\tau)\right]\right)\mathrm{d}\tau\,; \quad t\ge 0; \quad \mathrm{y}(t)=\mathrm{x}^{(2)}(t) \\ &\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{d}t^2}=\int_0^t \mathrm{h}(t-\tau)\left(\mathrm{L}\left[\mathrm{x}(\tau)\right]\right)\mathrm{d}\tau\,, \quad \mathrm{x}(t)\equiv \mathrm{w}(x,t), \quad \mathrm{L}\equiv\frac{\partial^2}{\partial x^2} \\ &\frac{\partial^2\,\mathrm{w}(x,t)}{\partial t^2}=\left(\mathrm{h}(t)\right)*\left(\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2}\right) \end{aligned} \tag{77}$$

We are not repeating the above steps, but we write the following for various memory kernels as in following expressions

$$\begin{aligned} &\mathrm{h}(t)=\mathrm{C}\delta(t) \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2} \\ &\mathrm{h}(t)=\mathrm{C}t^{-\alpha} \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}_\alpha\left({}_0I_t^{(1-\alpha)}\left[\frac{\partial^2}{\partial x^2}\mathrm{w}(x,t)\right]\right), \quad \mathrm{C}_\alpha=\mathrm{C}\Gamma(1-\alpha) \\ &\mathrm{h}(t)=\mathrm{C}\left((1+\upsilon(t-\tau)\right)^{-\alpha} \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\left(\sum_{n=0}^{\infty}\mathrm{a}_n\left({}_0I_t^{n+1}\left(\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2}\right)\right)\right) \\ &\qquad\qquad \mathrm{a}_0=1, \quad \mathrm{a}_1=-\alpha\upsilon, \quad \mathrm{a}_2=\alpha(\alpha+1)\upsilon^2\ldots \\ &\mathrm{h}(t)=\mathrm{C}E_\alpha(-\lambda t^\alpha) \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2}\right]\right) \\ &\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-\kappa t} \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2}\right]\right) \\ &\mathrm{h}(t)=\mathrm{C}\mathrm{e}^{-(\kappa t)^\alpha} \qquad \frac{\partial^2\mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2\mathrm{w}(x,t)}{\partial x^2}\right]\right) \end{aligned} \tag{78}$$

Discussions and Observations

a. Genesis of constituent equations for physical laws classical (memory-less) and anomalous (with memory)

We have discussed two types of laws, classically described by first order differentials. For zero memory case i.e. with memory kernel (singular) delta function, i.e. $\mathrm{h}(t)\propto\delta(t)$ we get from the evolution equation i.e. $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$ with a) $\mathrm{x}(t)=\mathrm{f}^{(1)}(t)$ b) $\mathrm{y}(t)=\mathrm{x}^{(1)}(t)$ the following constituent equations

$$\begin{aligned} &\mathrm{y}(t)\propto\frac{\mathrm{df}(t)}{\mathrm{d}t} \qquad \mathrm{y}(t)\propto \mathrm{f}^{(1)}(t) \\ &\frac{\mathrm{dx}(t)}{\mathrm{d}t}\propto\mathrm{x}(t) \qquad \mathrm{x}^{(1)}(t)\propto\mathrm{x}(t) \end{aligned} \tag{79}$$

The memory less cases as in (79) for case a) $\mathrm{y}(t)\propto\mathrm{f}^{(1)}(t)$, the response $\mathrm{y}(t)$ vanishes as soon as the excitation $\mathrm{f}^{(1)}(t)$ vanishes. For example the step voltage to an ideal loss less capacitor gives impulse

current at the time of application of voltage, i.e. current through ideal loss less capacitor is zero after the rate of change of voltage vanishes. This is memory less case.
Well, if we observe current lingering even after the voltage change has vanished, we say it is anomalous relaxation, but note that this is memory-based relaxation of dielectric, depicted in Figure-5

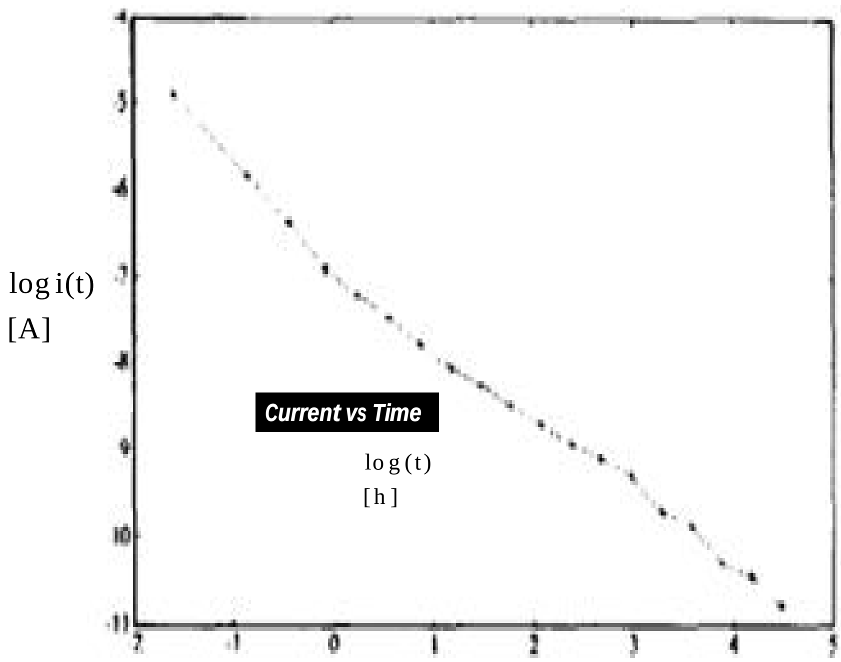

Figure-5: At time zero a voltage of 100V is connected to a 0.47uF metalized paper dielectric capacitor; in log-log scales average slope is -0.86. Thus exponent of relaxation current is non-integer. IEEE Trans on Dielectrics and Insulation, 1,826. (1994), S. Westerlund, L. Ekstam .

For the case b) $\mathrm{x}^{(1)}(t) \propto \mathrm{x}(t)$, say for radioactive decay case, we note that this gives pure exponential function as decay. Whereas the non-exponential cases of decay-call the anomalous decay, is based on memory. In Figure-3 such non-exponential decay are depicted.

As Corollary to the second case i.e. b), we derived the Laws of diffusion equations and wave equations, by setting $\mathrm{y}(t) \leftarrow \mathrm{x}^{(1)}(t)$ and $\mathrm{x}(t) \leftarrow \mathrm{L}\left[\mathrm{x}(t)\right]$, and $\mathrm{y}(t) \leftarrow \mathrm{x}^{(2)}(t)$ respectively, with $\mathrm{L} \equiv \frac{\partial^2}{\partial x^2}$

$$\frac{\partial[\mathrm{x}]}{\partial t} \propto \frac{\partial^2[\mathrm{x}]}{\partial x^2} \qquad \frac{\partial^2[\mathrm{x}]}{\partial t^2} \propto \frac{\partial^2[\mathrm{x}]}{\partial x^2} \tag{80}$$

These basic laws (79) and (80) are zero-memory case, with memory kernel in the evolution equation being a delta function. There is genesis to these fundamental equations (79) (80) of physical laws-as described above. The (80) gives memory less diffusion equation, and memory-less wave equation. For classical diffusion equation we have for a delta function at origin, manifesting as a Gaussian plume. Depicted in Figure-6, [64]. The anomalous diffusion, is non-Gaussian plume that is with memory-given by fractional time diffusion equation-for singular power law memory case. In (80) we have wave equation, that is classically without any memory kernel. The impulse at origin becomes two impulses travelling in plus and minus directions with half amplitude-this is pure travelling wave-Figure-7. With memory-based wave equation, we have diffused travelling waves with tails in front and back with a maximum value (Figure-7), [64]. These are anomalous travelling waves given by solution to Time Fractional Wave Equation. The generalization is studied/developed by Mainardi-by M-Wright Function, (generalization to Gaussian Function)-which is solution to Time Fractional Diffuso-Wave Equation [64].

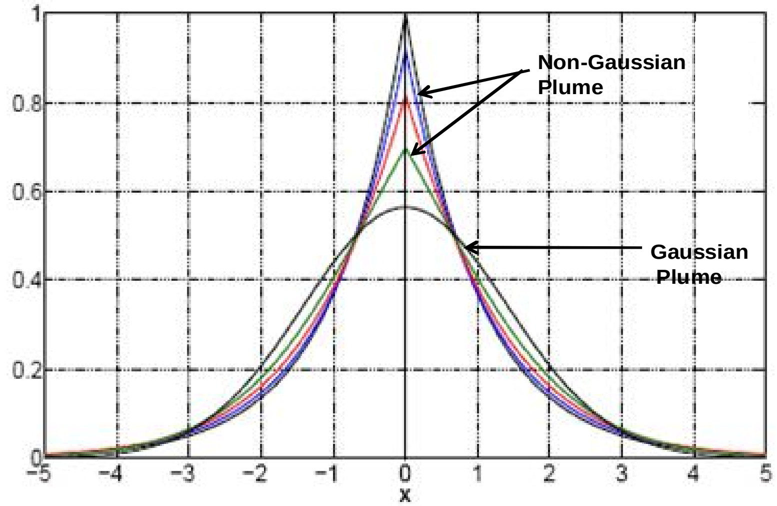

Figure-6: The Gaussian Plume and Non-Gaussian Plume for classical memory-less diffusion vis-à-vis diffusion with memory

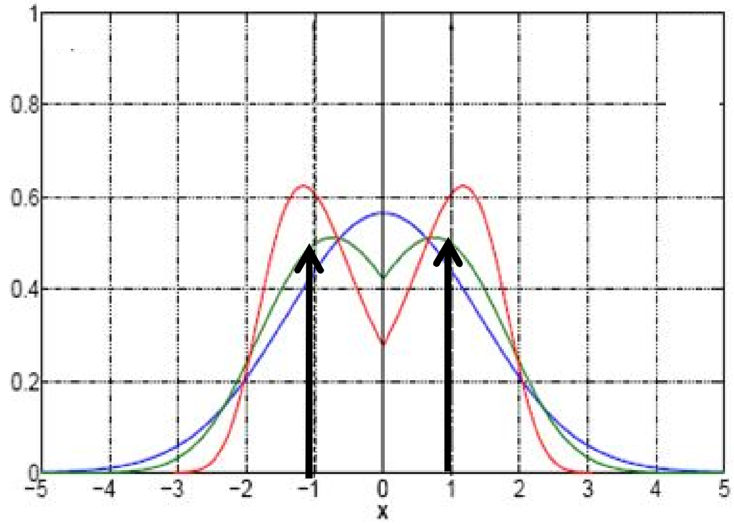

Figure-7: Pure traveling waves in a memory-less wave equations vis-à-vis diffused travelling waves for memory based wave equation

The first one in (79) is Newton's second law of motion, relating force acting on a particle is proportional to rate of change of velocity i.e. $f = m\upsilon^{(1)}$. This is similar to our first constituent equation, i.e. $y = Cf^{(1)}$. These laws as $y = Cf^{(1)}$ are fundamental ones, derived from Hamilton's principle. Since Newton's time, classical mechanics has been elegantly reformulated as a single unifying principle known as Hamilton's principle. Following Feynman [30] and Landau [31], Hamilton's principle is frequently called "the principle of least action." The action is defined as $S \equiv \int_{t_1}^{t_2} (\mathrm{KE} - \mathrm{PE})\mathrm{d}t$. Where KE is Kinetic Energy and PE is potential energy. The action between time t_1 and t_2 is viewed as position of particle at time t_1 call it x_1 and position of particle at time t_2 call it x_2. According to the least action formulation of classical mechanics, a particle moves along the path for which the action S is a minimum. The law $f = m\upsilon^{(1)}$ is got by minimizing the action S in a linearly varying potential field, with $f = -\frac{\mathrm{d(PE)}}{\mathrm{d}x}$. This is second law of Newton's law of motion. The first law says that velocity is constant when potential field is constant or zero, and in that case, action is a straight line. We stress here the classical laws $\mathrm{y} = \mathrm{Cf}^{(1)}$ for zero-memory case is derived from Hamilton's principle of least action.

The case of capacitor current proportional to rate of change of applied voltage, or inductor voltage proportional to rate of change of current through inductor or stress of a Newtonian viscous element proportional to rate of change of strain etc all are derived from Hamilton's principle of least action in a potential field. We are not discussing the variation calculus and Hamilton's principle. But what we observe that we get the constituent equation $\mathrm{y} = \mathrm{Cf}^{(1)}$ for a zero-memory case where memory kernel is $\mathrm{h}(t) = \mathrm{C}\delta(t)$ from evolution equation i.e. $\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$ with $\mathrm{x}(t) = \mathrm{f}^{(1)}(t)$.

Most of the physical laws that are extracted from experimental observations are expressed mathematically in terms of differential equations. A simple example is when an observable y is a function of a single variable t and the experimental observations indicate a relationship between y and its first and/or higher differentials with respect to t. Such a relationship is a differential equation, and the solution of the equation gives the function y and determines how it varies with t . Suppose there is a fixed amount of radioactive isotope, which decays to a neighboring stable nucleus. The radioactive material is divided into batches of different mass and the number of decays from each batch measured over the same time interval. The data show that the numbers of decays are proportional to the mass of material used, whatever the time interval over which the measurements on the different samples were made, and show that as the time interval is varied the numbers of decays are proportional to the size of that interval. These observations suggest that the number of decays dN in an infinitesimal time interval $\mathrm{d}t$ is proportional to the number of radioactive atoms present and to the interval $\mathrm{d}t$. That is $\mathrm{dN} = -\lambda \mathrm{N}\mathrm{d}t$ or $\mathrm{dN}/\mathrm{N} = -\lambda \mathrm{d}t$ where λ is the proportionality constant. The minus sign is present because the decays dN represent a reduction in the number N of nuclei. This is the classical law $\mathrm{x}^{(1)}(t) = \mathrm{Cx}(t)$, that is zero-memory case with memory kernel as $\mathrm{h}(t) \propto \delta(t)$ in our evolution equation i.e. $\mathrm{x}^{(1)}(t) = \mathrm{h}(t) * \mathrm{x}(t)$ that we have dealt in general terms and analyzed with various memory kernels singular and non-singular thereafter. We have also discussed the corollary to this case to get classical diffusion and wave equations and in these cases; too we have used various memory kernels to get various other types of diffusion and wave equations.

b. Summary

The Table-1 gives the summary of the evolution equation $\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$ with $\mathrm{x}(t) = \mathrm{f}^{(1)}(t)$, studied for various memory kernels. Table-2 gives summary of the evolution equation $\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$ with $\mathrm{y}(t) = \mathrm{x}^{(1)}(t)$ studied for various memory kernels. The Table-3 and table-4 gives various types of Diffusion and wave equations, with various memory kernels.

S.No.	**Function of Memory Kernel**	**Type**	**Memory Kernel Function** $\mathrm{h}(t)$	**Constitutive Equation of** $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)$ $n=0,1,2,\ldots;\quad 0<\alpha<1$	**Relaxation of** $\mathrm{y}(t)$ **for unit step input of** $\mathrm{f}(t)$
1	Delta Function	Singular	$\mathrm{C}\delta(t)$	$\mathrm{y}(t)=\mathrm{C}\mathrm{f}^{(1)}(t)$ $\mathrm{y}(t)=\mathrm{C}\left({}_0I_t^0\mathrm{f}^{(1)}(t)\right)$	$\mathrm{C}\delta(t)$
2	Power Law	Singular	$\mathrm{C}t^{-\alpha}$ $0<\alpha<1$	$\mathrm{y}(t)=\mathrm{C}_\alpha\mathrm{f}^{(\alpha)}(t);\quad \mathrm{C}_\alpha=\mathrm{C}\left(\Gamma(1-\alpha)\right)$ $\mathrm{y(t)}=\mathrm{C}_\alpha\left({}_0I_t^{1-\alpha}\mathrm{f}^{(1)}(t)\right)$	$\mathrm{C}t^{-\alpha}$
3	Non-singular Power Law	Non-Singular	$\mathrm{C}(1+\upsilon t)^{-\alpha}$	$\mathrm{y}(t)=\mathrm{C}\sum_{n=1}^{\infty}\mathrm{a}_n\left({}_0I_t^n\mathrm{f}^{(1)}(t)\right)$ $\mathrm{a}_1=1,\ \mathrm{a}_2=-\alpha\upsilon,$ $\mathrm{a}_3=(\alpha)(\alpha+1)\upsilon^2\ldots$	$\mathrm{C}(1+\upsilon t)^{-\alpha}$
4	Mittag-Leffler	Non-Singular	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$	$\mathrm{y}(t)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+1}\mathrm{f}^{(1)}(t)\right)$ $\mathrm{b}_0=1,\ \mathrm{b}_1=-\lambda;\ldots$ $\mathrm{b}_n=(-1)^n\lambda^n\ \ldots..$	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$
5	Exponential	Non-Singular	$\mathrm{C}\mathrm{e}^{-\kappa t}$	$\mathrm{y}(t)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{c}_n\left({}_0I_t^{n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)$ $\mathrm{c}_0=1,\quad \mathrm{c}_1=-\kappa,\quad \mathrm{c}_3=\kappa^2$ $\ldots\mathrm{c}_n=(-1)^n\kappa^n$	$\mathrm{C}\mathrm{e}^{-\kappa t}$
6	Stretched-Exponential	Non-Singular	$\mathrm{C}\mathrm{e}^{-(\kappa t)^\alpha}$	$\mathrm{y}(t)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{d}_n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)$ $\mathrm{d}_n=(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$	$\mathrm{C}\mathrm{e}^{-(\kappa t)^\alpha}$

Table-1: Summary of results of various singular and Non-singular Memory Kernels for evolution equation $\mathrm{y}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{f}^{(1)}(t)\right)$

S.No.	Function of Memory Kernel	Type	Memory Kernel Function $\mathrm{h}(t)$	Constitutive Equation of $\mathrm{x}^{(1)}(t)=\mathrm{h}(t)*\mathrm{x}(t)$	Relaxation of $\mathrm{x}(t)$ from initial value $\mathrm{x}(0)$
1	Delta Function	Singular	$\mathrm{C}\delta(t)$	$\mathrm{x}^{(1)}(t)=\mathrm{Cx}(t)$ $\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\left({}_0I_t^1\mathrm{x}(t)\right)$	$\mathrm{x}(0)\,\mathrm{e}^{\mathrm{C}t}$
2	Power Law	Singular	$\mathrm{C}t^{-\alpha}$ $1<\alpha\le 2$	${}_0^CD_t^{\upsilon}\mathrm{x}(t)=\mathrm{C}_\alpha\mathrm{x}(t)$ $\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}_\alpha\left({}_0I_t^{\upsilon}\mathrm{x}(t)\right)$ $\mathrm{C}_\alpha=\mathrm{C}\left(\Gamma(1-\alpha)\right),\quad \upsilon=2-\alpha$	$\mathrm{x}(0)E_\upsilon\left(\mathrm{C}_\alpha t^{\upsilon}\right)$
3	Non-singular Power Law	Non-Singular	$\mathrm{C}(1+\upsilon t)^{-\alpha}$	$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{a}_n\left({}_0I_t^{n+2}\,\mathrm{x}(t)\right)$ $\mathrm{a}_0=1,\quad \mathrm{a}_1=-\alpha\upsilon$ $\mathrm{a}_2=\alpha(\alpha+1)\upsilon^2...$	X
4	Mittag-Leffler	Non-Singular	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$	$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+2}\left[\mathrm{x}(t)\right]\right)$ $\mathrm{b}_n=(-1)^n\lambda^n$	$\mathrm{x}(0)\mathcal{L}^{-1}\left\{\frac{s^\alpha+\lambda}{s^{\alpha+1}+\lambda s-\mathrm{C}s^{\alpha-1}}\right\}$
5	Exponential	Non-Singular	$\mathrm{Ce}^{-\kappa t}$	$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{c}_n\left({}_0I_t^{n+2}\left[\mathrm{x}(t)\right]\right)$ $\mathrm{c}_n=(-1)^n\kappa^n$	$\mathrm{x}(0)\mathcal{L}^{-1}\left\{\frac{s+\kappa}{s^2+\kappa s-\mathrm{C}}\right\}$
6	Stretched-Exponential	Non-Singular	$\mathrm{Ce}^{-(\kappa t)^\alpha}$	$\mathrm{x}(t)-\mathrm{x}(0)=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{d}_n\left({}_0I_t^{\alpha n+2}\left[\mathrm{x}(t)\right]\right)$ $\mathrm{d}_n=(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$	X

Table-2: Summary of results of various singular and Non-singular Memory Kernels for the evolution equation $\mathrm{x}^{(1)}(t)=\mathrm{h}(t)*\mathrm{x}(t)$

S.No.	Function of Memory Kernel	Type	Memory Kernel $h(t)$	Constitutive Equation of Diffusion $c^{(1)}(t)=h(t)*\left(\frac{\partial^2 c}{\partial x^2}\right)$
1	Delta-Function	Singular	$C\delta(t)$	$\frac{\partial c(x,t)}{\partial t}=C\frac{\partial^2 c(x,t)}{\partial x^2}$ $c(x,t)-c(x,0)=C\left({}_0I_t^1\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$
2	Power-Law	Singular	$Ct^{-\alpha}$	$C_\alpha=C(\Gamma(1-\alpha))$ $\frac{\partial c(x,t)}{\partial t}=C_\alpha\left({}_0I_t^{(1-\alpha)}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$ ${}_0^C D_t^{(2-\alpha)}c(x,t)=C_\alpha\frac{\partial^2 c(x,t)}{\partial x^2}$
3	Non-Singular Power law	Non-Singular	$C(1+\upsilon t)^{-\alpha}$	$\frac{\partial c(x,t)}{\partial t}=C\left(\sum_{n=0}^{\infty}a_n\left({}_0I_t^{n+1}\left(\frac{\partial^2 c(x,t)}{\partial x^2}\right)\right)\right)$ $c(x,t)-c(x,0)=C\left(\sum_{n=0}^{\infty}a_n\left({}_0I_t^{n+2}\left(\frac{\partial^2 c(x,t)}{\partial x^2}\right)\right)\right)$ $a_0=1,\quad a_1=-\alpha\upsilon,\quad a_2=\alpha(\alpha+1)\upsilon^2...$
4	Mittag-Leffler	Non-Singular	$CE_\alpha(-\lambda t^\alpha)$	$b_n=(-1)^n\lambda^n$ $\frac{\partial c(x,t)}{\partial t}=C\sum_{n=0}^{\infty}b_n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$ $c(x,t)-c(x,0)=C\sum_{n=0}^{\infty}b_n\left({}_0I_t^{\alpha n+2}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$
5	Exponential	Non-Singular	$Ce^{-\kappa t}$	$c_n=(-1)^n\kappa^n$ $\frac{\partial c(x,t)}{\partial t}=C\sum_{n=0}^{\infty}c_n\left({}_0I_t^{n+1}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$ $c(x,t)-c(x,0)=C\sum_{n=0}^{\infty}c_n\left({}_0I_t^{n+2}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$
6	Stretched Exponential	Non-Singular	$Ce^{-(\kappa t)^\alpha}$	$d_n=(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$ $\frac{\partial c(x,t)}{\partial t}=C\sum_{n=0}^{\infty}d_n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$ $c(x,t)-c(x,0)=C\sum_{n=0}^{\infty}d_n\left({}_0I_t^{\alpha n+2}\left[\frac{\partial^2 c(x,t)}{\partial x^2}\right]\right)$

Table-3: Various diffusion equations constituted

S.No.	Function of Memory Kernel	Type	Memory Kernel $\mathrm{h}(t)$	Constitutive Equation of Diffusion $\mathrm{w}^{(2)}(t)=\mathrm{h}(t)*\left(\frac{\partial^2 \mathrm{w}}{\partial x^2}\right)$
1	Delta-Function	Singular	$\mathrm{C}\delta(t)$	$\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}$
2	Power-Law	Singular	$\mathrm{C}t^{-\alpha}$	$\mathrm{C}_\alpha=\mathrm{C}\left(\Gamma(1-\alpha)\right)$ $\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}_\alpha\left({}_0I_t^{(1-\alpha)}\left[\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}\right]\right)$
3	Non-Singular Power law	Non-Singular	$\mathrm{C}(1+\upsilon t)^{-\alpha}$	$\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\left(\sum_{n=0}^{\infty}\mathrm{a}_n\left({}_0I_t^{n+1}\left(\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}\right)\right)\right)$ $\mathrm{a}_0=1,\quad \mathrm{a}_1=-\alpha\upsilon,\quad \mathrm{a}_2=\alpha(\alpha+1)\upsilon^2\ldots$
4	Mittag-Leffler	Non-Singular	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$	$\mathrm{b}_n=(-1)^n\lambda^n$ $\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}\right]\right)$
5	Exponential	Non-Singular	$\mathrm{C}e^{-\kappa t}$	$\mathrm{c}_n=(-1)^n\kappa^n$ $\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{c}_n\left({}_0I_t^{n+1}\left[\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}\right]\right)$
6	Stretched Exponential	Non-Singular	$\mathrm{C}e^{-(\kappa t)^\alpha}$	$\mathrm{d}_n=(-1)^n\left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$ $\frac{\partial^2 \mathrm{w}(x,t)}{\partial t^2}=\mathrm{C}\sum_{n=0}^{\infty}\mathrm{d}_n\left({}_0I_t^{\alpha n+1}\left[\frac{\partial^2 \mathrm{w}(x,t)}{\partial x^2}\right]\right)$

Table-4: Various wave equations constituted

c. Experimental evidences-about anomalous laws (with memory)

We have given few examples of Memory Kernel that gives constitutive expression for relaxation quantity $\mathrm{y}(t)$, for evolution equation $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{f}^{(1)}(t)$. The memory-less relaxation (zero-memory) is via Memory Kernel with delta function $\mathrm{h}(t)\propto\delta(t)$, gives a classical constitutive formula, i.e. $\mathrm{y}(t)\propto\mathrm{f}^{(1)}(t)$. This we can write as zero order integration as $\mathrm{y}(t)\propto {}_0I_t^0\left[\mathrm{f}^{(1)}(t)\right]$.

The memory kernel if formulated via a singular power-law kernel then we have a fractional derivative of Caputo type relating relaxation quantity $\mathrm{y}(t)$ and impressed excitation $\mathrm{f}(t)$, as $\mathrm{y}(t)\propto\mathrm{f}^{(\alpha)}(t)$. This we can also write as $\mathrm{y}(t)\propto {}_0I_t^{1-\alpha}\left[\mathrm{f}^{(1)}(t)\right]$, in terms of fractional integration. This power-law memory kernel is singular in nature and non-differentiable at start. We note that for singular memory kernels (zero-memory case, power law memory case), we get only one term in the RHS of the constituent equation. We make modification, and try to write a non-singular power-law memory kernel and derive its constitutive equation for relaxation quantity $\mathrm{y}(t)$. We observe that here we get weighted sum of integrations of input excitation i.e. the rate of change of excitation quantity as $\mathrm{y}(t)\propto\sum_{n=1}^{\infty}\mathrm{a}_n\left({}_0I_t^n\mathrm{f}^{(1)}(t)\right)$, i.e. infinite sum. This we get all together different from the singular kernels results, for classical as well as fractional cases i.e. $\mathrm{y}(t) \propto {}_0I_t^0\left[\mathrm{f}^{(1)}(t)\right]$ and $\mathrm{y}(t) \propto {}_0I_t^{1-\alpha}\left[\mathrm{f}^{(1)}(t)\right]$ respectively. We extend this analysis with memory-kernel, which is Mittag-Leffler function. This kernel is non-singular at origin but the derivative at origin does not exist. With this, we get the constitutive equation as $\mathrm{y}(t) \propto \sum_{n=0}^{\infty}\mathrm{b}_n\left({}_0I_t^{\alpha n+1}\mathrm{f}^{(1)}(t)\right)$, i.e. again infinite sum. We note that the structure of this expression is much away from that $\mathrm{y}(t) \propto {}_0I_t^0\left[\mathrm{f}^{(1)}(t)\right]$ the classical case with zero-memory and the $\mathrm{y}(t) \propto {}_0I_t^{1-\alpha}\left[\mathrm{f}^{(1)}(t)\right]$ the fractional one. This comprises of infinite series of fractional integrations, may thus be mathematically fine but we may not be getting physical sense.

Thereafter we take the memory kernel as pure exponential decay function, which is non-singular and everywhere differentiable function. With this, we construct a constitutive equation given as $\mathrm{y}(t) \propto \sum_{n=0}^{\infty}\mathrm{c}_n\left({}_0I_t^{n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)$ that is infinite series sum of integer order repeated integrations-and is very much off from the capacitor dynamics classical case or fractional case i.e. $\mathrm{y}(t) \propto {}_0I_t^0\left[\mathrm{f}^{(1)}(t)\right]$ or $\mathrm{y}(t) \propto {}_0I_t^{1-\alpha}\left[\mathrm{f}^{(1)}(t)\right]$ respectively.

We modify this non-singular memory kernel to a stretched exponential function. This function is non-singular. We get constitutive equation here that is weighted sum-series of the fractional integrations of input function i.e. rate of change of applied voltage given as $\mathrm{y}(t) \propto \sum_{n=0}^{\infty}\mathrm{d}_n\left({}_0I_t^{\alpha n+1}\left[\mathrm{f}^{(1)}(t)\right]\right)$, i.e. infinite series of integrations of various orders.

However, mathematically it is fine, to have non-singular memory kernels yet physical applicability of the constitutive expressions obtained is questionable; presently because we are used to the classical law $\mathrm{y}(t) \propto \mathrm{f}^{(1)}(t)$ or $\mathrm{y}(t) \propto {}_0I_t^0\left[\mathrm{f}^{(1)}(t)\right]$ and fractional law $\mathrm{y}(t) \propto \mathrm{f}^{(\alpha)}(t)$ or $\mathrm{y}(t) \propto {}_0I_t^{1-\alpha}\left[\mathrm{f}^{(1)}(t)\right]$ for some real physical systems. Presently we are unable to give interpretation to the weighted sum series of integrations and fractional integrations of rate of change of voltage that appears for capacitor dynamics when we take non-singular memory kernel.

We note that though classical textbook capacitors currents $\mathrm{i}(t)$ to applied voltage $\mathrm{v}(t)$ is expressed as in $\mathrm{i}(t) \propto \mathrm{v}^{(1)}(t)$, yet in reality they have power-law decay current, when excited by a step-voltage for an uncharged capacitor. This is well established by Curie-von Schweidler law the current relaxation is $\mathrm{i}(t) \sim t^{-\alpha}$; $0<\alpha<1$ [1]-[4], [28], when a constant step voltage is applied to an uncharged capacitor. Therefore, the memory-kernel associated with relaxation dynamics is $\mathrm{h}(t) \sim t^{-\alpha}$; that is singular power-law function, is applicable. Here the fractional derivative appears in constitutive expression i.e. $\mathrm{i}(t) \propto \mathrm{v}^{(\alpha)}(t)$; $0<\alpha<1$, or $\mathrm{i}(t) \propto {}_0I_t^{1-\alpha}\mathrm{v}^{(1)}(t)$ [5]-[12], [17]-[20] [22], [28].

In Rheology studies, there is a visco-elastic element, described as $\sigma \propto \varepsilon^{(\alpha)}$, where σ is stress function of time and ε as strain function with α as fractional order of derivative (Caputo type) $0<\alpha<1$. This fractional dynamic is similar to the fractional capacitor case as described above, i.e. $\mathrm{i} \propto \mathrm{v}^{(\alpha)}$. From linear viscoelaticity theory, the relaxation modulus $\mathrm{g}(t)$ is obtained by property $\tilde{\sigma}(s) = s\mathrm{G}(s)\tilde{\varepsilon}(s)$. This expression is written in Laplace transformed domain. Where $\mathrm{G}(s) = \mathcal{L}\left\{\mathrm{g}(t)\right\}$, $\tilde{\sigma}(s) = \mathcal{L}\left\{\sigma(t)\right\}$ and

$\varepsilon(s) = \mathcal{L}\left\{\tilde{\varepsilon}(t)\right\}$. This is Scott-Blair model, and gives a 'relaxation modulus' $\mathrm{g}(t) \propto t^{-\alpha}$, that is the stress function to unit step strain function [32], [33]. This constitutive equation $\sigma(t) \propto \varepsilon^{(\alpha)}(t)$ i.e. Scott-Blair model also thus fits into the evolution equation, $\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{f}^{(1)}(t)$ with $\mathrm{y}(t) \equiv \sigma(t)$ and $\mathrm{f}(t) \equiv \varepsilon(\mathrm{t})$, having singular power-law memory kernel; i.e. $\mathrm{h}(t) \propto t^{-\alpha}$. The classical Newtonian viscous element is $\sigma \propto \varepsilon^{(1)}$ and pure spring element is $\sigma \propto \varepsilon$. Thus, Scott-Blair model is in-between pure-spring and pure viscous element. The relaxation modulus of pure Newtonian viscous element is $\mathrm{g}(t) \propto \delta(t)$ i.e. delta function. The pure Newtonian element is zero memory case with $\mathrm{h}(t) \propto \delta(t)$. The use of fractional derivative in Rheological studies for non-Newtonian fluids is found in, [34]-[37].

The second physical phenomenon that is of growth and decay was using evolution equation $\mathrm{x}^{(1)}(t) = \mathrm{h}(t) * \mathrm{x}(t)$, with various memory kernels, (summarized in Table-2). This gives very interesting observations like, use of singular kernels like zero-memory case gives exponential relaxation of $\mathrm{x}(t)$ observed for pure radioactive decay or pure neutron population growth, that is $\mathrm{x}(t) \sim \mathrm{e}^{\mathrm{C}t}$; with constituent equation as $\mathrm{x}^{(1)}(t) \propto \mathrm{x}(t)$. The singular power law kernel gives a constituent law i.e. $\mathrm{x}^{(\alpha)}(t) \sim \mathrm{x}(t)$ with relaxation function as $\mathrm{x}(t) = E_{2-\alpha}(\mathrm{C}_\alpha t^{2-\alpha})$, this is non-exponential decay/growth.

The observation is similar to the earlier case when the memory kernel is non-singular; giving constituent equation for decay or growth phenomena as weighted sum series of integer order or fractional order integrals. These constituent equations are difficult to solve to get relaxation function $\mathrm{x}(t)$. The classical growth or decay is exponential function, we say that is Debye type, the anomalous relaxation is non-Debye type can be modeled by various non-exponential relaxation functions [38]-[41].

Extending the basic evaluation equation $\mathrm{y}(t) = \mathrm{h}(t) * \mathrm{x}(t)$, writing $\mathrm{L}[\mathrm{x}] \leftarrow \mathrm{x}$ and x as variable of x and t, with Linear Operator as $\mathrm{L} \equiv \frac{\partial^2}{\partial x^2}$ we constitute various forms of diffusion equations with $\mathrm{y} \leftarrow \mathrm{x}^{(1)}$, and various forms of wave equations with $\mathrm{y} \leftarrow \mathrm{x}^{(2)}$, with singular and non-singular memory kernels. The observations remain same as in the previous cases.

Conclusions

Several observation points relaxation laws, which show memory effect, and are termed by all as 'anomalous relaxations'. The ideality is zero-memory case laws; that we studied in our courses from school days. We have discussed here how the laws, change if we consider memory kernel in basic causality equation-given by convolution expression. We have considered singular and non-singular memory kernels. We address a very relevant question that is 'if we have in reality a singular memory kernel or a non-singular memory kernel', for system dynamics for process evolution that is governed by convolution integral. This presentation shows if we are having a singular memory kernel, then we observe that the laws are close to classical laws, only the classical derivative or integral operation is replaced by fractional derivative or fractional integral operators. Though mathematically non-singular memory kernels are possible, yet the constitutive equation for system relaxation dynamics does not give the useful information, results in constituent equations that are weighted infinite series sums of multiple integer order integrals or fractional order integrals. This maybe because we are unable presently assigns physical sense to mathematically correct constitutive equations for system relaxation, due to non-singular memory kernel. Therefore, may we not ask that is singular memory kernel be the order of Nature? If so then presently we are unable to give physical insight into reasons of singularity of memory kernels that we showed to have given reality in constituent equations for physical laws that we described.

Appendix-1

Fractional Calculus-with singular Kernel in Convolution

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration [6], [26], [29] of order $\upsilon \in \mathbb{R}^{+}$ is defined as

$$ {}_0I_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)\mathrm{d}\tau \qquad \text{A1}$$

Where $\Gamma(\upsilon)$ is Euler's Gamma function, is generalization of factorial function [23], [29], we have $\Gamma(\upsilon) = (\upsilon-1)!$. The formula A1 is appearing as generalization of Cauchy's multiple integration formula of m fold integration [6], [26], [29] where $m \in \mathbb{N}$ given as follows

$$ {}_0I_t^{m}\left[f(t)\right] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)\mathrm{d}\tau ; \quad m = 1, 2, 3, ... \qquad \text{A2}$$

The fractional derivative of order υ for $0 < \upsilon < 1$ by Riemann-Liouville (RL) formula [6], [26], [29] is

$$ {}_0D_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(1-\upsilon)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\tau)^{-\upsilon} f(\tau)\mathrm{d}\tau ; \quad 0 < \upsilon < 1 \qquad \text{A3}$$

The A3 is fractionally integrating the function by order $(1-\upsilon)$ by formula A1 and then followed by one-whole differentiation. There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative [6], [26], [29] for fractional order $0 < \upsilon < 1$ is given as

$$ {}_0^CD_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(1-\upsilon)}\int_0^t (t-\tau)^{-\upsilon} f^{(1)}(\tau)\mathrm{d}\tau ; \quad 0 < \upsilon < 1 \qquad \text{A4}$$

Thus for A4 we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order $1-\upsilon$, by formula A1. The Caputo and Riemann-Liouville (RL) fractional derivative are related [6], [26], [29] by

$$ {}_0D_t^{\upsilon}\left[f(t)\right] = {}_0^CD_t^{\upsilon}\left[f(t)\right] + \frac{f(0)}{\Gamma(1-\upsilon)}t^{-\upsilon} ; \quad 0 < \upsilon < 1 \qquad \text{A5}$$

We mention that both the fractional derivatives, i.e. RL and Caputo are equal when initial value is zero i.e. $f(0) = 0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^{\beta}\left[\mathrm{K}\right] = \frac{\mathrm{K}}{\Gamma(1-\beta)}t^{-\beta}$. Whereas, the Caputo fractional derivative of a constant is zero i.e. ${}_0^CD_t^{\beta}\left[\mathrm{K}\right] = 0$, [6], [26], [29].

The fractional integration and fractional differentiation of delta function [6], [26], [29] is as follows

$$ {}_0I_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(\upsilon)}t^{\upsilon-1} ; \quad {}_0D_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(-\upsilon)}t^{-\upsilon-1} , \qquad 0 < \upsilon < 1 \qquad \text{A6}$$

Fractional derivative and fractional integration of power function $f(t) = \mathrm{K}t^{p}$ [6], [26], [29] is

$$ {}_0I_t^{\upsilon}\mathrm{K}t^{p} = \mathrm{K}\frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}t^{p+\upsilon} , \qquad {}_0D_t^{\upsilon}\mathrm{K}t^{p} = \mathrm{K}\frac{\Gamma(p+1)}{\Gamma(p-\upsilon+1)}t^{p-\upsilon} , \quad p > -1 \qquad \text{A7}$$

Laplace Transform of Fractional Differ-integration

The Laplace transform of fractional integral operation is

$$ \mathcal{L}\left\{ {}_0I_t^{\upsilon} f(t)\right\} = s^{-\upsilon}F(s) \qquad \text{A8}$$

Laplace transform of Caputo derivative for fractional order $0<\upsilon<1$ is

$$\mathcal{L}\left\{{}_0^C D_t^{\upsilon} f(t)\right\} = s^{\upsilon}F(s) - s^{\upsilon-1}f(0) \qquad \text{A9}$$

Mittag-Leffler Function

Like in classical calculus, we have exponential function; similarly, in fractional Calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(z)^n}{\Gamma(\alpha n+\beta)}, \quad z\in\mathbb{C}; \quad \mathrm{Re}[\alpha,\beta]>0 \qquad \text{A10}$$

For $\beta=1$ we have $E_{\alpha,1}(z)=E_{\alpha}(z)$; is called One-Parameter Mittag-Leffler function.

The Laplace transformation of Mittag-Leffler function is

$$\mathcal{L}\left\{E_{\alpha}(\lambda t^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}-\lambda} \qquad \text{A11}$$

We observe that for $E_{\alpha}(-bt^{\alpha})\big|_{\alpha=1} = \mathrm{e}^{-bt}$, and $E_{\alpha}(-at^{\alpha})\big|_{\alpha=2} = \cos\sqrt{a}t$.

We point here that $f(t)=E_{\alpha}(\lambda t^{\alpha})$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^{\alpha} f(t) = \lambda f(t)$; and $f(t)=t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$ is eigen-function for fractional differential equation with RL fractional derivative, i.e. ${}_0D_t^{\alpha} f(t)=\lambda f(t)$ [6], [26], [29]. We will be using these concepts of fractional calculus in our discussion.

Appendix-2

Fractional Calculus with Non-Singular Kernel

We have seen in the Appendix-1 the classical Fractional Calculus is based on singular power law as kernel in Riemann-Liouville fractional integration formula. Here we review the Fractional Calculus where the kernel function in fractional integration is non-singular one; this calculus is also called Prabhakar Calculus. The aim of this section is to point out similarity of what we applied in all our previous sections.

Fractional Caputo Derivative with Non-Singular kernel

The new definition of Fractional derivative with non-singular kernel is proposed in [56], [57] that we describe in this section. From Appendix-1, we write the formulation of Caputo derivative, as ${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{(1)}(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha}$, for fractional order $0<\alpha<1$. This is applicable if the function $f(t)$ has one-whole derivative $f^{(1)}(t)$ in the interval under consideration. As per our discussions, this Caputo derivative is fractional integration of order $1-\alpha$ for function $f^{(1)}(t)$ i.e. ${}_a^C D_t^\alpha f(t) = {}_a I_t^{1-\alpha} f^{(1)}(t)$. Here the kernel of integration is 'power function' and is a singular function at $t=0$ i.e. $\mathrm{k}(t) \sim t^{-\alpha}$. In terms of the convolution integral ${}_a^C D_t^\alpha f(t)$ is ${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\left(\mathrm{k}(t) \right) * \left(f^{(1)}(t) \right) \right)$.

The kernel if considered as $\mathrm{k}(t) \sim \exp\left(-\frac{\alpha}{1-\alpha} t\right)$, instead usual 'power function' then we have Caputo-Fabrizo (CF) definition given as

$$ {}_a^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) f^{(1)}(\tau)\,\mathrm{d}\tau; \quad 0<\alpha<1 \qquad \text{A12} $$

Where $M(\alpha)$ is normalization constant $M(0)=M(1)=1$.

Now if we have a kernel as 'higher transcendental function' Mittag-Leffler $\mathrm{k}(t) \sim E_\alpha\left(-\frac{\alpha}{1-\alpha} t^\alpha\right)$, $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ (14) then we have Atangana-Baleanu-Caputo (ABC) definition given as

$$ {}_a^{ABC} D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \left(E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\right)\left(f^{(1)}(\tau)\right)\mathrm{d}\tau, \quad 0<\alpha<1 \qquad \text{A13} $$

Where $B(\alpha)$ is normalization constant $B(0)=B(1)=1$.

We note that our classical definition of fractional derivative (RL or Caputo) is connected to Riemann-Liouville fractional integration, with kernel in convolution integration as 'power function'. We observe that in those classical definitions, the kernel of fractional integration is $\sim t^{-\alpha}$ which is singular at $t=0$; whereas the definitions (A12) and (A13) have kernel of convolution integration as non-singular functions-described by exponential and Mittag-Leffler functions respectively.

One may verify, (A12) by expanding $\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)$ in series and then using Cauchy's formula for repeated integration, that this expression is weighted series sum of various orders of integer integrals. Similarly by expanding $E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)$ in (A13) one may verify that this is actually is weighted series sum of various orders of fractional integrals of RL type.

The Three-Parameter Mittag-Leffler Function $f(z) = E_{\alpha,\beta}^{\gamma}(z)$ (Prabhakar Function)

This function is modification of Two-Parameter Mittag-Leffler function done by T R Prabhakar in 1971, [47] defined in series form as following

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \qquad \text{A14}$$

with $z\in\mathbb{C}$, $\alpha,\beta,\gamma\in\mathbb{C}$ and $\mathrm{Re}\left[\alpha\right]>0$. In (A14) we have $(\gamma)_k$ as Pochhammer Number, the rising factorial and is $(\gamma)_k=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. This (A14) is entire function. The Laplace Transformation of the Prabhakar function is following

$$\mathcal{L}\left\{t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha})\right\}=\frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+\lambda)^{\gamma}};\quad \mathrm{Re}\left[s\right]>\left|\lambda\right|^{1/\alpha} \qquad \text{A15}$$

From (92) and (93) we note that $E_{\alpha,\beta}(z)=E^{1}_{\alpha,\beta}(z)$ and $E_{\alpha}(z)=E^{1}_{\alpha,1}(z)$.

Prabhakar Integral

Using the function $\mathrm{k}^{\gamma}_{\alpha,\beta}(t)=t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda t^{\alpha})$ as kernel of convolution integral, we can define Prabhakar Integral [47] as ${}_a\mathbf{I}^{\gamma}_{(\alpha,\beta,\lambda);t}\left[f(t)\right]=\left(\mathrm{k}^{\gamma}_{\alpha,\beta}(t)\right)*\left(f(t)\right)$, described as following expression

$${}_a\mathbf{I}^{\gamma}_{(\alpha,\beta,\lambda);t}\left[f(t)\right]=\int_a^t\left((t-\tau)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda(t-\tau)^{\alpha})\right)\left(f(\tau)\right)\mathrm{d}\tau \qquad \text{A16}$$

We note that Prabhakar integral (A16) is having Kernel of integration different from power law function ($\sim t^{\upsilon-1}$) that we used as Kernel in Riemann-Liouville Fractional Integration formula.

Prabhakar Integral as series-sum of Riemann-Liouville fractional integrals

We know the Riemann-Liouville fractional integral as ${}_aI^{\upsilon}_t f(t)=\frac{1}{\Gamma(\upsilon)}\int_a^t(t-\tau)^{\upsilon-1}f(\tau)\mathrm{d}\tau$ Let us expand (A16) with inserting (A14) as in following steps [42]-[49]

$$\begin{aligned}{}_a\mathbf{I}^{\gamma}_{(\alpha,\beta,\lambda);t}\left[f(t)\right]&=\int_a^t\left((t-\tau)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\lambda(t-\tau)^{\alpha})\right)\left(f(\tau)\right)\mathrm{d}\tau\\&=\int_a^t\left((t-\tau)^{\beta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{(\lambda(t-\tau)^{\alpha})^k}{k!}\right)\left(f(\tau)\right)\mathrm{d}\tau\\&=\sum_{k=0}^{\infty}\left(\frac{(\gamma)_k\lambda^k}{k!}\left(\frac{1}{\Gamma(\alpha k+\beta)}\right)\int_a^t(t-\tau)^{\alpha k+\beta-1}f(\tau)\mathrm{d}\tau\right)\\&=\sum_{k=0}^{\infty}\frac{(\gamma)_k\lambda^k}{k!}\left({}_aI^{\alpha k+\beta}_t f(t)\right)=\sum_{k=0}^{\infty}\frac{\lambda^k\Gamma(\gamma+k)}{k!\Gamma(\gamma)}\left({}_aI^{\alpha k+\beta}_t f(t)\right)\end{aligned} \qquad \text{A17}$$

We used $I^{\upsilon}_t f(t)=\frac{1}{\Gamma(\upsilon)}\int_a^t(t-\tau)^{\upsilon-1}f(\tau)\mathrm{d}\tau$ Using (A17) we can write the following in expanded form

$$
\begin{aligned}
{}_a\mathbf{I}^{\gamma}_{(\alpha,\beta,\lambda);t}\left[f(t)\right] &= \sum_{k=0}^{\infty}\frac{(\gamma)_k\lambda^k}{k!}\left({}_aI^{\alpha k+\beta}f(t)\right);\quad (\gamma)_k=\frac{\Gamma(\upsilon+k)}{\Gamma(\upsilon)};\quad \Gamma(\upsilon+1)=\upsilon\Gamma(\upsilon)\\
&= {}_aI_t^{\beta}f(t)+\gamma\lambda\left({}_aI_t^{\alpha+\beta}f(t)\right)+\frac{\gamma(\gamma+1)\lambda^2}{2!}\left({}_aI_t^{2\alpha+\beta}f(t)\right)\\
&\quad+\frac{\gamma(\gamma+1)(\gamma+2)\lambda^3}{3!}\left({}_aI_t^{3\alpha+\beta}f(t)\right)+\dots
\end{aligned}
\qquad \text{A18}
$$

In (A18) putting, $\gamma=\beta=1$ we have following

$$
\begin{aligned}
{}_a\mathbf{I}^{1}_{(\alpha,1,\lambda);t}\left[f(t)\right] &= \int_a^t\left(E_\alpha(\lambda(t-\tau)^\alpha)\right)\left(f(\tau)\right)\mathrm{d}\tau\\
&= {}_aI_t^{1}f(t)+\lambda\left({}_aI_t^{\alpha+1}f(t)\right)+\lambda^2\left({}_aI_t^{2\alpha+1}f(t)\right)\\
&\quad+\lambda^3\left({}_aI_t^{3\alpha+1}f(t)\right)+\dots
\end{aligned}
\qquad \text{A19}
$$

We note that Prabhakar integral defined via non-singular kernel $\mathrm{k}^1_{\alpha,1}(t)=E_\alpha(\lambda t^\alpha)$ is sum of classical Riemann-Liouville fractional integration with power law kernel.

We see the similarities of these fundamentals of Prabhakar Integrals described in when we dealt with several types of non-singular memory kernels $\mathrm{h}(t)$ in our relaxation or evolution equation i.e. $\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}(t)$. This Fractional Calculus with non-singular memory kernel is developed and applied by many pioneer workers [42]-[57].

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