Inserting various Memory Kernels in basic Evolution Equation in Process Dynamics-and formation of new constituent equations for various Physical Laws-(different from Classical formulas)

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One-Day Seminar on Theme
“Discrete mathematical modeling in Physics Chemistry and Biology with applications in emerging fields”
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Dedicating this talk to
Prof. Abdon Atangana & Prof Jordan Hristov

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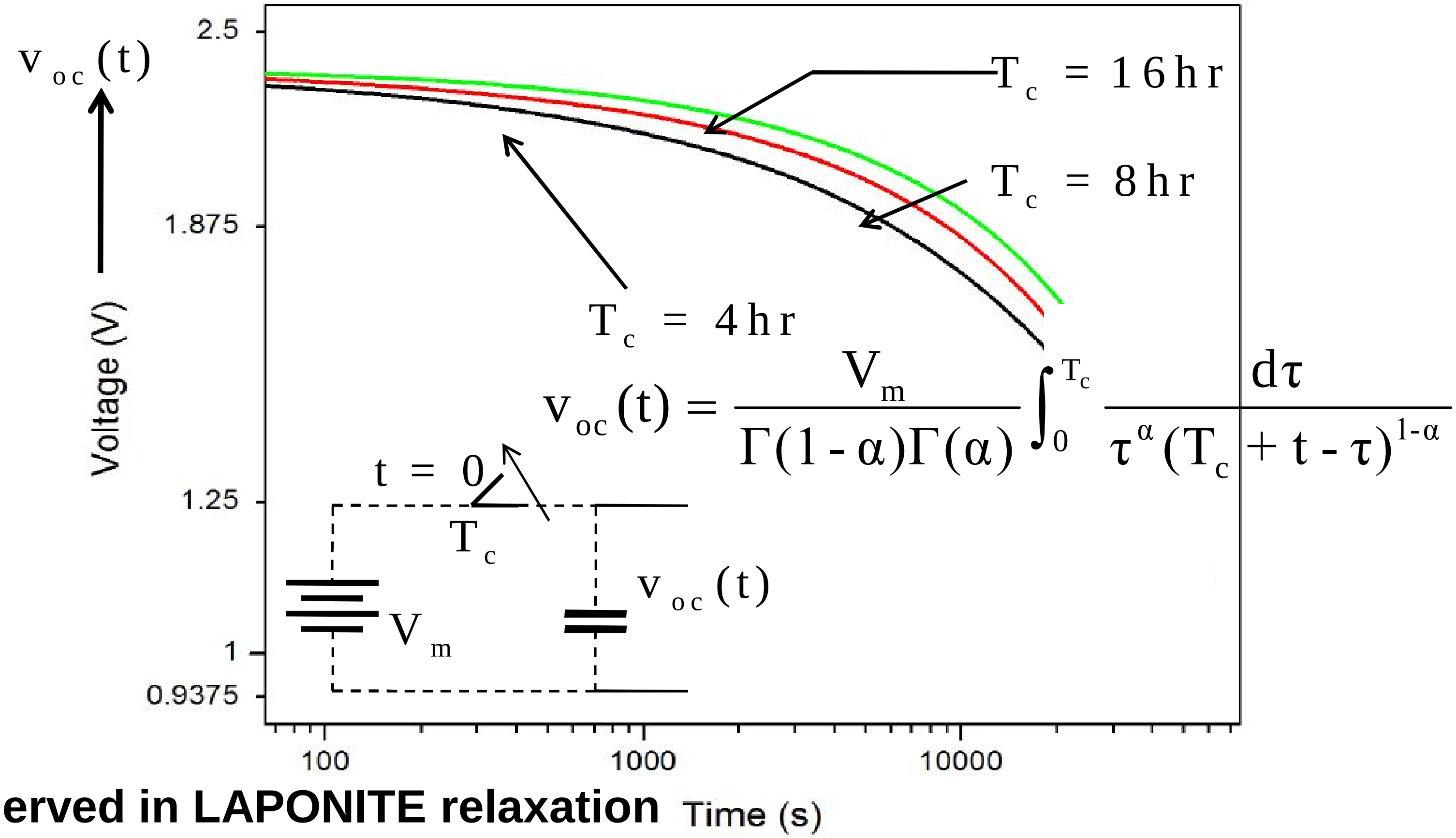
Pioneers in using and developing non-singular kernel in fractional derivative and using the for various physical process like circuits, heat equations, viscoelasticity etc

Evidence of Memory Observed in Relaxation process - (memorizing how long super-capacitor is kept on charging)

A supercapacitor (SC) kept to charge from a constant voltage V_m for time T_c then kept at open circuited condition- The self-discharge decay of OC voltage $v_{oc}(t)$ <u>remembers the history</u> of its charging time T_c

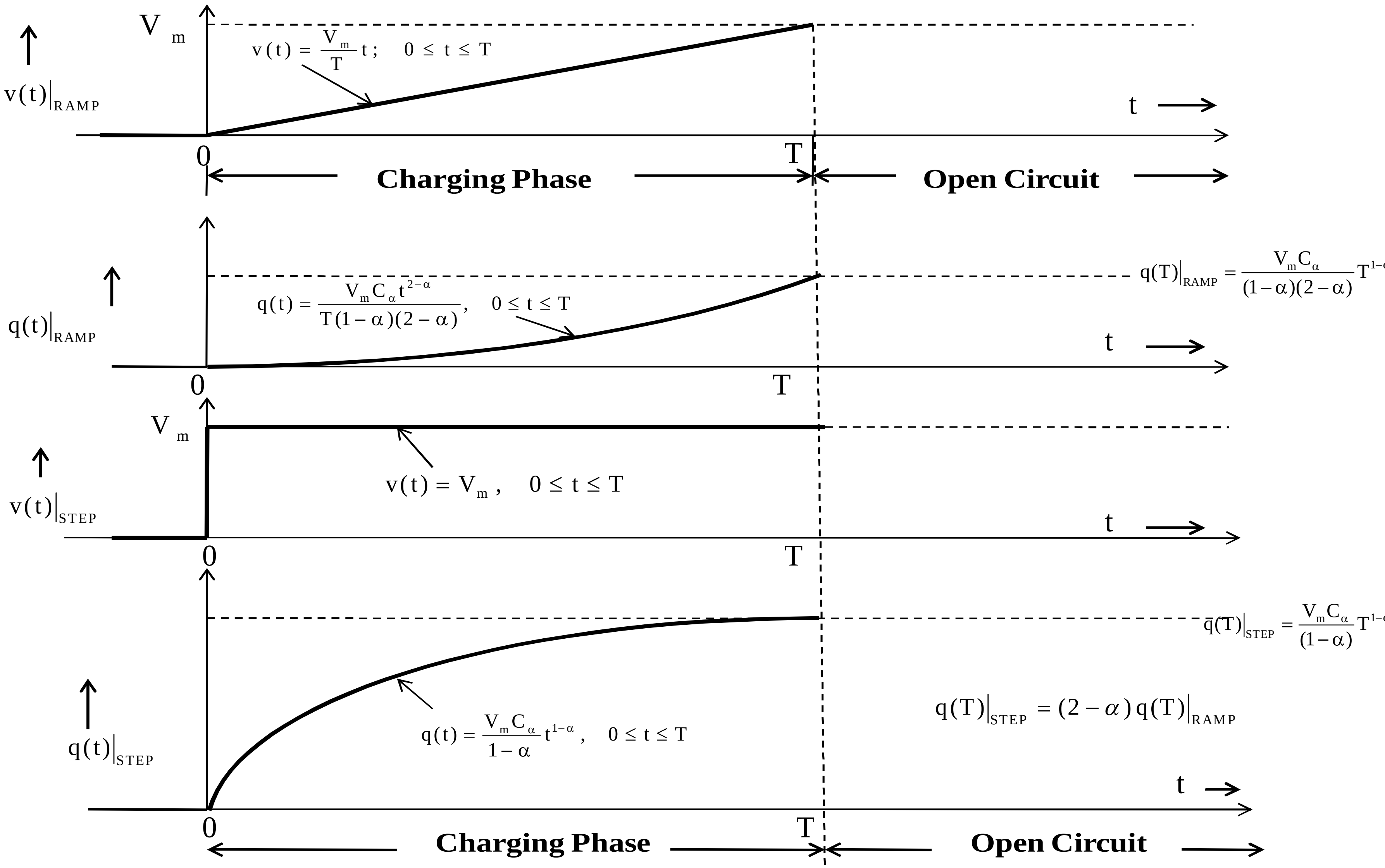


$$v_{oc}(t) = \frac{V_m}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{T_c} \frac{d\tau}{\tau^{\alpha}(T_c + t - \tau)^{1-\alpha}}$$

This is also observed in LAPONITE relaxation

Ideal loss less capacitor (memory-less) does not have OC voltage droop depending on charging time, here irrespective of history OC voltage retains at same value

Evidence of Memory Observed in Relaxation (remembering what type charging function one uses for SC)



Step input pumps more charge that ramp input–for memory less case in both cases charge is same

What is and what type of memory-in dead matter?

A dead matter are: dielectric, visco-elastic, Electric Double layer, Constant Phase Element (CPE), magnetic, electrostatic materials, liquid cooling etc. show memory

Systems or processes responds to a stimulus-i.e. cause $x(t)$ **gives some effect** $y(t)$

Each process has characteristic response function $h(t)$ **-a property of system.**

We have system/process reacts as per causality principle we simply write as:

$$y(t)=\int_{-\infty}^{\infty}h(t-t')x(t')dt'=h(t)*x(t)$$ **Convolution operation**

$$y(t)=\int_0^t h(t-t')x(t')dt'=\int_t^0 h(\tau)x(t-\tau)(-d\tau)=\int_0^t h(\tau)x(t-\tau)d\tau;\quad t\geq 0$$

We term the function $h(t)$ **as memory kernel, and say cause** $x(t)$ **we take as rate of change of voltage** $v^{(1)}(t)$ **with effect** $y(t)$ **as current call** $i(t)$ **; we have constituent equation as**

$$i(t)=\int_0^{\infty}h(t-t')v^{(1)}(t')dt'$$

Say take $h(t)=C_1\delta(t)$ **we get Classical Capacitor Law** $i(t)=C_1v^{(1)}(t)$

Classically we will observe impulse function as current, when we apply a step input voltage to capacitor. The cause i.e. rate of change vanishes and so does our effect-and we see no current after the voltage change has vanished.
This is Zero Memory Case

Derivation of causality is given in detailed notes

Zero-memory case laws
(memory kernel as delta function)

We have effect related as convolution of cause and memory kernel as

$$y(t)=\big(h(t)\big)*\big(x(t)\big)=\int_0^t h(t-t')\,x(t')\,\mathrm{d}t'; \qquad t\geq 0$$

zero memory case laws are:, with $h(t)$ **proportional to delta function i.e.** $h(t)\propto\delta(t)$

Capacitor law $h(t)=C_1\delta(t),\quad x(t)=v^{(1)}(t),\quad y(t)=i(t)$ $\qquad i(t)=C_1\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$

Radioactive Law $h(t)=-\lambda\delta(t),\quad x(t)=N(t),\quad y(t)=N^{(1)}(t)$ $\qquad \dfrac{\mathrm{d}N(t)}{\mathrm{d}t}=-\lambda N(t)$

Diffusion Equation $h(t)=\mathbb{D}\delta(t),\quad x(t)=\dfrac{\partial^2 c}{\partial x^2},\quad y(t)=x^{(1)}(t)$ $\qquad \dfrac{\partial c}{\partial t}=\mathbb{D}\dfrac{\partial^2 c}{\partial x^2}$

Wave Equation $h(t)=c\delta(t),\quad x(t)=\dfrac{\partial^2 w}{\partial x^2},\quad y(t)=x^{(2)}(t)$ $\qquad \dfrac{\partial^2 w}{\partial t^2}=c\dfrac{\partial^2 w}{\partial x^2}$

Newton Fluid Law $h(t)=\eta\delta(t),\quad x(t)=\varepsilon^{(1)}(t),\quad y(t)=\sigma(t)$ $\qquad \sigma(t)=\eta\dfrac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}$

These are all <u>classical expressions</u> of laws that we know

Memory based relaxations- (the anomalous effects!)

For a system if the effect lingers even if the cause has vanished, we say that the system/process relaxes with memory-i.e. say if the current in dielectric lingers after the rate of change of voltage/electric field has vanished,

or in a radioactive decay/growth we observe non exponential function,

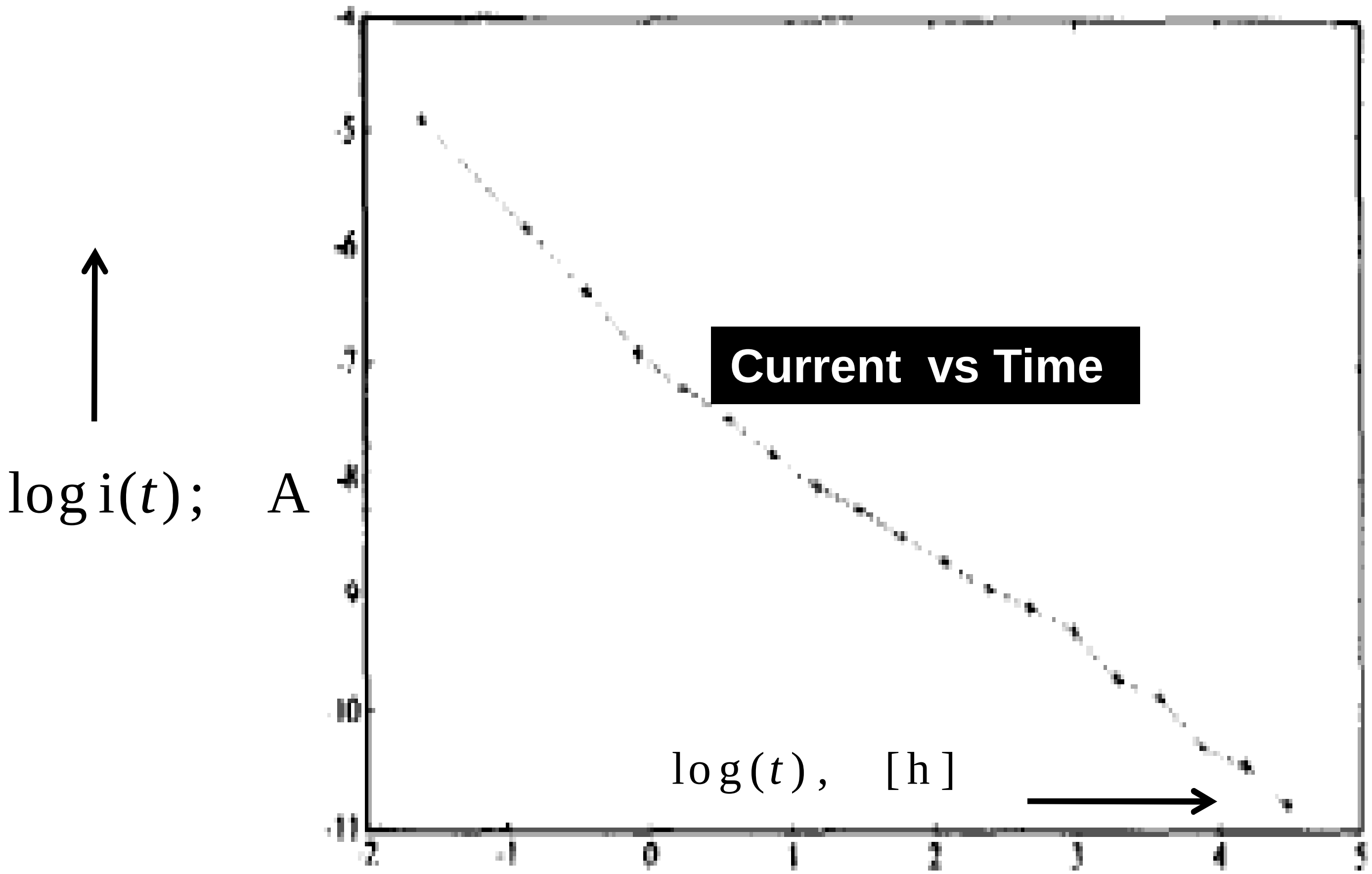
or we get non-Gaussian plume in diffusion etc.

-are called <u>anomalous processes.</u>

These processes are relaxing with memory.

<u>But what typeof memory is a good topic to study</u>

Experimental evidence a case of memory based relaxation in dielectric



At time zero a voltage of 100V is connected to a 0.47uF metalized paper dielectric capacitor; in log-log scales average slope is -0.86. Thus exponent of relaxation current is non-integer (<u>Note the current is not impulse function</u>)

IEEE Trans on Dielectrics and Insulation, 1,826. (1994), S. Westerlund, L.Ekstam

Types of memory kernels-that we are considering

We have memory decaying with time

We will insert various decaying kernel in the equation

$$y(t) = \big(h(t)\big) * \big(x(t)\big) = \int_0^t h(t-t')x(t')dt'; \quad t \geq 0$$

Power Law Singular Kernel $\quad h(t) = C\,t^{-\alpha}; \quad 0 < \alpha < 1$

Power Law Non-Singular Kernel $\quad h(t) = C(1+kt)^{-\alpha}; \quad 0 < \alpha < 1, \quad k > 0, \quad t \geq 0$

Decaying Mittag Leffler Non-Singular Kernel $\quad h(t) = C\,E_\alpha(-\lambda t^\alpha); \quad 0 < \alpha < 1, \quad t \geq 0$

Decaying Exponential Non-Singular Kernel $\quad h(t) = C\,e^{-\kappa t}, \quad t \geq 0, \quad \kappa > 0; \quad C > 0$

Decaying Stretched Exponential Non-Singular Kernel $\quad h(t) = Ce^{-(\kappa t)^\alpha}, \quad t \geq 0, \quad \kappa > 0; \quad 0 < \alpha < 1; \quad C > 0$

In this presentation we will not take all of them but will discuss few cases of above apply in capacitors and radioactive decay/growth cases

There could be several other types of memory kernels

Derivation of laws with all these kernels is given in detailed notes

Memory kernel as singular power-law in capacitor law

$$y(t) = h(t) * x(t) = \int_0^t h(t-t')x(t')dt'; \quad t \geq 0$$

$$h(t) = Ct^{-\alpha}, \quad x(t) = v^{(1)}(t), \quad y(t) = i(t)$$

$$\begin{aligned} i(t) &= \left(h(t)\right) * \left(v^{(1)}(t)\right) \\ &= \int_0^t \left(h(t-\tau)\right)\left(v^{(1)}(\tau)\right)d\tau; \quad h(t) = Ct^{-\alpha}, \quad 0 < \alpha < 1; \quad t \geq 0 \\ &= \int_0^t \left(C(t-\tau)^{-\alpha}\right)\left(v^{(1)}(\tau)\right)d\tau \\ &= C\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t \left((t-\tau)^{-\alpha}\right)\left(v^{(1)}(\tau)\right)d\tau\right), \quad C_\alpha = C\left(\Gamma(1-\alpha)\right) \\ &= C_\alpha\left({}_0^C D_t^\alpha v(t)\right) = C_\alpha v^{(\alpha)}(t) = C_\alpha\left({}_0 I_t^{1-\alpha}\left[v^{(1)}(t)\right]\right) \end{aligned}$$

We used integral formula of Caputo Fractional Derivative, i.e.

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \left((t-\tau)^{-\alpha}\right)\left(f^{(1)}(\tau)\right)d\tau, \quad 0 < \alpha < 1$$

For zero memory case we have for $\alpha = 1$ **, the classical capacitor law i.e.**

$$i(t) = C_1 v^{(1)}(t) = C_1\left({}_0 I_t^0\left[v^{(1)}(t)\right]\right), \quad h(t) = C_1\delta(t)$$

Current for a step-voltage input for capacitor having singular power law memory kernel

We have derived constitutive law for capacitor having singular power law kernel as

$$\mathrm{i}(t) = \mathrm{C}_\alpha \mathrm{v}^{(\alpha)}(t) = \mathrm{C}_\alpha \left({}_0^C D_t^\alpha \mathrm{v}(t) \right)$$

$$\mathcal{L}\left\{\mathrm{i}(t)\right\} = \mathrm{C}_\alpha \mathcal{L}\left\{ {}_0^C D_t^\alpha \mathrm{v}(t) \right\}; \quad 0 < \alpha < 1, \quad \mathrm{v}(t) = u(t)$$

$$\mathrm{I}(s) = \mathrm{C}_\alpha \left(s^\alpha \mathrm{V}(s) - s^{\alpha-1}\mathrm{v}(0) \right); \quad \mathrm{V}(s) = \frac{1}{s}, \quad \mathrm{v}(0) = 0$$

$$\mathrm{I}(s) = \mathrm{C}_\alpha s^{\alpha-1}$$

using $\mathcal{L}\left\{t^p\right\} = \dfrac{\Gamma(p+1)}{s^{p+1}}$ **we get**

$$\mathrm{i}(t) = \mathcal{L}^{-1}\left\{\mathrm{I}(s)\right\} = \mathcal{L}^{-1}\left\{\mathrm{C}_\alpha s^{\alpha-1}\right\}$$

$$\mathrm{i}(t) = \frac{\mathrm{C}_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, \quad 0 < \alpha < 1, \qquad t \geq 0$$

$$\mathrm{C}_\alpha = \mathrm{C}\,\Gamma(1-\alpha) \qquad \mathrm{i}(t) = \mathrm{C}\, t^{-\alpha}; \quad 0 < \alpha < 1$$

This relaxation current is as singular power law is as per <u>(UDL) Universal Dielectric Law of Curie-von-Schweidler</u>

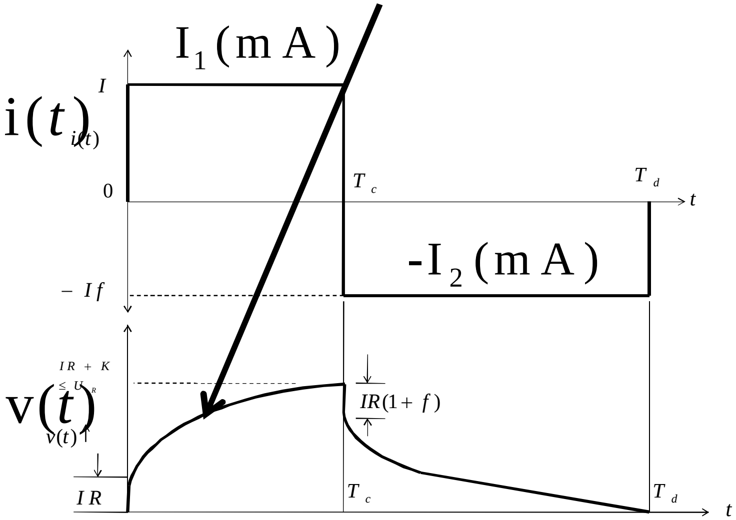
For a constant charging current the voltage across capacitor is not changing linearly proportional to time !

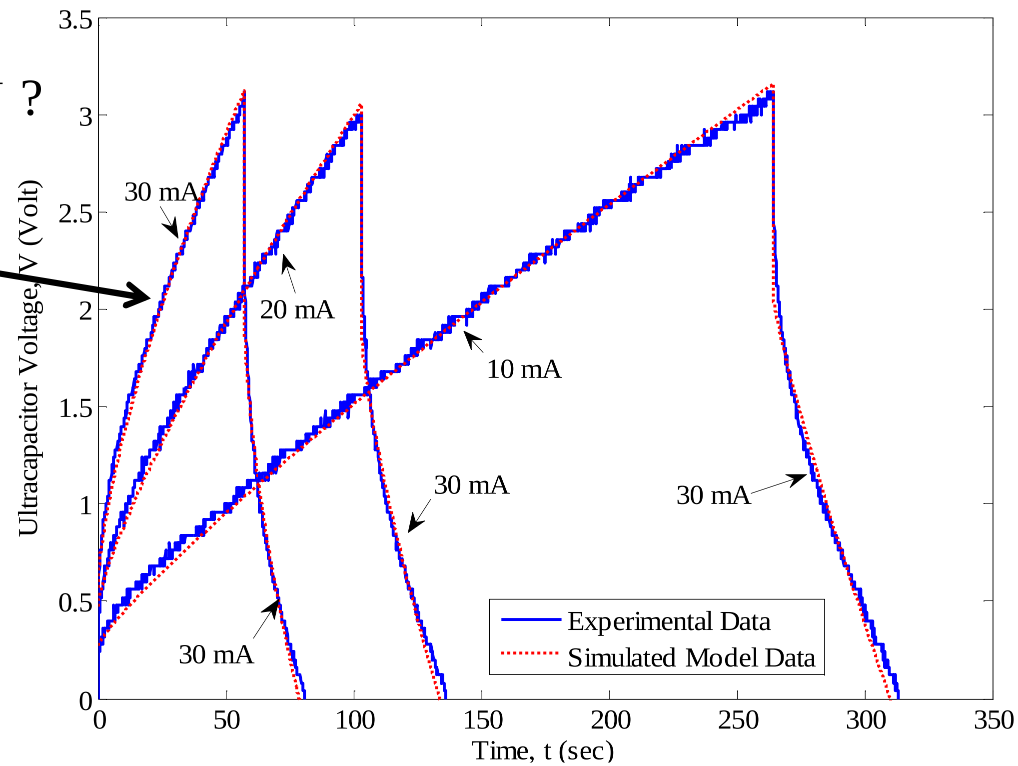
We have $\mathrm{v}(t) \propto t^{\alpha}\,; \quad 0 < \alpha < 1; \quad \mathrm{i}(t) = \mathrm{I\,mA} \qquad \mathrm{v}(t) \neq \frac{1}{\mathrm{C}_1}\int_{t_1}^{t_2} \mathrm{i}(\tau)\mathrm{d}\tau$

So $\mathrm{i}(t) \neq \mathrm{C}_1 \frac{\mathrm{d\,v}(t)}{\mathrm{d}t}$ **does fractional derivative relates the voltage-current?**

Is $\mathrm{i}(t) = \mathrm{C}_{\alpha} \left. \frac{\mathrm{d}^{\alpha}\mathrm{v}(t)}{\mathrm{d}t^{n}} \right|_{t_1}^{t}, \quad \alpha \in \mathbb{R}^{+}\,?$

<u>Not usual linear rise</u> for $\mathrm{v}(t)$ **constant current**





<u>Observed Non-linear charging voltage profile</u>

Memory kernel as non-singular power-law function for capacitor law

$$\mathrm{h}(t)=\mathrm{C}(1+kt)^{-\alpha}\;;\quad 0<\alpha<1,\quad k>0,\quad t\geq 0$$

$$\mathrm{i}(t)=\left(\mathrm{h}(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)$$

$$=\int_0^t \mathrm{C}\left((1+k(t-\tau)^{-\alpha}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau$$

$$=\mathrm{C}\int_0^t\left(\binom{-\alpha}{0}\left(k(t-\tau)\right)^0+\binom{-\alpha}{1}\left(k(t-\tau)\right)+\binom{-\alpha}{2}\left(k(t-\tau)\right)^2+\ldots\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau$$

$$=\mathrm{C}\left(\int_0^t\left(\mathrm{v}^{(1)}(\tau)\right)+\tfrac{(-\alpha)}{1!}\int_0^t\left(k(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau+\tfrac{(-\alpha)(-\alpha-1)}{2!}\int_0^t\left(k(t-\tau)\right)^2\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau+\ldots\right)$$

$$=\mathrm{C}\left({}_0I_t^1\mathrm{v}^{(1)}(t)+(-\alpha)k\left({}_0I_t^2\mathrm{v}^{(1)}(t)\right)+(-\alpha)(-\alpha-1)k^2\left({}_0I_t^3\mathrm{v}^{(1)}(t)\right)+\ldots\ldots\ldots\right)$$

$$=\mathrm{C}\sum_{n=1}^{\infty}\mathrm{a}_n\left({}_0I_t^n\mathrm{v}^{(1)}(t)\right),\quad \mathrm{a}_1=1,\quad \mathrm{a}_2=-\alpha k,\quad \mathrm{a}_3=(\alpha)(\alpha+1)k^2\ldots.$$

We used is Cauchy's formula for multiple integration i.e.

$${}_0I_t^m g(t)=\tfrac{1}{(m-1)!}\int_0^t(t-\tau)^{m-1}g(\tau)\,\mathrm{d}\tau$$

Note that for unit step input voltage $\mathrm{v}(t)=u(t)$ **with** $\mathrm{v}^{(1)}(t)=\delta(t)$ **the current we get as** $\mathrm{i}(t)=\mathrm{C}(1+kt)^{-\alpha}$

Derivation is given in detailed notes

For other memory kernel as non-singular function

We use the same method as done in previous page by expanding the function $(1+kt)^{-\alpha}$ as infinite series in the formula $\mathrm{i}(t) = \left(\mathrm{h}(t)\right) * \left(\mathrm{v}^{(1)}(t)\right)$ for the following

$$h(t) = E_{\alpha}(-\lambda t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-\lambda t^{\alpha})^{n}}{\Gamma(\alpha n + 1)}, \qquad t \geq 0$$

$$h(t) = \mathrm{e}^{-\kappa t} = \sum_{n=0}^{\infty} \frac{(-\kappa t)^{n}}{n!}$$

$$h(t) = \mathrm{e}^{-(\kappa t)^{\alpha}} = \sum_{n=0}^{\infty} \frac{\left((-\kappa t)^{\alpha}\right)^{n}}{n!}$$

and do manipulations to use multiple integration formula, and fractional integration formulas to write various constituent laws as depicted next page

Detailed derivation is given in notes

Various capacitor laws memory based vis-à-vis zero memory

S.No	Memory kernel k(t)	Type	Constitutive equation of capacitor	Impedance function in Laplace domain
1	Delta Function $C\delta(t)$	Singular	$\mathrm{i}(t) = \mathrm{C}\mathrm{v}^{(1)}(t)$ $\mathrm{i}(t) = \mathrm{C}\left({}_0I_t^0 \mathrm{v}^{(1)}(t)\right)$	$\mathrm{Z}(s) = \frac{1}{s\mathrm{C}}$
2	Power Law $Ct^{-\alpha}$ $0 < \alpha < 1$	Singular	$\mathrm{i}(t) = \mathrm{C}_\alpha \mathrm{v}^{(\alpha)}(t)$ $\mathrm{C}_\alpha = \mathrm{C}\left(\Gamma(1-\alpha)\right)$ $\mathrm{i}(t) = \mathrm{C}_\alpha\left({}_0I_t^{1-\alpha}\mathrm{v}^{(1)}(t)\right)$	$\mathrm{Z}(s) = \frac{1}{s^\alpha \mathrm{C}_\alpha}$
3	Non-singular Power Law $\mathrm{C}(1+\lambda t)^{-\alpha}$ $0 < \alpha < 1$	Non-Singular	$\mathrm{i}(t) = \mathrm{C}\sum_{n=1}^{\infty} w_n \left({}_0I_t^n \mathrm{v}^{(1)}(t)\right)$ $w_1 = 1, w_2 = -\alpha\lambda,$ $w_3 = (\alpha)(\alpha+1)\lambda^2, \ldots$	$\mathrm{Z}(s) = \frac{1}{\mathrm{C}\left(\sum_{n=1}^{\infty} w_n s^{1-n}\right)}$
4	Mittag-Leffler $CE_\alpha(-\lambda t^\alpha)$ $0 < \alpha < 1$	Non-Singular	$\mathrm{i}(t) = \mathrm{C}\sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1}\mathrm{v}^{(1)}(t)\right)$ $w_n = (-1)^n\lambda^n$ $\mathrm{v}(t) = \frac{1}{\mathrm{C}}\mathrm{i}(t) + \frac{\lambda}{\mathrm{C}}\left({}_0I_t^\alpha \mathrm{i}(t)\right)$	$\mathrm{Z}(s) = \frac{1}{\mathrm{C}\sum_{n=0}^{\infty} w_n s^{-\alpha n}}$ $\mathrm{Z}(s) = \frac{1}{\mathrm{C}} + \frac{1}{\left(\frac{\mathrm{C}}{\lambda}\right)s^\alpha}$
5	Exponential $\mathrm{C}e^{-\lambda t}$ $0 < \alpha < 1$	Non-Singular	$\mathrm{i}(t) = \mathrm{C}\sum_{n=0}^{\infty} w_n \left({}_0I_t^{n+1}\mathrm{v}^{(1)}(t)\right)$ $w_n = (-1)^n\lambda^n$ $\mathrm{v}(t) = \frac{1}{\mathrm{C}}\mathrm{i}(t) + \frac{\lambda}{\mathrm{C}}\int_0^t \mathrm{i}(\tau)d\tau$	$\mathrm{Z}(s) = \frac{1}{\mathrm{C}\sum_{n=0}^{\infty} w_n s^{-n}}$ $\mathrm{Z}(s) = \frac{1}{\mathrm{C}} + \frac{1}{s\left(\frac{\mathrm{C}}{\lambda}\right)}$
6	Stretched-Exponential $\mathrm{C}e^{-(\lambda t)^\alpha}$ $0 < \alpha < 1$	Non-Singular	$\mathrm{i}(t) = \mathrm{C}\sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)$ $w_n = (-1)^n\left(\frac{\lambda^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$	$\mathrm{Z}(s) = \frac{1}{\mathrm{C}\sum_{n=0}^{\infty} w_n s^{-\alpha n}}$

Discrete & Continuous Dynamical Systems Series-S doi: 10.3934/dcdss.2020032

Recent experimental validation of q = c * v

$$y(t) = \int_0^t h(t-t')x(t')dt'$$

$$q(t) = c(t) * v(t)$$

$$c(t) = C\,t^{-\alpha}$$

$$q(t) = \int_0^t C(t-t')^{-\alpha}\, v(t')dt'$$

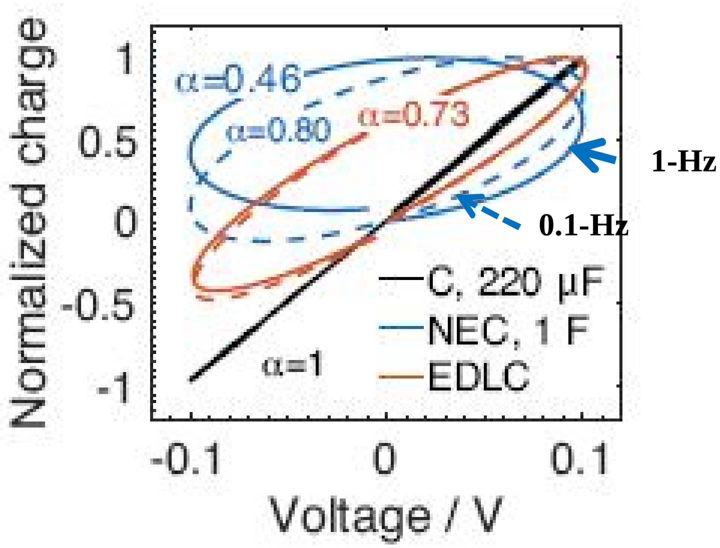
$$v(t) = V_m \cos\omega_0 t, \quad c(t) = C t^{-\alpha}, \quad C_\alpha = C\Gamma(1-\alpha)$$

$$i(t) = V_m C_\alpha \omega_0^\alpha \cos\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right)$$

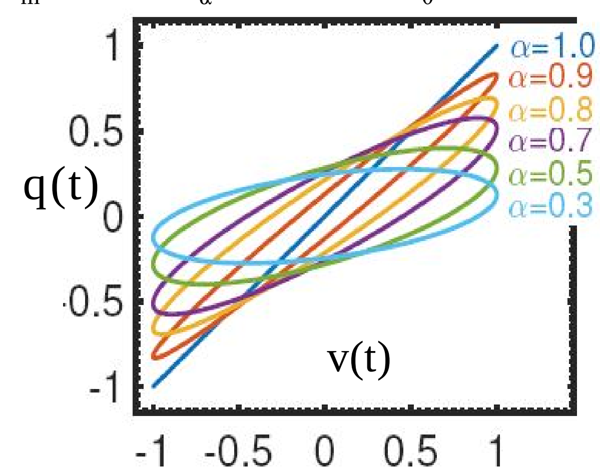
$$q(t) = V_m C_\alpha \omega_0^{\alpha-1} \cos\left(\omega_0 t - \tfrac{(1-\alpha)\pi}{2}\right)$$

$$q(t) = \omega_0^{\alpha-1} C_\alpha \left(\sin\left(\tfrac{\alpha\pi}{2}\right) + \cos\left(\tfrac{\alpha\pi}{2}\right)\sqrt{\left(\tfrac{V_m}{v(t)}\right)^2 - 1}\right)(v(t)) = (C(v))\,v(t)$$

$$C(v) = \omega_0^{\alpha-1} C_\alpha \left(\sin\left(\tfrac{\alpha\pi}{2}\right) + \cos\left(\tfrac{\alpha\pi}{2}\right)\sqrt{\left(\tfrac{V_m}{v(t)}\right)^2 - 1}\right)$$



$V_m = 1V, \quad C_\alpha = 1F/s^{1-\alpha}, \quad \omega_0 = 1 Rad/s$



One experiment validating the formula q = c * v in simulation and experiment on CPE, EDLC, ideal capacitor

"Nonlinear charge-voltage relation in constant phase element"- Preprint submitted to Journal of The Electrochemical Society – Communications Courtesy A S Elwakil et al.

Radioactive decay/growth classical law with zero memory

The classical law is:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \mathrm{C}x(t); \quad x(t) = x(0)e^{\mathrm{C}t}$$

$$x(t) - x(0) = \mathrm{C}\int_0^t x(\tau)\mathrm{d}\tau$$

$$x(t) - x(0) = \mathrm{C}\left({}_0I_t^1 x(t)\right)$$

The above classical law is a zero-memory case depicted below

$$y(t) = \int_0^t h(t-\tau)x(\tau)\mathrm{d}\tau; \quad t \geq 0; \quad y(t) = x^{(1)}(t)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_0^t h(t-\tau)x(\tau)\mathrm{d}\tau$$

$$h(t) = \mathrm{C}\delta(t)$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \int_0^t h(t-\tau)x(\tau)\mathrm{d}\tau$$

$$= \int_0^t \mathrm{C}\delta(t-\tau)x(\tau)\mathrm{d}\tau = \mathrm{C}x(t)$$

$$x(t) = x(0)e^{\mathrm{C}t}; \quad t \geq 0$$

Radioactive decay/growth law with singular power law memory kernel

$$\frac{\mathrm{dx}(t)}{\mathrm{d}t} = \int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau\,, \quad \mathrm{h}(t) = \mathrm{C}t^{-\alpha}\,, \quad 1 < \alpha \le 2$$

Write $\mathrm{C}\left(\Gamma(1-\alpha)\right) = \mathrm{C}_\alpha$

$$= \int_0^t \left(\mathrm{C}(t-\tau)^{-\alpha}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau = \mathrm{C}\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t \left((t-\tau)^{-\alpha}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau\right)$$

$$= \mathrm{C}_\alpha\left({}_0I_t^{(1-\alpha)}\mathrm{x}(t)\right)$$

Using fractional integration formula we get this

Integrating once we get $\mathrm{x}(t) - \mathrm{x}(0) = \mathrm{C}_\alpha\left({}_0I_t^{(2-\alpha)}\mathrm{x}(t)\right)$ **Operate RL fractional** ${}_0D_t^{2-\alpha}$

to get ${}_0D_t^{(2-\alpha)}\left(\mathrm{x}(t) - \mathrm{x}(0)\right) = \mathrm{C}_\alpha\left({}_0D_t^{(2-\alpha)}\,{}_0I_t^{(2-\alpha)}\mathrm{x}(t)\right)$

Using the relation ${}_0D_t^{(2-\alpha)}\left(\mathrm{x}(t) - \mathrm{x}(0)\right) = {}_0^CD_t^{(2-\alpha)}\mathrm{x}(t)$ **RL to Caputo for** $0 < (2-\alpha) < 1$

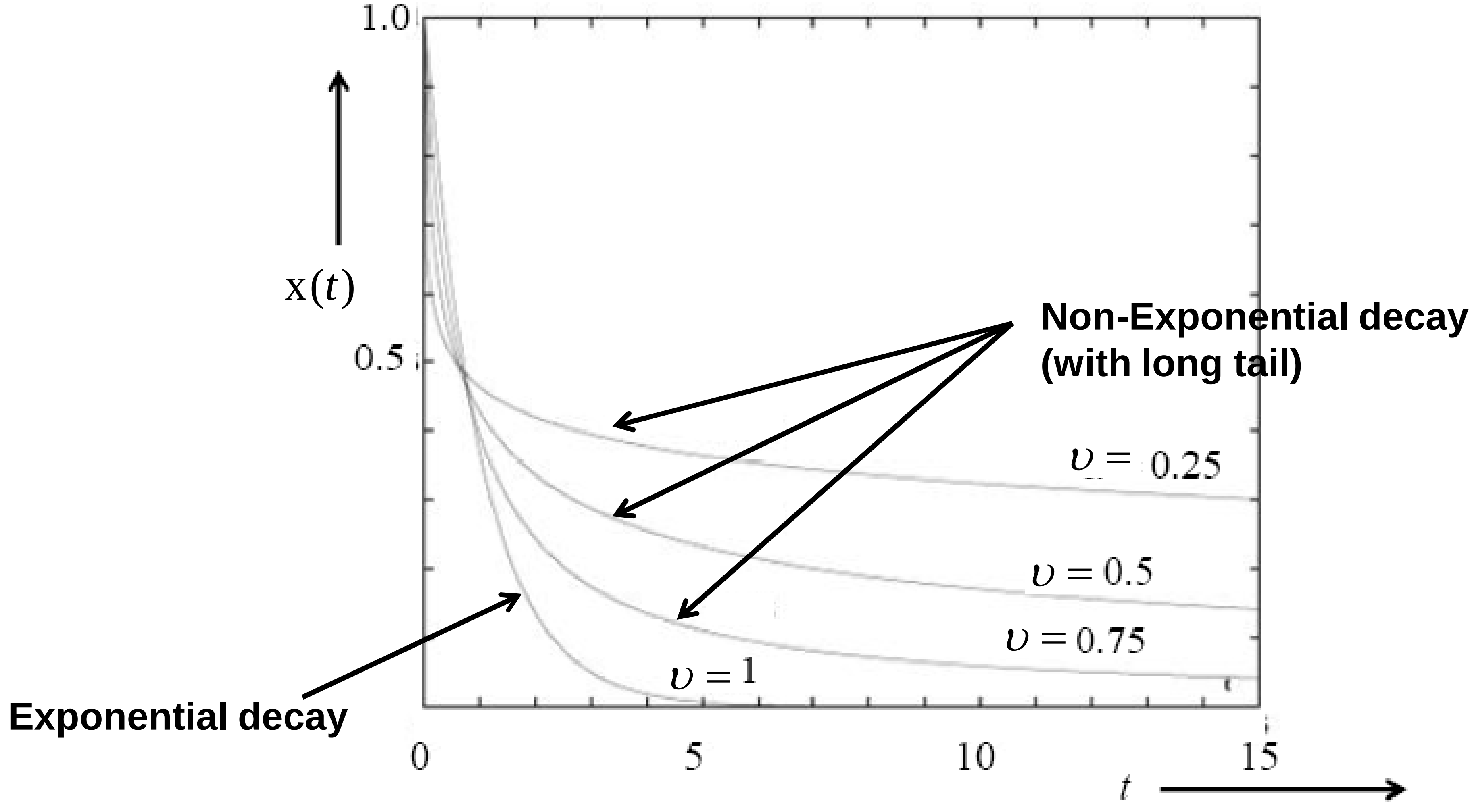
$${}_0^CD_t^{(2-\alpha)}\mathrm{x}(t) = \mathrm{C}_\alpha\mathrm{x}(t)\,, \quad 1 < \alpha < 2$$

Solution to above FDE is $\mathrm{x}(t) = \mathrm{x}(0)E_{2-\alpha}\left(\mathrm{C}_\alpha t^{(2-\alpha)}\right), \quad 1 < \alpha < 2$

Monotonically decaying Mittag-Leffler function

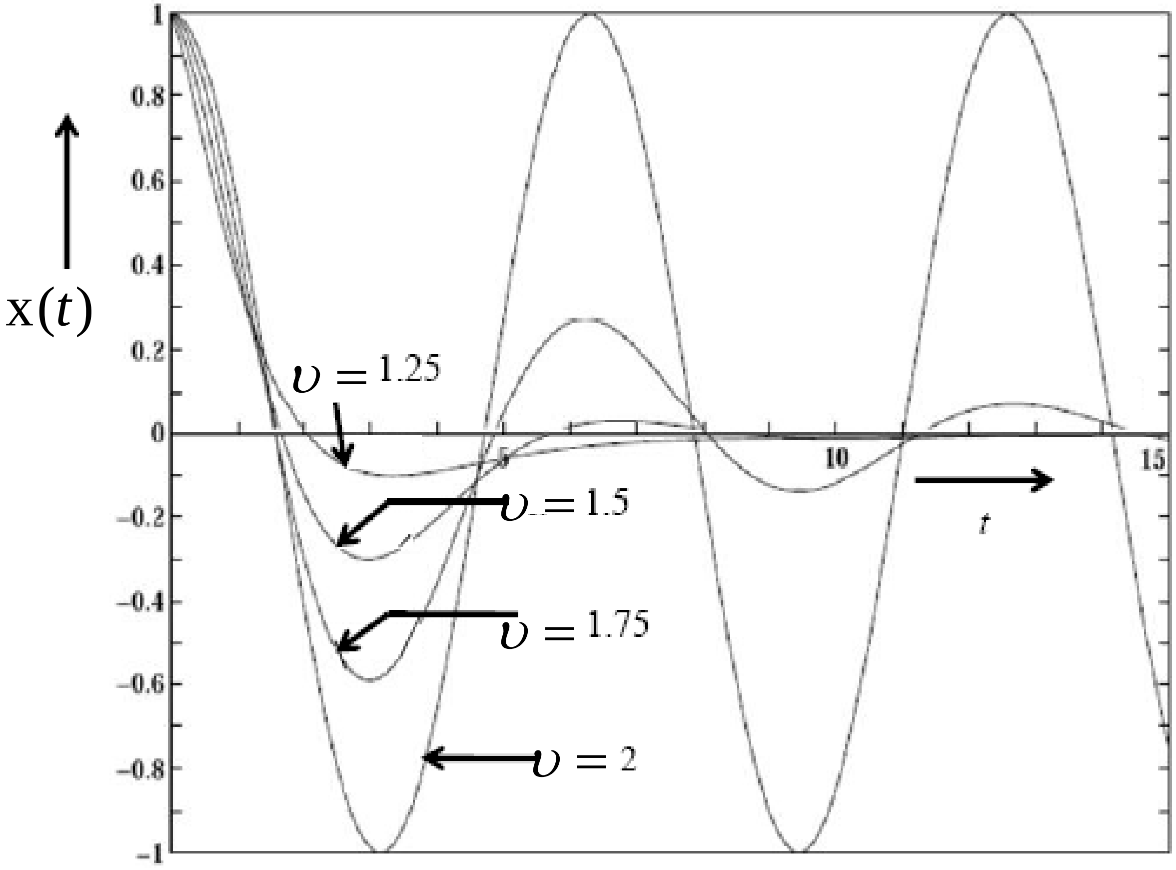
$$x(t) = x(0)E_{2-\alpha}\left(C_\alpha t^{(2-\alpha)}\right), \quad 1 < \alpha < 2$$

$$2-\alpha = \upsilon, \quad C_\alpha = -k, \quad x(0) = 1 \qquad x(t) = E_\upsilon(-kt^\upsilon), \quad t > 0; \quad k = 1; \quad 0 \le \upsilon \le 1$$



Oscillatory decaying Mittag-Leffler function

$$x(t) = E_\upsilon(-kt^\upsilon), \quad t > 0; \quad k = 1; \quad 1 < \upsilon \leq 2$$



Radioactive decay/growth law with non-singular kernel

$$\mathrm{y}(t) = \left(\mathrm{h}(t)\right) * \left(\mathrm{x}(t)\right); \quad \mathrm{h}(t) = \mathrm{C}\,\mathrm{e}^{-\kappa t}; \quad \mathrm{y}(t) = \mathrm{x}^{(1)}(t)$$

$$\frac{\mathrm{d}\,\mathrm{x}(t)}{\mathrm{d}t} = \mathrm{C}\int_0^t \left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau$$

$$= \mathrm{C}\int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\mathrm{x}(\tau)\right)\mathrm{d}\tau$$

$$= \mathrm{C}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t (t-\tau)^n\,\mathrm{x}(\tau)\,\mathrm{d}\tau$$

$$= \mathrm{C}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t (t-\tau)^n\,\mathrm{x}(\tau)\,\mathrm{d}\tau\right)$$

$$= \mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{x}(t)\right]\right)$$

$$\mathrm{x}(t) - \mathrm{x}(0) = \mathrm{C}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+2}\left[\mathrm{x}(t)\right]\right)$$

Solution $\dfrac{\mathrm{dx}(t)}{\mathrm{d}t} = \mathrm{C}\mathrm{e}^{-\kappa t} * \mathrm{x}(t)$

$$s\,\mathrm{X}(s) - \mathrm{x}(0) = \mathrm{C}\left(\frac{1}{s+\kappa}\right)\mathrm{X}(s)$$

$$\mathrm{x}(t) = \mathrm{x}(0)\mathcal{L}^{-1}\left\{\frac{s+\kappa}{s^2+\kappa s-\mathrm{C}}\right\}$$

$$\mathrm{C} = -20; \quad \kappa = 12$$

$$\mathrm{x}(t) = \mathrm{x}(0)\left(\tfrac{5}{4}\mathrm{e}^{-2t} - \tfrac{1}{4}\mathrm{e}^{-10t}\right)$$

Various radioactive decay/growth laws memory based

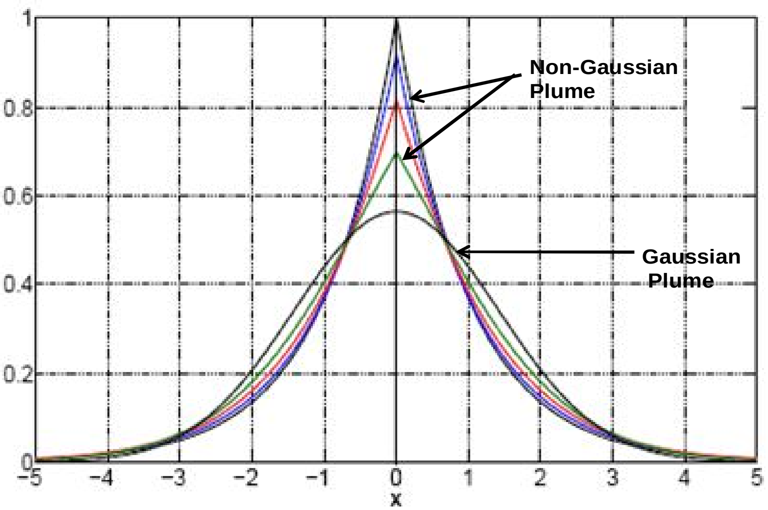
S.No.	Function of Memory Kernel	Type	Memory Kernel Function $h(t)$	Constitutive Equation of $x^{(1)}(t) = h(t) * x(t)$	Relaxation of $x(t)$ from initial value $x(0)$
1	Delta Function	Singular	$C\delta(t)$	$x^{(1)}(t) = Cx(t)$ $x(t) - x(0) = C\left({}_0I_t^1 x(t)\right)$	$x(0)e^{Ct}$
2	Power Law	Singular	$Ct^{-\alpha}$ $1 < \alpha \le 2$	${}_0^C D_t^{\upsilon} x(t) = C_\alpha x(t)$ $x(t) - x(0) = C_\alpha \left({}_0I_t^{\upsilon} x(t)\right)$ $C_\alpha = C\left(\Gamma(1-\alpha)\right), \quad \upsilon = 2-\alpha$	$x(0)E_\upsilon\left(C_\alpha t^\upsilon\right)$
3	Non-singular Power Law	Non-Singular	$C(1+\upsilon t)^{-\alpha}$	$x(t) - x(0) = C\sum_{n=0}^{\infty} a_n \left({}_0I_t^{n+2}\, x(t)\right)$ $a_0 = 1, \quad a_1 = -\alpha\upsilon$ $a_2 = \alpha(\alpha+1)\upsilon^2 \ldots$	X
4	Mittag-Leffler	Non-Singular	$CE_\alpha(-\lambda t^\alpha)$	$x(t) - x(0) = C\sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+2}\left[x(t)\right]\right)$ $b_n = (-1)^n \lambda^n$	$x(0)\mathcal{L}^{-1}\left\{\frac{s^\alpha+\lambda}{s^{\alpha+1}+\lambda s - Cs^{\alpha-1}}\right\}$
5	Exponential	Non-Singular	$Ce^{-\kappa t}$	$x(t) - x(0) = C\sum_{n=0}^{\infty} c_n \left({}_0I_t^{n+2}\left[x(t)\right]\right)$ $c_n = (-1)^n \kappa^n$	$x(0)\mathcal{L}^{-1}\left\{\frac{s+\kappa}{s^2+\kappa s - C}\right\}$
6	Stretched-Exponential	Non-Singular	$Ce^{-(\kappa t)^\alpha}$	$x(t) - x(0) = C\sum_{n=0}^{\infty} d_n \left({}_0I_t^{\alpha n+2}\left[x(t)\right]\right)$ $d_n = (-1)^n \left(\frac{\kappa^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)$	X

Derivation is given in detailed notes

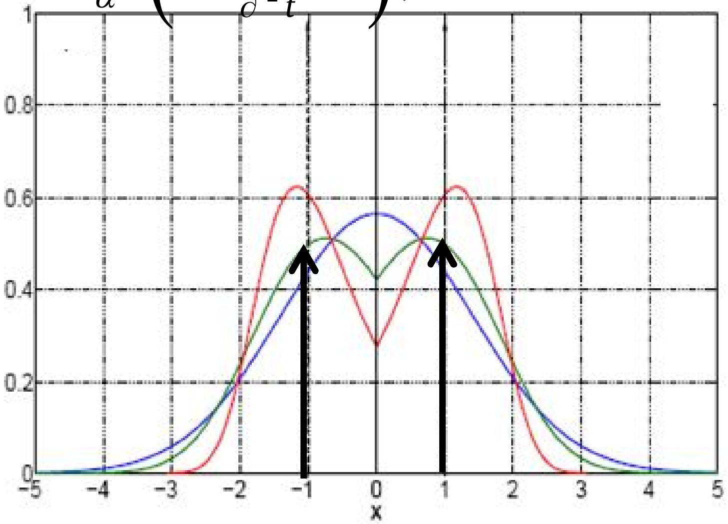
Diffusion/wave equation with and without memory

Time Fractional Diffuso-Wave Equation (TFDWE)

$$ {}^{C}D_t^{2\upsilon} c(x,t) = C_\alpha \left(\frac{\partial^2 c(x,t)}{\partial^2 t} \right); \quad 0 < \upsilon \le 1 $$



(a)



(b)

a. The Gaussian Plume and Non-Gaussian Plume for classical memory-less diffusion vis-à-vis diffusion with memory

b. Pure traveling waves in a memory-less wave equations vis-à-vis diffused travelling waves for memory based wave equation

This TFDWE is for singular power law memory kernel

"http://arxiv:org/pdf/1004.2950.pdf M Wright function in fractional diffusion process: a tutorial survey; F Mainrandi, A. Mura, G Pagnini

Comments

Observations say that dead matter does have memory

In those memory based responses for a process or system, fractional calculus is useful to describe constituent laws

However the memory with singular power law decay as kernel, gives the constituent laws in close conjugation to classical (memory less) laws: where the integer order derivative (or integral) gets replaced by fractional counterpart!

To have singularity in the natural dynamics makes us uncomfortable-presently as it is difficult to visualize singularity

Mathematically possible to have non-singular memory kernel, & using those we get the laws which are having infinite sum of integration operations

This is interesting research work to establish natural phenomena based on non-singular memory kernel, and to have physical interpretability-for say impedance, equivalent circuit representation etc.

So we are in dark if the nature follows singular or non-singular memory?

However scientists/engineers are working on this non-singular systems as recently

Please refer detailed notes given on this presentation-for detailed discussions derivations and references.

Acknowledgement

Thanking Prof. Sujata Tarafdar and Prof Ashish Nandy for giving me opportunity to speak on a topic which leaves more queries and confusion! However, topic is a developing one-as we cannot deny that dead matter is memory-less !

I acknowledge several students of Physics here in Jadavpur Univ. and at other Institutes, to have taken up this subject of Fractional Calculus-in their research work.

End of Part-1