Fractional Calculus in Left Handed Maxwell System

Shantanu Das

Reactor Control Division, B.A.R.C
Mumbai-400085
Email: shantanu@barc.gov.in

Dedicating this paper to Prof. Victor Vaselago and Prof. Michelle Caputo

ABSTRACT: This paper gives a new insight to fractional solutions that exists in electromagnetic wave propagation in doubly positive as well as doubly negative media. Here the cross product operation is fractionized by new geometrical method and then derived in a simple way the expressions for fractional curl, requiring fractional derivative. This gives a simple understanding of the process, and is very helpful in various practical cases of electromagnetic wave propagation in doubly positive and doubly negative media, as demonstrated. The formula derived in this paper are useful for several applications and applied in various vector fields with this new geometrical explanation are thus helpful in understanding the utility of fractional cross product and fractional curl in practical cases. This developed methodology is new.

Keywords: Cross-product, curl, fractional derivative, operator algebra, fractionizing, Left Handed Maxwell System, Left Handed Material (LHM), Doubly Positive System (DPS), Doubly Negative Material (DNG), backward wave

1. INTRODUCTION

It was Prof. Victor. G. Vaselago, in 1967, who theoretically studied the propagation of Electromagnetic waves with medium with both negative values $\varepsilon < 0$ and $\mu < 0$, called Doubly Negative Medium (DNG); opened great possibilities of, having negative refractive index materials with reversal of Snell's law, Cherenkov radiation etc. In the year 1967 Prof. Michelle Caputo gave a definition of fractional derivative, which has practical relevance, as this definition requires integer order initial states to solve fractional differential equations; which otherwise requires fractional order initial-states. We see in this paper how fractional derivative is used to form fractional order curl and apply them for electromagnetic wave propagation in DNG material. This paper is dedicated to both of the pioneers Prof. Vaselago and Prof. Caputo. The derivation via geometric technique, is new and gives insight to the physics of various aspects of electromagnetic wave propagation; especially polarization of waves. This new developed formulas are applied in Left Handed Maxwell Systems (LHM) and interesting physical interpretation is drawn. This new developed technique will be helpful in describing 'Faraday Rotator' made out of meta-material, and various other experimental devises on LHM; for future science.

We have demonstrated practically the negative refraction by artificially creating DNG prisms in Ka-Band and X-Band, via novel structural inclusions and designs. Most interesting feature of these DNG probably is that of Poynting vector's direction of time harmonic monochromatic plane wave is anti-parallel with the flow (direction) of phases. We have deliberated this in detail in other papers as to how the backward wave in DNG has risen to fame, giving a left handed triad with $\bar{\mathbf{E}}$, $\bar{\mathbf{H}}$ and $\bar{\mathrm{k}}$ vectors as compared to Doubly Positive Material (DPS) medium. The fact that we need to have in DNG system $\angle(\bar{\mathbf{E}}\times\bar{\mathbf{H}}) = \angle(-\bar{\mathrm{k}})$; as compared to DPS where $\angle(\bar{\mathbf{E}}\times\bar{\mathbf{H}}) = \angle(\bar{\mathrm{k}})$, makes us to take cross product with left hand, in DNG-opens up new possibilities in electromagnetic theory. The fractional curl arriving out of generalization of calculus namely 'fractional' calculus, has found applications in especially electromagnetic theory; where the solution to the Maxwell's equation and its dual is mapped in between-the two solutions. Namely if, the electric field and magnetic field $(\bar{\mathbf{E}},\eta\bar{\mathbf{H}})$ vectors are the solution to the Maxwell equations the dual solutions are $(\eta\bar{\mathbf{H}},-\bar{\mathbf{E}})$ vectors. The symbol η denotes impedance of the medium. The; $\mathrm{curl}(\bar{\mathbf{E}}) \propto \eta\bar{\mathbf{H}}$ and $\mathrm{curl}(\eta\bar{\mathbf{H}}) \propto -\bar{\mathbf{E}}$. If we note here, the original solution $(\mathbf{E},\eta\overline{\mathbf{H}})$ and its dual solution $(\eta\overline{\mathbf{H}},-\overline{\mathbf{E}})$ are in fact; one is rotated by 90^0 to get the other. This opens up a thought that what is in between the right angular rotation; the answer is that it is fractionally rotated field vectors by angle $\alpha(90^0)$ $0\le\alpha\le 90^0$; given by fractional curl that is $\mathrm{curl}^{\alpha}\overline{\mathbf{E}}$ or $\mathrm{curl}^{\alpha}(\eta\overline{\mathbf{H}})$; which are fractional fields or fractional dual solutions. Also what happens if the material is DNG and how this duality and fractional fields in the material where both $\varepsilon<0$, and $\mu<0$; giving backward wave propagation, is investigated here.

In this paper we arrive at fractionizing of cross product operation by means of geometrical arguments, and then we derive expression for fractional curl. This approach is new and gives a deeper insight of fractioning steps and pictorially explains the derivation; not dealt earlier. The section 2 gives idea of dual solutions to the Maxwell's equation as wave propagation in a wave guide (original) with perfect electric conductor (PEC) as boundaries and its dual waveguide giving dual solution where the boundaries are perfect magnetic conductor (PMC). Section-3 deals with preliminaries of fractional calculus, the Laplace and Fourier transform of fractional differ-integration, with examples of fractional differ-integrals of some useful functions relevant to this paper. The section 4 deals with geometrical derivation of fractionizing process of derivation of cross-product for orthogonal unit vectors. This section gives expressions for fractional cross products of different combinations of unit vectors. The section-5 utilizes all these and derives the general expression for obtaining cross product for any two vectors in $\mathbb{R}^3$ space; via geometrical derivation and this section, also deals with obtaining the fractional curl via fractionized cross product in $\overline{\mathrm{k}}$ space in $\mathbb{R}^3$; where $\overline{\mathrm{k}}$ space that is $(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}})$ is Fourier transformed for spatial coordinate $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ space in $\mathbb{R}^3$. Section-6 gives the operational algebra method to obtain fractionizing of linear operator (in this case cross product); via eigenvalue and eigenvector of operator. This also verifies the geometrically derived fractional cross product formulae. Section-7 deals with few applications of getting fractional curl for given vector fields of electromagnetic waves, with its physical interpretation. Section-8/9 deals with application of the obtained method of to find fractional dual fields in Left Handed Maxwell System and comparison to DPS media; with physical interpretation of obtained result. In this paper we use fractional curl symbol as $(\nabla\times)^{\alpha}\overline{\mathbf{A}}$ or sometimes as ($\mathrm{curl}^{\alpha}\{\overline{\mathbf{A}}\}$). Our discussion for geometrical derivations and applications remain for α as real, but one example we have shown the extension of the derived formula for complex order curl that is ($\mathrm{curl}^{\alpha+\mathrm{i}\beta}\{\overline{\mathbf{A}}\}$); and have given the physical interpretation for right and left circularly polarized electromagnetic wave in DPS and LHM system. In this paper the time harmonic part $e^{\mathrm{i}\omega t}$ is implicitly assumed. The detailed derivations for each case has made the length of the paper large, unfortunately these details are required as one does not get elsewhere.

2. DUALITY PRINCIPLE IN ELECTROMAGNETIC THEORY

The sources free Maxwell equations with $k=\omega\sqrt{\mu\varepsilon}$; defining wave-vector (wave-number) and the $\eta=\sqrt{\varepsilon/\mu}=|\overline{\mathbf{E}}|/|\overline{\mathbf{H}}|$ defining the intrinsic impedance of medium; for DPS with $\varepsilon>0$ and $\mu>0$ are

$$\mathrm{curl}(\eta\overline{\mathbf{H}})=-(\mathrm{i}k)\overline{\mathbf{E}}\qquad \mathrm{curl}(\overline{\mathbf{E}})=(\mathrm{i}k)\eta\overline{\mathbf{H}}\qquad \mathrm{div.}(\eta\overline{\mathbf{H}})=0\qquad \mathrm{div.}(\overline{\mathbf{E}})=0 \tag{1}$$

Also we can write for DPS, a time harmonic source free Maxwell's equation as that is same as (1)

$$\frac{1}{(\mathrm{i}k)}\Big[(\mathrm{i}\overline{\mathrm{k}}\times)(\eta\overline{\mathbf{H}})\Big]=-\overline{\mathbf{E}}\qquad \frac{1}{(\mathrm{i}k)}\Big[(\mathrm{i}\overline{\mathrm{k}}\times)(\overline{\mathbf{E}})\Big]=\eta\overline{\mathbf{H}}\qquad (\mathrm{i}\overline{\mathrm{k}})\bullet(\eta\overline{\mathbf{H}})=0\qquad (\mathrm{i}\overline{\mathrm{k}})\bullet(\overline{\mathbf{E}})=0 \tag{2}$$

Duality means if $(\overline{\mathbf{E}},\eta\overline{\mathbf{H}})$ is one set of solution to (1) (2), then other set $(\eta\overline{\mathbf{H}},-\overline{\mathbf{E}})$ is its dual solution. The concept of original and dual solutions is depicted in figure-1. The figure-1 depicts Electromagnetic wave travelling inside two plates as TM mode, in a multiple reflection by two-boundary Perfectly Electric Conductors (PEC), placed in $x-z$ planes. This is wave guide type of propagation; and with these two PEC as boundaries be changed to Perfectly Magnetic Conductors (PMC) we get TE mode of propagation inside the guided space, which is dual solution of electric and magnetic vectors. Here we note that $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ vectors of dual solutions (TE) gets rotated about $\overline{\mathrm{k}}$ axis anticlockwise by 90^0. So we can write dual solution as

$$\bar{\mathbf{E}}_{\text{dual}} = \frac{1}{(ik)}\left[(i\bar{k}\times)\bar{\mathbf{E}}\right] \qquad (\eta\bar{\mathbf{H}}_{\text{dual}}) = \frac{1}{(ik)}\left[(i\bar{k}\times)(\eta\bar{\mathbf{H}})\right]$$

$$\bar{\mathbf{E}}_{\text{TE}} = \frac{1}{(ik)}\left[(i\bar{k}\times)\bar{\mathbf{E}}_{\mathbf{TM}}\right] \qquad (\eta\bar{\mathbf{H}}_{\text{TE}}) = \frac{1}{(ik)}\left[(i\bar{k}\times)(\eta\bar{\mathbf{H}}_{\mathbf{TM}})\right] \tag{3}$$

$$\bar{\mathbf{E}}_{\text{dual}} = \frac{1}{(ik)}\left[(\nabla\times)\bar{\mathbf{E}}\right] \qquad (\eta\bar{\mathbf{H}}_{\text{dual}}) = \frac{1}{(ik)}\left[(\nabla\times)(\eta\bar{\mathbf{H}})\right]$$

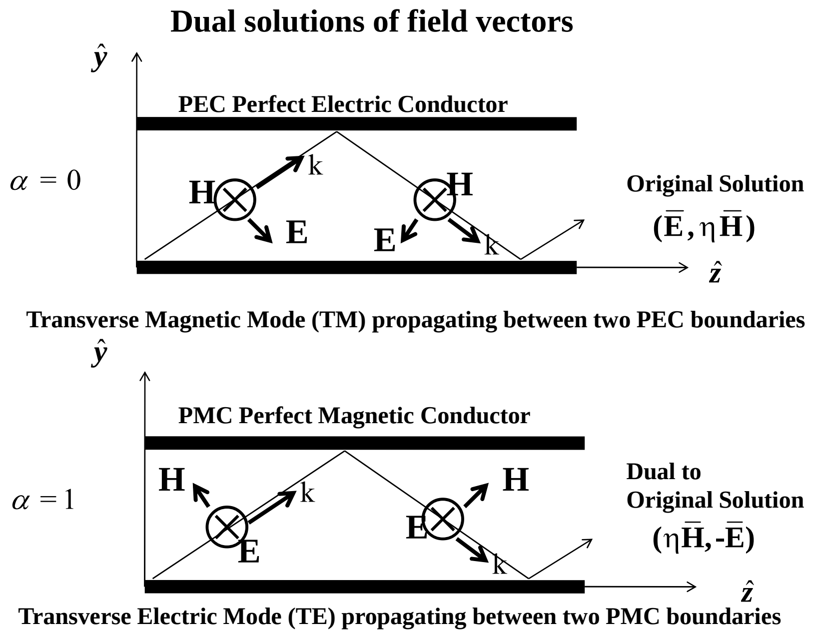


Figure-1: Original solution and dual to original solution

Therefore, taking cross-product of $\bar{k}$ with original $\bar{\mathbf{E}}$ of TM mode gives us direction of $\bar{\mathbf{E}}$ for dual solution TE mode. Similarly taking cross of $\bar{k}$ with original $\bar{\mathbf{H}}$ of TM mode gives us direction of $\bar{\mathbf{H}}$ for dual solution TE mode. Therefore, we may ask a question; is it possible to have in-between original TE and dual TM mode of propagation in this wave guide. Mathematically why restrict the rotation to 90^0; definitely can have rotation in between this that is ($\alpha 90^0$); to have in-between TE and TM modes. Will it always possible to have PEC and PMC as boundaries; what happens if the boundary is mix of PEC and PMC. This opens up a possibility of defining fractional fields that is in between original solutions and its dual solutions. Then expression (4) therefore gives the field vectors of in between original and dual solution.

$$\bar{\mathbf{E}}_f = (ik)^{-\alpha}\left[(\nabla\times)^{\alpha}\bar{\mathbf{E}}\right] \qquad (\eta\bar{\mathbf{H}}_f) = (ik)^{-\alpha}\left[(\nabla\times)^{\alpha}(\eta\bar{\mathbf{H}})\right]$$

$$\bar{\mathbf{E}}_f = (ik)^{-\alpha}\left[(i\bar{k}\times)^{\alpha}\bar{\mathbf{E}}\right] \qquad (\eta\bar{\mathbf{H}}_f) = (ik)^{-\alpha}\left[(i\bar{k}\times)^{\alpha}(\eta\bar{\mathbf{H}})\right] \tag{4}$$

Defining fractional field as explained above, we get in electromagnetic theory, with $(\bar{\mathbf{E}},\eta\bar{\mathbf{H}})$ as electric field and magnetic fields as original solution to the Maxwell equations; then we get fractional fields as in (4) The fractional parameter α, when equals zero, $\alpha = 0$; we obtain, $\bar{\mathbf{E}}_f = \bar{\mathbf{E}}$ and $\bar{\mathbf{H}}_f = \bar{\mathbf{H}}$; the original field. For $\alpha = 1$ in (4) and with (1) (2) we have $\bar{\mathbf{E}}_f = \eta\bar{\mathbf{H}}$ and $\bar{\mathbf{H}}_f = -\bar{\mathbf{E}}$; which is dual solution to (1) and (2).

If we make sign of $\bar{k}$ in (1) (2) as negative, indicating the phase flow is opposite; we give equations for DNG; thus Maxwell equations for source free region of DNG material we write as (5)

$$\frac{1}{(\mathrm{i}k)}\left[(\mathrm{i}\bar{\mathrm{k}}\times)(\eta\bar{\mathbf{H}})\right]=\bar{\mathbf{E}} \qquad \frac{1}{(\mathrm{i}k)}\left[(\mathrm{i}\bar{\mathrm{k}}\times)(\bar{\mathbf{E}})\right]=-\eta\bar{\mathbf{H}} \qquad (\mathrm{i}\bar{\mathrm{k}})\bullet(\eta\bar{\mathbf{H}})=0 \qquad (\mathrm{i}\bar{\mathrm{k}})\bullet(\bar{\mathbf{E}})=0 \quad (5)$$

Note k does not have sign, as this is magnitude of $\bar{\mathrm{k}}$. The expression (4) gives fractional fields in DNG too we assume here. The LHM has one set as $(\bar{\mathbf{E}},\eta\bar{\mathbf{H}})$ and its dual solution as $(-\eta\bar{\mathbf{H}},\bar{\mathbf{E}})$ .Our aim thus is to get what is fractional cross product and fractional derivative.

3. FRACTIONAL CALCULUS PRELIMINARIES

The method of getting fractional curl requires some preliminaries, of fractional calculus, since it has operator ∇^{α} ; which manifests as fractional derivative $\partial^{\alpha}/\partial x^{\alpha}\equiv {}_{-\infty}D_x^{\alpha}$, requires fractional derivative operation. The fractional calculus is generalization of integer order calculus, and is three centuries old. The readers may refer books listed in reference to get feel of the subject. The fractional integration is defined as (generalization of $n-$ th order repeated integration when n is not integer) that is

$${}_aD_x^{\beta}f(x)=\frac{1}{\Gamma(-\beta)}\int_a^x(x-u)^{-\beta-1}f(u)\mathrm{d}u \qquad \beta<0 \qquad x>a \qquad (6)$$

The Gamma function in (6) generalizes the factorial which appears in formula of repeated integer order integration. For fractional differentiation $\alpha>0$, we use Riemann-Liouvelli formulation by choosing an integer m such that $(\alpha-m)$ becomes just negative and we can apply the fractional integration (6) that is first obtain ${}_aD_x^{\alpha-m}f(x)$, then differentiate, repeatedly $\mathrm{d}^m/\mathrm{d}x^m$, to get

$${}_0D_x^{\alpha}f(x)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\,{}_aD_x^{\alpha}f(x) \qquad \alpha>0 \qquad (7)$$

The Caputo fractional derivative is just opposite process of (7), where we first take integer order m -th derivative of the function and then followed up by fractional integration α -folds. When there are no initial states, or the initial states are at rest, both the definitions of fractional derivatives are same.

Various fractional derivatives (rather fractional differ-integrations are provided in the listed references). The fractional derivative of sinusoidal functions, with lower terminal as $-\infty$ is found to be ${}_{-\infty}D_t^{\alpha}\sin\lambda t=\lambda^{\alpha}\sin\left(\lambda t+(\alpha\pi/2)\right)$, ${}_{-\infty}D_t^{\alpha}\cos\lambda t=\lambda^{\alpha}\cos\left(\lambda t+(\alpha\pi/2)\right)$. The other relevant functions with lower terminal $-\infty$, has fractional derivatives as ${}_{-\infty}D_t^{\alpha}e^{\lambda t}=\lambda^{\alpha}e^{\lambda t}$, ${}_{-\infty}D_t^{\alpha}e^{\lambda t+\mu}=\lambda^{\alpha}e^{\lambda t+\mu}$; which we shall be using in this paper. Whereas, when the lower terminal of fractional derivative is other than $-\infty$, then the functional form alters, and we get the expressions in form of higher transcendental functions; for order of differ-integration α,v non-integer. Like ${}_0D_x^{v}e^{ax}=E_x(-v,a)$; ${}_0D_x^{v}\cos ax=C_x(-v,a)$; ${}_0D_x^{v}\sin ax=S_x(-v,a)$; where $S_x(-v,a)$, $C_x(-v,a)$ and $E_x(-v,a)$ are higher sine, higher cosine and Miller-Ross functions; (the Higher Transcendental functions). Only for case of steady state that is fractional derivative starting from $-\infty$, we get same functional form as we get in integer order case, for these functions. In our paper we are dealing with steady state case.

The Laplace Transform of fractional derivative-integral operation is

$$\mathcal{L}\left\{{}_0D_x^{\alpha}f(x)\right\}=s^{\alpha}\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\left[{}_0D_x^{\alpha-1-k}f(x)\right]_{\text{at}\quad x=0} \qquad (8)$$

Where Laplace Transform defined as

$$\mathcal{L}\left\{f(x)\right\}\overset{\text{def}}{=}\int_0^{\infty}\mathrm{d}x\{e^{-sx}f(x)\} \qquad (9)$$

In (8) the order of differ-integration $\alpha\in\mathbb{R}$; and the integer $n\in\mathbb{Z}$ such that $(n-1)<\alpha\le n$. In expression (8) when $\alpha<0$, that is operation is fractional integration, the term involving summation becomes zero for any function, $f(x)$ with available Laplace Transform. Also one can have similar to Laplace Transform of fractional differ-integrals of $f(x)$; a Fourier Transform of fractional differ-integral operation. A function $f(x)$, which is "well-behaved" at $x=-\infty$, we can have

$$\mathcal{F}\left\{ {}_{-\infty}D_x^{\alpha} f(x)\right\} = (\mathrm{i}\omega)^{\alpha}\mathcal{F}\left\{ f(x)\right\} \tag{10}$$

And therefore we have fractional derivative/integral operation as inverse Fourier transformed one

$${}_{-\infty}D_x^{\alpha} f(x) = \mathcal{F}^{-1}\left\{(\mathrm{i}\omega)^{\alpha}\mathcal{F}\{ f(x)\}\right\} \tag{11}$$

Where the Fourier and Inverse Fourier Transform is depicted as following

$$\mathcal{F}\left\{ f(x)\right\} = F(\omega) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \mathrm{d}x\{e^{\mathrm{i}\omega x} f(x)\} \qquad f(x) = \mathcal{F}^{-1}\left\{F(\omega)\right\} \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega\{e^{-\mathrm{i}\omega x} F(\omega)\} \tag{12}$$

From (8) onwards we are using fractional differ-integration operation. In some cases (especially for steady state systems with lower terminal of differ-integration $a=-\infty$) the Fourier Transformation method is another way to find fractional derivative/fractional integration of function $f(x)$. That is

(i) Obtain the Fourier Transform of $f(x)$ as $F(\omega)$.

(ii) Then this transformed $F(\omega)$ in frequency ω domain we multiply by $(\mathrm{i}\omega)^{\alpha}$, where $\alpha \in \mathbb{R}$.

(iii) The resulting function $(\mathrm{i}\omega)^{\alpha} F(\omega)$ we inverse Fourier transform, to get ${}_{-\infty}D_x^{\alpha} f(x)$.

With above preliminaries we proceed now how to get fractional curl. For a vector $\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}}) \equiv (f_x, f_y, f_z) = f_x\hat{\boldsymbol{x}} + f_y\hat{\boldsymbol{y}} + f_z\hat{\boldsymbol{z}}$ in spatial coordinate $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$, the curl of $\overline{\mathbf{F}}$ or $(\nabla\times)\overline{\mathbf{F}}$ can be spatially Fourier Transformed into $\overline{\mathrm{k}}$ domain, with coordinates as $(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}})$ where $\overline{\mathrm{k}} \equiv (k_x,k_y,k_z) = k_x\hat{\boldsymbol{x}}_{\boldsymbol{k}} + k_y\hat{\boldsymbol{y}}_{\boldsymbol{k}} + k_z\hat{\boldsymbol{z}}_{\boldsymbol{k}}$. In the $\overline{\mathrm{k}}$ domain, the curl operator can be expressed as cross product of vector $(\mathrm{i}\overline{\mathrm{k}})$ with vector $\overline{\mathbf{F}}_k$; where $\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}}) = \mathcal{F}\left\{\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\}$, that is

$$\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}}) = \mathcal{F}\left\{\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\hat{\boldsymbol{x}}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{z}}\{\overline{\mathbf{F}}(\hat{x},\hat{y},\hat{z})e^{-\mathrm{i}k_x\hat{\boldsymbol{x}}-\mathrm{i}k_y\hat{\boldsymbol{y}}-\mathrm{i}k_z\hat{\boldsymbol{z}}}\} \tag{13}$$

The spatial Fourier Transform of $(\nabla\times)\overline{\mathbf{F}}$ in $\overline{\mathrm{k}}$ domain can be expressed as

$$\mathcal{F}\left\{\nabla\times\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\} = (\mathrm{i}\overline{\mathrm{k}})\times\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}}) \tag{14}$$

So in order to fractionalize the curl operation what we need is obtain fractional cross product $(\mathrm{i}\overline{\mathrm{k}}\times)^{\alpha}\overline{\mathbf{F}}_k$ in $\overline{\mathrm{k}}$ domain, then Fourier invert the result back to $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ domain to get $(\nabla\times)^{\alpha}\overline{\mathbf{F}}$.

4. GEOMETRICAL DERIVATION OF FRACTIONAL CROSS PRODUCT OF ORTHOGONAL UNIT VECTORS VIA ROTATION

The operation curl is an operation 'cross-product'. The cross product of two vectors $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ is simply given as $(\overline{\mathbf{A}}\times)\overline{\mathbf{B}} = \left|\overline{\mathbf{A}}\right|\left|\overline{\mathbf{B}}\right|\sin\theta$; where θ is the angle between the two vectors; and the cross-product is directed perpendicular to both $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$. Refer figure-2 where the concept is elaborated geometrically. The figure does not provide for magnitude, but direction is shown. One can geometrically interpret this cross product as rotation of the vector anticlockwise by an angle of 90 degree about an axis made by vector direction, which is $\left|\overline{\mathbf{A}}\right|\sin\theta$; which is orthogonal to projection of $\overline{\mathbf{A}}$ on $\overline{\mathbf{B}}$. The projection of $\overline{\mathbf{A}}$ on $\overline{\mathbf{B}}$ is $\left|\overline{\mathbf{A}}\right|\cos\theta$. If these are orthogonal vectors, then rotation of $\overline{\mathbf{B}}$ is by 90 degree about the axis $\overline{\mathbf{A}}$. The question is; what if we fractionally rotate the vector $\overline{\mathbf{B}}$ anticlockwise about the same axis, by angle $\alpha(90^0)$ will we get fractional cross-product. The paper deals with the geometrical derivation of the fractional rotation to arrive at expressions for fractionizing the cross product operation. First we take simple case of orthogonal unit vectors $(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}})$ and obtain step by step the fractional cross products $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\hat{\boldsymbol{x}}_{\boldsymbol{k}}$, $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\hat{\boldsymbol{y}}_{\boldsymbol{k}}$ and $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\hat{\boldsymbol{z}}_{\boldsymbol{k}}$; where $0\le\alpha\le 1$, a real number. The number

α though restricted here is only for explanation in first quadrant, however can have any value. A value greater than four will be repeating the rotation, as will be clear subsequently. Generally α can also be a complex number, we will see its application later.

Cross-Product of Two Vectors

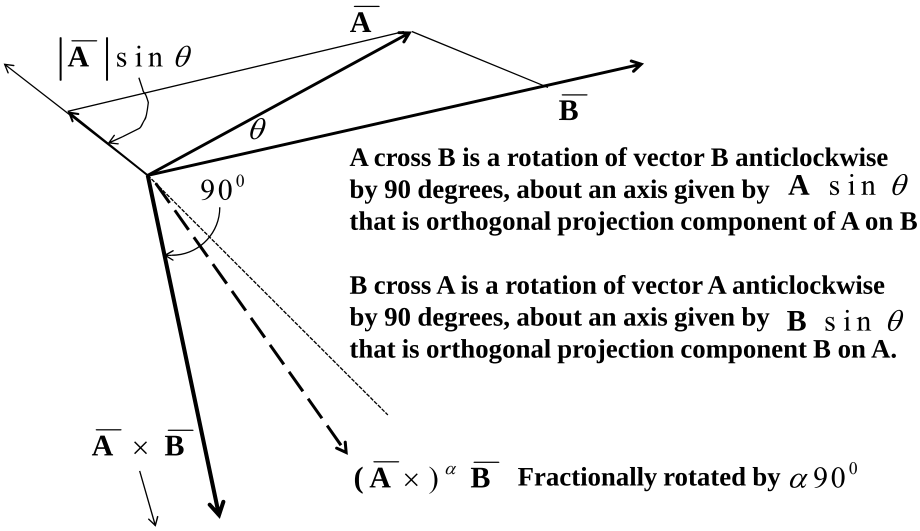


Figure-2: Cross Product as rotation of one vector by 90 degree

The $|\hat{\boldsymbol{x}}_k| = |\hat{\boldsymbol{y}}_k| = |\hat{\boldsymbol{z}}_k| = 1$; as they are unit vectors and are orthogonal to each other. Following relations are obvious

$$(\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{y}}_k \qquad (\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{y}}_k = -\hat{\boldsymbol{x}}_k \qquad (\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{z}}_k = 0 \tag{15}$$

Let us take the first one $(\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{y}}_k$; the right hand's fingers pointing towards unit vector $\hat{\boldsymbol{z}}_k$; when bent towards $\hat{\boldsymbol{x}}_k$; the thumb points towards $\hat{\boldsymbol{y}}_k$; thus we get this result of cross product. If we rotate, 'clockwise', our thumb after obtaining cross product $(\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{x}}_k$ in the $x_k - y_k$ plane, by an angle $(1-\alpha)(\pi/2)$ the thumb will trace a unit circular arc; and that unit vector after this-rotation will have projections in $\hat{\boldsymbol{x}}_k$ axis and $\hat{\boldsymbol{y}}_k$ axis; as $[\cos(\alpha\pi/2)]\hat{\boldsymbol{x}}_k + [\sin(\alpha\pi/2)]\hat{\boldsymbol{y}}_k$. If we write the fractional cross product as

$$(\hat{\boldsymbol{z}}_k\times)^{\alpha}\,\hat{\boldsymbol{x}}_k = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_k \tag{16}$$

Will it be generally true? To verify the above explanation via rotation, put $\alpha = 1$, in above expression we get $(\hat{\boldsymbol{z}}_k\times)^1\,\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{y}}_k$; while putting $\alpha = 0$ we get $(\hat{\boldsymbol{z}}_k\times)^0\,\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k$. Thus we have recovered normal (integer-order) cross product, and identity operation. So far this rotation method has worked. Now let us try for doing the same logic of rotation for getting $(\hat{\boldsymbol{z}}_k\times)^{\alpha}\,\hat{\boldsymbol{y}}_k$. The normal cross product is $(\hat{\boldsymbol{z}}_k\times)\hat{\boldsymbol{y}}_k = -\hat{\boldsymbol{x}}_k$; verify this via right-hand rule, and we see the thumb points towards $-\hat{\boldsymbol{x}}_k$ axis; after we first point the right hand's finger towards $\hat{\boldsymbol{z}}_k$ axis and then fold the fingers towards $\hat{\boldsymbol{y}}_k$ axis. Now again if we rotate the thumb pointing towards $-\hat{\boldsymbol{x}}_k$ axis 'clockwise' by an angle $(1-\alpha)(\pi/2)$ in $x_k - y_k$ plane; we write the projection of this as fractional cross product

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}} \tag{17}$$

The verification of this is similarly done by putting $\alpha=1$ in (17) to get $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{1}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}=-\hat{\boldsymbol{x}}_{\boldsymbol{k}}$ and identity operation $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{0}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\hat{\boldsymbol{y}}_{\boldsymbol{k}}$. Obviously the following is also true regarding self fractional cross product that is

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{z}}_{\boldsymbol{k}}=0 \tag{18}$$

The argument of geometrically rotating the 'thumb' clockwise and then getting the projections on the principal axis can also be stated that the vector on which we are taking cross-product operation is rotated 'anticlockwise' by an angle $\alpha\pi/2$ and then the projection of this rotation to the principal axis is the fractional cross product! This statement is verified as for (16) rotate the $\hat{\boldsymbol{x}}_{\boldsymbol{k}}$ by $\alpha\pi/2$ anticlockwise; about the axis $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ and take this vector's component orthogonal to $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$-axis's plane that is x_k-y_k plane, to get $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{x}}_{\boldsymbol{k}}$. This is depicted in figure-3. The same rotation via anticlockwise angle $\alpha\pi/2$ of $\hat{\boldsymbol{y}}_{\boldsymbol{k}}$ in plane orthogonal to $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ (and also we can say about $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$) gives fractional cross product $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}$; as in (17). If $\alpha=1$ the rotation is 90^{0} anticlockwise about the vector taking the cross product that is $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ in these cases. We can have same argument and write for fractional cross operation for $\hat{\boldsymbol{y}}_{\boldsymbol{k}}$ by vector $\hat{\boldsymbol{x}}_{\boldsymbol{k}}$; where the vector $\hat{\boldsymbol{y}}_{\boldsymbol{k}}$ rotates about axis $\hat{\boldsymbol{x}}_{\boldsymbol{k}}$ anticlockwise by an angle $\alpha\pi/2$, in y_k-z_k plane

$$(\hat{\boldsymbol{x}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

The other similar combinations follow for fractional cross products of orthogonal unit vectors are

$$(\hat{\boldsymbol{x}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{z}}_{\boldsymbol{k}}=\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

$$(\hat{\boldsymbol{y}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{z}}_{\boldsymbol{k}}=\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

$$(\hat{\boldsymbol{y}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{x}}_{\boldsymbol{k}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

5. FRACTIONAL CROSS PRODUCT OF ANY VECTOR PAIR

From the previous section we learnt that the fractional cross product is rotation by a fractional angle $\alpha\pi/2$ of the vector on which the cross product operation is carried on, and the rotation about the axis of the vector which is doing the cross-product operation. The rotation operation being a linear operation we can use superposition to get the general fractional cross product's expression. Let us take an example of a vector $\overline{\mathbf{A}}=(1)\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(1)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(1)\hat{\boldsymbol{z}}_{\boldsymbol{k}}=(1,1,1)$, if we want to take fractional cross product of this $\overline{\mathbf{A}}$, with unity vector $\hat{z}_k$, that is $(\hat{z}_k\times)^{\alpha}\,\overline{\mathbf{A}}$; we can use linearity of rotation operation to get from (16) and (17); by adding them the following

$$(\hat{z}_k\times)^{\alpha}\,\overline{\mathbf{A}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}} \tag{19}$$

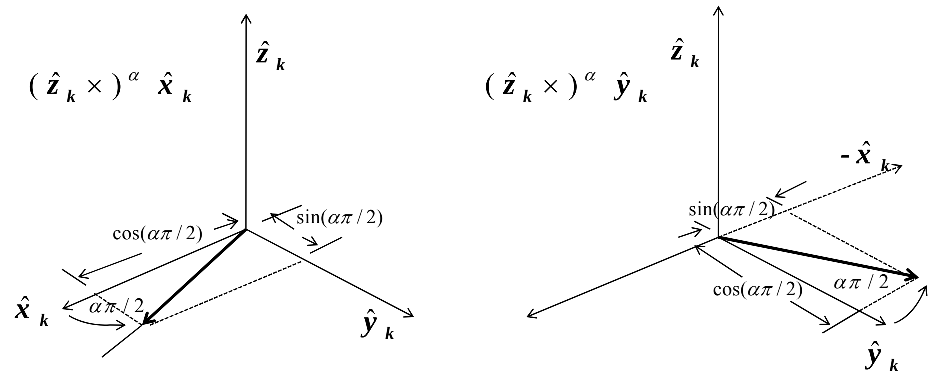
What is done is to have (19) we did superposition of $(\hat{z}_k\times)^{\alpha}\,\hat{x}_k$, $(\hat{z}_k\times)^{\alpha}\,\hat{y}_k$ and $(\hat{z}_k\times)^{\alpha}\,\hat{z}_k$ discussed in previous section; fractional rotation too is linear operation. Since $\overline{\mathbf{A}}$ has the unit vector components in three directions we derived this very easily, by adding the (16) (17) and (18). Rearranging (19) we get

$$(\hat{z}_k\times)^{\alpha}\,\overline{\mathbf{A}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)+\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}} \qquad (20)$$

Putting $\alpha=1$ we get from (6), $(\hat{z}_k\times)^1\,\overline{\mathbf{A}}=(-1,1,0)$; which is same as normal cross product , demonstrated in (21)

$$(\hat{z}_k\times)\overline{\mathbf{A}}=(-1)\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(1)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}}=(-1,1,0)=\det\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}} & \hat{\boldsymbol{y}}_{\boldsymbol{k}} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\\ 0 & 0 & 1\\ 1 & 1 & 1\end{bmatrix} \qquad (21)$$

Fractional cross product of orthogonal unit vectors



Anticlockwise rotation of $\hat{x}_k$ about $\hat{z}_k$

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{x}}_{\boldsymbol{k}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

Anticlockwise rotation of $\hat{y}_k$ about $\hat{z}_k$

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\,\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

Figure-3: Fractional cross product of unit vectors

Interestingly the expression (20) is not returning identity operator when we put $\alpha=0$, we are getting $(\hat{z}_k\times)^0\,\overline{\mathbf{A}}=(1,1,0)$; should have given $\overline{\mathbf{A}}=(1,1,1)$. Therefore (20) is only valid if $\alpha\neq 0$. To modify (20) we use δ_α a function valuing one for $\alpha=0$ zero otherwise for $\alpha\neq 0$; and write (20) as

$$(\hat{z}_k\times)^{\alpha}\,\overline{\mathbf{A}}=\left[\cos\left(\frac{\alpha\pi}{2}\right)-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)+\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(\delta_\alpha)\hat{\boldsymbol{z}}_{\boldsymbol{k}} \qquad (22)$$

We can also reframe (20) as matrix representation for $\alpha\neq 0$; as follows;

$$(\hat{z}_k\times)^{\alpha}\,\overline{\mathbf{A}}=\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0]\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad (23)$$

If instead of $\overline{\mathbf{A}}=(1,1,1)$, we have $\overline{\mathbf{F}}_k=(F_{xk},F_{yk},F_{zk})=F_{xk}\hat{\boldsymbol{x}}_{\boldsymbol{k}}+F_{yk}\hat{\boldsymbol{y}}_{\boldsymbol{k}}+F_{zk}\hat{\boldsymbol{z}}_{\boldsymbol{k}}$, then with argument of linearity we can write fractional cross product for $\alpha\neq 0$ as follows

$$(\hat{z}_k\times)^\alpha\overline{\mathbf{F}}_k=\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0]\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}\tag{24}$$

Putting $\alpha=1$ in (24), we get the normal cross product $(\hat{z}_k\times)\overline{\mathbf{F}}_k$, and its verification via classical definition in (25)

$$\begin{aligned}(\hat{z}_k\times)\overline{\mathbf{F}}_k&=\begin{bmatrix}0 & \hat{\boldsymbol{x}}_{\boldsymbol{k}}(-1) & 0\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}(1) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}=(-F_{yk})\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(F_{xk})\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}}\\ (\hat{z}_k\times)\overline{\mathbf{F}}_k&=(\hat{z}_k\times)(F_{xk},F_{yk},F_{zk})=\det\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}} & \hat{\boldsymbol{y}}_{\boldsymbol{k}} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\\ 0 & 0 & 1\\ F_{xk} & F_{yk} & F_{zk}\end{bmatrix}=(-F_{yk})\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(F_{xk})\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}}\end{aligned}\tag{25}$$

We write for $(\hat{x}_k\times)^\alpha\overline{\mathbf{F}}_k,\alpha\neq 0$ with similar reasoning as we did for (24) (25)

$$(\hat{x}_k\times)^\alpha\overline{\mathbf{F}}_k=\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}\tag{26}$$

Putting $\alpha=1$, in (26) we get

$$\begin{aligned}(\hat{x}_k\times)\overline{\mathbf{F}}_k&=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \hat{\boldsymbol{y}}_{\boldsymbol{k}}(-1)\\ 0 & \hat{\boldsymbol{z}}_{\boldsymbol{k}}(1) & 0\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}=(0)\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(-F_{zk})\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(F_{yk})\hat{\boldsymbol{z}}_{\boldsymbol{k}}\\ (\hat{x}_k\times)\overline{\mathbf{F}}_k&=(\hat{x}_k\times)(F_{xk},F_{yk},F_{zk})=\det\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}} & \hat{\boldsymbol{y}}_{\boldsymbol{k}} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\\ 1 & 0 & 0\\ F_{xk} & F_{yk} & F_{zk}\end{bmatrix}=(0)\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(-F_{zk})\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(F_{yk})\hat{\boldsymbol{z}}_{\boldsymbol{k}}\end{aligned}\tag{27}$$

We similarly write for $(\hat{y}_k\times)^\alpha(\overline{\mathbf{F}}_k),\alpha\neq 0$ the following as done for above cases

$$(\hat{y}_k\times)^\alpha\overline{\mathbf{F}}_k=\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[\cos(\alpha\pi/2)\right] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[\sin(\alpha\pi/2)\right]\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0]\\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[-\sin(\alpha\pi/2)\right] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0] & \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[\cos(\alpha\pi/2)\right]\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}\tag{28}$$

Putting $\alpha=1$, in (28) we obtain

$$(\hat{y}_k\times)\overline{\mathbf{F}}_k=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\begin{bmatrix}F_{xk}\\F_{yk}\\F_{zk}\end{bmatrix}=(F_{zk})\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(-F_{xk})\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

$$(\hat{y}_k\times)\overline{\mathbf{F}}_k=(\hat{y}_k\times)(F_{xk},F_{yk},F_{zk})=\det\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}&\hat{\boldsymbol{y}}_{\boldsymbol{k}}&\hat{\boldsymbol{z}}_{\boldsymbol{k}}\\0&1&0\\F_{xk}&F_{yk}&F_{zk}\end{bmatrix}=(F_{zk})\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(-F_{xk})\hat{\boldsymbol{z}}_{\boldsymbol{k}} \tag{29}$$

From the above derivations using (24) (26) and (28) superposing them together we get fractional cross product of vector $(\hat{x}_k,\hat{y}_k,\hat{z}_k)$ with the vector $\overline{\mathbf{F}}_k=(F_{xk},F_{yk},F_{zk})$ that is for $\alpha\neq 0$, where $|\hat{x}_k|=|\hat{y}_k|=|\hat{z}_k|=1$ or $(\hat{x}_k,\hat{y}_k,\hat{z}_k)=(1)\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(1)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(1)\hat{\boldsymbol{z}}_{\boldsymbol{k}}$; for $\alpha\neq 0$

$$\begin{aligned}
&((x_k,y_k,z_k)\times)^{\alpha}(F_{xk},F_{yk},F_{zk})\\
&\quad=\hat{\boldsymbol{x}}_{\boldsymbol{k}}[0]+\hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[F_{yk}\cos\left(\frac{\alpha\pi}{2}\right)-F_{zk}\sin\left(\frac{\alpha\pi}{2}\right)\right]+\hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[F_{yk}\sin\left(\frac{\alpha\pi}{2}\right)+F_{zk}\cos\left(\frac{\alpha\pi}{2}\right)\right]\\
&\quad=\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[F_{xk}\cos\left(\frac{\alpha\pi}{2}\right)+F_{zk}\sin\left(\frac{\alpha\pi}{2}\right)\right]+\hat{\boldsymbol{y}}_{\boldsymbol{k}}[0]+\hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[-F_{xk}\sin\left(\frac{\alpha\pi}{2}\right)+F_{zk}\cos\left(\frac{\alpha\pi}{2}\right)\right]\\
&\quad=\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[F_{xk}\cos\left(\frac{\alpha\pi}{2}\right)-F_{yk}\sin\left(\frac{\alpha\pi}{2}\right)\right]+\hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[F_{xk}\sin\left(\frac{\alpha\pi}{2}\right)+F_{yk}\cos\left(\frac{\alpha\pi}{2}\right)\right]+\hat{\boldsymbol{z}}_{\boldsymbol{k}}[0]
\end{aligned} \tag{30}$$

Rearranging (30) we write compactly for $((x_k,y_k,z_k)\times)^{\alpha}(F_{xk},F_{yk},F_{zk})\,;\alpha\neq 0$

$$\begin{aligned}
&((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)^{\alpha}(F_{xk},F_{yk},F_{zk})\\
&\quad=\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[2F_{xk}\cos\left(\alpha\pi/2\right)+(F_{zk}-F_{yk})\sin\left(\alpha\pi/2\right)\right]\\
&\qquad+\hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[2F_{yk}\cos\left(\alpha\pi/2\right)+(F_{xk}-F_{zk})\sin\left(\alpha\pi/2\right)\right]\\
&\qquad+\hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[2F_{zk}\cos\left(\alpha\pi/2\right)+(F_{yk}-F_{xk})\sin\left(\alpha\pi/2\right)\right]
\end{aligned} \tag{31}$$

Putting $\alpha=1$ in (31) we get

$$\begin{aligned}
&((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)\overline{\mathbf{F}}_k=\hat{\boldsymbol{x}}_{\boldsymbol{k}}(F_{zk}-F_{yk})+\hat{\boldsymbol{y}}_{\boldsymbol{k}}(F_{xk}-F_{zk})+\hat{\boldsymbol{z}}_{\boldsymbol{k}}(F_{yk}-F_{xk})\\
&((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)(F_{xk},F_{yk},F_{zk})=\\
&\qquad\det\begin{bmatrix}\hat{x}_k&\hat{y}_k&\hat{z}_k\\1&1&1\\F_{xk}&F_{yk}&F_{zk}\end{bmatrix}=\hat{\boldsymbol{x}}_{\boldsymbol{k}}(F_{zk}-F_{yk})+\hat{\boldsymbol{y}}_{\boldsymbol{k}}(F_{xk}-F_{zk})+\hat{\boldsymbol{z}}_{\boldsymbol{k}}(F_{yk}-F_{xk})
\end{aligned} \tag{32}$$

Which is normal cross product operation

Let us have a vector field which has variation in only $\hat{\boldsymbol{z}}$ direction can be described as Fourier Transformed vector field as $\overline{\mathbf{F}}_k(z)=F_x\hat{\boldsymbol{x}}_{\boldsymbol{k}}+F_y\hat{\boldsymbol{y}}_{\boldsymbol{k}}+F_z\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ in $\overline{\mathrm{k}}$ domain. We have to find gradient in z -direction only, since the vector field has variation only in that direction. So in $\overline{\mathrm{k}}$ domain, we need to first take fractional cross product of the vector field with $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ axis, or find $(\boldsymbol{z}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k$, which we have already noted in (24); and we multiply by $(\mathrm{i}k)^{\alpha}$ to obtain the following

$$(\mathrm{i}k\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k=\begin{bmatrix}\hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[\cos\left(\alpha\pi/2\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[-\sin\left(\alpha\pi/2\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0](ik)^{\alpha}\\ \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\sin\left(\alpha\pi/2\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[\cos\left(\alpha\pi/2\right)\right](ik)^{\alpha} & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0](ik)^{\alpha}\\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha}\end{bmatrix}\begin{bmatrix}F_{xk}\\ F_{yk}\\ F_{zk}\end{bmatrix} \tag{33}$$

On Fourier inverting the (33) we will get the fractional curl for $\alpha\neq 0$

$$(\nabla_z\hat{\boldsymbol{z}}\times)^{\alpha}\overline{\mathbf{F}}=\begin{bmatrix}\hat{\boldsymbol{x}}\left[\cos\left(\alpha\pi/2\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{\boldsymbol{x}}\left[-\sin\left(\alpha\pi/2\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{\boldsymbol{x}}[0]\\ \hat{\boldsymbol{y}}\left[\sin\left(\alpha\pi/2\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{\boldsymbol{y}}\left[\cos\left(\alpha\pi/2\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{\boldsymbol{y}}[0]\\ \hat{\boldsymbol{z}}[0] & \hat{\boldsymbol{z}}[0] & \hat{\boldsymbol{z}}[0]\end{bmatrix}\begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix} \tag{34}$$

Putting $\alpha=1$, we get, and are same as normal curl as described below

$$\begin{gathered}(\nabla_z\hat{\boldsymbol{z}}\times)\overline{\mathbf{F}}=\begin{bmatrix}0 & \hat{\boldsymbol{x}}(-{}_{-\infty}D_z^{1}) & 0\\ \hat{\boldsymbol{y}}({}_{-\infty}D_z^{1}) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}=\hat{\boldsymbol{x}}(-{}_{-\infty}D_z^{1}f_y)+\hat{\boldsymbol{y}}({}_{-\infty}D_z^{1}f_x)\\ (\nabla_z\hat{\boldsymbol{z}}\times)\overline{\mathbf{F}}=\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ 0 & 0 & {}_{-\infty}D_z^{1}\\ f_x & f_y & f_z\end{bmatrix}=\hat{\boldsymbol{x}}(-{}_{-\infty}D_z^{1}f_y)+\hat{\boldsymbol{y}}({}_{-\infty}D_z^{1}f_x)\end{gathered} \tag{35}$$

We can rewrite from (33), for $\alpha\neq 0$

$$\begin{aligned}(\mathrm{i}k_z)^{\alpha}\overline{\mathbf{F}}_k&=(\mathrm{i}k_z)^{\alpha}\left[F_{xk}\cos\left(\alpha\pi/2\right)-F_{yk}\sin\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\\ &\quad(\mathrm{i}k_z)^{\alpha}\left[F_{xk}\sin\left(\alpha\pi/2\right)+F_{yk}\cos\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(\mathrm{i}k_z)^{\alpha}(0)F_{zk}\hat{\boldsymbol{z}}_{\boldsymbol{k}}\end{aligned} \tag{36}$$

Invert Fourier Transform of (36) will give

$$\begin{aligned}(\nabla_z\times)^{\alpha}\overline{\mathbf{F}}&=\hat{\boldsymbol{x}}\left[\cos\left(\alpha\pi/2\right)({}_{-\infty}D_z^{\alpha}f_x)-\sin\left(\alpha\pi/2\right)({}_{-\infty}D_z^{\alpha}f_y)\right]+\\ &\quad\hat{\boldsymbol{y}}\left[\sin\left(\alpha\pi/2\right)({}_{-\infty}D_z^{\alpha}f_x)+\cos\left(\alpha\pi/2\right)({}_{-\infty}D_z^{\alpha}f_y)\right]+\hat{\boldsymbol{z}}\,(0)({}_{-\infty}D_z^{\alpha}f_z)\end{aligned} \tag{37}$$

Similarly we write the other curls of fractional order as follows

$$\begin{aligned}(\mathrm{i}k_y)^{\alpha}\overline{\mathbf{F}}_k&=(\mathrm{i}k_y)^{\alpha}\left[F_{xk}\cos\left(\alpha\pi/2\right)+F_{zk}\sin\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+(\mathrm{i}k_y)^{\alpha}(0)F_{yk}\hat{\boldsymbol{y}}_{\boldsymbol{k}}\\ &\quad+(\mathrm{i}k_y)^{\alpha}\left[-F_{xk}\sin\left(\alpha\pi/2\right)+F_{zk}\cos\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}\end{aligned} \tag{38}$$

Invert Fourier Transform of (38) will give

$$\begin{aligned}(\nabla_y\times)^{\alpha}\overline{\mathbf{F}}&=\hat{\boldsymbol{x}}\left[\cos\left(\alpha\pi/2\right)({}_{-\infty}D_y^{\alpha}f_x)+\sin\left(\alpha\pi/2\right)({}_{-\infty}D_y^{\alpha}f_z)\right]+\hat{\boldsymbol{y}}(0)({}_{-\infty}D_y^{\alpha}f_y)\\ &\quad+\hat{\boldsymbol{z}}\left[-\sin\left(\alpha\pi/2\right)({}_{-\infty}D_y^{\alpha}f_x)+\cos\left(\alpha\pi/2\right)({}_{-\infty}D_y^{\alpha}f_z)\right]\end{aligned} \tag{39}$$

$$\begin{aligned}(\mathrm{i}k_x)^{\alpha}\overline{\mathbf{F}}_k&=\hat{\boldsymbol{x}}_{\boldsymbol{k}}(ik_x)^{\alpha}(0)F_{xk}+(\mathrm{i}k_x)^{\alpha}\left[F_{yk}\cos\left(\alpha\pi/2\right)-F_{zk}\sin\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}+\\ &\quad(\mathrm{i}k_x)^{\alpha}\left[-F_{yk}\sin\left(\alpha\pi/2\right)+F_{zk}\cos\left(\alpha\pi/2\right)\right]\hat{\boldsymbol{z}}_{\boldsymbol{k}}\end{aligned} \tag{40}$$

Invert Fourier Transform of (40) will give

$$\begin{aligned}(\nabla_x\times)^{\alpha}\overline{\mathbf{F}}&=\hat{\boldsymbol{x}}(0)({}_{-\infty}D_x^{\alpha}f_x)+\hat{\boldsymbol{y}}\left[\cos\left(\alpha\pi/2\right)({}_{-\infty}D_x^{\alpha}f_y)-\sin\left(\alpha\pi/2\right)({}_{-\infty}D_x^{\alpha}f_z)\right]+\\ &\quad\hat{\boldsymbol{z}}\left[\sin\left(\alpha\pi/2\right)({}_{-\infty}D_x^{\alpha}f_y)+\cos\left(\alpha\pi/2\right)({}_{-\infty}D_x^{\alpha}f_z)\right]\end{aligned} \tag{41}$$

Adding (37), (39) and (41) which are components of $(\nabla\times)^{\alpha}\overline{\mathbf{F}}$ where $\nabla\equiv\hat{\boldsymbol{x}}\nabla_x+\hat{\boldsymbol{y}}\nabla_y+\hat{\boldsymbol{z}}\nabla_z$ and $\overline{\mathbf{F}}\equiv\hat{\boldsymbol{x}}f_x+\hat{\boldsymbol{y}}f_y+\hat{\boldsymbol{z}}f_z$; with slight rearrangement we obtain for $\alpha\neq 0$, the following

$$
\begin{aligned}
(\nabla\times)^{\alpha}\overline{\mathbf{F}} &= \hat{\boldsymbol{x}}\left[\left(\frac{\partial^{\alpha} f_x}{\partial y^{\alpha}}+\frac{\partial^{\alpha} f_x}{\partial z^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_z}{\partial y^{\alpha}}-\frac{\partial^{\alpha} f_y}{\partial z^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right] \\
&+\hat{\boldsymbol{y}}\left[\left(\frac{\partial^{\alpha} f_y}{\partial x^{\alpha}}+\frac{\partial^{\alpha} f_y}{\partial z^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_x}{\partial z^{\alpha}}-\frac{\partial^{\alpha} f_z}{\partial x^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right] \\
&+\hat{\boldsymbol{z}}\left[\left(\frac{\partial^{\alpha} f_z}{\partial x^{\alpha}}+\frac{\partial^{\alpha} f_z}{\partial y^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_y}{\partial x^{\alpha}}-\frac{\partial^{\alpha} f_x}{\partial y^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right]
\end{aligned}
\quad (42)
$$

By putting $\alpha = 1$, in (42) we get normal integer (first) order curl as follows, and its two standard formulations are also noted in (44)

$$
(\nabla\times)\overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right)+\hat{\boldsymbol{y}}\left(\frac{\partial f_x}{\partial z}-\frac{\partial f_z}{\partial x}\right)+\hat{\boldsymbol{z}}\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)
$$

$$
(\nabla\times)\overline{\mathbf{F}} = \begin{bmatrix} 0 & -\hat{\boldsymbol{x}}\frac{\partial}{\partial z} & \hat{\boldsymbol{x}}\frac{\partial}{\partial y} \\ \hat{\boldsymbol{y}}\frac{\partial}{\partial z} & 0 & -\hat{\boldsymbol{y}}\frac{\partial}{\partial x} \\ -\hat{\boldsymbol{z}}\frac{\partial}{\partial y} & \hat{\boldsymbol{z}}\frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{bmatrix} \quad (43)
$$

We can write the fractional curl from (42) and (43), for $\alpha \neq 0$ as

$$
(\nabla\times)^{\alpha}\overline{\mathbf{F}} = \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\left(\det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x^{\alpha} & \partial_y^{\alpha} & \partial_z^{\alpha} \\ f_x & f_y & f_z \end{bmatrix}\right)+\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\left(\begin{bmatrix} \nabla_{yz}^{\alpha} & 0 & 0 \\ 0 & \nabla_{xz}^{\alpha} & 0 \\ 0 & 0 & \nabla_{xy}^{\alpha} \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{x}}f_x \\ \hat{\boldsymbol{y}}f_y \\ \hat{\boldsymbol{z}}f_z \end{bmatrix}\right) \quad (44)
$$

$$
\nabla_{yz}^{\alpha} \equiv \partial_y^{\alpha}+\partial_z^{\alpha} \qquad \nabla_{xz}^{\alpha} \equiv \partial_x^{\alpha}+\partial_z^{\alpha} \qquad \nabla_{xy}^{\alpha} \equiv \partial_x^{\alpha}+\partial_y^{\alpha} \quad (45)
$$

Where; we have used a short symbol $\partial^{\alpha}/\partial x^{\alpha} \equiv \partial_x^{\alpha}$; the lower limit is $-\infty$ always in this paper.

6. FRACTIONIZING OF LINEAR OPERATOR BY RULES OF OPERATOR ALGEBRA

In this section we apply rules of fractioning of a linear operator and verify our geometrically derived method against this rule. Fractionizing rules for linear operator from Operator algebra are as follows

1. The domain and range of the linear operator should have the same dimensions. Specifically a linear operator as $\mathbf{L}$ should map any element from the space $\mathbb{R}^n$ into the space $\mathbb{R}^n$. That is $\mathbf{L}:\mathbb{R}^n \to \mathbb{R}^n$.
2. The new fractional operator $\mathbf{L}^{\alpha}$ is considered as fractional operator obtained from $\mathbf{L}$ if
   a. For $\alpha = 0$ we get the identity operator or unit operator that is $\mathbf{I}$
   b. For $\alpha = 1$ we obtain the original operator that is $\mathbf{L}$
   c. For α_1 and α_2 the commutation should hold as $\mathbf{L}^{\alpha_1}\mathbf{L}^{\alpha_2} = \mathbf{L}^{\alpha_2}\mathbf{L}^{\alpha_1} = \mathbf{L}^{\alpha_1+\alpha_2}$

Following are the step of construction of $\mathbf{L}^{\alpha}$ from given $\mathbf{L}$

(a). Find eigenvalues and eigenvectors of the operator $\mathbf{L}$ in space $\mathbb{R}^n$. So we have $\mathbf{L}.\mathbf{V}_m = \lambda_m\mathbf{V}_m$. Where for $m = 1,2,3,...n$, λ_m, $\mathbf{V}_m$ are eigenvalue and eigenvectors respectively of the operator $\mathbf{L}$.

(b) If $\mathbf{V}_m$'s form a complete basis in the space $\mathbb{R}^n$ any vector in $\mathbb{R}^n$ can be expressed in the linear combination of $\mathbf{V}_m$'s. Thus $\mathbf{F} \in \mathbb{R}^n$ can be constructed as $\mathbf{F} = \sum_{m=1}^{n} a_m \mathbf{V}_m$, where a_m 's are coefficients of expansion

(c) The fractional operator $\mathbf{L}^\alpha$ shall have same eigenvector $\mathbf{V}_m$ as that of operator $\mathbf{L}$ but the eigenvalues are $(\lambda_m)^\alpha$. Therefore $\mathbf{L}^\alpha.\mathbf{V}_m = (\lambda_m)^\alpha \mathbf{V}_m$. When this fractional operator operates on arbitrary $\mathbf{F}$, we can write

$$\mathbf{L}^\alpha.\mathbf{F} = \mathbf{L}^\alpha \sum_{m=1}^{n} a_m \mathbf{V}_m = \sum_{m=1}^{n} a_m \mathbf{L}^\alpha.\mathbf{V}_m = \sum_{m=1}^{n} a_m (\lambda_m)^\alpha \mathbf{V}_m$$

Let us take example of fractionizing, the cross product operator $\mathbf{L} = (\hat{\boldsymbol{z}}_k \times)$, obviously a linear operator, in space $\mathbb{R}^3$. Let $\mathbf{V}_m$, $m = 1,2,3$, be the eigenvector of the linear operator with eigenvalue as λ_m ; $m = 1,2,3$, then $\mathbf{L}.\mathbf{V}_m = (\hat{\boldsymbol{z}}_k \times)\mathbf{V}_m = \lambda_m \mathbf{V}_m$.

We have to obtain these three eigenvalues and eigenvectors. Let $\mathbf{V}_m = (x)\hat{\boldsymbol{x}}_k + (y)\hat{\boldsymbol{y}}_k + (z)\hat{\boldsymbol{z}}_k$; so we have the following

$$\mathbf{L}.\mathbf{V}_m = (\hat{\boldsymbol{z}}_k \times)(x,y,z) = \hat{\boldsymbol{x}}_k(-zy) + \hat{\boldsymbol{y}}_k(xz) + \hat{\boldsymbol{z}}_k(0) = \lambda_m \mathbf{V}_m = (\lambda_m x)\hat{\boldsymbol{x}}_k + (\lambda_m y)\hat{\boldsymbol{y}}_k + (\lambda_m z)\hat{\boldsymbol{z}}_k$$

This gives following sets of equations;

$$\lambda_m x = -zy \qquad \lambda_m y = xz \qquad \lambda_m z = 0 \qquad m = 1,2,3$$

Gives, for $z = 1$ (is one eigenvector), the eigenvalue is $\lambda_3 = 0$, and thus we have remaining two eigenvalues from: $\lambda_m^2 = -1$ or $\lambda_m = \pm \mathrm{i}$, so $\lambda_1 = +\mathrm{i}$ and $\lambda_2 = -\mathrm{i}$, are the three eigenvalues. For $\lambda_1 = +\mathrm{i}$, we have, for $z = 1$, $\mathrm{i}x = -y$ and $\mathrm{i}y = x$, so choose $x = 1$ and $y = -\mathrm{i}$. Thus; $\mathbf{V}_1 = [(1)\hat{\boldsymbol{x}}_k - (\mathrm{i})\hat{\boldsymbol{y}}_k]$, becomes first eigenvector. Similarly for; $\lambda_2 = -\mathrm{i}$, the second eigenvector as; $\mathbf{V}_2 = [(1)\hat{\boldsymbol{x}}_k + (\mathrm{i})\hat{y}_k]$ and for $\lambda_3 = 0$, we have $\mathbf{V}_3 = 0\hat{\boldsymbol{x}}_k + 0\hat{\boldsymbol{y}}_k + (1)\hat{\boldsymbol{z}}_k$ as the third eigenvector. These eigenvectors are $(1,-\mathrm{i},0)$, $(1,\mathrm{i},0)$ and $(0,0,1)$.

Say a vector $\overline{\mathbf{F}}_k = F_x\hat{\boldsymbol{x}}_k + F_y\hat{\boldsymbol{y}}_k + F_z\hat{\boldsymbol{z}}_k$ can be represented as also via the linear combination of these eigenvectors obtained $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$; so we write $\overline{\mathbf{F}}_k = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3$.

Therefore,

$$\begin{aligned}\overline{\mathbf{F}}_k = F_x\hat{\boldsymbol{x}}_k + F_y\hat{\boldsymbol{y}}_k + F_z\hat{\boldsymbol{z}}_k &= a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 \\ &= a_1(\hat{\boldsymbol{x}}_k - \mathrm{i}\hat{\boldsymbol{y}}_k) + a_2(\hat{\boldsymbol{x}}_k + \mathrm{i}\hat{\boldsymbol{y}}_k) + a_3(\hat{\boldsymbol{z}}_k) \\ &= (a_1 + a_2)\hat{\boldsymbol{x}}_k + (-\mathrm{i}a_1 + \mathrm{i}a_2)\hat{\boldsymbol{y}}_k + (a_3)\hat{\boldsymbol{z}}_k\end{aligned}$$

We thus obtain from above

$$a_1 + a_2 = F_x \qquad -\mathrm{i}a_1 + \mathrm{i}a_2 = F_y \qquad a_3 = F_z$$

We have the coefficients a_m $m = 1,2,3$, as

$$a_1 = \left(\frac{F_x + \mathrm{i}F_y}{2}\right) \qquad a_2 = \left(\frac{F_x - \mathrm{i}F_y}{2}\right) \qquad a_3 = F_z$$

Therefore we get the vector as;

$$\overline{\mathbf{F}}_k = \left(\frac{F_x + \mathrm{i}F_y}{2}\right)(\hat{\boldsymbol{x}}_k - \mathrm{i}\hat{\boldsymbol{y}}_k) + \left(\frac{F_x - \mathrm{i}F_y}{2}\right)(\hat{\boldsymbol{x}}_k + \mathrm{i}\hat{\boldsymbol{y}}_k) + F_z(\hat{\boldsymbol{z}}_k)$$

From above fractioning rule we have

$\mathbf{L}^\alpha.\overline{\mathbf{F}}_k = (\hat{\boldsymbol{z}}_k \times)^\alpha \overline{\mathbf{F}}_k = \sum_{m=1}^{3} (\lambda_m)^\alpha a_m \mathbf{V}_m$, applying this rule we get the following for our operator

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k = (+\mathrm{i})^{\alpha}\frac{(F_x+\mathrm{i}F_y)}{2}(\hat{\boldsymbol{x}}_{\boldsymbol{k}}-\mathrm{i}\hat{\boldsymbol{y}}_{\boldsymbol{k}})+(-\mathrm{i})^{\alpha}\frac{(F_x-\mathrm{i}F_y)}{2}(\hat{x}_k+\mathrm{i}\hat{\boldsymbol{y}}_{\boldsymbol{k}})+(0)^{\alpha}F_z\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

Let us try to evaluate $(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\hat{\boldsymbol{x}}_{\boldsymbol{k}}$; from above obtained method. This implies the following

$$\overline{\mathbf{F}}_k = (1)\boldsymbol{x}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{y}}_{\boldsymbol{k}}+(0)\hat{\boldsymbol{z}}_{\boldsymbol{k}}; \qquad \overline{\mathbf{F}}_k=\hat{x}_k \qquad F_x=1 \qquad F_y=F_z=0$$

$$(\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k = (\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)\hat{\boldsymbol{x}}_{\boldsymbol{k}} = (+\mathrm{i})^{\alpha}\left(\frac{1}{2}\right)(\hat{\boldsymbol{x}}_{\boldsymbol{k}}-\mathrm{i}\hat{\boldsymbol{y}}_{\boldsymbol{k}})+(-\mathrm{i})^{\alpha}\left(\frac{1}{2}\right)(\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\mathrm{i}\hat{\boldsymbol{y}}_{\boldsymbol{k}})$$

$$=\left[\frac{(\mathrm{i})^{\alpha}}{2}+\frac{(-\mathrm{i})^{\alpha}}{2}\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[-\frac{(\mathrm{i})^{\alpha}\mathrm{i}}{2}+\frac{(-\mathrm{i})^{\alpha}\mathrm{i}}{2}\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

Use $(\pm\mathrm{i})^{\alpha}\equiv e^{\pm\mathrm{i}(\alpha\pi/2)}=\cos(\alpha\pi/2)\pm\mathrm{i}\sin(\alpha\pi/2)$. Thus first term of above is $[\cos(\alpha\pi/2)]\hat{\boldsymbol{x}}_{\boldsymbol{k}}$. Simplifying second term we get

$$\left[-\frac{(+\mathrm{i})^{\alpha}\mathrm{i}}{2}+\frac{(-\mathrm{i})^{\alpha}\mathrm{i}}{2}\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\left[-\frac{(+\mathrm{i})^{\alpha}(\mathrm{i})}{2}-\frac{(-\mathrm{i})^{\alpha}(-\mathrm{i})}{2}\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\left[-\frac{\mathrm{i}^{\alpha+1}}{2}-\frac{(-\mathrm{i})^{\alpha+1}}{2}\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

$$=\frac{1}{2}\left[-e^{\mathrm{i}(\alpha+1)\pi/2}-e^{-\mathrm{i}(\alpha+1)\pi/2}\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}=-\cos\left(\frac{(\alpha+1)\pi}{2}\right)\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

$$=-\cos\left(\frac{\pi}{2}+\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{y}}_{\boldsymbol{k}}=\sin\left(\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

Therefore

$$\overline{\mathbf{F}}_k=\hat{\boldsymbol{x}}_{\boldsymbol{k}} \qquad (\boldsymbol{z}_{\boldsymbol{k}}\times)^{\alpha}\,\overline{\mathbf{F}}_k=\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_{\boldsymbol{k}}+\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_{\boldsymbol{k}}$$

This is what we have got via geometrical method as in expressed in previous section in (16).

7. FRACTIONAL CURL OF ELECTROMAGNETIC VECTOR FIELD FROM FRACTIONIZED CROSS PRODUCT OBTAINED GEOMETRICALLY

Let us take a case of stationary wave as $\overline{\mathbf{F}}(z)=\hat{\boldsymbol{x}}(\cos z)+\hat{\boldsymbol{y}}(0)+\hat{\boldsymbol{z}}(0)$; i.e. a vector field directed towards x-axis with variation in z direction. Thus this vector field has gradient in z-direction. This is vibrating string fixed at $z=-\pi/2$ and $z=+\pi/2$, vibrating in $x-y$ plane but directed at x-axis. The normal curl of this 'stationary standing wave' of vibrating string gives (35) as $+y$ directed with a variation in z direction $(\nabla_z\hat{\boldsymbol{z}}\times)\overline{\mathbf{F}}=(\nabla_z\hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}\cos z)=\hat{\boldsymbol{y}}(\sin z)=\hat{\boldsymbol{y}}\cos(z+[\pi/2])$. Physically we have rotated the vector field anticlockwise; in $x-y$ plane and about z-axis by 90^0, and advanced its spatial phase in z direction by $\pi/2$. Anticlockwise is looking from positive z direction, the initial 12; o clock now is at 9; o clock position. The fractional curl operation will therefore be a fractional rotation of this vector field in $x-y$ plane by an angle $\alpha(90^0)$ about the z axis, in anticlockwise direction; and the vector field will advanced by fractional angle $\alpha\pi/2$. Nevertheless ${}_{-\infty}D_z^{\alpha}\cos z=\cos(z+\alpha\pi/2)$, is fractional derivative of cosine with lower terminal at $-\infty$. Thus we get 'tilted' fractional curl field of this vector field; and this tilted curl will have 'shadows' on the $x-z$ plane and $y-z$ plane. We write this as that are shadows of the tilted curl having components in these planes as in (46)

$$\begin{aligned}&\left[\hat{\boldsymbol{x}}\cos(\alpha\pi/2)+\hat{\boldsymbol{y}}\sin(\alpha\pi/2)\right]\cos(z+\{\alpha\pi/2\})=\\&\left[\hat{\boldsymbol{x}}\cos(\alpha\pi/2)+\hat{\boldsymbol{y}}\sin(\alpha\pi/2)\right]\{{}_{-\infty}D_z^{\alpha}f_x(z)\}=(\nabla_z\hat{\boldsymbol{z}}\times)^{\alpha}\overline{\mathbf{F}}\end{aligned} \tag{46}$$

The (46) is same from (34) as $f_y=0$ and $f_x=0$ for our vector field example. The fractional curl of this standing wave is depicted in figure-4.

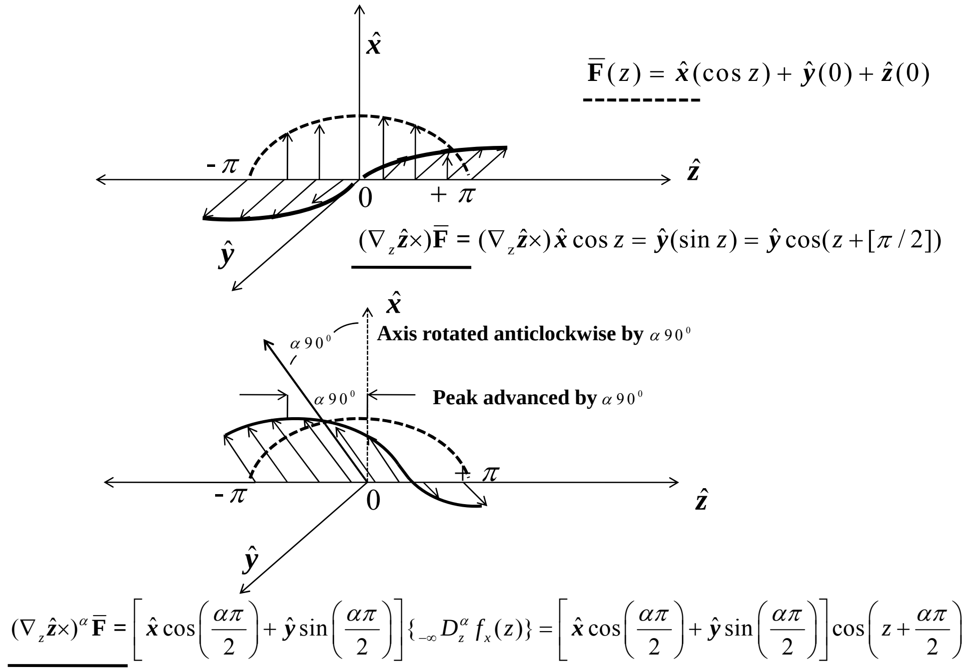


Figure 4: Demonstration of fractional curl of a standing wave

Let be our example a case of complex vector field $\overline{\mathbf{F}} = \hat{\boldsymbol{x}}(Ae^{ik_0 z})$; implying this is x -directed field, varying in z direction. Thus $f_x = Ae^{ik_0 z}$, $f_y = 0$ and $f_z = 0$. This is a travelling wave directed towards x and travelling harmonically in space in z direction. For this vector field we have $\partial^\alpha F_x / \partial x^\alpha = \partial^\alpha F_x / \partial y^\alpha = 0$, and $\nabla_{xy}^\alpha \overline{\mathbf{F}} = 0$. The operator ∇_{yz}^α becomes $\partial^\alpha / \partial z^\alpha$ and ∇_{xz}^α becomes $\partial^\alpha / \partial z^\alpha$. Using these simplification we write (44) for this vector field as

$$(\nabla\times)^\alpha \overline{\mathbf{F}} = \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \partial_z^\alpha \\ f_x & 0 & 0\end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\partial_z^\alpha & 0 & 0 \\ 0 & \partial_z^\alpha & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}f_x \\ 0 \\ 0\end{bmatrix}$$
$$= \hat{\boldsymbol{y}}\sin(\alpha\pi/2)\left[\partial_z^\alpha f_x\right] + \hat{\boldsymbol{x}}\cos(\alpha\pi/2)\left[\partial_z^\alpha f_x\right] \qquad (47)$$

Using ${}_{-\infty}D_t^\alpha e^{\lambda t} = \partial^\alpha e^{\lambda t} / \partial^\alpha t = \lambda^\alpha e^{\lambda t}$; we write the fractional curl as

$$\text{curl}^\alpha \overline{\mathbf{F}} = (\nabla\times)^\alpha \{\hat{\boldsymbol{x}}(Ae^{ik_0 z})\} = (ik_0)^\alpha A\left[\hat{\boldsymbol{x}}\cos(\alpha\pi/2) + \hat{\boldsymbol{y}}\sin(\alpha\pi/2)\right]e^{ik_0 z} \qquad (48)$$

The (43) is also a travelling wave in z direction but rotated anticlockwise about z axis in $x - y$ plane-a tilted travelling wave; or polarized one. The fractional field by using $\overline{\mathbf{F}}_f = (ik)^{-\alpha}[(\nabla\times)^\alpha \overline{\mathbf{F}}]$, as discussed in expression (4) for this field we have

$$\overline{\mathbf{F}}_f = (ik)^{-\alpha}(\nabla\times)^\alpha \{\hat{\boldsymbol{x}}(Ae^{ik_0 z})\} = A\left[\hat{\boldsymbol{x}}\cos(\alpha\pi/2) + \hat{\boldsymbol{y}}\sin(\alpha\pi/2)\right]e^{ik_0 z} \qquad (49)$$

The next example we take as a vector field z directed but having variation in $x - y$ plane as $\overline{\mathbf{F}} = \hat{\boldsymbol{z}}F(x,y) = \hat{\boldsymbol{z}}(e^{i\lambda x + i\mu y})$; meaning $f_x = f_y = 0$ and $f_z = F(x,y) = e^{i\lambda x + i\mu y}$. We have from (44) and (45) the fractional curl of this vector field as follows. This is z directed wave, travelling 'obliquely' in $x - y$ plane making an angle φ, with x axis, such that $\tan\varphi = \mu / \lambda$

$$(\nabla\times)^{\alpha}\{\hat{\boldsymbol{z}}F(x,y)\}=\left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial_x^{\alpha} & \partial_y^{\alpha} & 0\\ 0 & 0 & F(x,y)\end{bmatrix}$$
$$+\left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ 0 & 0 & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}0\\ 0\\ \hat{\boldsymbol{z}}F(x,y)\end{bmatrix} \tag{50}$$
$$=\left(\sin\alpha\pi/2\right)\left[\hat{\boldsymbol{x}}\partial_y^{\alpha}F(x,y)-\hat{\boldsymbol{y}}\partial_x^{\alpha}F(x,y)\right]$$
$$+\left(\cos\alpha\pi/2\right)\left[\partial_x^{\alpha}F(x,y)+\partial_y^{\alpha}F(x,y)\right]\hat{\boldsymbol{z}}$$

To find 'partial fractional derivative' $\partial^{\alpha}F(x,y)/\partial x^{\alpha}$; we resort to a trick. The vector field, we write in terms of polar coordinates as: $F(x,y)=e^{\mathrm{i}\lambda\hat{x}+\mathrm{i}\mu\hat{y}}=F(\hat{\boldsymbol{r}})=e^{\mathrm{i}|r|.\hat{\boldsymbol{r}}}$; where $|r|=\sqrt{\lambda^2+\mu^2}$, and $\hat{\boldsymbol{r}}$ is vector in radial direction, that is $\hat{\boldsymbol{r}}=\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}$, where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ are unit vectors of Cartesian coordinates. The vector $|r|.\hat{\boldsymbol{r}}=\lambda\hat{\boldsymbol{x}}+\mu\hat{\boldsymbol{y}}$ .The fractional derivative of $F(\hat{\boldsymbol{r}})$ by using ${}_{-\infty}D_t^{\alpha}e^{\lambda t}=\partial^{\alpha}e^{\lambda t}/\partial^{\alpha}t=\lambda^{\alpha}e^{\lambda t}$, is $\partial^{\alpha}F(\hat{\boldsymbol{r}})/\partial\hat{\boldsymbol{r}}^{\alpha}=\partial^{\alpha}(e^{\mathrm{i}|r|.\hat{\boldsymbol{r}}})/\partial\hat{\boldsymbol{r}}^{\alpha}=(\mathrm{i}|r|)^{\alpha}e^{\mathrm{i}|r|.\hat{\boldsymbol{r}}}$. This fractional derivative is in $\hat{\boldsymbol{r}}$ direction; and component of this in x axis will give, the $\partial^{\alpha}F(x,y)/\partial x^{\alpha}$. To do this we have to multiply the obtained fractional derivative in $\hat{\boldsymbol{r}}$ direction by projection on x axis that is $\lambda/(|r|)$. To get $\partial^{\alpha}F(x,y)/\partial y^{\alpha}$, we need the projection in y axis; which we get by multiplying the obtained derivative by $\mu/|r|$, elaborated below;

$$\partial_x^{\alpha}F(x,y)=\frac{\partial^{\alpha}(e^{\mathrm{i}|r|.\hat{\boldsymbol{r}}})}{\partial\hat{x}^{\alpha}}=\frac{\partial^{\alpha}(e^{\mathrm{i}\lambda\hat{x}+\mathrm{i}\mu\hat{y}})}{\partial\hat{x}^{\alpha}}=\mathrm{i}^{\alpha}|r|^{\alpha}e^{\mathrm{i}(\lambda\hat{x}+\mu\hat{y})}\frac{\lambda}{|r|}=\mathrm{i}^{\alpha}\lambda\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha-1}e^{\mathrm{i}(\lambda\hat{x}+\mu\hat{y})} \tag{51}$$

$$\partial_y^{\alpha}F(x,y)=\frac{\partial^{\alpha}(e^{\mathrm{i}|r|.\hat{\boldsymbol{r}}})}{\partial\hat{y}^{\alpha}}=\frac{\partial^{\alpha}(e^{\mathrm{i}\lambda\hat{x}+\mathrm{i}\mu\hat{y}})}{\partial\hat{y}^{\alpha}}=\mathrm{i}^{\alpha}|r|^{\alpha}e^{\mathrm{i}(\lambda\hat{x}+\mu\hat{y})}\frac{\mu}{|r|}=\mathrm{i}^{\alpha}\mu\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha-1}e^{\mathrm{i}(\lambda\hat{x}+\mu\hat{y})} \tag{52}$$

For verification put $\alpha=1$, we get integer order partial derivative as $\partial F(x,y)/\partial\hat{x}=\mathrm{i}\lambda e^{\mathrm{i}(\lambda\hat{x}+\mu\hat{y})}$ .For calculation of z component of (50) we have following steps of adding vectorically the quantities in $[...]$; since individually (51) and (52) are projections on x and y axis respectively. By doing so we have $|\mathrm{i}^{\alpha}r^{\alpha-1}\lambda|+|\mathrm{i}^{\alpha}r^{\alpha-1}\mu|$, which is; $\mathrm{i}^{\alpha}|r|^{\alpha-1}(|\lambda|+|\mu|)$. Recognizing $|\lambda|+|\mu|=|r|=\sqrt{\lambda^2+\mu^2}$; the component of z in (50) that is $[...]$ is $\mathrm{i}^{\alpha}|r|^{\alpha}$ or $\mathrm{i}^{\alpha}\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha}$. So from (51) (52) and this derivation for z component of the $[...]$ of (50); we completely write the fractional curl of this vector field as

$$(\nabla\times)^{\alpha}\left(\hat{\boldsymbol{z}}e^{\mathrm{i}\lambda x+\mathrm{i}\mu y}\right)=\hat{\boldsymbol{x}}.\mathrm{i}^{\alpha}\mu\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha-1}\sin\left(\alpha\pi/2\right)e^{\mathrm{i}\lambda x+\mathrm{i}\mu y}$$
$$-\hat{\boldsymbol{y}}.\mathrm{i}^{\alpha}\lambda\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha-1}\sin\left(\alpha\pi/2\right)e^{\mathrm{i}\lambda x+\mathrm{i}\mu y} \tag{53}$$
$$+\hat{\boldsymbol{z}}.\mathrm{i}^{\alpha}\left(\sqrt{\lambda^2+\mu^2}\right)^{\alpha}\cos\left(\alpha\pi/2\right)e^{\mathrm{i}\lambda x+\mathrm{i}\mu y}$$

In above example, for $\overline{\mathbf{F}}=\hat{\boldsymbol{z}}.e^{ik_x\hat{x}+ik_y\hat{y}}$ $k_x=\lambda$ and $k_y=\mu$, and $k=\sqrt{k_x^2+k_y^2}=\sqrt{\lambda^2+\mu^2}$, we can have fractional field as following

$$\overline{\mathbf{F}}_f = (\mathrm{i}k)^{-\alpha}(\nabla\times)^{\alpha}\left(\hat{\boldsymbol{z}}e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}}\right) = (\mathrm{i}k)^{-\alpha}\,\hat{\boldsymbol{x}}.\mathrm{i}^{\alpha}k_y\left(k\right)^{\alpha-1}\sin\left(\alpha\pi/2\right)e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}} +$$

$$(\mathrm{i}k)^{-\alpha}(-\hat{\boldsymbol{y}}.\mathrm{i}^{\alpha})k_x\left(k\right)^{\alpha-1}\sin\left(\alpha\pi/2\right)e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}} +$$

$$(\mathrm{i}k)^{-\alpha}\,\hat{\boldsymbol{z}}.\mathrm{i}^{\alpha}\left(k\right)^{\alpha}\cos\left(\alpha\pi/2\right)e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}}$$

$$\overline{\mathbf{F}}_f = \left[\hat{\boldsymbol{x}}.\left(k_y/k\right)\sin\left(\alpha\pi/2\right) + \hat{\boldsymbol{y}}.\left(k_x/k\right)\left[-\sin\left(\alpha\pi/2\right)\right] + \hat{\boldsymbol{z}}\cos\left(\alpha\pi/2\right)\right]e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}}$$

For, $\alpha = 0$, $\overline{\mathbf{F}}_f = \hat{\boldsymbol{z}}.e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}} = \mathbf{F}$ the original one; the $\alpha = 1$ gives $\overline{\mathbf{F}}_f = \left[\hat{\boldsymbol{x}}(k_y/k) - \hat{\boldsymbol{y}}(k_x/k)\right]e^{\mathrm{i}k_x\hat{x}+\mathrm{i}k_y\hat{y}}$. This wave is also travelling wave oblique incident in $x-y$ plane, with angle $\varphi = \tan^{-1}(k_y/k_x)$ comprising of two components directed towards x and y directions both in same oblique direction φ, with no component directed in z direction as was in case of original case, the original has been rotated about $\overline{\mathrm{k}}$ axis (propagation direction) to $x-y$ plane from original z axis.

From above example if we have field of the form $\overline{\mathbf{E}} = ze^{\mathrm{i}k(x\cos\varphi+y\sin\varphi)}$; then fractional curl of this $\overline{\mathbf{E}}$ field is; as following the steps for obtaining (53)

$$\begin{aligned}(\nabla\times)^{\alpha}\overline{\mathbf{E}} &= \mathrm{curl}^{\alpha}\left\{\hat{\boldsymbol{z}}.e^{\mathrm{i}k(x\cos\varphi+y\sin\varphi)}\right\}\\ &= (\mathrm{i}k)^{\alpha}\left[\hat{\boldsymbol{x}}.\sin\left(\alpha\pi/2\right)\sin\varphi - \hat{\boldsymbol{y}}.\sin\left(\alpha\pi/2\right)\cos\varphi + \hat{\boldsymbol{z}}.\cos\left(\alpha\pi/2\right)\right]e^{\mathrm{i}k(x\cos\varphi+y\sin\varphi)}\end{aligned} \tag{54}$$

The fractional field of this Electric field we write as

$$\begin{aligned}\overline{\mathbf{E}}_f &= (\mathrm{i}k)^{-\alpha}(\nabla\times)^{\alpha}\overline{\mathbf{E}}\\ &= \left[\hat{\boldsymbol{x}}.\sin\left(\alpha\pi/2\right)\sin\varphi - \hat{\boldsymbol{y}}.\sin\left(\alpha\pi/2\right)\cos\varphi + \hat{\boldsymbol{z}}.\cos\left(\alpha\pi/2\right)\right]e^{\mathrm{i}k(x\cos\varphi+y\sin\varphi)}\end{aligned} \tag{55}$$

Let us take a traveling wave given by, vector field $\overline{\mathbf{F}} = E_0(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})e^{ik\hat{z}}$. This is a case of (right circularly) polarized vector field, with radius as E_0 in $x-y$ plane, but travelling in z direction. So we have $f_z = 0$, $f_x = E_0e^{ikz}$ and $f_y = \mathrm{i}E_0e^{ikz}$. Therefore we write the fractional curl for $\alpha \neq 0$ as

$$\begin{aligned}(\nabla\times)^{\alpha}\overline{\mathbf{F}} &= \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ 0 & 0 & \partial_z^{\alpha}\\ E_0e^{ikz} & \mathrm{i}E_0e^{ikz} & 0\end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\partial_z^{\alpha} & 0 & 0\\ 0 & \partial_z^{\alpha} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}E_0e^{ikz}\\ \hat{\boldsymbol{y}}E_0e^{ikz}\\ 0\end{bmatrix}\\ &= \hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^{\alpha}}{\partial z^{\alpha}}\mathrm{i}E_0e^{ikz}\right] - \hat{\boldsymbol{y}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^{\alpha}}{\partial z^{\alpha}}E_0e^{ikz}\right]\\ &\quad + \hat{\boldsymbol{x}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^{\alpha}}{\partial z^{\alpha}}E_0e^{ikz}\right] + \hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^{\alpha}}{\partial z^{\alpha}}\mathrm{i}E_0e^{ikz}\right]\end{aligned} \tag{56}$$

Using ${}_{-\infty}D_t^{\alpha}e^{\lambda t} = \partial^{\alpha}e^{\lambda t}/\partial^{\alpha}t = \lambda^{\alpha}e^{\lambda t}$, $\pm\mathrm{i} = e^{\pm\mathrm{i}\pi/2}$ and $\cos\chi \pm \mathrm{i}\sin\chi = e^{\pm\mathrm{i}\chi}$, we get the following

$$(\nabla\times)^{\alpha}\overline{\mathbf{F}} = E_0\{\hat{\boldsymbol{x}}[-\mathrm{i}\sin(\alpha\pi/2)] + \hat{\boldsymbol{y}}\sin(\alpha\pi/2) + \hat{\boldsymbol{x}}\cos(\alpha\pi/2) + \hat{\boldsymbol{y}}[i\cos(\alpha\pi/2)]\}(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$\begin{aligned}
&= \left[\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\sin\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{x}} + \left(\sin\frac{\alpha\pi}{2} + \mathrm{i}\cos\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} \\
&= \left[\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\sin\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{x}} + \left(\cos\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\} + \mathrm{i}\sin\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} \\
&= \left[\left(e^{-\mathrm{i}\frac{\alpha\pi}{2}}\right)\hat{\boldsymbol{x}} + \left(e^{\mathrm{i}\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\}}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} \qquad (57) \\
&= \left[\left\{e^{\left(-\mathrm{i}\frac{\pi}{2}\right)}\right\}^{\alpha}\hat{\boldsymbol{x}} + \left\{e^{i\frac{\pi}{2}}\right\}\left\{e^{\left(-\mathrm{i}\frac{\pi}{2}\right)}\right\}^{\alpha}\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} = \left[(-\mathrm{i})^{\alpha}\hat{\boldsymbol{x}} + (\mathrm{i})(-\mathrm{i})^{\alpha}\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} \\
&= (-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}
\end{aligned}$$

If the vector field is left circularly polarized then we have $E_0(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})e^{\mathrm{i}kz}$ fractional curl, via above steps as $\mathrm{curl}^{\alpha}\left\{(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (+\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$.

The fractional fields for this circularly polarized traveling wave is

$$\overline{\mathrm{F}}_f = (\mathrm{i}k)^{-\alpha}\mathrm{curl}^{\alpha}\left\{(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (\mp\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz} = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}(kz\mp\{\alpha\pi/2\})}. \qquad (58)$$

For a complex order curl that is $\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\}$, very interesting result appears. Let us use all the steps of (56) and (57) but replace α by imaginary number $\mathrm{i}\beta$. We will get the following

$$\begin{aligned}
\mathrm{curl}^{\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} &= (-\mathrm{i})^{\mathrm{i}\beta}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz} \\
&= \left(e^{-i\frac{\pi}{2}}\right)^{i\beta}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz} = (\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\beta\pi/2}E_0\right\}(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz}
\end{aligned} \qquad (59)$$

Using (56) and (57) we write complex order curl of right circularly polarized vector field as

$$\begin{aligned}
\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} &= (-\mathrm{i})^{\mathrm{i}\beta}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz} \\
&= (-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\beta\pi/2}E_0\right\}(\mathrm{i}k)^{\alpha+\mathrm{i}\beta}e^{\mathrm{i}kz}
\end{aligned} \qquad (60)$$

Therefore the fractional field of (60) for right and left circularly polarized vector field is

$$\begin{aligned}
\overline{\mathbf{F}}_{f(\pm)} &= (\mathrm{i}k)^{-(\alpha+i\beta)}\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (\mp\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}kz} \\
&= \left\{e^{\mp\mathrm{i}(\pi/2)}\right\}^{\alpha}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}kz} = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}\left(kz\mp\frac{\alpha\pi}{2}\right)}
\end{aligned} \qquad (61)$$

The fractional field obtained by complex order curl operation gives interpretation as following. The real part of the fractional order ($\alpha > 0$) gives a spatial phase lead in case of right circularly polarized vector field, gives a spatial phase lag for left circularly polarized vector. In other words the real part of fractional operator modifies the phase of the vector field. Whereas the imaginary part of the operator ($\beta > 0$) modifies the radius of the amplitude. For the right circularly polarized vector field it increases the amplitude (in this case radius in $x - y$ plane); and for left circularly polarized vector field the fractional field reduces the radius of the vector field's amplitude.

Consider a right and left circularly polarized transverse EM uniform plane wave propagating along $+z$ direction has Electric and magnetic fields as $\overline{\mathbf{E}}_{\pm} = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}$ and $\eta\overline{\mathbf{H}}_{\pm} = (\pm\mathrm{i})(\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}$. The fractional field we get from (57)-(61) with (4) as obtained for the vector $\overline{\mathbf{F}}_{f(\pm)}$, the result is, for real order; $\overline{\mathbf{E}}_{f(\pm)} = (\mp\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}$ and $\eta\overline{\mathbf{H}}_{f(\pm)} = (\mp\mathrm{i})^{\alpha+1}(\hat{\boldsymbol{x}} \pm \hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}$; for $0 < \alpha < 1$. The quantity $(\mp\mathrm{i})^{\alpha}$ is eigenvalue for the operator $(ik)^{-\alpha}\{(\overline{\mathrm{k}}\times)^{\alpha}\}$ or $(ik)^{-\alpha}\mathrm{curl}^{\alpha}$, and the original circularly polarized electric and magnetic fields are eigenvectors for this operator; very interesting observation.

Let us take the formal example of a standing wave formation in electromagnetic theory. A standing wave is formed from two TEM uniform plane waves propagating in opposite directions, having electric and magnetic fields as

$$\begin{aligned}\overline{\mathbf{E}} &= \hat{\boldsymbol{x}}\left[E_0 e^{-\mathrm{i}kz} - E_0 e^{+\mathrm{i}kz}\right] = -\hat{\boldsymbol{x}}2\mathrm{i}E_0\sin(kz)\\ \eta\overline{\mathbf{H}} &= \hat{\boldsymbol{y}}\left[E_0 e^{-\mathrm{i}kz} + E_0 e^{+\mathrm{i}kz}\right] = \hat{\boldsymbol{y}}2E_0\cos(kz)\end{aligned} \tag{62}$$

The standing waves are formed as a result of a TEM plane wave perpendicularly incident, travelling in z direction, on a 'conducting' sheet in $x-y$ plane located at $z=0$, is reflected from the sheet, with 100% reflection. The incident and reflected waves thus form a standing wave as depicted in (62). Using the formula of fractional curl (44) and (45) and then formula of fractional field (4); as done for the above examples we get the result as

$$\begin{aligned}\overline{\mathbf{E}}_f &= -2\mathrm{i}e^{\left(-\mathrm{i}\frac{\alpha\pi}{2}\right)}E_0\sin\left(kz+\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}.\cos\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}.\sin\frac{\alpha\pi}{2}\right]\\ \eta\overline{\mathbf{H}}_f &= 2e^{\left(-\mathrm{i}\frac{\alpha\pi}{2}\right)}E_0\cos\left(kz+\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}.\left(-\sin\frac{\alpha\pi}{2}\right)+\hat{\boldsymbol{y}}.\cos\frac{\alpha\pi}{2}\right]\end{aligned} \tag{63}$$

There is fractional (transverse) impedance of the system (63) transverse to propagation z directions of two plane waves (62) forming standing wave,, as ratio the of x component of fractional electric field to y component of the magnetic field, and vice-versa as noted below. We get from (63) the value of fractional transverse impedance as

$$Z_f = -\frac{E_{f-\hat{x}}}{H_{f-\hat{y}}} = \frac{E_{f-\hat{y}}}{H_{f-\hat{x}}} = \mathrm{i}\eta\tan\left(kz+\frac{\alpha\pi}{2}\right) \qquad Z_f\Big|_{z=0} = \mathrm{i}\eta\tan\left(\frac{\alpha\pi}{2}\right) \tag{64}$$

For $\alpha=0$, the surface impedance at $z=0$ is $Z_f=\mathrm{i}(0)$, meaning a flat surface in $x-y$ plane located at $z=0$ is a perfect electric conductor (PEC). While for $\alpha=1$, the surface impedance is $Z_f=\mathrm{i}(\infty)$; the surface is thus perfect magnetic conductor (PMC). For values $0<\alpha<1$, the impedance of surface is in between PEC and PMC, and we get fractional fields of the standing wave.

8. FRACTIONAL ELECTROMAGNETIC FIELDS IN LEFT HANDED MAXWELL SYSTEM

Here we will apply all that we did in earlier sections. Consider the half space $z>0$ is media with DPS properties with $\varepsilon>0$ and $\mu>0$; while the half space $z<0$ is with Left Handed Maxwell Media (LHM) that is with DNG properties of $\varepsilon<0$ and $\mu<0$. The plane $x-y$ at $z=0$ is a source of time harmonic current (a current sheet) directed in x direction with unit current density as $\overline{\mathbf{J}}=\hat{\boldsymbol{x}}.\delta(z)$. This current sheet radiates time harmonic electromagnetic fields in form of uniform plane waves, and field must propagate away from the current sheet source. The DPS region $z>0$, has the propagation wave vector $\overline{\mathrm{k}}$ is directed away from the source while for LHM region $z<0$, the wave vector $\overline{\mathrm{k}}$ is directed towards the source. The DPS region thus has fields

$$\overline{\mathbf{E}} = \hat{\boldsymbol{x}}.E_0 e^{\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}} = \hat{\boldsymbol{y}}.E_0 e^{\mathrm{i}kz} \tag{65}$$

For the LHM region, the fields are (66) forming the triad $\overline{\mathbf{E}},\overline{\mathbf{H}},\overline{\mathrm{k}}$ via left hand (cross product via left hand), compared to (65) where the right handed triad is obtained. Refer figure-5.

$$\overline{\mathbf{E}} = \hat{\boldsymbol{x}}.E_0 e^{\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}} = -\hat{\boldsymbol{y}}.E_0 e^{\mathrm{i}kz} \tag{66}$$

Using (44) (45) and (4) we get fractional solution from original EM fields for DPS (65) as (67); via following steps for $\overline{\mathbf{F}}=\eta\overline{\mathbf{H}}=\hat{y}E_0 e^{\mathrm{i}kz}$, for $z>0$

$$(\nabla\times)^{\alpha}\overline{\mathbf{F}}=\left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ 0 & 0 & \partial_z^{\alpha}\\ 0 & E_0e^{ikz} & 0\end{bmatrix}+\cos\frac{\alpha\pi}{2}\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ & & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}0\\ \hat{\boldsymbol{y}}E_0e^{ikz}\\ 0\end{bmatrix}$$

$$=\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2}\left[-\frac{\partial^{\alpha}(E_0e^{ikz})}{\partial z^{\alpha}}\right]+\hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\left[\frac{\partial^{\alpha}(E_0e^{ikz})}{\partial z^{\alpha}}\right]$$

$$=\hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-E_0(ik)^{\alpha}e^{ikz}\right]+\hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[E_0(ik)^{\alpha}e^{ikz}\right]$$

Thus the fractional field is $\overline{\mathbf{F}}_f=(ik)^{-\alpha}(\nabla\times)^{\alpha}\overline{\mathbf{F}}$; which is (67). Note that in this case the direction of gradient in z direction is the same as flow of phases $\overline{\mathrm{k}}$

$$\overline{\mathbf{F}}_f=E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{ikz}$$

$$\overline{\mathbf{E}}_f=E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{ikz}\qquad \eta\overline{\mathbf{H}}_f=E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{ikz}\tag{67}$$

That is rotation of original field vectors by an angle $\alpha(90^0)$ anticlockwise in $x-y$ plane about the positive z axis. The fractional dual solutions for the LHM region applying (44) and (45) and (4) for (66); via the following steps for $\overline{\mathbf{F}}=\eta\overline{\mathbf{H}}=-\hat{\boldsymbol{y}}E_0e^{ikz}$,region of LHM $z<0$;

$$(\nabla\times)^{\alpha}\overline{\mathbf{F}}=\left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ 0 & 0 & \partial_z^{\alpha}\\ 0 & (-E_0e^{ikz}) & 0\end{bmatrix}+\left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ 0 & 0 & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}0\\ \hat{\boldsymbol{y}}(-E_0e^{ikz})\\ 0\end{bmatrix}$$

$$=\hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^{\alpha}(-E_0e^{ikz})}{\partial z^{\alpha}}\right]+\hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^{\alpha}(-E_0e^{ikz})}{\partial z^{\alpha}}\right]$$

$$=\hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[(ik)^{\alpha}E_0e^{ikz}\right]-\hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[(ik)^{\alpha}E_0e^{ikz}\right]$$

In above the $z<0$; but the direction of gradient (in increasing z), points towards the direction of phase flow $\overline{\mathrm{k}}$, that is also in increasing direction of z ; therefore we have retained the sign of ∂z ; while taking the fractional derivative; as we did for (67). The fractional field is therefore is $\overline{\mathbf{F}}_f=(ik)^{-\alpha}\mathrm{curl}^{\alpha}\overline{\mathbf{F}}$

$\overline{\mathbf{F}}_f=-E_0\left[\hat{\boldsymbol{x}}\left(-\sin\{\alpha\pi/2\}\right)+\hat{\boldsymbol{y}}\left(\cos\{\alpha\pi/2\}\right)\right]e^{ikz}$, which is noted in (68);

$$\overline{\mathbf{E}}_f=E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{ikz}\qquad \eta\overline{\mathbf{H}}_f=-E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{ikz}\tag{68}$$

That is rotation of original field vectors by an angle $\alpha(90^0)$ anticlockwise in $x-y$ plane about the positive z axis. Therefore for both DPS and LHM regions the fractional dual solutions are obtained via rotation anticlockwise by $\alpha(90^0)$, in $x-y$ plane, about $+z$ axis.

If the region $z<0$ instead of LHM be of DPS media then we have the plane waves in the DPS media for $z<0$ as

$$\overline{\mathbf{E}}=\hat{\boldsymbol{x}}.E_0e^{-ikz}\qquad \eta\overline{\mathbf{H}}=-\hat{\boldsymbol{y}}.E_0e^{-ikz}\tag{69}$$

In (69) since now represents DPS media the $\overline{\mathrm{k}}$ will be pointing away from sheet $z=0$ towards the minus z direction. Compared to the (65), the propagation will be opposite; and thus we have (69) with $\overline{\mathrm{k}}$ as $-\overline{\mathrm{k}}$. Therefore in this case to calculate fractional derivative D_z^{α} , we need to take negative $-\partial z$; or we write

$D_z^\alpha \equiv \partial^\alpha /(-1)\partial z^\alpha$ Let us take for example $\overline{\mathbf{F}} = \eta\overline{\mathbf{H}} = -\hat{\boldsymbol{y}}E_0 e^{-\mathrm{i}kz}$; and we find fractional field for this via following steps

$$(\nabla\times)^\alpha \overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^\alpha (E_0 e^{-\mathrm{i}kz})}{(-1)\partial z^\alpha}\right] + \hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^\alpha (E_0 e^{-\mathrm{i}kz})}{(-1)\partial z^\alpha}\right]$$

$$= \hat{\boldsymbol{x}}\left(\sin\{\alpha\pi/2\}\right)\left[-(-1)(-\mathrm{i}k)^\alpha E_0 e^{-\mathrm{i}kz}\right] + \hat{\boldsymbol{y}}\left(\cos\{\alpha\pi/2\}\right)\left[(-1)(-\mathrm{i}k)^\alpha E_0 e^{-\mathrm{i}kz}\right]$$

The fractional field is $\overline{\mathbf{F}}_f = (\mathrm{i}k)^{-\alpha}(\nabla\times)^\alpha \mathbf{F}$ gives the following which is noted in (70);

$$\overline{\mathbf{F}}_f = (-1)(-1)^\alpha E_0\left[\hat{x}\left(-\sin\{\alpha\pi/2\}\right) + \hat{y}\left(\cos\{\alpha\pi/2\}\right)\right]e^{-\mathrm{i}kz}$$

And applying the formulas $(-1) = e^{\mathrm{i}\pi}$ we obtain for DPS at the region $z < 0$ the fractional fields as;

$$\begin{aligned} \overline{\mathbf{E}}_f &= (-1)^\alpha E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}kz} = E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}(kz-\alpha\pi)} \\ \eta\overline{\mathbf{H}}_f &= -(-1)^\alpha E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}kz} = -E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}(kz-\alpha\pi)} \end{aligned} \tag{70}$$

Meaning that for fractional field in the region $z < 0$ with DPS the original solution does get a rotation in the $x - y$ plane in anticlockwise angle $\alpha\pi/2$ but added is a spatial phase of $\alpha\pi$. Let us compare the region; $z < 0$ if it were made of DPS material then we have original fields as

$$\overline{\mathbf{E}} = \hat{\boldsymbol{x}}.E_0 e^{-\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}} = -\hat{\boldsymbol{y}}.E_0 e^{-\mathrm{i}kz}$$

The fractional fields as

$$\overline{\mathbf{E}}_f = (-1)^\alpha E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}}_f = -(-1)^\alpha E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{-\mathrm{i}kz}$$

If the same region $z < 0$ is LHM with DNG material the original field is

$$\overline{\mathbf{E}} = \hat{\boldsymbol{x}}.E_0 e^{\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}} = -\hat{\boldsymbol{y}}.E_0 e^{\mathrm{i}kz}$$

The fractional fields are

$$\overline{\mathbf{E}}_f = E_0\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}}_f = -E_0\left[-\hat{\boldsymbol{x}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}\right]e^{\mathrm{i}kz}$$

In both the cases of DPS and LHM media, in the region $z < 0$ the fractional field is obtained as rotation of the original fields by anticlockwise by angle $\alpha(90)^0$ about the $z-$ axis, (looking from positive side); but DPS and LHM with DNG media differ by additional spatial phase $\alpha\pi$.

Let us have a LHM media with DNG, and have right and left circularly polarized transverse electromagnetic (TEM) uniform plane wave propagating in the positive z direction; this case will have fields as follows

$$\overline{\mathbf{E}}_\pm = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0 e^{-\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}}_\pm = (\mp\mathrm{i})(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0 e^{-\mathrm{i}kz} \tag{71}$$

Repeating the steps of (57)-(61), with (71) we get the fractional of dual solution as

$$\overline{\mathbf{E}}_{f(\pm)} = (\pm\mathrm{i})^\alpha(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0 e^{-\mathrm{i}kz} \qquad \eta\overline{\mathbf{H}}_{f(\pm)} = -(\pm\mathrm{i})^{\alpha+1}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0 e^{-\mathrm{i}kz} \tag{72}$$

In this case as compared with the DPS case of (57)-(61); the eigenvalues for LHM, DNG media is $(\pm\mathrm{i})^\alpha$ with eigenvectors as circularly polarized original TEM wave (71).

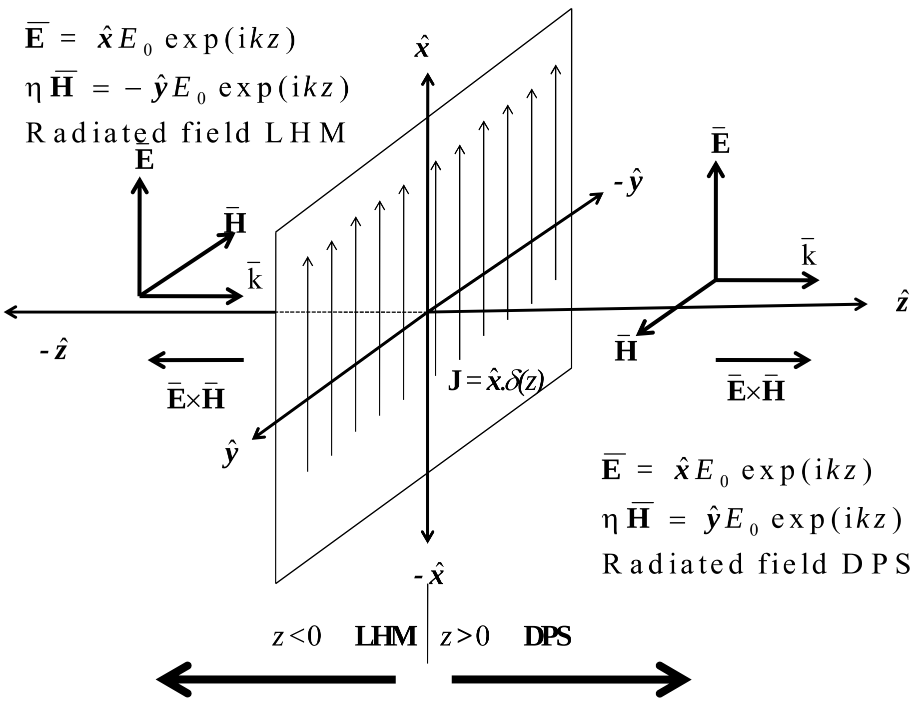


Figure-5 Radiated field from current sheet, propagation in DPS and LHM

9. A CASE OF EM WAVE TRAVELLING FROM DPS REGION TO LHM REGION WITH OBLIQUE INCIDENCE AND ITS FRACTIONAL FIELDS

Here we take a case of a TE wave incident from half space of DPS, $z<0$ to half space of DNG, $z>0$. The original fields in DPS media let it be

$$\overline{\mathbf{E}} = \hat{y}e^{\mathrm{i}k_x x + \mathrm{i}k_z z} \qquad \eta\overline{\mathbf{H}} = \left[-\hat{\boldsymbol{x}} + \hat{\boldsymbol{z}}\right]e^{\mathrm{i}k_x x + \mathrm{i}k_z z} \qquad k = \sqrt{k_x^{\ 2} + k_z^{\ 2}} \qquad \overline{\mathrm{k}} = k_x\hat{\boldsymbol{x}} + k_z\hat{\boldsymbol{z}} \tag{73}$$

The propagation is shown in figure-6. The reflection from boundary is neglected, assuming entire power is transmitted towards LHM region, the figure-6, also shows negative refraction. The method of (50) to (55) is applied to get the fractional solutions; which are demonstrated in following steps for $z<0$.

Take the exponent as: $\mathrm{i}\overline{\mathrm{k}}.\overline{r} = \mathrm{i}k_x x + \mathrm{i}k_y z = (\mathrm{i}\overline{\mathrm{k}}).(\overline{r})$. The wave is propagating in $z<0$ region, in direction of $\overline{\mathrm{k}}$ and in increasing x and z or say in the direction of increasing $\overline{r}$. Therefore the gradient $\mathrm{d}/\mathrm{d}\overline{r}$ is positive. And to take partial derivatives we take in this case $\mathrm{d}^\alpha/\mathrm{d}\overline{r}^\alpha$ and take respective projections on x and z coordinates those are k_x/k and k_z/k respectively. This gives in the following the values of $\partial^\alpha/\partial x^\alpha$ and $\partial^\alpha/\partial z^\alpha$ in the calculations respectively.

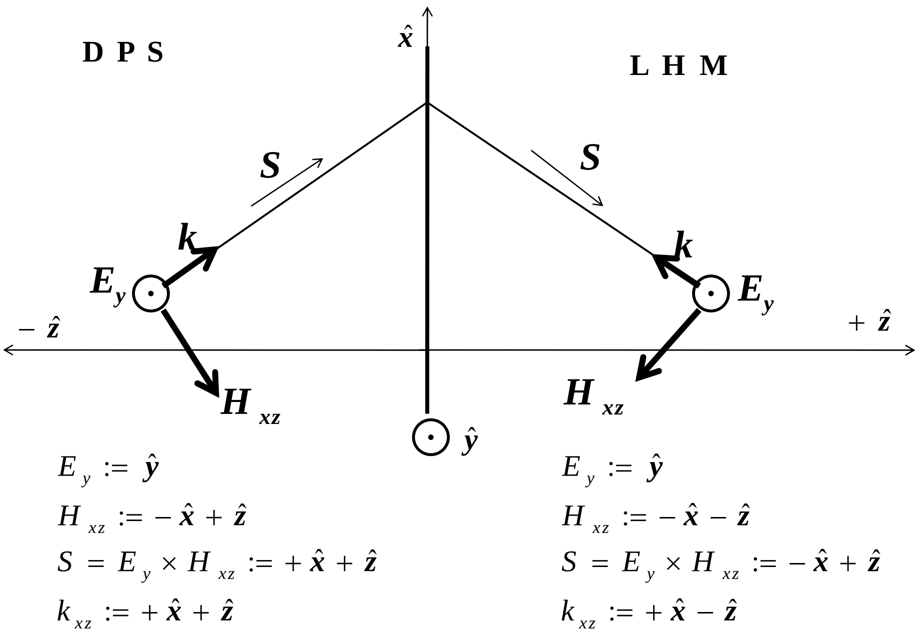


Figure-6: An oblique TE wave from DPS media bending towards LHM media showing negative refraction

$$
\begin{aligned}
(\nabla\times)^{\alpha}\bar{\mathbf{E}} &= \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x^{\alpha} & 0 & \partial_z^{\alpha} \\ 0 & e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}} & 0 \end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix} \nabla_{yz}^{\alpha} & 0 & 0 \\ 0 & \nabla_{xz}^{\alpha} & 0 \\ 0 & 0 & \nabla_{xy}^{\alpha} \end{bmatrix}\begin{bmatrix} (0)\hat{\boldsymbol{x}} \\ (e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}})\hat{\boldsymbol{y}} \\ (0)\hat{\boldsymbol{z}} \end{bmatrix} \\
&= \left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(-\frac{\partial^{\alpha}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial z^{\alpha}}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial^{\alpha}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial x^{\alpha}}\right)\right] + \left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left(\frac{\partial^{\alpha}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial x^{\alpha}} + \frac{\partial^{\alpha}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial z^{\alpha}}\right)\right] \\
&= \left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}(\mathrm{i}k)^{\alpha}\left(-\frac{k_z}{k}\right)e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right] + \left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left((\mathrm{i}k)^{\alpha}\frac{k_x}{k}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}} + (\mathrm{i}k)^{\alpha}\frac{k_z}{k}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)\right] \\
&\quad + \left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}(\mathrm{i}k)^{\alpha}\left(\frac{k_x}{k}\right)e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right] \\
&= \left[\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(-\frac{k_z}{k}\right)\right] + \hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right) + \left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(\frac{k_x}{k}\right)\right]\right](\mathrm{i}k)^{\alpha}e^{\mathrm{i}\bar{\mathrm{k}}.\bar{r}}
\end{aligned}
$$

In above derivation for the (partial) fractional derivative, the projection on respective coordinates is taken. Also in the above derivation for the expression of y component the two terms of partial fractional derivative are vectorically added. Thus fractional electric field is for $z<0$ DPS media is, which is noted in (74)

$$
\bar{\mathbf{E}}_f = \left[\hat{\boldsymbol{x}}\left(-\frac{k_z}{k}\sin\frac{\alpha\pi}{2}\right) + \hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right) + \hat{\boldsymbol{z}}\left(\frac{k_x}{k}\sin\frac{\alpha\pi}{2}\right)\right]e^{ik_x x + ik_z z}
$$

Let us find the fractional magnetic field via similar way

$$(\nabla\times)^{\alpha}\eta\bar{\mathbf{H}} = \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial_x^{\alpha} & 0 & \partial_z^{\alpha}\\ -e^{\mathrm{i}\bar{k}.\bar{r}} & 0 & e^{\mathrm{i}\bar{k}.\bar{r}}\end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ 0 & 0 & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}(-e^{\mathrm{i}\bar{k}.\bar{r}})\hat{\boldsymbol{x}}\\ (0)\hat{\boldsymbol{y}}\\ (e^{\mathrm{i}\bar{k}.\bar{r}})\hat{\boldsymbol{z}}\end{bmatrix}$$

$$= \left(\sin\frac{\alpha\pi}{2}\right)\left[-\hat{\boldsymbol{y}}\left(\frac{\partial^{\alpha}e^{\mathrm{i}\bar{k}.\bar{r}}}{\partial x^{\alpha}} - \frac{\partial(-e^{\mathrm{i}\bar{k}.\bar{r}})}{\partial z^{\alpha}}\right)\right] + \left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(\frac{\partial^{\alpha}(-e^{\mathrm{i}\bar{k}.\bar{r}})}{\partial z^{\alpha}}\right)\right]$$

$$+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(\frac{\partial^{\alpha}e^{\mathrm{i}\bar{k}.\bar{r}}}{\partial x^{\alpha}}\right)\right]$$

$$= \left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}(\mathrm{i}k)^{\alpha}\left(-\frac{k_z}{k}\right)e^{\mathrm{i}\bar{k}.\bar{r}}\right] - \left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left((\mathrm{i}k)^{\alpha}\frac{k_x}{k}e^{\mathrm{i}\bar{k}.\bar{r}} + (\mathrm{i}k)^{\alpha}\frac{k_z}{k}e^{\mathrm{i}\bar{k}.\bar{r}}\right)\right]$$

$$+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}(\mathrm{i}k)^{\alpha}\left(\frac{k_x}{k}\right)e^{\mathrm{i}\bar{k}.\bar{r}}\right]$$

$$= \left[\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(-\frac{k_z}{k}\right)\right] - \hat{\boldsymbol{y}}\left(\sin\frac{\alpha\pi}{2}\right) + \left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(\frac{k_x}{k}\right)\right]\right](\mathrm{i}k)^{\alpha}e^{\mathrm{i}\bar{k}.\bar{r}}$$

Therefore the fractional Magnetic field is

$$\eta\bar{\mathbf{H}}_f = \left[-\hat{\boldsymbol{x}}\frac{k_z}{k}\cos\frac{\alpha\pi}{2} - \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{z}}\frac{k_x}{k}\cos\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x + \mathrm{i}k_z z}$$

Which is noted in (74).

$$\begin{aligned}\bar{\mathbf{E}}_f &= \left[-\hat{\boldsymbol{x}}\frac{k_z}{k}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{z}}\frac{k_x}{k}\sin\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x + \mathrm{i}k_z z}\\ \eta\bar{\mathbf{H}}_f &= \left[-\hat{\boldsymbol{x}}\frac{k_z}{k}\cos\frac{\alpha\pi}{2} - \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2} + \hat{\boldsymbol{z}}\frac{k_x}{k}\cos\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x + \mathrm{i}k_z z}\end{aligned} \tag{74}$$

This fractional solution is rotation of original solution, about direction of propagation k axis, anticlockwise by angle $\alpha(90^0)$, for DPS media

For the region $z > 0$, the LHM region with DPS media we have the fields as

$$\bar{\mathbf{E}} = \hat{\boldsymbol{y}}e^{\mathrm{i}k_x - \mathrm{i}k_z z} = \hat{\boldsymbol{y}}e^{-(-\mathrm{i}k_x x + \mathrm{i}k_z z)} \qquad \eta\bar{\mathbf{H}} = \left[-\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}\right]e^{\mathrm{i}k_x x - \mathrm{i}k_z z} = \left[-\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}}\right]e^{-(-\mathrm{i}k_x x + \mathrm{i}k_z z)}$$

$$k^2 = k_x^2 + k_z^2 \qquad \bar{\mathrm{k}} = -k_x\hat{\boldsymbol{x}} + k_z\hat{\boldsymbol{z}} \qquad e^{\mathrm{i}k_x x - \mathrm{i}k_z z} = e^{-\mathrm{i}\bar{k}.\bar{r}}$$

The following elaborate steps we repeat again, and observe that for fractional gradient in x that is for evaluation of ∂_x^{α} we apply one negative sign. That is because the propagation of power is in positive x and positive z in the LHM region (refer figure-6) along which we take gradient but x is decreasing in this direction. Therefore the minus sign appears for ∂_x^{α}.

$$(\nabla\times)^{\alpha}\bar{\mathbf{E}}=\left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial_x^{\alpha} & 0 & \partial_z^{\alpha}\\ 0 & e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}} & 0\end{bmatrix}+\left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ 0 & 0 & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}(0)\hat{\boldsymbol{x}}\\ (e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})\hat{\boldsymbol{y}}\\ (0)\hat{\boldsymbol{z}}\end{bmatrix}$$

$$=\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(-\frac{\partial^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial z^{\alpha}}\right)+\hat{\boldsymbol{z}}\left(\frac{\partial^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{(-1)\partial x^{\alpha}}\right)\right]+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left(\frac{\partial^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{(-1)\partial x^{\alpha}}+\frac{\partial^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}}{\partial z^{\alpha}}\right)\right]$$

$$=\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}(-\mathrm{i}k)^{\alpha}\left(-\frac{k_z}{k}\right)e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right]+$$

$$+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left((-\mathrm{i}k)^{\alpha}\left(-\frac{k_x}{k}\right)e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}+(-\mathrm{i}k)^{\alpha}\frac{k_z}{k}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)\right]$$

$$+\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}(-\mathrm{i}k)^{\alpha}\left(-\frac{k_x}{k}\right)e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right]$$

$$=(-1)^{\alpha}\left[\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(-\frac{k_z}{k}\right)\right]+\hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)+\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(-\frac{k_x}{k}\right)\right]\right](\mathrm{i}k)^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}$$

In above we have taken negative gradient for x, i.e. $\partial^{\alpha}/(-1)\partial x^{\alpha}$; the projections on x and z axis are k_x/k and k_z/k to compute $\partial^{\alpha}/\partial x^{\alpha}$ and $\partial^{\alpha}/\partial z^{\alpha}$ in above; also for the $\hat{y}$ components we used $\bar{\mathrm{k}}=-k_x\hat{x}+k_z\hat{z}=k$. The fractional electric field is thus as follows noted in (75).

$$\bar{\mathbf{E}}_f=(-1)^{\alpha}\left[\hat{\boldsymbol{x}}\left(-\frac{k_z}{k}\sin\frac{\alpha\pi}{2}\right)+\hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)+\hat{\boldsymbol{z}}\left(-\frac{k_x}{k}\sin\frac{\alpha\pi}{2}\right)\right]e^{\mathrm{i}k_x x-\mathrm{i}k_z z}$$

Now let us calculate the fractional magnetic field as we did for DPS

$$(\nabla\times)^{\alpha}\eta\bar{\mathbf{H}}=\left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ D_x^{\alpha} & 0 & D_z^{\alpha}\\ -e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}} & 0 & -e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\end{bmatrix}+\left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix}\nabla_{yz}^{\alpha} & 0 & 0\\ 0 & \nabla_{xz}^{\alpha} & 0\\ 0 & 0 & \nabla_{xy}^{\alpha}\end{bmatrix}\begin{bmatrix}(-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})\hat{\boldsymbol{x}}\\ (0)\hat{\boldsymbol{y}}\\ (-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})\hat{\boldsymbol{z}}\end{bmatrix}$$

$$=\left(\sin\frac{\alpha\pi}{2}\right)\left[-\hat{\boldsymbol{y}}\left(\frac{\partial^{\alpha}(-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})}{(-1)\partial x^{\alpha}}-\frac{\partial(-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})}{\partial z^{\alpha}}\right)\right]+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(\frac{\partial^{\alpha}(-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})}{\partial z^{\alpha}}\right)\right]$$

$$+\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(\frac{\partial^{\alpha}(-e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}})}{(-1)\partial x^{\alpha}}\right)\right]$$

$$=\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}(-\mathrm{i}k)^{\alpha}\left(-\frac{k_z}{k}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)\right]-\left(\sin\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{y}}\left((-\mathrm{i}k)^{\alpha}\left(\frac{k_x}{k}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)+(-\mathrm{i}k)^{\alpha}\frac{k_z}{k}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)\right]$$

$$+\left(\cos\frac{\alpha\pi}{2}\right)\left[-\hat{\boldsymbol{z}}(-\mathrm{i}k)^{\alpha}\left(-\frac{k_x}{k}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}\right)\right]$$

$$=(-1)^{\alpha}\left[-\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{x}}\left(\frac{k_z}{k}\right)\right]-\hat{\boldsymbol{y}}\left(\sin\frac{\alpha\pi}{2}\right)-\left(\cos\frac{\alpha\pi}{2}\right)\left[\hat{\boldsymbol{z}}\left(\frac{k_x}{k}\right)\right]\right](\mathrm{i}k)^{\alpha}e^{-\mathrm{i}\bar{\mathrm{k}}.\bar{r}}$$

The fractional magnetic field is

$$\eta\bar{\mathbf{H}}_f=-(-1)^{\alpha}\left[\hat{\boldsymbol{x}}\frac{k_z}{k}\cos\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{z}}\frac{k_x}{k}\cos\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x-\mathrm{i}k_z z}$$

Fractional dual solutions for LHM DNG media in region $z>0$ are

$$\bar{\mathbf{E}}_f = (-1)^{\alpha}\left[-\hat{\boldsymbol{x}}\frac{k_z}{k}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\cos\frac{\alpha\pi}{2}-\hat{\boldsymbol{z}}\frac{k_x}{k}\sin\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x-\mathrm{i}k_z z}$$

$$\eta\bar{\mathbf{H}}_f = -(-1)^{\alpha}\left[\hat{\boldsymbol{x}}\frac{k_z}{k}\cos\frac{\alpha\pi}{2}+\hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}+\hat{\boldsymbol{z}}\frac{k_x}{k}\cos\frac{\alpha\pi}{2}\right]e^{\mathrm{i}k_x x-\mathrm{i}k_z z} \quad (75)$$

Fractional dual solutions may be obtained by rotation of the original solution via angle $\alpha(90^0)$ with addition of spatial phase of $\alpha\pi$.

10. CONCLUSIONS

In this paper we have, derived the process of, fractioning of cross product operation, and used the concept to derive fractional curl; via a very new geometrical means. This development has given insight of mechanism of fractional operator. In this paper we noted down elaborate formulations that are useful for applications especially in electrodynamics, and applied the same in several examples, of DPS and LHM medium. For DPS we have $(\bar{\mathbf{E}},\eta\bar{\mathbf{H}})$ and $(-\eta\bar{\mathbf{H}},\bar{\mathbf{E}})$ as original and its dual solution whereas that for LHM $(\bar{\mathbf{E}},\eta\bar{\mathbf{H}})$ and $(-\eta\bar{\mathbf{H}},\bar{\mathbf{E}})$ is two sets of solution to the Maxwell equation. The fractional solutions are obtained in both the systems via the formulated technique in this paper for fractional curl but it is also pointed out that fractional dual solutions for LHM is with additional multiplication of phase factor $\alpha\pi$. This new method will be useful for many problems with LHM or DPS, especially for polarization, reflection studies, impedance studies, boundary condition; where fractional solutions are reality.

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