Backward Wave Hidden Momentum and Negative Square root for Left Handed Maxwell System-by Thought Experiment

Dedicating this new thought to my beloved mother Purabi Das

Shantanu Das (shantanu@barc.gov.in)
Scientist, Reactor Control Division, Bhabha Atomic Research Centre (BARC) Mumbai

Key words

Backward wave, hidden momentum, meta-material, left handed cross product, negative root

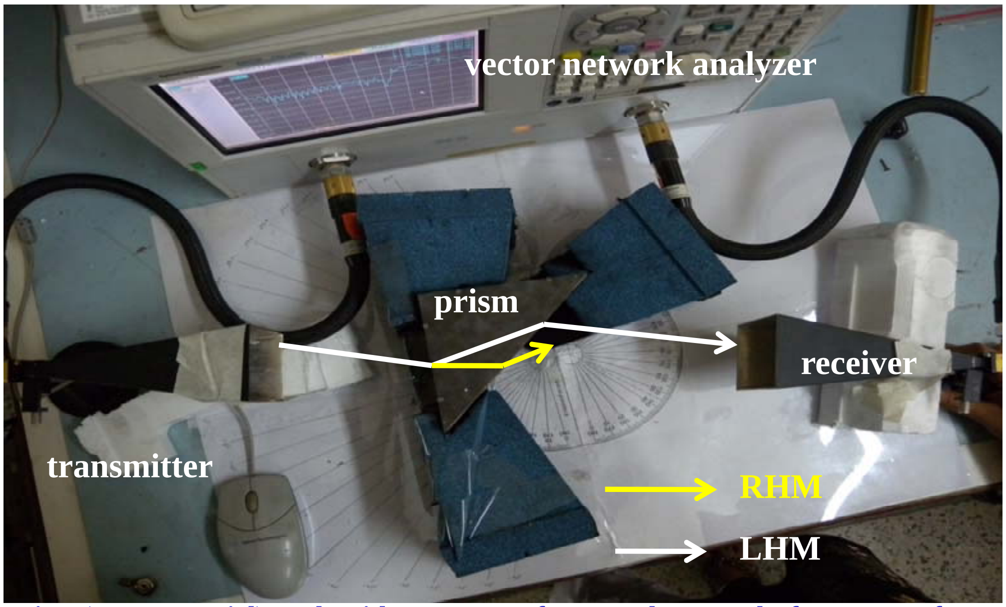
Abstract

Why I have called my project as "Left Handed Maxwell Systems" is due to counterintuitive nature of the cross product I need to take in the Maxwell equations, to satisfy that wave vector as opposite to the Poynting vector when the dielectric permittivity and magnetic permeability are both negatives; that is giving me a media of refractive index negative. This paper explains these phenomena, with mathematical and possible physical explanations-via thought experiment; elaborating concept of 'backward wave' and 'hidden momentum' and 'negative root'. Though several approaches to explain these counterintuitive phenomena have been evolving, yet it is interesting if in the meta-material parlance this theory be founded, via thought experiments!

1. Introduction

I am doing a project with a team (with SAMEER Kolkata Scientists guided by Professors from University of Calcutta and other institutions) called "Left Handed Maxwell Systems", have demonstrated negative refractive index 'meta-material' plasmonic structures in Ka-band (33 GHz). In our experimental investigation, we have made these plasmonic meta-material prisms of 45, 30 and 15 degrees to get enhanced transmittance of more than 15 dB from background; at negative angles indicating a refractive index of about -1.8. The experimental set up with meta-material prism is depicted in figure-1, and the difference between Right Handed Material (RHM) normal prism, and Left Handed Material (LHM) for reversal of refraction is compared in the same figure-as produced by ray diagram. Why I have called the project as "Left Handed Maxwell Systems" is due to counterintuitive nature of the cross product I need to take in the Maxwell equations, to satisfy that wave vector is opposite to the Poynting vector when the dielectric permittivity and magnetic permeability are both negatives; (that is giving me a media of refractive index negative). The meta-material is though new to all of us; but was first conceived by Sir Jagadeesh Chandra Basu (Sir J C Bose) in 1898. The meta-material 'theory' is really counterintuitive-several interesting explanations are given in reference section. In this paper I tried to explain the evolution of 'backward wave' in meta-material resonating elements;

and tried to explain why shall I call my project a "Left Handed Maxwell" System; by deriving via simple reasoning that the cross product of wave vector with electric or magnetic field vectors needs be taken via 'left hand'. Here I also elaborate via thought experiment, the physics of 'concept of hidden momentum' arising out of no movement of the resonating element of the meta-material structure, that too in reverse direction to the 'energy flux flow'.



Prism (meta-material) made with LR a type of SRR and WA stacks for 33 G Hz frequency, the excitation is TE10 mode of EM from 26GHz to 40GHz, with *E* parallel with the wire structure (that is the negative EPSILON sheet interlaced SRR), *H* perpendicular to SRR

Figure-1 Experimental set up to demonstrate reversal of Snell's law

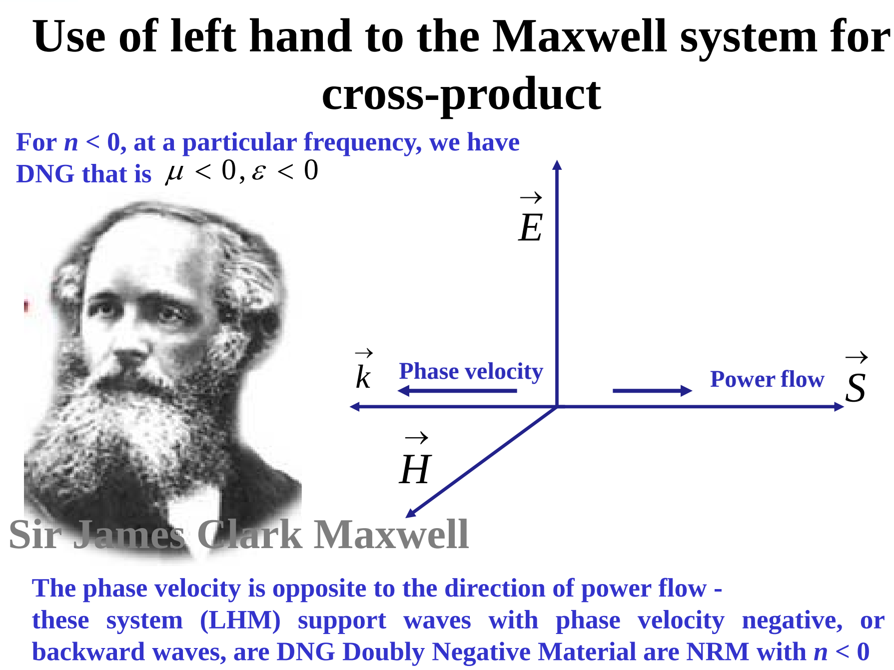


Figure-2: The Left Handed Maxwell Systems

2. How the negative mu and negative epsilon are realized, and makes "Left Handed Maxwell System"

Our refractive index needs to be anomalous for negative indexed systems of meta-material, and most importantly that the interaction of EM waves to the system media should be with 'bound-free' charges. I have metals where I have free electrons-so metals can give me negative index of refraction! At UV frequency the metal's free electrons behave as 'plasma', that free electrons does not have ω_0, in the equation of motion of bound electron, that is:

$$m_e\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\gamma\frac{\mathrm{d}x}{\mathrm{d}t}+\omega_0^2x\right)=F=-eE_0e^{\mathrm{i}\omega_c t}$$

The above equation is classical equation of motion of bound rotating (oscillating) charge with oscillating frequency in the bound potential as ω_0 and having charge of $-e$ coulombs, with mass m_e with damping γ , under oscillating incident electric field $E_0e^{\mathrm{i}\omega_c t}$. So, for unbounded charge with $\omega_0=0$ I can write the EM interaction with inclusion of damped motion of free electrons as

$$m_e\ddot{x}+m_e\gamma\dot{x}=-eE_0e^{\mathrm{i}\omega_c t}$$
$$m_ex(-\omega^2+\mathrm{i}\omega\gamma)=-eE_0e^{\mathrm{i}\omega_c t}$$
$$x=\frac{-eE_0e^{\mathrm{i}\omega_c t}}{m_e(-\omega^2+\mathrm{i}\omega\gamma)}$$

I have used $\mathrm{d}/\mathrm{d}t$ as $\mathrm{i}\omega$ in above to get displacement's solution. Here I assume that wave length of the EM shining the material (metal) is substantially larger than path length of electron, so that effectively the electron sees a spatially constant field, and velocities are low so that I can forgo magnetic fields and its effect. This gives me 'polarization' per unit volume and I write that as

$$P = \varepsilon\varepsilon_0 E_0 e^{\mathrm{i}\omega_c t} - \varepsilon_0 E_0 e^{\mathrm{i}\omega_c t} = (\varepsilon - 1)\varepsilon_0 E_0 e^{\mathrm{i}\omega_c t} = -Nex$$

Where N is the number density of free electrons and all of them contributing to the polarization. On this polarization expression substituting the displacement expression for x, with algebraic manipulations and also I use the expression for plasma frequency as $\omega_{ep}{}^2 = (Ne^2)/m_e\varepsilon_0$; I obtain 'dispersion' expression for the 'effective dielectric' constant of the metals as

$$\varepsilon = 1 - \frac{\omega_{ep}^2}{\omega^2 - \mathrm{i}\gamma\omega}$$

So I have a negative dielectric permittivity below plasma ω_{ep} frequency for metals. This plasma frequency in case of metal is in UV ranges. So naturally metals behave as epsilon negative materials (ENG) below plasma frequency. I can bring down this plasma frequency to G Hz range of X, Ka band by making the number density very low-by using thin metal wires embedded sparsely in a dielectric (or in air). The ENG behavior I owe it to 'electric polarization'. Well with ENG I can realize negative index as $n^2 = -\varepsilon_r$ for $\omega_c < \omega_{ep}$; and waves will be of 'evanescent' bounded waves in ENG (here the wave vector is imaginary in direction of propagation). I can also comment in the ENG region the response of the material up to plasma frequency is 'out-of-phase' to the driving field. I can approximate this epsilon's dispersion as $\varepsilon_r \cong 1 - (\omega_{ep}{}^2/\omega^2)$. In short I can say that this dispersion effect of dielectric constant is 'resonance' effect.

I will briefly now state the artificial realizations of the 'negative' magnetic permeability. These are resonating elements. The classical structure to realize the negative mu is split ring resonator SRR. The SRR are concentric rings split at one at 0^0 and the inner ring separated by gap d, split at 180^0 relative to the first one. The SRR works on principle of the magnetic field of EM radiation, which drives a resonant LC element (circuit) through inductance, and resulting in dispersive magnetic permeability as

$$\mu = 1 - \frac{\pi r^2/a^2}{1 - (3d/\mu_0\varepsilon_0\varepsilon\pi^2\omega^2 r^3) + \mathrm{i}(2\rho/\mu_0\omega r)}$$
$$= 1 - \frac{F\omega^2}{\omega^2 - \omega_R{}^2 - \mathrm{i}\Gamma\omega}$$

With filling factor is $F = \pi r^2/a^2$, the 'magnetic plasma' $\omega_{mp}{}^2 = (3d)/\left[\mu_0\varepsilon_0\varepsilon\pi^2 r^3\right]$; circuit resonance frequency is $\omega_R{}^2 = (3d)/\left[(1-F)\mu_0\varepsilon_0\varepsilon\pi^2 r^3\right]$. Say I design the SRR with $d = 0.2\mathrm{mm}$, radius of outer ring as $r = 1.5\mathrm{mm}$, lattice spacing for SRR repeated structure as $a = 5\mathrm{mm}$, the copper resistivity $\rho \cong 0$, with ε the dielectric permittivity of gap (capacitance) I get magnetic plasma frequency as $7.56\mathrm{GHz}$ and the SRR circuit resonance frequency as $6.41\mathrm{GHz}$. The dispersion in mu I can approximate as, $\mu_r = 1 - F\omega^2/(\omega^2 - \omega_R^2)$, with magnetic plasma frequency as $\omega_{mp} = \omega_R/\sqrt{1-F}$; at this mu is zero and below this the value is negative.

The 'artificially' structured magnetic activity I obtain a negative value from the circuit resonance frequency ω_R to 'magnetic' plasma frequency ω_{mp}. This negative realization is 'resonance' where very high resonating EM fields are obtained. Therefore response of SRR, near the resonance ω_R, and up to ω_{mp} I expect 'out of phase' response to the driving EM field! This way I can get negative mu material (MNG).

This ENG and MNG together if I tune (design) I get a region where both mu and epsilon are negatives; and that I get a region as DNG (Double Negative), a region where the refractive index is negative, and media DNG supports propagation of EM waves, but in backward phase!

3. Cross Product with Left Hand?

Does it help me to make 'Left Handed Maxwell' system having realized artificially a media with ENG $\varepsilon < 0$ and MNG $\mu < 0$? Let me take a plane wave travelling in $+z$ direction, with electric field E_R pointing towards $+x$ direction, the magnetic field H_R pointing in $+y$ direction, a travelling wave($E_R = E_0 e^{(\mathrm{i}k_R z - \mathrm{i}\omega t)}$) in normal dispersive media, that is with $\varepsilon_R > 0$ and $\mu_R > 0$ the media properties. This is a wave travelling in RHM right handed media! Well the Poynting flux with right hand cross product is (the power flow direction), that is $S_R = E_R \times H_R$, in $+z$ direction. The Maxwell's equations in RHM are

$$\nabla \times E_R = -\mathrm{i}\omega\mu_R H_R$$
$$\nabla \times H_R = +\mathrm{i}\omega\varepsilon_R E_R$$

In this RHM the wave vector k_R is in direction of propagation that is in $+z$ direction, and let me re write the RHM Maxwell's equations in terms of wave vector as (comes from above curls)

$$k_R \times E_R = +\omega\mu_R H_R$$
$$k_R \times H_R = -\omega\varepsilon_R E_R$$

Above cross product I obtain via right hand.

Now as I pointed out that the media what I made is a resonating structure, especially the realization of the MNG via SRR. This resonance gives me a response of very high electric and magnetic fields at near about resonance. As I pointed out the response at the electric and magnetic resonance in the region of ENG and MNG will give 'phase opposition' to the excitation EM signal; so I should get a strong E_L in $-x$ direction (opposite to E_R) and strong H_L in $-y$ direction (opposite to H_R). The near resonance response of the DNG media with ENG and MNG is very strong, so resultant response is with E_L and H_L only (the incident is negligible compared to these giant fields, and I obtain 'out of phase' response!). Now if I check with the right hand the cross product $E_L \times H_L$, gives me the power flow S_L in the original direction of $+z$ directed!

Now if I re-write the Maxwell's equation by putting the media properties as negatives, that is by putting the values as, $\mu_L < 0$, and $\varepsilon_L < 0$ I get the curl expressions for NRM DNG media as

$$\nabla \times E_L = +\mathrm{i}\omega\mu_L H_L$$
$$\nabla \times H_L = -\mathrm{i}\omega\varepsilon_L E_L$$

I cannot use k_R to make the equations with MNG and ENG so I write inequality and equality as

$$k_R \times E_L \neq -\omega\mu_L H_L = +\omega\mu_L H_L$$
$$k_R \times H_L \neq +\omega\varepsilon_L E_L = -\omega\varepsilon_L E_L$$

The, vector k_R is in $+z$ direction, so if I use right hand and take cross with E_L my thumb points towards $-y$ or in H_L's direction and not in desired $-H_L$'s way! Similarly I do right handed cross of k_R with H_L, I find my thumb towards $+x$ that is towards $-E_L$ as against desired direction E_L. From here I infer that k_R is not the direction of 'wave-vector'!

Let me reverse the wave vector and point it towards $-z$ direction (opposite to k_R), and I call it k_L and write the equations as desired (off course the following will come from the revised curls)

$$k_L \times E_L = -\omega\mu_L H_L$$
$$k_L \times H_L = +\omega\varepsilon_L E_L$$

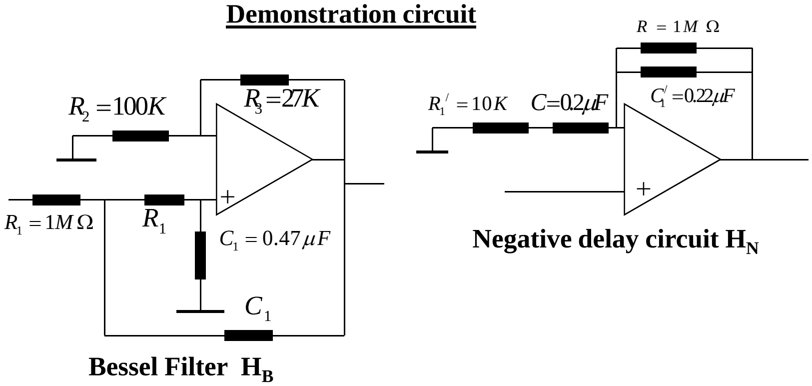
I find that k_L with E_L and H_L is following the right handed cross product rule to satisfy above. The fact is $+x$, $+y$ and $+z$ follows right handed cross product rule, but $-x$, $-y$ and $-z$ will follow the 'left handed cross product' rule. But the above cross product equations for NRM DNG media satisfy right handed cross product rule since I have made 'phase inversion' of E and H too. So where is left handed cross product? The answer lies in my reference coordinate system, that is E in $+x$, H in $+y$ and let me write the following.

Now if I write a general for DNG or NRM media that k is opposite (that is in $-z$ direction) to $S = E \times H$ ($+z$ direction) with, E in $+x$ and H in $+y$ direction the Maxwell's equations for $\varepsilon < 0$ and $\mu < 0$ are

$$\begin{matrix} \nabla \times E = +\mathrm{i}\omega\mu H \\ \nabla \times H = -\mathrm{i}\omega\varepsilon E \end{matrix} \quad \text{and} \quad \begin{matrix} k \times E = -\omega\mu H \\ k \times H = +\omega\varepsilon E \end{matrix}$$

The reference remains E as pointing towards $+x$, H pointing towards $+y$, and travelling in $+z$ as for source incident field of RHM. The Poynting flux follows the right handed cross product while the k E H triad follows a cross product with left hand! This is why called this system with ENG and MNG (DNG) an NRM media a 'Left Handed Maxwell's system'. Figure-2 elaborates this left handed cross product. The k opposite to S means 'backward wave'. The energy moves forward in $+z$ direction, but the phases crest & troughs move the opposite that is towards $-z$ direction.

Demonstration circuit



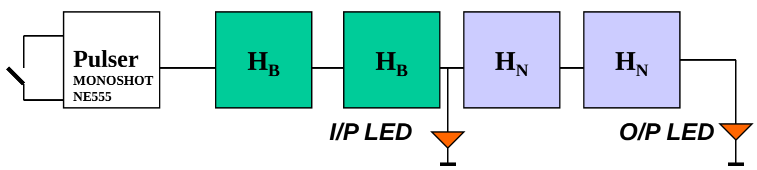
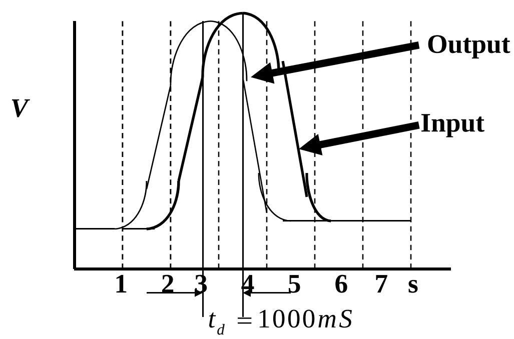


Figure-3 Demonstration circuit of backward wave in LHM idea of faster than light propagation?

Demonstration Circuit pulses observations and comments



Output LED glows before input LED.
CRO gives the measure of negative delay about one second.

Figure-4 CRO output showing peak of output appears before 1 second as compared to input peak

4. The Backward Wave Realization with its Physical Generation

I can demonstrate backward wave, by a circuit presented in figure-3. Here I have emulated LHM via circuit techniques, where the output LED glows before the input LED; giving idea of faster than light propagation! The CRO record of about one second pulse peak advancement is depicted for this circuit in figure-4. The same or rather similar effect is obtained when I make Periodically Loaded Transmission Line (PLTL) depicted in figure-5. The values of the Transmission line are depicted in Table-1, and the 'faster than light' effect is shown in figure-6. The figure-7 depicts its dispersion characteristics ($\omega-\beta$) dispersion diagram, the region where I get effect of Negative Group Velocity ($\mathrm{d}\omega/\mathrm{d}\beta<0$), though $\omega/\beta>0$, the region where phase and group velocity are of opposite signs, is NRM region. Is it so? I will justify this anomaly in the last section as to actually I should have NPV and positive group velocity; after all figure-7 is in first quadrant! Wait for last section, for explanation of this anomaly.

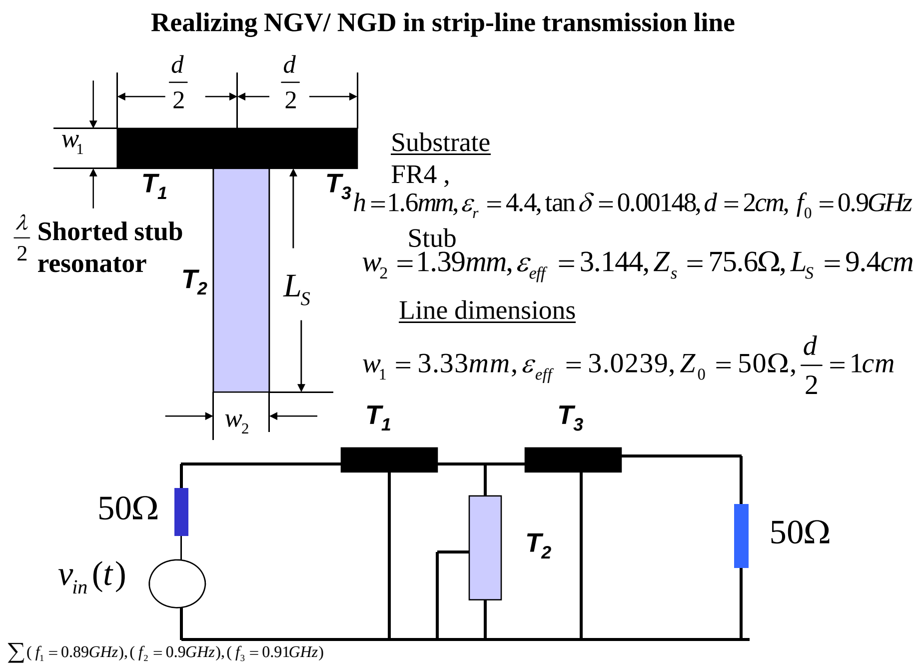


Figure-5 Periodically Loaded Transmission Line PLTL to make LHM

Negative Group Velocity/ Negative Group Delay in time domain

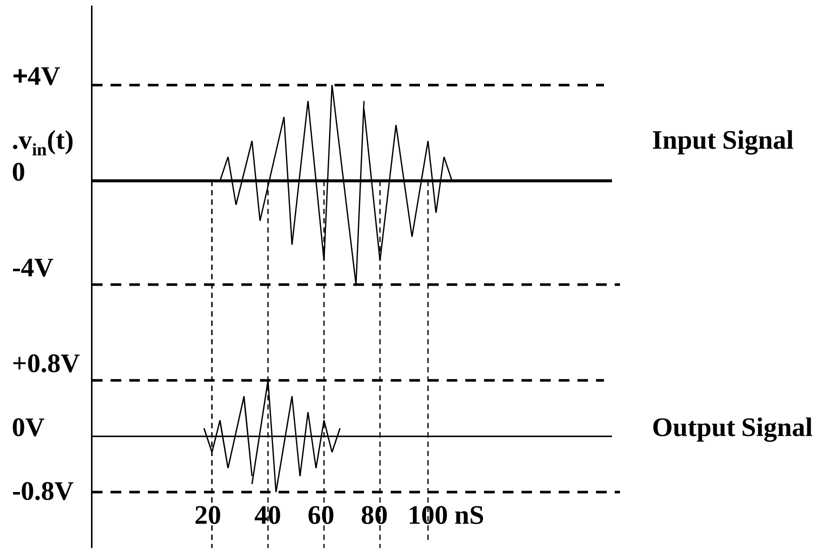


Figure-6 Input output response to have feel of 'faster than light' propagation through LHM

NGV /NGD from dispersion diagram of TL-RLC Shunt structure

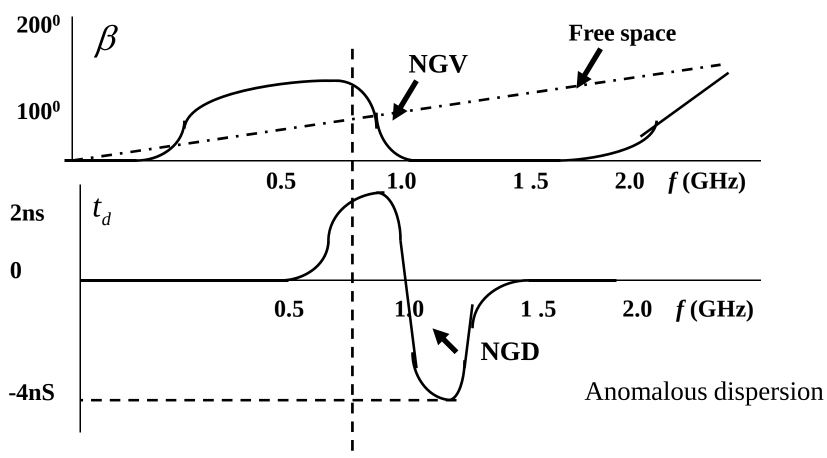


Figure-7 Dispersion diagram of PLTL showing region of Negative Group Delay and Negative Group Velocity

Values of electrical parameters of the strip-line sections

Line	R Ω / m	L nH / m	G $\mu S / m$	C pF / m	length (cm)
T_1	5.0928	304.15	10200	121.66	1.000
T_2	11.079	446.84	6500	78.18	9.4
T_3	5.09 28	304.15	10200	121.66	1.000

Table-1 Showing Parameters Values of PLTL

5. Concept of 'Hidden Momentum' in Backward Wave

Now I shall try and explain the physical behavior of resonating element. Refer figure 8, depicting lattice of meta-material comprising of the wire-array structure and (square) split ring resonator. The excitation or driving plane wave has magnetic field directed towards the $+y$ axis, the driving plane waves travel in $+z$ direction, with electric field in the $+x$ direction.

In the SRR via Lenz's law there will be induced currents direction shown in the figure, to oppose the driving field, and at resonance there is very strong 'phase opposition' to the driving field. Thus a resultant field will be in $-y$ direction. Similarly the ENG realized by the wire-array will give a strong phase opposed electric field response, giving resultant electric field in the $-x$ direction. The same I had explanted in justifying the term 'Left Handed Maxwell's' Systems; that these meta-materials are. The figure-8 depicts the response fields as of opposite phase, and very large-call them giant fields.

Thus here a magnetic dipole is formed, what it does under action of E_L is our 'thought experiment' in this section. As I may guess this action of electric field to the dipole moment can have associated energy flux and a momentum even if the dipoles are not moving! A hidden momentum then!

Backward Wave & Hidden Momentum

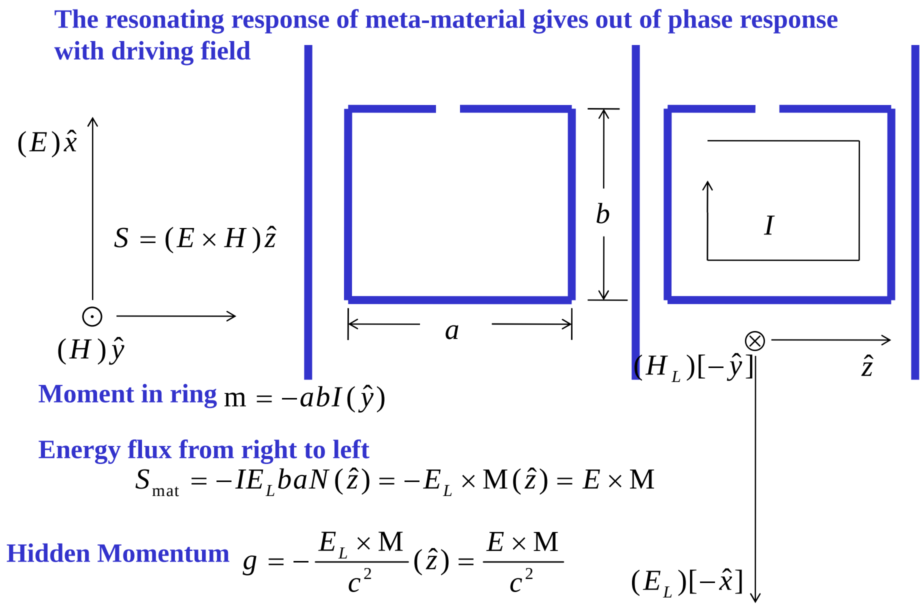


Figure-8 Physics of Backward Waves and Hidden Momentum.

I can write the magnetic moment as $\mathrm{m} = -abI(\hat{y})$, directed towards $-y$ direction. Now carefully look at the figure-8, where I have shown the current I in the resonating ring, the direction of movement of positive charges. The electric field (the resultant one) that is E_L pointed in $-x$, does 'positive' work on the positive charges that goes down at the right arm of the resonating square ring; and does a negative work at the left arm, while the positive charges are climbing up the left side arm. So a positive charge (say q_e) moving from right side of the square ring to the left side has extra energy, which I can quantify as $q_e E_L b$ (with respect to situation when the charge moves from left to right of the ring).

This argument generates an 'energy flux' at a point (and pointing towards $-z$ direction). In each magnetic dipole this energy flux is there, and says there are N such rings, where dipoles are formed. I can state that in each ring the energy flux at a point per unit time is $IE_L b$. This is my flow quantity of energy crossing dz. So the energy flux per ring (integrate $\mathrm{d}z$ from 0 to a) I get, as $-IE_L ba(\hat{z})$ directed towards $-z$. Thus total energy flux from N magnetic dipoles I can write as $S_{\mathrm{mat}} = -IE_L baN(\hat{z}) = -E_L \times \mathrm{M} = E \times \mathrm{M}$. This M is magnetization of the medium. Therefore if I have a magnetization M as in the ring resonator, with source electric field E, there is a 'hidden momentum' density $g = (E \times \mathrm{M})/c^2$, directed towards $-z$ direction.

The momentum of charges that move towards the left of the ring is higher because they pose higher energy. The ratio between momentums to energy of relativistic particle is v/c^2. The hidden momentum concept is therefore a purely relativistic effect as I have not considered $m\mathrm{v}$ type of momentum! This 'hidden momentum' is phase reversal, thus in meta-materials with

NRM there is a concept of 'backward wave' the phases travel in backward direction to the power flow direction! This is happening due to negative composite properties at resonance, and not due to Bragg's (anomalous) scattering like in photonic crystals.

I have explained just above the hidden momentum that is due to interaction of electric field to the magnetization of resonating split-rings. Is there a converse that is any other momentum of the EM radiation, as effect of 'magnetic field'? I will now see this interaction. The electric field acts on charges (free charges) of wire-array, as they drive them up down with them. When I consider say positive free charges, and the resonance effect of NRM then the displacement as well as velocity will be directed opposite so I can take velocity of the free positive charge in $-x$ direction. The induced B_L is resultant due to phase opposition in $-y$ direction of H_L (as explained in the figure-8). Thus there force on the charge as $F = q_e \mathrm{v} \times B_L$ and is directed in $+z$ direction. If I take $B_L = E_L / c$, I get $F = q_e(\mathrm{v}E_L)/c$, directed in $+z$ direction. But $q_e E_L$ is electric force on the charges, times v the velocity is work done per unit time. Thus I can state that the force gives a 'pushing momentum' directed in $+z$ direction-this is also related to a mechanical pressure of radiation; which I must state is positive by this argument in NRM.

Thus I see a hidden momentum, and a pushing momentum for EM radiation which are directed opposite in NRM media.

6. Why for Negative Index take negative root of product of two negative quantities?

Let me first draw attention that the refractive index $n = \mathrm{Re}\, n + \mathrm{i}\,\mathrm{Im}\, n$ is derived with electrodynamics principles in text books, is a complex quantity; the imaginary part of the refractive index comes because of 'damping' term in equation in motion (oscillator) of charges. The basic property of media that is dielectric permittivity and magnetic permeability gives me the index of refraction, that is, $n = \sqrt{\varepsilon\mu}$ for the medium. Well in the resonance of epsilon and mu, near electric and magnetic plasma frequency, I get negative epsilon and negative mu. So I say my index of refraction is; $n < 0$, a negative number. That is $n = -\sqrt{(-\varepsilon)(-\mu)}$; surprising! The mathematicians will scold me on this issue, how should I take a negative square root of product of two negative numbers?

Let me start with a naïve approach. In actual cases I have loss tangents for dielectric and for magnetic permeability; signifying amount of losses in those material. Therefore actually I can write negative values of epsilon and mu as $|\varepsilon|e^{\mathrm{i}\pi}$ and $|\mu|e^{\mathrm{i}\pi}$, which are actually complex numbers. With this representation I can go ahead and say that $n = \sqrt{|\varepsilon||\mu|e^{\mathrm{i}\pi}} = -\sqrt{\varepsilon\mu}$. A very raw explanation, that I must take negative root. Mathematicians are not satisfied. I say this is a very raw and naïve approach, but opens up possibility of further arguments, to consider negative roots.

Well, I must have second naïve argument, with respect to the 'radiation' of power, as my wave propagates. The power radiated inside any media (be it NRM) depends on the 'wave-impedance' Z , and is always shall be positive for power radiating away, that is $Z = \sqrt{\mu / \varepsilon} > 0$.

Let me write $Z = \sqrt{\mu / \varepsilon} = \sqrt{\mu\mu / \varepsilon\mu} = \mu / n > 0$; for $\mu > 0$ and $n > 0$, is normal refractive index positive. But it also tells me that for negative epsilon and mu I should have $n < 0$, the index I should have negative sign! Also writing the same as $Z = \sqrt{\mu\varepsilon / \varepsilon\varepsilon} = n / \varepsilon > 0$, tells me that I should choose the sign of refractive index as 'negative', for negative epsilon. Therefore, for a system which must radiate power away, the wave impedance indicates that for DNG material (epsilon and mu both negatives) I must choose the sign of refractive index as negative. Still mathematicians are not satisfied. However, I must state that these two naïve approaches have given me free hand to put a minus sign in front of refractive index!

The complex wave vector and wave propagation

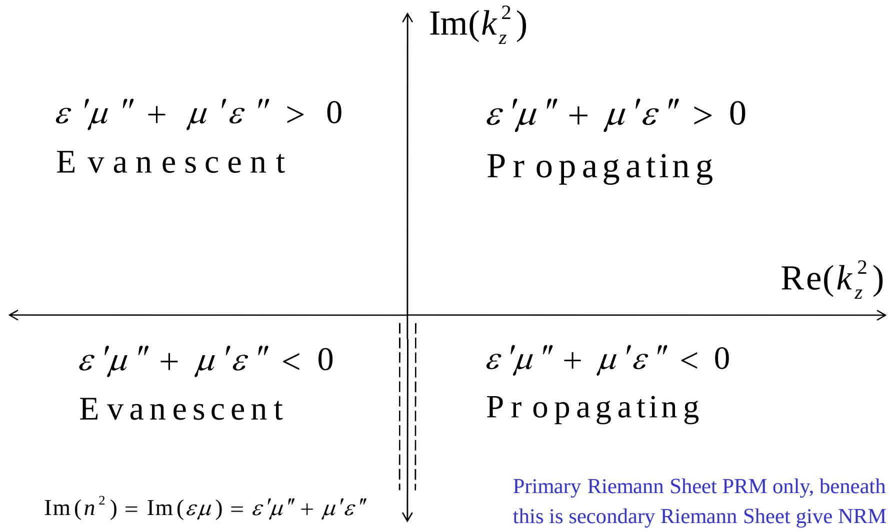


Figure-9 Complex plane for wave vector square

First let me write the Maxwell' equation for electric field as $\nabla^2 E + k^2 E = 0$, for a source free region; is nothing but a wave equation. Note here that the Maxwell's equation does not state about the sign of wave vector (wave number k). Where, $k^2 = \omega^2 \varepsilon\mu = \omega^2 n^2$, is the wave number, what physicists call, and wave vector an engineer will state as, to specify the propagation of EM waves. The solution to the Maxwell's equation is $E = E_0 e^{-\mathrm{i}k \bullet r}$, a case of plane wave as none of its variable change in the plane perpendicular to wave vector k. The wave vector is also $k^2 = k_x^2 + k_y^2 + k_z^2$, and also $k = \beta - \mathrm{i}\alpha$; with β as propagation coefficient, and α the attenuation constant. The plane wave travelling in $+z$ direction I have thus, the electric field as $E = E_0 e^{-\mathrm{i}\beta z} e^{-\alpha z}$, a decaying spatially oscillating wave in a lossy media. Well in lossless case I have $\alpha = 0$ and thus $k = \beta$. This was general revisit to concept of travelling wave, nevertheless the associated perpendicular H field does travel similarly (with magnitude divided by impedance of media), that is $Z = E / H = \sqrt{\mu / \varepsilon}$.

Let me now develop this concept of the 'complex wave vector' while radiation propagates inside the media positive or be it negatively refracting. Consider an EM wave vector $(k_x, 0, k_z)$ with propagation in $+z$ direction incident from free space, that is $(-\infty < z < 0)$. At $z = 0$, I have a semi infinite media from $(0 < z < \infty)$, other than free space with material properties as ε and μ . Due to invariance in x -direction, I preserve the k_x across the boundary. The propagating component k_z is found from rule given as following

$$k_z = \pm\sqrt{\varepsilon\mu \frac{\omega^2}{c^2} - k_x^2}; \qquad k_z^2 = \varepsilon\mu \frac{\omega^2}{c^2} - k_x^2$$

Now the physical choice is to be made for sign of the square root above. It may so happen that the second media the semi infinite one, could be propagating media if the $k_x^2 < \mathrm{Re}(\varepsilon\mu\omega^2 / c^2)$, or $\mathrm{Re}\, k_z^2 > 0$ which gives me k_z as real; or the media may be supporting decaying evanescent waves, while k_z is imaginary, that is when I have condition of $k_x^2 > \mathrm{Re}(\varepsilon\mu\omega^2 / c^2)$ or $\mathrm{Re}\, k_z^2 < 0$. This enables me to draw a plane depicting plane for k_z^2 as $k_z^2 = (\mathrm{Re}\, k_z^2) + \mathrm{i}(\mathrm{Im}\, k_z^2)$.

If I look at expression of k_z^2 , I find the real and imaginary part comes from $\mathrm{Im}(\varepsilon\mu) = \mathrm{Im}\left[(\varepsilon' + \mathrm{i}\varepsilon'')(\mu' + \mathrm{i}\mu'')\right] = \varepsilon'\mu'' + \mu'\varepsilon''$, when I take these epsilon and mu as complex quantities. The four quadrants based on propagating and evanescent I have depicted in figure-9. Let me now write the dielectric permittivity and magnetic permeability of a 'lossy' medium as $\varepsilon \equiv \varepsilon' + \mathrm{i}\varepsilon''$ and $\mu \equiv \mu' + \mathrm{i}\mu''$; complex quantities indeed, with real part with prime and imaginary part as double-prime. The convention I take for absorbing media when $\varepsilon'' > 0$ and $\mu'' > 0$; for 'amplifying' media I take the imaginary parts of both epsilon and mu as less than zero (negative). I consider a 'plane wave' proportional to $e^{\mathrm{i}(nk_0)z}$, traveling in 'absorbing' media, (k_0 is free space wave vector) and do the following arithmetic

$$\begin{aligned}
n &= \pm\sqrt{\varepsilon\mu} = \pm\sqrt{(\varepsilon' + \mathrm{i}\varepsilon'')(\mu' + \mathrm{i}\mu'')} \\
&= \pm\sqrt{(\varepsilon'\mu' - \varepsilon''\mu'') + \mathrm{i}(\varepsilon'\mu'' + \mu'\varepsilon'')} \\
&\cong \pm\sqrt{\varepsilon'\mu' + \mathrm{i}(\varepsilon'\mu'' + \mu'\varepsilon'')} \\
&\cong \pm\sqrt{\varepsilon'\mu'}\left(1 + \frac{\mathrm{i}(\varepsilon'\mu'' + \mu'\varepsilon'')}{2\varepsilon'\mu'}\right) \\
&\cong \pm\left(\sqrt{\varepsilon'\mu'} + \frac{\mathrm{i}}{2}\frac{\varepsilon'\mu'' + \mu'\varepsilon''}{\sqrt{\varepsilon'\mu'}}\right)
\end{aligned}$$

I get, from above $n = \pm(\mathrm{Re}\, n + \mathrm{i}\,\mathrm{Im}\, n)$, what I got in the section of refractive index, a complex one (that one due to damped harmonic motion of charges).

If the media is absorbing, with $\varepsilon'' > 0$ and $\mu'' > 0$ also has negative epsilon and negative mu, that is $\varepsilon' < 0$, and $\mu' < 0$, I have the index for NRM as

$$n = \pm\left(\sqrt{\varepsilon'\mu'} - \frac{\mathrm{i}}{2}\frac{\varepsilon'\mu'' + \mu'\varepsilon''}{\sqrt{\varepsilon'\mu'}}\right) = \pm\mathrm{Re}\, n \mp \mathrm{i}\,\mathrm{Im}\, n$$

If I choose positive sign of above, then the plane wave in semi infinite absorbing NRM get the form as following (I have only written the complex part)

$$\exp(\mathrm{i}nk_0 z) = \exp(\mathrm{i}k_0 z\sqrt{\varepsilon'\mu'}) + \exp\left(\frac{\varepsilon'\mu'' + \mu'\varepsilon''}{2\sqrt{\varepsilon'\mu'}}\right)k_0 z$$

The above states that as the distance, z grows, the spatially oscillating plane waves 'grows' in amplitude inside negative refractive indexed material (NRM)! On contrary, waves should decay in the dispersive media; thus I cannot select positive root; this enables me to select the negative root (and only negative root) for refractive index of DNG, NRM. I write negative refractive index, for doubly negative material therefore as, with negative sign as following

$$n = -\left(\sqrt{\varepsilon'\mu'} - \frac{\mathrm{i}}{2}\frac{\varepsilon'\mu'' + \mu'\varepsilon''}{\sqrt{\varepsilon'\mu'}}\right) = -\sqrt{\varepsilon'\mu'} + \frac{\mathrm{i}}{2}\frac{\varepsilon'\mu'' + \mu'\varepsilon''}{\sqrt{\varepsilon'\mu'}}$$

Could I have satisfied mathematicians now, for selecting negative root for negative epsilon and negative mu? Perhaps.

Now I draw attention towards wave vector k_z, which inside a medium for travelling wave is actually nk_0. Therefore, the imaginary part of the refractive index that is mainly from the quantity $\varepsilon'\mu'' + \mu'\varepsilon''$, determines the nature of propagation (properties) in any medium (absorbing, amplifying, positively refracting or negatively refracting). This quantity is also $\mathrm{Im}(n^2) = \mathrm{Im}\left[(\varepsilon' + \mathrm{i}\varepsilon'')(\mu' + \mathrm{i}\mu'')\right] = \varepsilon'\mu'' + \mu'\varepsilon''$. The quantity $\mathrm{Im}\, k_z^2$ in a media is thus proportional to $\mathrm{Im}(n^2) = \varepsilon'\mu'' + \mu'\varepsilon''$. This I have placed in four quadrants of k_z^2 plane depicted in figure-9

For 'absorbing' medium the wave amplitudes at infinities has to disappear. For 'amplifying' medium one should carefully form the discussion. The only conditions are that evanescent waves remain decaying, propagating ones remains propagating and no 'information' can flow from infinities towards source! This ensures that the 'near field' features of a source cannot be probed at a large distance merely by putting the source in an amplifying medium.

Now if I take square root of k_z^2 plane, I will divide the plane into two Riemann sheets. The figure-9 depicts one plane, that is primary Riemann sheet, with propagating region and evanescent region of EM waves (that depends on $\mathrm{Re}(k_z^2)$, as I have explained earlier in this section), with regions of positive $\mathrm{Im}(k_z^2)$ indicating absorbing media, and negative

$\mathrm{Im}(k_z{}^2)$ indicating region of amplifying media. If I say that $\mathrm{Re}(\varepsilon)$ and $\mathrm{Re}(\mu)$ are positive, I can say that the primary Riemann sheet corresponds to positively refracting media (PRM). That is what I have printed in figure-9 too. Also this figure show a branch cut by 'dotted' lines which I have to unfold to take square root of this plane $k_z{}^2$; rather 'inconvenient' branch cut!

This inconvenient branch cut -90^0 and 270^0 in the plane of $k_z{}^2$ gives me range of arguments (angles) for primary Riemann sheet as $\angle k_z{}^2 = \theta$, with $-\pi/2 < \theta < 3\pi/2$, beneath this sheet the secondary Riemann sheet with range of angles as $\angle k_z{}^2 = \theta$, with $3\pi/2 < \theta < 7\pi/2$. If I take square root of this plane of figure-9, I write the following

$$k_z = \pm\sqrt{k_z^2} = \pm\sqrt{\left|k_z^2\right| e^{\mathrm{i}\theta}} = \begin{cases} \sqrt{k_z^2}\, e^{\mathrm{i}\theta/2} \\ \sqrt{k_z^2}\, e^{\mathrm{i}\theta/2 + \mathrm{i}\pi} \end{cases}$$

The second Riemann sheet corresponds to NRM with angles of $3\pi/4 < \angle k_z < 7\pi/4$. This is depicted in figure-10

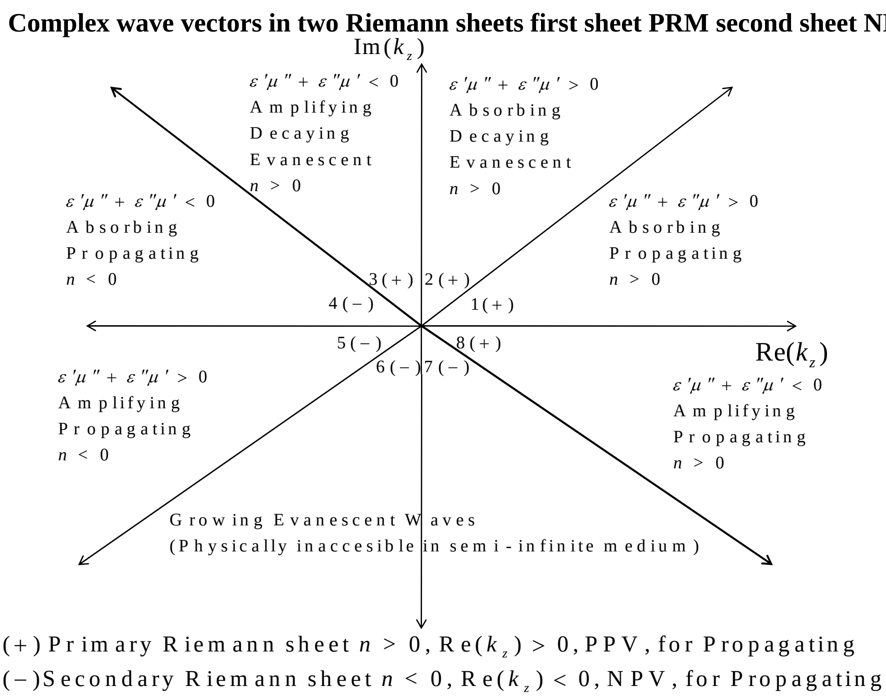


Figure-10: Taking square root of plane of $k_z{}^2$ to get plane of k_z

I now give explanation for the two Riemann sheet zones, as it appeared in figure-10. Region 1 and 8 corresponds to propagating waves in (positively refracting media) PRM that are absorbing or amplifying respectively. Region 6 and 7 corresponds to growing evanescent waves that build up at infinities, which are unphysical in the semi-infinite medium (though important in truncated NRM slab!). Decaying evanescent waves fall with region 2 if

$$\mathrm{Im}(n^2) = \mathrm{Im}(\varepsilon\mu) = \varepsilon'\mu'' + \varepsilon''\mu' > 0$$

and in the region 3 if $\varepsilon'\mu'' + \varepsilon''\mu' < 0$. Note the Poynting vector points away from the source (interface) if medium are absorbing overall and actually towards the source (interface) if media is amplifying overall. For the case of evanescent waves in amplifying media my choice of Poynting vector those points towards the source (interface in this case). This however does not violate the causality as the Poynting vector energy flow decays exponentially to zero at infinity and no information flows from infinity. The counter-intuitive behavior does not imply that source has turned into sink-rather indicates that there would be large (infinitely large unsaturated linear gain) accumulation of energy density (intense field enhancements) near a source. Now propagating waves in ENG MNG, simultaneously, that is DNG in region 4 and 5 depending on whether $\varepsilon'\mu'' + \varepsilon''\mu' < 0$ or $\varepsilon'\mu'' + \varepsilon''\mu' > 0$ respectively corresponding to absorbing and amplifying media respectively. In both cases negative square root need be chosen, this is start of second Riemann sheet. In case of normal incidence the sign of wave vector (k_z) and sign of index of refraction (n), are same. The quantity $\varepsilon'\mu'' + \varepsilon''\mu'$ determines the energy flow. In dissipative media $\mathrm{Im}(k_z) < 0$ for propagating waves, which reduces to $\mathrm{Im}(n) > 0$ for normal incidence. Thus one can reasonable talk of Negative Phase Velocity (NPV) rather Negative Group Velocity (NGV). In PRM I have positive phase velocity (PPV) and positive group velocity. The definition of absorbing and amplifying gets reversed in the secondary Riemann sheet, which I summarized in figure-11.

For NRM in secondary Riemann sheet absorbing and amplifying reversed

$$n \cong \pm[(\varepsilon'\mu')^{1/2} + \{i/2\}\{\varepsilon'\mu'' + \mu'\varepsilon''\}/(\varepsilon'\mu')^{1/2}] = \pm[\mathrm{Re}(n) + i\,\mathrm{Im}(n)]$$

$$n = \begin{cases} \mathrm{Re}(n) + i\,\mathrm{Im}(n) \\ -\mathrm{Re}(n) - i\,\mathrm{Im}(n) \end{cases}$$

$$\mathrm{Im}(n) \cong \varepsilon'\mu'' + \mu'\varepsilon''$$

$$\mathrm{Im}(k_z) \cong \varepsilon'\mu'' + \varepsilon''\mu' = \begin{cases} > 0 & \text{dissipative PRM} \\ < 0 & \text{amplifying PRM} \end{cases}$$

$$\mathrm{Im}(k_z) \cong \varepsilon'\mu'' + \varepsilon''\mu' = \begin{cases} < 0 & \text{dissipative NRM} \\ > 0 & \text{amplifying NRM} \end{cases}$$

For dissipative , absorbing NRM with ENG and MNG simultaneously, we get , $\mathrm{Re}(n) < 0$, $\mathrm{Re}(k_z) < 0$
$\varepsilon'\mu'' + \varepsilon''\mu' < 0$ $\mathrm{Im}(k_z) < 0$ NPV , opposite definitions of absorbing and amplifying as compared to PRM. Thus we get $\mathrm{Im}(n) > 0$

Figure 11: The secondary Riemann sheet details for NRM

When I plot the $\omega - k$ or $\omega - \beta$ diagram ($k = \beta - i\alpha$), the dispersion diagram, the anomalous dispersion I call as negative slope indicating $\mathrm{d}\omega/\mathrm{d}k < 0$ or $\mathrm{d}\omega/\mathrm{d}\beta < 0$, gives me to talk about negative group velocity (NGV). Well, I must say the anomalous dispersion as observed in figure-7, states about negative group delay or feel of negative group velocity-since the diagram is made in first quadrant (k and β positives). The above on plane of k_z (figure-10) tells me that in NRM, the $k_z < 0$, to have NPV. Thus this figure-7 giving idea of NGV and Negative Delay is actually in the second quadrant where k and β are negative-take image of this diagram about vertical axis thus to settle this anomaly.

7. Conclusion

The counterintuitive phenomena of Negative Refractive Index experiments have thrown open several challenges in the normal thought process; be addressed by thought experiments. Someway to explain the counterintuitive questions regarding meta materials thought experiments are helpful, to some extend gave satisfactory answers to left handed cross product, hidden momentum and backward wave, the negative square root emphasis, and several others can be formulated here. This area of thought experiment in meta-material parlance is new enriching field.

Acknowledgement

This work is supported fully by Board of Research in Nuclear Science (BRNS) Department of Atomic Energy (DAE): the project is called **LEFT HANDED MAXWELL SYSTEMS**. I acknowledge the encouragement received by **Sri B B Biswas** Head Reactor Control Division, **Dr B N Jagtap** Head AMP Div. BARC, **Sri G P Srivastava** Dir. E & I Group, **Dr R K Sinha** Director BARC, and **Dr S Banerjee** Chairman AEC; for inspiring me to do counterintuitive science along with Emeritus Professor of Univ. of Jadavpur, **Dr Deepak Ranjan Poddar**.

The group for my "**LEFT HANDED MAXWELL SYSTEMS"** project comprises of

Sougata Chaterjee (sougata.sameer@gmail.com), SAMEER (Research Scientist)
Amitesh Kumar (amiteshkumarsss@yahoo.co.in), SAMEER (Research Scientist)
Paulami Sarkar(paulami.sameer@gmail.com), SAMEER (Scientist-B)
Arijit Mazumder(arijit.majumder@gmail.com) , SAMEER (Scientist-E)
Dr Ananta Lal Das (ald.pdirector@gmail.com) Director, Society for Applied Microwave Electronics Engineering & Research (SAMEER).
Prof. Subal Kar Fulbright Fellow (subalkar@hotmail.com) **Guide & Consultant of LEFT HANDED MAXWELL SYSTEMS Project.** Institute of Radio Physics & Electronics (IRPE) University of Calcutta, Kolkata, I acknowledge them all.

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