Inverse Laplace transformation by analytical method without usual contour integration-by Berberan Santos method : with few solved examples

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Abstract

The Laplace transform technique is a very important aspect in solution of differential equations. What is tough-is the taking inverse Laplace transform, for a transfer function of a system, via usual contour integration and residue calculus. It is fine if tables give the result of inverse Laplace transform but if not we resort to classical technique of contour integration-and residue calculus, using Jordan lemma of complex analysis subject. Sometimes this contour integration is tough to do. More so for the transfer functions arising out of fractional order systems, we get irrational transfer functions, which may have poles spread across multiple Riemann-sheets-where we have to resort to concepts of branch cuts and branch points, - whereas this method of Berberan Santos gives ease. Here we describe the method devised by Mario. N. Berberan-Santos where the analytical method is obtained to find out inverse Laplace transformation, without going for usual contour integration method. The result is placed in integral representation, the closed form answers of which is difficult to get. However, this integral representation can be used to plot graphically by using numerical integration techniques. These integral representations are very useful, and can replace functions that are represented by infinite series and have Gamma functions as coefficients-which become difficult to numerically solve by computer for large number of terms. This analytical technique is very useful in our approach of using generalized calculus for studying anomalous visco-elastic (non-Newtonian) systems, and also for electrical systems with fractional order elements, where the transfer functions we get for irrational ones. Berberan-Santos method is newly developed technique, and still to be used by scientists and engineers.

Keywords
Berberan-Santos method, Integral representation, Bromwitch Path, Analytical function

1. Introduction

Our study here is for dealing a new technique for doing inverse Laplace transform. We will develop the formulas from the theory developed by Mario. N. Berberan-Santos [1], [2]. We will derive that first and then apply for various transfer functions $\mathrm{G(s)}$ to inverse Laplace transform to get $\mathrm{g(t)}$ .We will write integral representations of the $\mathcal{L}^{-1}\left\{\mathrm{G(s)}\right\}$, that will suffice, and this is actual integral representations for the function $\mathrm{g(t)}$. In [1], [2], the inverse Laplace transform of function time domain-complex relaxation function $\mathrm{I(t)}$ is evaluated i.e. $\mathcal{L}^{-1}\left\{\mathrm{I(t)}\right\}$ to get $\mathrm{H(k)}$ that is distribution function for relaxation rates. Here in [1], [2] the relation is $\mathcal{L}^{-1}\{I(t)\} = H(k)$, we take it for $\mathcal{L}^{-1}\{G(s)\} = g(t)$. We note [1], [2] are pioneering work by Mario. N. Berberan-Santos, which is yet to find application in regular applied science and engineering research. We will apply this technique developed by Mario. N. Berberan Santos to find the inverse Laplace transform of functions which are $G(s) = (s-a)^{-1}$, $G(s) = s(s^2+1)^{-1}$, $G(s) = e^{-\lambda_0 s}$, $G(s) = s^{-\alpha}$, $G(s) = u(s)$, $G(s) = s^{-1}$, $G(s) = e^{-(s/s_0)^{\beta}}$, $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$, $G(s) = \left(1+(s/a)^{\alpha}\right)^{-1}$, $G(s) = s^{\alpha-1}(s^{\alpha}+1)^{-1}$, $G(s) = ks^{-1}(s^{\alpha}+k)^{-1}$, $G(s) = \frac{s^b - s^a}{\ln(s)}$, to get corresponding $g(t)$, in integral representation.

The formulas for integral representation for $g(t) = \mathcal{L}^{-1}\{G(s)\}$ that Berberan-Santos derived are following sets (i), (ii) and (iii)

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\left(\cos(\omega t)\right)\Big) - \Big(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\Big)d\omega \quad \text{(i)}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\Big(\cos\left(\omega t + \theta(\omega)\right)\Big)d\omega; \qquad \text{(ii)}$$
$$\rho(\omega) = \left|G(\sigma_0+i\omega)\right| \qquad \theta(\omega) = \angle G(\sigma_0+i\omega)$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \begin{cases} \dfrac{2e^{\sigma_0 t}}{\pi}\displaystyle\int_0^{\infty}\Big(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)\Big)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases}$$
$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \begin{cases} -\dfrac{2e^{\sigma_0 t}}{\pi}\displaystyle\int_0^{\infty}\Big(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\Big)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad \text{(iii)}$$

We will derive all these formulas, which are equivalent and used them in our examples of $G(s)$ to get $g(t)$ as integral representation. In this note we will see via this Berberan-Santos method, integral representations for e^{at}, $\cos(t)$, Delta function at $t = \lambda_0$ i.e. $\delta(t-\lambda_0)$, t^{-n}, $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\delta(t)$, unit step function $u(t)$, and Mittag-Leffler functions $E_{\alpha,\alpha}(-a^{\alpha}t^{\alpha})$, $E_{\alpha}(-t^{\alpha})$.

2. The Laplace integral

The Laplace transform $G(s)$ of a function i.e. $g(t)$ in time domain defined $t \ge 0$ and for $t < 0$ $g(t) = 0$, defined as following integral transform relation [3] i.e. called Laplace integral (provided it exits)

$$G(s) \overset{\text{def}}{=} \int_0^{\infty}\left(g(t)\right)e^{-st}dt$$
$$G(s) = \mathcal{L}\{g(t)\} \qquad (1)$$

This is standard integral transform of a function $g(t)$ from a time domain ($t \ge 0$) to a complex frequency domain i.e. $s = \sigma + i\omega$; $i = \sqrt{-1}$; where real part is significant in the transient response and the imaginary part of the frequency corresponds to 'steady-state' response; in classical 'Control Science'. Here $g(t)$ is 'inverse Laplace transform' of $G(s)$, and we write $\mathcal{L}^{-1}\{G(s)\} = g(t)$ and $\mathcal{L}\{g(t)\} = G(s)$. We need to find $g(t) = \mathcal{L}^{-1}\{G(s)\}$, that is following integral, on a vertical line in $s = \sigma + i\infty$ plane, called Bromwich-path (Appendix)

$$g(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(G(s)\right)e^{st}ds \qquad (2)$$

Usually the solution of (2) is obtained via contour integration [3]. Here we do analytical technique to find (2) without using the usual contour integration method. We present in the Appendix the classical theory of inverse Laplace technique via contour integration, the choice of σ, and conditions where (2) is valid.

3. Analytical inversion of Laplace transform- the Berberan-Santos method-derivation

We describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. Here we take function $G(s)$, with $s \in \mathbb{C}$ call it $s = \sigma + i\omega = \mathrm{Re}[s] + i\,\mathrm{Im}[s]$. Here the real part i.e. σ is constant as a vertical line calls it $\sigma = \sigma_0$ a constant. Where the σ_0 is a real number being such that $G(s)$ has some form of singularity on the line $\mathrm{Re}[s] = \sigma_0$ but analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}[s] > \sigma_0$. Refer Appendix about the theory of choosing $\mathrm{Re}[s] = \sigma_0$.

We take $\sigma_0 = 0$, means the function $G(s)$ is analytic for $\mathrm{Re}[s] > 0$ and may have singularities on the line $\sigma_0 = 0$. For example let us take transfer function $G(s) = \frac{1}{s}$, the inverse Laplace of this is $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = u(t)$, that is standard unit step function at $t = 0$. We note that $G(s) = \frac{1}{s}$ has singularity at $s = 0$ that is a simple pole, but is analytic in complex right half plane, i.e. $\mathrm{Re}[s] > 0$. Thus in this case, we have $\sigma_0 = \varepsilon > 0$, and can further proceed with limit as $\varepsilon \downarrow 0^+$ with the application of formulas. As second example we take $G(s) = \frac{a}{s^2+a^2} = \mathcal{L}\{\sin at\}$. This transfer function $G(s)$ has poles at $s = \pm ia$ (i.e. singularity at $s = 0 \pm ia$); means in this case $\sigma_0 = \varepsilon > 0$, and proceed with limit $\varepsilon \downarrow 0^+$. As a third example we take $G(s) = \frac{b}{(s+a)^2+b^2} = \mathcal{L}\left\{e^{-at}\sin bt\right\}$. This has singularity at $s = -a \pm ib$, in which case we have $\sigma_0 > -a$.

Performing the variable change on (2) to $s = \sigma_0 + i\omega$ we get following steps

$$\begin{aligned} g(t) &= \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(G(s)\right)e^{st}ds; \qquad s = \sigma_0 + i\omega \\ &= \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(G(\sigma_0 + i\omega)\right)\left(e^{(\sigma_0+i\omega)t}\right)\left(d(\sigma_0 + i\omega)\right); \qquad d\sigma_0 = 0 \\ &= \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \left(G(\sigma_0 + i\omega)\right)\left(e^{t\sigma_0}e^{it\omega}\right)\left(id\omega\right) \\ &= \frac{e^{\sigma_0 t}}{2\pi}\int_{-\infty}^{+\infty} \left(G(\sigma_0 + i\omega)\right)e^{i\omega t}d\omega \end{aligned} \tag{3}$$

Writing $e^{it\omega} = \cos\omega t + i\sin\omega t$ we get the following form

$$g(t) = \frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(G(\sigma_0 + i\omega)\right)\cos(\omega t)d\omega + i\int_{-\infty}^{+\infty}\left(G(\sigma_0 + i\omega)\right)\sin(\omega t)d\omega\right) \tag{4}$$

Write complex function i.e. $G(\sigma_0 + i\omega)$ as following

$$G(\sigma_0 + i\omega) = \mathrm{Re}\left[G(\sigma_0 + i\omega)\right] + i\mathrm{Im}\left[G(\sigma_0 + i\omega)\right] \tag{5}$$

and place in above expression (4) to get the following expression

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\left(\cos(\omega t)\right)\right) - \left(\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega\right) \\ &+ i\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\left(\cos(\omega t)\right)\right) + \left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega\right) \end{aligned} \tag{6}$$

Given that $g(t)$ is a real function, we get the following (i.e. equating the imaginary part to zero)

$$\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\left(\cos(\omega t)\right) + \mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega\right) = 0 \tag{7}$$

Thus the above expression (6) for $g(t)$ reduces to following (i.e. considering only real part) we get

$$g(t) = \frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)d\omega\right) \tag{8}$$

But we have from (1); $G(s) = \int_0^\infty \left(g(t)\right)e^{-st}dt$ and by putting $s = \sigma_0 + i\omega$ we get following

$$G(\sigma_0 + i\omega) = \int_0^\infty \left(g(t)\right)e^{-t(\sigma_0 + i\omega)}dt = \int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)dt - i\int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)dt \tag{9}$$

This gives following

$$\mathrm{Re}\left[G(\sigma_0 + i\omega)\right] = \int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)dt \qquad \mathrm{Im}\left[G(\sigma_0 + i\omega)\right] = -\int_0^\infty e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)dt \tag{10}$$

We observe from (10) $\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]$ is even function in variable ω and $\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]$ is odd function in variable ω . Therefore we can write

$$\int_{-\infty}^{\infty} d\omega \mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) = 2\int_0^\infty \mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t)d\omega \tag{11}$$

and

$$\int_{-\infty}^{\infty} d\omega\left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega = 2\int_0^{\infty}\left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega \quad (12)$$

That is because even times even is even function, and odd times odd is even function.
With use of (11) and (12) we get the following

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\left(\cos(\omega t)\right)\right) - \left(\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\left(\sin(\omega t)\right)\right)d\omega \quad (13)$$

This (13) is basic formula of analytical Laplace inversion by Berberan-Santos method. We will use formula (13) in following discussion and use to solve several cases. Write (13) in polar form as described below

$$\begin{aligned} G(\sigma_0+i\omega) &= \rho(\omega)e^{i\theta(\omega)} \\ &= \rho(\omega)\left(\cos\left(\theta(\omega)\right)+i\sin\left(\theta(\omega)\right)\right) \\ \rho(\omega) = \left|G(\sigma_0+i\omega)\right| & \qquad \theta(\omega) = \angle G(\sigma_0+i\omega) \end{aligned} \quad (14)$$

to get following formulas

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\{G(s)\} &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\left(\cos\theta(\omega)\right)\cos(\omega t) - \rho(\omega)\left(\sin\theta(\omega)\right)\sin(\omega t)\right)d\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\left(\cos\left(\omega t+\theta(\omega)\right)\right)d\omega \end{aligned} \quad (15)$$

The (15) is formula in polar form

4. Simplification of formula for cases $g(t) = 0$ for $t < 0$ to get $g(t) = \mathcal{L}^{-1}\{G(s)\}$ for $t \geq 0$

We have basic formula that is (13); for a system where $g(t) = 0$ when $t < 0$. Take for simplification $\sigma_0 = 0$, then we write from (13) the following

$$g(t) = \frac{1}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega \quad (16)$$

For $t < 0$ we write we have $g(t) = 0$, and writing $t \equiv -t$ we get following

$$\begin{aligned} g(-t) = 0 &= \frac{1}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega(-t)) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega(-t)\right)d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) + \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega \end{aligned} \quad (17)$$

Adding (16) and (17) we get

$$g(t) = \frac{2}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)\right)d\omega \quad (18)$$

Subtracting (16) and (17) we get

$$g(t) = -\frac{2}{\pi}\int_0^\infty \left(\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)d\omega \tag{19}$$

Now introducing general σ_0, we get simplified formula as follows

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \begin{cases} \dfrac{2e^{\sigma_0 t}}{\pi}\displaystyle\int_0^\infty \left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t)\right)d\omega & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \begin{cases} -\dfrac{2e^{\sigma_0 t}}{\pi}\displaystyle\int_0^\infty \left(\mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)d\omega & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{20}$$

5. Inverse Laplace transform for $G(s) = (s - a)^{-1}$

Take $G(s) = (s - a)^{-1}$. This transfer function has a singularity a $s = a$; that is a pole. The function $G(s) = (s - a)^{-1}$ is analytic at right of the vertical line, $\mathrm{Re}[s] > a$ put $s = \sigma_0 + i\omega$, with $\sigma_0 > a$ to write following

$$G(s) = \frac{1}{s - a}, \quad s = \sigma_0 + i\omega \qquad G(s) = \frac{1}{\sigma_0 + i\omega - a}$$

$$G(\sigma_0 + i\omega) = \frac{1}{(\sigma_0 - a) + i\omega} = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} - i\frac{\omega}{(\sigma_0 - a)^2 + \omega^2} \tag{21}$$

From (21) we write real and imaginary parts as follows

$$\mathrm{Re}\left[G(\sigma_0 + i\omega)\right] = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} \qquad \mathrm{Im}\left[G(\sigma_0 + i\omega)\right] = -\frac{\omega}{(\sigma_0 - a)^2 + \omega^2} \tag{22}$$

Applying (13) we get following

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)d\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t)d\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)d\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}(\sigma_0 - a)}{\pi}\int_0^\infty \frac{\cos(\omega t)d\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)d\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t) + \omega\sin(\omega t)}{(\sigma_0 - a)^2 + \omega^2}d\omega \end{aligned} \tag{23}$$

From Laplace tables we know that $\mathcal{L}^{-1}\left\{(s - a)^{-1}\right\} = e^{at}$. So we have integral representation of e^{at} as following

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t) + \omega\sin(\omega t)}{(\sigma_0 - a)^2 + \omega^2}d\omega = e^{at} \tag{24}$$

For $a = -b, \quad b > 0$, we choose $\sigma_0 = 0 > -b$, to write for $\mathcal{L}^{-1}\left\{(s+b)^{-1}\right\} = e^{-bt}$ the following

$$g(t) = \mathcal{L}^{-1}\left\{(s+b)^{-1}\right\} = \frac{1}{\pi}\int_0^\infty \frac{b\cos(\omega t) + \omega\sin(\omega t)}{b^2+\omega^2}d\omega = e^{-bt} \tag{25}$$

In polar form we write using (15) the following for

$$\begin{aligned}
&G(s) = \frac{1}{s-a}, \qquad s = \sigma_0 + i\omega \\
&G(\sigma_0 + i\omega) = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} - i\frac{\omega}{(\sigma_0 - a)^2 + \omega^2} \\
&\theta(\omega) = \angle G(\sigma_0 + i\omega) = \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right) \\
&\rho(\omega) = \left|G(\sigma_0 + i\omega)\right| = \tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}}
\end{aligned} \tag{26}$$

We apply (15) to get following

$$\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\left(\cos\left(\omega t + \theta(\omega)\right)\right)d\omega \\
&= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}}\right)\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right)\right)d\omega
\end{aligned} \tag{27}$$

Thus we have another integral representation for e^{at} as

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(\sigma_0 - a)^2 + \omega^2}}\right)\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega}{(\sigma_0 - a)}\right)\right)d\omega = e^{at} \tag{28}$$

For a case $G(s) = (s+1)^{-1}$. The expression $G(s) = (s+1)^{-1}$ has singularity at $s = -1$. We choose $\sigma_0 = 0 > -1$ we write the following as integral representation for e^{-t}

$$\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \frac{\cos\omega t + \omega\sin\omega t}{1+\omega^2}d\omega = e^{-t} \\
&\frac{1}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(1+\omega^2}}\right)\cos\left(\omega t + \tan^{-1}(-\omega)\right)d\omega = e^{-t}
\end{aligned} \tag{29}$$

For the case $g(t) = 0$ for $t < 0$ we have from (20) $g(t) = \frac{2e^{\sigma_0 t}}{\pi}\int_0^\infty \mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t)d\omega$, we write

$$\begin{aligned}
&g(t) = \mathcal{L}^{-1}\left\{(s-a)^{-1}\right\} \\
&g(t) = \begin{cases} \dfrac{2e^{\sigma_0 t}}{\pi}\displaystyle\int_0^\infty \left(\tfrac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2}\right)\cos(\omega t)d\omega = e^{at} & , t \ge 0 \\ 0 & , t < 0 \end{cases}
\end{aligned} \tag{30}$$

Above is integral representation for e^{at} for $t \ge 0$

For $G(s) = (s+1)^{-1}$ we have $\sigma_0 = 0$, and we write for $g(t)$ for $t \ge 0$

$$\begin{aligned}
&g(t) = \mathcal{L}^{-1}\left\{(s+1)^{-1}\right\} \\
&g(t) = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\tfrac{1}{1+\omega^2}\right)\cos(\omega t)d\omega = e^{-t} & , t \ge 0 \\ 0 & , t < 0 \end{cases}
\end{aligned} \tag{31}$$

Above (31) is integral representation for e^{-t} for $t \geq 0$.

6. Inverse Laplace transform of $G(s) = s(s^2+1)^{-1}$

This $G(s) = s(s^2+1)^{-1}$ has singularity at $s = 0 \pm i\omega$. Put $s = \sigma_0 + i\omega$ in $G(s)$ with $\sigma_0 = 1 > 0$ to get following

$$G(s) = \frac{s}{s^2+1}; \qquad s = 1 + i\omega$$

$$G(1+i\omega) = \frac{1+i\omega}{(1+i\omega)^2+1} = \frac{1+i\omega}{(2-\omega^2)+2i\omega} = \frac{(1+i\omega)\left((2-\omega^2)-2i\omega\right)}{4+\omega^4} = \frac{(2+\omega^2)-i\omega^3}{4+\omega^4} \tag{32}$$

The above expression has singularity at $s = \pm i$ at line $\mathrm{Re}[s] = 0$. Thus we choose $\sigma_0 = 1 > 0$, in this case, we get

$$\mathrm{Re}\left[G(1+i\omega)\right] = \frac{2+\omega^2}{4+\omega^4} \qquad \mathrm{Im}\left[G(1+i\omega)\right] = -\frac{\omega^3}{4+\omega^4} \tag{33}$$

Using (13) we write the following

$$g(t) = \mathcal{L}^{-1}\left\{s(s^2+1)^{-1}\right\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega$$

$$= \frac{e^t}{\pi}\int_0^\infty \left(\mathrm{Re}\left[G(1+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(1+i\omega)\right]\sin(\omega t)\right)d\omega$$

$$= \frac{e^t}{\pi}\int_0^\infty \left(\frac{2+\omega^2}{4+\omega^4}\cos(\omega t) + \frac{\omega^3}{4+\omega^4}\sin(\omega t)\right)d\omega \tag{34}$$

$$= \frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos(\omega t)d\omega}{4+\omega^4} + \frac{e^t}{\pi}\int_0^\infty \frac{\omega^3\sin(\omega t)d\omega}{4+\omega^4}$$

$$= \frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t + \omega^3\sin\omega t}{4+\omega^4}d\omega$$

The known Laplace transform is $\mathcal{L}^{-1}\left\{s/(s^2+1)\right\} = \cos(t)$, so we have integration representation of $\cos(t)$ as

$$\frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t + \omega^3\sin\omega t}{4+\omega^4}d\omega = \cos(t) \tag{35}$$

In polar form we have

$$G(1+i\omega) = \frac{1+i\omega}{(1+i\omega)^2+1} = \frac{(2+\omega^2)}{4+\omega^4} - i\frac{\omega^3}{4+\omega^4} = \rho(\omega)e^{i\theta(\omega)}$$

$$\rho(\omega) = \frac{\sqrt{\omega^6+\omega^4+4\omega^2+4}}{\omega^4+4}; \quad \theta(\omega) = \tan^{-1}\left(-\frac{\omega^3}{\omega^2+2}\right) \tag{36}$$

Using (15) we write following

$$g(t) = \mathcal{L}^{-1}\left\{s(s^2+1)^{-1}\right\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\left(\cos\left(\omega t + \theta(\omega)\right)\right)d\omega$$
$$= \frac{e^{t}}{\pi}\int_0^{\infty}\left(\tfrac{\sqrt{\omega^6+\omega^4+4\omega^2+4}}{\omega^4+4}\right)\left(\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega^3}{\omega^2+2}\right)\right)\right)d\omega \quad (37)$$

We have another integral representation of $\cos(t)$ as follows

$$\frac{e^{t}}{\pi}\int_0^{\infty}\left(\tfrac{\sqrt{\omega^6+\omega^4+4\omega^2+4}}{\omega^4+4}\right)\left(\cos\left(\omega t + \tan^{-1}\left(-\tfrac{\omega^3}{\omega^2+2}\right)\right)\right)d\omega = \cos(t) \quad (38)$$

Using the simplified formula (20) as $g(t) = 0$ for $t < 0$ we write the following for $g(t)$ for $t \geq 0$

$$\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{s(s^2+1)^{-1}\right\} = \cos(t)$$

$$\mathrm{Re}\left[G(1+i\omega)\right] = \frac{2+\omega^2}{4+\omega^4}$$

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = \begin{cases} \dfrac{2e^{t}}{\pi}\displaystyle\int_0^{\infty}\left(\mathrm{Re}\left[G(1+i\omega)\right]\cos(\omega t)\right)d\omega & t \geq 0 \\ 0 & t < 0 \end{cases} \quad (39)$$
$$= \begin{cases} \dfrac{2e^{t}}{\pi}\displaystyle\int_0^{\infty}\left(\tfrac{2+\omega^2}{4+\omega^4}\right)\cos(\omega t)d\omega = \cos(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Another integral representation is $\frac{2e^t}{\pi}\int_0^{\infty} d\omega\left(\frac{2+\omega^2}{4+\omega^4}\right)\cos(\omega t) = \cos(t)$, for $t \geq 0$.

7. Inverse Laplace transform of $G(s) = e^{-\lambda_0 s}$

For the function $G(s) = e^{-\lambda_0 s}$ as no singularity at $s \geq 0$ so choose $\sigma_0 = 0$. Thus we write in complex variable the following

$$G(s) = e^{-\lambda_0 s} \qquad s = \sigma_0 + i\omega; \qquad \sigma_0 = 0$$
$$G(\sigma_0 + i\omega) = G(i\omega) = e^{-i\omega\lambda_0} \quad (40)$$
$$= \cos(\omega\lambda_0) - i\sin(\omega\lambda_0)$$

Thus we have

$$\mathrm{Re}\left[G(\sigma_0+i\omega)\right] = \cos(\omega\lambda_0) \qquad \mathrm{Im}\left[G(\sigma_0+i\omega)\right] = -\sin(\omega\lambda_0) \quad (41)$$

Applying the formula (13) we get the following

$$g(t) = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega$$
$$= \frac{1}{\pi}\int_0^{\infty}\left(\cos(\omega\lambda_0)\cos(\omega t) + \sin(\omega\lambda_0)\sin(\omega t)\right)d\omega \quad (42)$$
$$= \frac{1}{\pi}\int_0^{\infty}\cos\left(\omega(t-\lambda_0)\right)d\omega$$

From standard Laplace tables we have $\mathcal{L}^{-1}\left\{e^{-s\lambda_0}\right\}=\delta(t-\lambda_0)$, thus we get integral representation for $\delta(t-\lambda_0)$

$$\frac{1}{\pi}\int_0^\infty \cos\left(\omega(t-\lambda_0)\right)d\omega=\delta(t-\lambda_0) \tag{43}$$

With $\lambda_0=0$ in (43) we get $\int_0^\infty \cos(\omega t)d\omega=\pi\delta(t)$.

In polar form we have $\rho(\omega)=1, \quad \theta(\omega)=-\omega\lambda_0$, for $G(i\omega)=\cos(\omega\lambda_0)-i\sin(\omega\lambda_0)$ using (15) we write

$$\begin{aligned} g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\left(\cos\left(\omega t+\theta(\omega)\right)\right)d\omega \\ &=\frac{1}{\pi}\int_0^\infty \left(\cos\left(\omega t-\omega\lambda_0\right)\right)d\omega \end{aligned} \tag{44}$$

This is same that we obtained via (13) in (42). Using simplified formula (20) we write

$$\begin{aligned} g(t)=\mathcal{L}^{-1}\left\{e^{-s\lambda_0}\right\}=\delta(t-\lambda_0)&=\begin{cases}\frac{2}{\pi}\int_0^\infty \left(\mathrm{Re}\left[e^{-i\omega\lambda_0}\right]\cos(\omega t)\right)d\omega & t\ge 0\\ 0 & t<0\end{cases} \\ &=\begin{cases}\frac{2}{\pi}\int_0^\infty \cos(\omega\lambda_0)\cos(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases} \end{aligned} \tag{45}$$

Alternatively

$$\begin{aligned} g(t)=\mathcal{L}^{-1}\left\{e^{-s\lambda_0}\right\}=\delta(t-\lambda_0)&=\begin{cases}-\frac{2}{\pi}\int_0^\infty \left(\mathrm{Im}\left[e^{-i\omega\lambda_0}\right]\sin(\omega t)\right)d\omega & t\ge 0\\ 0 & t<0\end{cases} \\ &=\begin{cases}\frac{2}{\pi}\int_0^\infty \sin(\omega\lambda_0)\sin(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases} \end{aligned} \tag{46}$$

8. Inverse Laplace transform for $G(s)=s^{-\alpha}$

The transfer function $G(s)=s^{-\alpha}$ does not have singularity at $s>0$; as for any $\mathrm{Re}\left[s\right]\in\mathbb{R}^+$ gets defined $s^{-\alpha}$. For $G(s)=s^{-\alpha}$ we take $\sigma_0=\varepsilon>0$, right of this line will not have any singularity. We take $G(s)=s^{-\alpha}$ and with $\sigma_0=\varepsilon$ making $\varepsilon\downarrow 0^+$ we write in limit $G(\sigma_0+i\omega)=(i\omega)^{-\alpha}$, the following

$$G(\sigma_0+i\omega)=(i\omega)^{-\alpha}=\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)-i\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right) \tag{47}$$

Applying the formula (13) we get the following

$$g(t) = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\big[G(\sigma_0+i\omega)\big]\cos(\omega t) - \mathrm{Im}\big[G(\sigma_0+i\omega)\big]\sin(\omega t)\Big)d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\Big(\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)\cos(\omega t) + \omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\sin(\omega t)\Big)d\omega \qquad (48)$$

$$= \frac{1}{\pi}\int_0^{\infty}\omega^{-\alpha}\cos\left(\omega t - \tfrac{\alpha\pi}{2}\right)d\omega$$

The known Laplace transform is $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ we write following

$$\frac{1}{\pi}\int_0^{\infty}\omega^{-\alpha}\cos\left(\omega t - \tfrac{\alpha\pi}{2}\right)d\omega = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} \qquad (49)$$

Putting $(\alpha-1) = -n$ we write integral representation of t^{-n} as following

$$\frac{\Gamma(1-n)}{\pi}\int_0^{\infty}\omega^{n-1}\cos\left(\omega t - \tfrac{(1-n)\pi}{2}\right)d\omega = t^{-n} \qquad (50)$$

With use of polar form (15) we will get same result as above.
Using (20) with $g(t) = 0$ for $t < 0$, we write $g(t)$ for $t \ge 0$ as following

$$g(t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} = \begin{cases} \frac{2}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\big[(i\omega)^{-\alpha}\big]\cos(\omega t)\Big)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \frac{2}{\pi}\int_0^{\infty}\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)\cos(\omega t)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad (51)$$

Alternatively we have following

$$g(t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{1}{\Gamma(\alpha)}t^{\alpha-1} = \begin{cases} -\frac{2}{\pi}\int_0^{\infty}\Big(\mathrm{Im}\big[(i\omega)^{-\alpha}\big]\sin(\omega t)\Big)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \frac{2}{\pi}\int_0^{\infty}\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\sin(\omega t)d\omega & t \ge 0 \\ 0 & t < 0 \end{cases} \qquad (52)$$

9. Inverse Laplace transform of $G(s) = u(s)$ unit step function at $s = 0$

Take $G(s) = 1$ choose $\sigma_0 = 0$, we have for $s = \sigma_0 + i\omega$, $G(\sigma_0 + i\omega) = 1 + i(0)$. Using (13) we write

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\Big(\mathrm{Re}\big[G(\sigma_0+i\omega)\big]\cos(\omega t)-\mathrm{Im}\big[G(\sigma_0+i\omega)\big]\sin(\omega t)\Big)d\omega$$
$$=\frac{1}{\pi}\int_0^\infty\big((1)\cos(\omega t)-(0)\sin(\omega t)\big)d\omega \qquad (53)$$
$$=\frac{1}{\pi}\int_0^\infty\cos(\omega t)d\omega$$

Knowing $\mathcal{L}^{-1}\{1\}=\delta(t)$ we write integral representation of $\delta(t)$ as

$$\frac{1}{\pi}\int_0^\infty\cos(\omega t)d\omega=\delta(t) \qquad (54)$$

The same that we got in (43) with $\lambda_0=0$.

10. Inverse Laplace transform of $G(s)=s^{-1}$

For $G(s)=s^{-1}$ we take $\sigma_0=\varepsilon>0$, right of this line will not have any singularity. We take $G(s)=s^{-1}$ and with $\sigma_0=\varepsilon$ making $\varepsilon\downarrow 0^+$ we write in limit; $G(\sigma_0+i\omega)=(i\omega)^{-1}=\omega^{-1}\big(0-i(1)\big)$. We use formula (13) to write following

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\Big(\mathrm{Re}\big[G(\sigma_0+i\omega)\big]\cos(\omega t)-\mathrm{Im}\big[G(\sigma_0+i\omega)\big]\sin(\omega t)\Big)d\omega$$
$$=\frac{1}{\pi}\int_0^\infty\big(\omega^{-1}(0)\cos(\omega t)+(1)\omega^{-1}\sin(\omega t)\big)d\omega \qquad (55)$$
$$=\frac{1}{\pi}\int_0^\infty\frac{\sin\omega t}{\omega}d\omega$$

We know that $\mathcal{L}^{-1}\{s^{-1}\}=u(t)$ i.e. Heaviside unit step function at $t=0$, so we write integral representation of $u(t)$ as

$$\frac{1}{\pi}\int_0^\infty\frac{\sin\omega t}{\omega}d\omega=u(t)=1$$
$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases} \qquad (56)$$

11. Inverse Laplace transform of $G(s)=e^{-(s/s_0)^\beta}$

This is function $G(s)=e^{-(s/s_0)^\beta}$ is not appearing in standard Laplace transform tables. For $G(s)=e^{-(s/s_0)^\beta}$, in complex variable we get the following with $s=\sigma_0+i\omega$ and with choosing $\sigma_0=\varepsilon$ and with limit $\varepsilon\downarrow 0^+$ we have following

$$\begin{aligned}G(i\omega) &= e^{-(i\omega/s_0)^\beta} = e^{-(\omega/s_0)^\beta (i)^\beta} \\ &= e^{-(\omega/s_0)^\beta\left[\cos\left(\frac{\beta\pi}{2}\right)+i\sin\left(\frac{\beta\pi}{2}\right)\right]} \\ &= e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]} e^{\left[-i\left(\frac{\omega}{s_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)\right]} \\ &= \left(\rho(\omega)\right) e^{i(\theta(\omega))}\end{aligned} \tag{57}$$

Where in this case we have in polar form representation for use of (15)

$$\begin{aligned}\left|G(\sigma_0+i\omega)\right| &= \rho(\omega) = e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]} \\ \angle G(\sigma_0+i\omega) &= \theta(\omega) = -\left(\frac{\omega}{s_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)\end{aligned} \tag{58}$$

Therefore we write inverse Laplace transform $\mathcal{L}^{-1}\left\{e^{-(s/s_0)^\beta}\right\}$ as $g(t)$ in following expression

$$\begin{aligned}g(t) = \mathcal{L}^{-1}\left\{e^{-(s/s_0)^\beta}\right\} &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\cos\left(\omega t + \left(\theta(\omega)\right)\right)d\omega \\ &= \frac{1}{\pi}\int_0^\infty e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(\omega t - \left(\frac{\omega}{s_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)\right)d\omega\end{aligned} \tag{59}$$

Doing change of variable as $x = \omega / s_0$ we obtain $s_0 dx = d\omega$ from (59) we write

$$g(t) = \mathcal{L}^{-1}\left\{e^{-(s/s_0)^\beta}\right\} = \frac{s_0}{\pi}\int_0^\infty e^{\left[-x^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(ts_0 x - x^\beta \sin\left(\frac{\beta\pi}{2}\right)\right)dx \tag{60}$$

12. Inverse Laplace transform of $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$

We take the transfer function as $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$. We have following steps to find inverse Laplace transform of transfer function $G(s) = \left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}$ by putting $s = \sigma_0 + i\omega$ with limit $\sigma_0 \downarrow 0^+$

$$\begin{aligned}G(i\omega) &= \frac{1}{\left(1+(1-\beta)\left(\frac{i\omega}{s_0}\right)\right)^{1/1-\beta}} \\ \left|G(i\omega)\right| &= \rho(\omega) = \left(1+\left(\frac{(1-\beta)\omega}{s_0}\right)^2\right)^{-1/2(1-\beta)} \\ \angle G(i\omega) &= \theta(\omega) = -\frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta}\end{aligned} \tag{61}$$

So we write the inverse Laplace transform i.e. $\mathcal{L}^{-1}\left\{\left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}\right\} = g(t)$ as following

$$g(t)=\mathcal{L}^{-1}\left\{\left(1+(1-\beta)\left(\tfrac{s}{s_0}\right)\right)^{-1/(1-\beta)}\right\}=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\rho(\omega)\right)\cos\left(\omega t+\left(\theta(\omega)\right)\right)d\omega;\qquad \sigma_0=0$$
$$=\frac{1}{\pi}\int_0^\infty\left[1+\left(\tfrac{(1-\beta)\omega}{s_0}\right)^2\right]^{-\frac{1}{2(1-\beta)}}\cos\left(\omega t-\frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta}\right)d\omega \tag{62}$$

With change of variable $y=\frac{(1-\beta)\omega}{s_0}$, and $\frac{s_0}{(1-\beta)}dy=d\omega$ we get following expression from (62)

$$g(t)=\mathcal{L}^{-1}\left\{\left(1+(1-\beta)\left(\tfrac{s}{s_0}\right)\right)^{-1/(1-\beta)}\right\}=\frac{s_0}{\pi(1-\beta)}\int_0^{¥}\left(1+y^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{t\tau_0 y-\tan^{-1}y}{1-\beta}\right)dy \tag{63}$$

13. Inverse Laplace transform for $G(s)=\left(1+(s/a)^\alpha\right)^{-1}$

We do inverse Laplace transform of simple power law in s domain i.e. $G(s)=\frac{1}{1+(s/a)^\alpha};\quad \alpha<1$ to get function $g(t)=\mathcal{L}^{-1}\left\{\left(1+\left(\frac{s}{a}\right)^\alpha\right)^{-1}\right\}$. Will be expressed via same rule as previous examples, using formula (13) .Put $s=0+i\omega$ i.e. choosing $\sigma_0=0$ as follows

$$G(i\omega)=\frac{1}{1+\left(\frac{i\omega}{a}\right)^\alpha}=\frac{1}{1+\left(\frac{\omega}{a}\right)^\alpha(i)^\alpha}$$
$$=\frac{1}{1+\left(\left(\frac{\omega}{a}\right)^\alpha e^{i\frac{\alpha\pi}{2}}\right)}=\frac{1}{1+\left(\left(\frac{\omega}{a}\right)^\alpha\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)} \tag{64}$$
$$=\frac{1}{\left(1+\left(\frac{\omega}{a}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\left(\frac{\omega}{a}\right)^\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)}$$

From above (64) we have following

$$\mathrm{Re}\left[G(i\omega)\right]=\frac{\left(\frac{\omega}{a}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}$$
$$\mathrm{Im}\left[G(i\omega)\right]=-\frac{\left(\frac{\omega}{a}\right)^\alpha\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1} \tag{65}$$

We do inverse Laplace transform i.e. $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$ with $G(s)=\frac{1}{1+(s/a)^\alpha}$ as follows using the formula (13), as follows

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\left(1+\left(\tfrac{s}{a}\right)^{\alpha}\right)^{-1}\right\} = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\left(\begin{array}{c}\left(\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\cos(\omega t) \\ +\left(\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\sin(\omega t)\end{array}\right)d\omega \qquad (66)\\
&= \frac{1}{\pi}\int_0^{\infty}\left(\frac{\left(\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\omega t+\left(\frac{\omega}{a}\right)^{\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\omega t}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)d\omega; \quad y=\frac{\omega}{a} \\
&= \frac{a}{\pi}\int_0^{\infty}\left(\frac{\left(y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos(ayt)+\left(y^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin(ayt)}{y^{2\alpha}+2y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)dy
\end{aligned}
$$

With formula (20) we write the following for $g(t)=0$ when $t<0$, $g(t)$ for $t\geq 0$

$$
g(t)=\mathcal{L}^{-1}\left\{\left(1+\left(\tfrac{s}{a}\right)^{\alpha}\right)^{-1}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\mathrm{Re}\left[G(i\omega)\right]\cos(\omega t)\right)d\omega & t\geq 0\\ 0 & t<0\end{cases}
$$

$$
=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases} \qquad (67)
$$

14. Integral representations of Mittag-Leffler function $E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)$ -via inverse Laplace transform of Berberan Santos Method

The Laplace transform of Mittag-Leffler function is $\mathcal{L}\left\{t^{p\alpha+\beta-1}E^{(p)}_{\alpha,\beta}\left(-(\lambda t)^{\alpha}\right)\right\}=\frac{s^{\alpha-\beta}(p)!}{(s^{\alpha}-\lambda)^{p+1}}$. Here take $p=0$, $\beta=\alpha$, $\lambda=-a^{\alpha}$, to write $\mathcal{L}\left\{t^{\alpha-1}E_{\alpha,\alpha}\left(-a^{\alpha}t)^{\alpha}\right)\right\}=\frac{1}{(s^{\alpha}+a^{\alpha})}=\frac{1}{a^{\alpha}(1+s^{\alpha}/a^{\alpha})}$. We have Two-parameter Mittag-Leffler function defined as $E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{(z)^m}{\Gamma(\alpha m+\beta)}$ we have the following

$$
g(t)=\mathcal{L}^{-1}\left\{\frac{1}{1+(s/a)^{\alpha}}\right\}=\begin{cases}a\left(at\right)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)\right) & t\geq 0\\ 0 & t<0\end{cases} \qquad (68)
$$

Therefore we write the following using (67) and (68)

$$
a(at)^{\alpha-1}E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases} \qquad (69)
$$

$$a(at)^{\alpha-1}E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases}$$
$$E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)=\begin{cases}\dfrac{2(at)^{1-\alpha}}{a\pi}\displaystyle\int_0^{\infty}\dfrac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases} \qquad (70)$$

From (70) by placing $\alpha=1$ and $a=1$ we get

$$E_{1,1}\left(-(t)\right)=\mathcal{L}^{-1}\left\{(s+1)^{-1}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\dfrac{1}{\omega^2+1}\cos(\omega t)d\omega=e^{-t} & t\geq 0\\ 0 & t<0\end{cases} \qquad (71)$$

We have got $\frac{2}{\pi}\int_0^{\infty}\frac{\cos(\omega t)d\omega}{\omega^2+1}=e^{-t}$ in (31).

Also from the derivation of (66) we can write the following

$$g(t)=\mathcal{L}^{-1}\left\{\frac{k}{k+s^{\alpha}}\right\}$$
$$=\frac{\sqrt[\alpha]{k}}{\pi}\int_0^{\infty}\left(\frac{\left(y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(yt\sqrt[\alpha]{k}\right)+\left(y^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{y^{2\alpha}+2y^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)dy;\quad y=\frac{\omega}{\sqrt[\alpha]{k}} \qquad (72)$$

15. Inverse Laplace transforms of $G(s)=s^{\alpha-1}(s^{\alpha}+1)^{-1}$ to get integral representation of $E_{\alpha}(-t^{\alpha})$

We will apply this to known Laplace pair of Mittag-Leffler function $\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}=s^{\alpha-1}(s^{\alpha}+1)^{-1}$. This comes from $\mathcal{L}\left\{t^{p\alpha+\beta-1}E_{\alpha,\beta}^{(p)}\left(-(\lambda t)^{\alpha}\right)\right\}=\frac{s^{\alpha-\beta}(p)!}{(s^{\alpha}-\lambda)^{p+1}}$ with $p=0$, $\beta=1$, $\lambda=1$. We write this in Laplace integral as follows

$$\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}=\int_0^{\infty}E_{\alpha}(-t^{\alpha})e^{-st}dt=\frac{s^{\alpha-1}}{1+s^{\alpha}}=\frac{s^{\alpha}}{s(s^{\alpha}+1)}$$
$$E_{\alpha}(-t^{\alpha})=\mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{1+s^{\alpha}}\right\} \qquad (73)$$

Here now we apply the Berberan-Santo technique on $G(s)=s^{\alpha-1}(s^{\alpha}+1)^{-1}$, to get in integral representation of inverse Laplace transformed result.
In this technique we write $s=\sigma_0+i\omega$ with $\sigma_0=0$. That is because we do not expect singularity in the right half plane of complex frequency s i.e. $\mathrm{Re}[s]>0$ for function $G(s)$ for its well meaning behavior. With this substitution we get the following steps
From above we write the following

$$G(s)=\frac{s^{\alpha-1}}{1+s^{\alpha}} \qquad s=0+i\omega$$

$$\begin{aligned}
G(i\omega) &= \frac{(i\omega)^{\alpha-1}}{1+(i\omega)^{\alpha}}=\frac{\omega^{\alpha-1}(i)^{\alpha-1}}{1+(i\omega)^{\alpha}}\\
&= \frac{\omega^{\alpha-1}(i)^{-1}\left(\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)}\\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)-i\cos\left(\frac{\alpha\pi}{2}\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)}\\
&= \omega^{\alpha-1}\frac{\left(\sin\left(\frac{\alpha\pi}{2}\right)-i\cos\left(\frac{\alpha\pi}{2}\right)\right)\left(\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)-i\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}-i\omega^{\alpha-1}\frac{\omega^{\alpha}+\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}
\end{aligned} \qquad (74)$$

We get from (74) the following components

$$\mathrm{Re}\left[G(i\omega)\right]=\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \qquad \mathrm{Im}\left[G(i\omega)\right]=-\frac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \qquad (75)$$

The inverse Laplace transform by applying (13) is following

$$E_{\alpha}(-t^{\alpha})=g(t)=\mathcal{L}^{-1}\left\{G(i\omega)\right\}$$

$$\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega\\
&= \frac{1}{\pi}\int_0^{\infty}\left(\left(\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\cos(\omega t)+\left(\frac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\sin(\omega t)\right)d\omega \qquad (76)\\
&= \frac{1}{\pi}\int_0^{\infty}-\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)\cos(\omega t)+\omega^{2\alpha-1}\sin(\omega t)+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}d\omega\\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}d\omega
\end{aligned}$$

From (76) we write integral representation of $E_{\alpha}(-z)$ as following (i.e. placing $t^{\alpha}=z$)

$$E_{\alpha}(-z)=\frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z}+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}d\omega \qquad (77)$$

Therefore we can write from above derivation (77) the following relation

$$g(t)=\mathcal{L}^{-1}\left\{\frac{s^{\alpha}}{s(1+s^{\alpha})}\right\}=\frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z}+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}d\omega, \qquad t^{\alpha}=z \qquad (78)$$

From formula (20) we write the following for $g(t)=E_{\alpha}(-t^{\alpha})$ for $t\geq 0$

$$g(t)=\mathcal{L}^{-1}\left\{\tfrac{s^\alpha}{s(1+s^\alpha)}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty \mathrm{Re}\left[\tfrac{(i\omega)^\alpha}{i\omega(1+(i\omega)^\alpha)}\right]\cos(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases}$$

$$g(t)=\mathcal{L}^{-1}\left\{\tfrac{s^\alpha}{s(1+s^\alpha)}\right\}=E_\alpha(-t^\alpha)=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty\left(\dfrac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\cos(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases} \tag{79}$$

16. Inverse Laplace transform of $G(s)=ks^{-1}(s^\alpha+k)^{-1}$

We take $\sigma_0=\varepsilon$, and in limit $\varepsilon\downarrow 0^+$ in $G(s)=ks^{-1}(s^\alpha+k)^{-1}$, to have $G(i\omega)=k(i\omega)^{-1}\left((i\omega)^\alpha+k\right)^{-1}$ which is also $G(i\omega)=k\omega^{-1}(\omega^\alpha i^{\alpha+1}+ik)^{-1}$.Thus we have following steps

$$\begin{aligned}G(i\omega)&=\frac{k}{(i\omega)\left((i\omega)^\alpha+k\right)}=\frac{k}{\omega\left(\omega^\alpha i^{\alpha+1}+ik\right)}\\&=\frac{k}{\omega\left(\omega^\alpha\left(\cos\left(\frac{(\alpha+1)\pi}{2}\right)+i\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right)+ik\right)\right)}\\&=\frac{k}{\omega\left(\omega^\alpha\cos\left(\frac{(\alpha+1)\pi}{2}\right)+i\left(\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\right)\right)}\\&=\frac{k\left(\omega^\alpha\cos\left(\frac{(\alpha+1)\pi}{2}\right)-i\left(\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\right)\right)}{\omega\left(\omega^{2\alpha}\cos^2\left(\frac{(\alpha+1)\pi}{2}\right)+\omega^{2\alpha}\sin^2\left(\frac{(\alpha+1)\pi}{2}\right)+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2\right)}\\&=\frac{k\left(\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)-i\left(\omega^{\alpha-1}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\right)\right)}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\end{aligned} \tag{80}$$

We get from (80) the following

$$\begin{aligned}\mathrm{Re}\left[G(i\omega)\right]&=\frac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\\\mathrm{Im}\left[G(i\omega)\right]&=-\frac{k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\end{aligned} \tag{81}$$

Now using the formula of Laplace inversion by Berberan-Santos method (13) we get the integral representation of $g(t)=\mathcal{L}^{-1}\left\{\frac{k}{s(s^\alpha+k)}\right\}$

$$g(t)=\mathcal{L}^{-1}\left\{G(i\omega)\right\}$$

$$g(t)=\mathcal{L}^{-1}\left\{\tfrac{k}{s(s^{\alpha}+k)}\right\}=\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[G(\sigma_0+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[G(\sigma_0+i\omega)\right]\sin(\omega t)\right)d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\left(\left(\frac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\right)\cos(\omega t)+\left(\frac{\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\right)\sin(\omega t)\right)d\omega$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\left(k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)\right)\cos(\omega t)+\left(k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2\right)\sin(\omega t)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}d\omega$$

$$=\frac{k}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}d\omega \tag{82}$$

By using (20) we write

$$g(t)=\mathcal{L}^{-1}\left\{\tfrac{k}{s(s^{\alpha}+k)}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\mathrm{Re}\left[\tfrac{k}{(i\omega)((i\omega)^{\alpha}+k)}\right]\cos(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases} \tag{83}$$

$$g(t)=\mathcal{L}^{-1}\left\{\tfrac{k}{s(s^{\alpha}+k)}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\left(\tfrac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\right)\cos(\omega t)d\omega & t\ge 0\\ 0 & t<0\end{cases}$$

17. Laplace inversion of function $G(s)=\frac{s^b-s^a}{\ln(s)}$

We take $G(s)=\frac{s^b-s^a}{\ln(s)}$ and put $s=\sigma_0+i\omega$ with $\sigma_0\downarrow 0^+$, this makes the following steps

$$
\begin{aligned}
G(i\omega) &= \frac{(i\omega)^b - (i\omega)^a}{\ln(i\omega)} \\
&= \frac{\omega^b\left(\cos\left(\frac{b\pi}{2}\right) + i\sin\left(\frac{b\pi}{2}\right)\right) - \omega^a\left(\cos\left(\frac{a\pi}{2}\right) + i\sin\left(\frac{a\pi}{2}\right)\right)}{\ln\omega + i\left(\frac{\pi}{2}\right)} \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right) + i\left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)}{\ln\omega + i\left(\frac{\pi}{2}\right)} \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right) + i\left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\left(\ln\omega - i\left(\frac{\pi}{2}\right)\right) \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\ln\omega\right) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}} \\
&\quad - i\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}
\end{aligned}
\tag{84}
$$

This gives real and imaginary parts as

$$
\begin{aligned}
\mathrm{Re}\left[G\left(i\omega\right)\right] &= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\ln\omega\right) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}} \\
\mathrm{Im}\left[G\left(i\omega\right)\right] &= -\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}
\end{aligned}
\tag{85}
$$

We apply (13) to get the following

$$
g(t) = \mathcal{L}^{-1}\left\{\frac{s^b - s^a}{\ln s}\right\}
$$

$$
g(t) = \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[G(\sigma_0 + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[G(\sigma_0 + i\omega)\right]\sin(\omega t)\right)d\omega \qquad \sigma_0 = 0
$$

$$
= \frac{1}{\pi}\int_0^\infty\left(\begin{aligned}&\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\ln\omega\right) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\cos(\omega t) \\ &+ \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\end{aligned}\right)d\omega
\tag{85}
$$

By using (20) we write

$$g(t)=\mathcal{L}^{-1}\left\{\frac{s^b-s^a}{\ln s}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty \mathrm{Re}\left[\frac{(i\omega)^b-(i\omega)^a}{\ln(i\omega)}\right]\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases}$$

$$g(t)=\mathcal{L}^{-1}\left\{\frac{s^b-s^a}{\ln s}\right\}=\begin{cases}\dfrac{2}{\pi}\displaystyle\int_0^\infty \left(\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right)-\omega^a\cos\left(\frac{a\pi}{2}\right)\right)(\ln\omega)+\left(\omega^b\sin\left(\frac{b\pi}{2}\right)-\omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2+\frac{\pi^2}{4}\right)}}\right)\cos(\omega t)d\omega & t\geq 0\\ 0 & t<0\end{cases} \tag{86}$$

Conclusion

The technique discussed give a possible way to represent inverse Laplace transform of function via integral representation, not by using conventional method of contour integration, residue calculus using Jordon lemma etc. The functions that were taken as example may arise in solution of differential equations, fractional differential equations and also continuous order differential equations. This new proposed method by Mario. N. Berberan-Santos, is very useful for studies in visco-elastic systems, fractional order circuits and systems, and also to get spectral spreads Histogram of relaxation rates in anomalous non-Debye relaxation processes. As demonstrated by various examples this method is also useful in deriving alternative integral representations of known functions. We note here the integral representations obtained here by this method of Berberan-Santos, may or may not be equal to integral representations that we get from usual contour integration method. However, till now not very much of usage of this method is amongst scientists and engineers. This has got tremendous scope in applied science and engineering. This new method is very good for various mathematical and science projects.

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Appendix

Theory of inverse Laplace transform by contour integration

A.1-Revising basics of Laplace transforms

For a function $x(t)$ that is zero for $t < 0$, and is defined for $t \geq 0$; the Laplace transform is defined by following (A1) integral. This is also termed as Laplace integral

$$X(s) = \int_0^{\infty} e^{-st} x(t) dt \tag{A1}$$

Here we have complex variable $s = \sigma + i\omega$, with this we have following step

$$X(s) = \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) dt \tag{A2}$$

The inverse problem is stated as; given $X(s)$ how to get $x(t)$.

Whereas the Fourier transform of $x(t)$ is $\hat{x}(\omega)$ defined as $\hat{x}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} x(t) dt$ and the inverse Fourier transform is the given by following integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{x}(\omega) d\omega \tag{A3}$$

In the Laplace transform expression i.e. $X(s) = \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) dt$ as obtained above (A2), let us make the following substitution

$$\begin{aligned} f(t) &= e^{-\sigma t} x(t); \quad t \geq 0 \\ &= 0 \qquad\qquad t < 0 \end{aligned} \tag{A4}$$

Where σ is a constant in (A4). Taking the Fourier transform of $f(t)$ we write the following

$$\begin{aligned} \hat{f}(\omega) &= \int_0^{\infty} e^{-i\omega t} f(t) dt \\ &= \int_0^{\infty} e^{-i\omega t} \left(e^{-\sigma t} x(t) \right) dt \\ &= \int_0^{\infty} e^{-(\sigma + i\omega)t} x(t) dt = X(\sigma + i\omega) \end{aligned} \tag{A5}$$

Now we take inverse Fourier transform of $X(\sigma + i\omega)$ as following

$$f(t) = e^{-\sigma t} x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{i\omega t} X(\sigma + i\omega) d\omega \tag{A6}$$

Therefore, in above (A6) we obtain $x(t) = e^{\sigma t} f(t)$ and we write following expression

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{\sigma t} \left(e^{i\omega t} X(\sigma + i\omega) \right) d\omega \\ &= \frac{1}{2\pi} \int_{\omega=-\infty}^{\omega=\infty} e^{(\sigma + i\omega)t} X(\sigma + i\omega) d\omega \end{aligned} \tag{A7}$$

Letting $s = \sigma + i\omega$ we have $ds = id\omega$, we get the following from above (A7)

$$\begin{aligned} x(t) &= \frac{1}{2\pi i} \int_{s=\sigma - i\omega}^{s=\sigma + i\omega} e^{st} X(s) ds; \quad t \geq 0 \\ &= 0; \quad t < 0 \end{aligned} \tag{A8}$$

A.2-The inverse Laplace transform via contour integration

The formula that we derived (A8) that is following, called inverse Laplace transform

$$x(t) = \frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega} e^{st}X(s)ds \qquad \text{(A9)}$$

We ask the following questions on this expression (A9)

1). How do we choose the real part of s i.e. σ

2). How do we calculate/evaluate the above integral (A9) in complex domain.

We already know that $x(t) = 0$ for $t < 0$; that will help to answer (1).

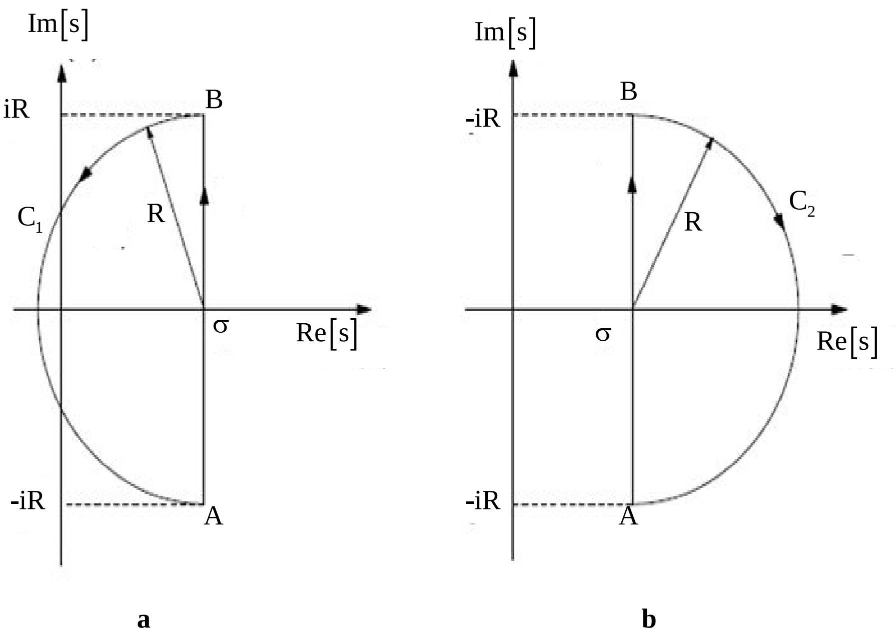

Figure-A1: Contour in complex-plane to evaluate inverse Laplace transforms

Consider Figure-A1a, for the closed contour is $A \to B \to C_1$. We write the contour integration as per residue theorem of complex analysis as following

$$\int_{A\to B\to C_1} e^{st}X(s)ds = \int_{A\to B} e^{st}X(s)ds + \int_{C_1} e^{st}X(s)ds$$

$$= \int_{\sigma-iR}^{\sigma+iR} e^{st}X(s)ds + \int_{C_1} e^{st}X(s)ds \qquad \text{(A10)}$$

$$= 2\pi i \sum_{\text{At poles}} \text{Residues}\left[e^{st}X(s)\right]$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_1$ of Figure-A1a. Now as $R \uparrow \infty$ the integral $\int_{\sigma-iR}^{\sigma+iR} e^{st}X(s)ds$ is the integral of interest that we require to evaluate inverse Laplace transform formula (A9). We note that the integral on the line $A \to B$ is the Bromwich integral, and is used for finding the inverse Laplace transform. We will use Jordan lemma (described shortly) which says for $t > 0$, $\lim_{R\uparrow\infty}\int_{C_1} e^{st}X(s)ds = 0$.

Therefore for $t > 0$ we write the following, as $R \uparrow \infty$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)ds = \sum_{\text{Poles at Left of } \sigma} \text{Residues}\left[e^{st}X(s)\right] \qquad \text{(A11)}$$

Consider Figure-A1b, the closed contour is $A \to B \to C_2$, we write the contour integration as following

$$\begin{aligned}\int_{A\to B\to C_2} e^{st}X(s)ds &= \int_{A\to B} e^{st}X(s)ds + \int_{C_2} e^{st}X(s)ds \\ &= \int_{\sigma-iR}^{\sigma+iR} e^{st}X(s)ds + \int_{C_2} e^{st}X(s)ds \qquad \text{(A12)} \\ &= -2\pi i \sum_{\text{At Poles}} \text{Residues}\left[e^{st}X(s)\right]\end{aligned}$$

We stress that residues are at the poles inside the closed contour $A \to B \to C_2$ of Figure-A1b. The negative sign in (A12) indicate that contour is taken in clock-wise direction. We will use Jordan lemma (described shortly) which says for $t < 0$, $\lim_{R\uparrow\infty}\int_{C_2} e^{st}X(s)ds = 0$. Thus we write for $t < 0$ the following

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)ds = -\sum_{\text{Poles at Right of } \sigma} \text{Residues}\left[e^{st}X(s)\right] \qquad \text{(A13)}$$

We know that this above integral (A13) must be zero, since for $t < 0$, we have $x(t) = 0$. Implying, we have $x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)ds = 0; \quad t < 0$. Thus LHS of (A13) is zero, making $\sum_{\text{Poles at Right of } \sigma} \text{Residues}\left[e^{st}X(s)\right] = 0$. This says that there are no poles at the right side of the line $\text{Re}[s] = \sigma$; Figure-A1b.

Therefore, the $\text{Re}[s] = \sigma$, the line AB (Figure-A1), or Bromwich line, must be chosen such that contour of Figure-A1b i.e. $A \to B \to C_2$ does not contain any poles of $e^{st}X(s)$ as $R \uparrow \infty$. Thus the contour of Figure-A1a, i.e. $A \to B \to C_1$ must have all poles of $e^{st}X(s)$. This gives answer to point (1) above. In addition, we note that since e^{st} is analytic everywhere (i.e. it has no poles in entire Complex-s plane), the poles of $e^{st}X(s)$ are same as that of $X(s)$. This gives reply to point (2) as posed above. Thus, we apply residue calculus of complex analysis and write the following

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}X(s)ds = \sum_{\text{all poles of } X(s)} \text{Residues}\left[e^{st}X(s)\right] \qquad \text{(A14)}$$

This is how we need evaluate the integral for obtaining inverse Laplace transform.

A.3-The Jordan lemma

While we discussed the inverse Laplace transform using contour integration and Residue calculus, we very well stated that we need a condition that is integral on arc C_1 in Figure-A1a; as following

$$\lim_{R\uparrow\infty}\int_{C_1} e^{st}X(s)ds = 0 \qquad t > 0 \qquad \text{(A15)}$$

While for $t > 0$ the points on this arc C_1 is given as

$$s = \sigma + Re^{i\theta}; \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \tag{A16}$$

This condition i.e. $\lim_{|s|\uparrow\infty} X(s) = 0$ means that for any $M_R > 0$ a radius R can be found such that $|X(s)| = |X(\sigma + Re^{i\theta})| < M_R$. By using the inequality i.e. $\left|\int_C f(s)ds\right| \le \int_C |f(s)|ds$, for this R we have the following expressions

$$\begin{aligned} \left|\int_{C_1} e^{st}X(s)ds\right| &\le \int_{C_1} \left|e^{st}X(s)\right|ds \\ \left|\int_{C_1} e^{st}X(s)ds\right| &\le M_R \int_{C_1} \left|e^{st}\right|ds \end{aligned} \tag{A17}$$

On the arc C_1 for $t > 0$, $s = \sigma + Re^{i\theta}$; $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ and $ds = iRe^{i\theta}d\theta$ .With these we write the following steps

$$\begin{aligned} \left|e^{st}\right| = \left|e^{(\sigma+Re^{i\theta})t}\right| &= \left|e^{(\sigma+R\cos\theta+iR\sin\theta)t}\right| \\ &= \left|e^{(\sigma+R\cos\theta)t}e^{iRt\sin\theta}\right|, \quad \left|e^{i(Rt\sin\theta)}\right| = 1 \\ &= \left|e^{(\sigma+R\cos\theta)t}\right|, \quad e^{(\sigma t+Rt\cos\theta)} > 0 \\ &= e^{\sigma t}e^{Rt\cos\theta} \end{aligned} \tag{A18}$$

Further, we write the steps from (A17) and (A18), with observation that $e^{(Rt)\cos\theta}$ is even function of θ as following

$$\begin{aligned} M_R \int_{C_1} \left|e^{st}ds\right| &= M_R \int_{\pi/2}^{3\pi/2} \left|e^{\sigma t}e^{Rt\cos\theta}iRe^{i\theta}d\theta\right| \\ &\le M_R Re^{\sigma t} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta = 2M_R Re^{\sigma t} \int_{\pi/2}^{\pi} e^{Rt\cos\theta}d\theta \end{aligned} \tag{A19}$$

By changing variable, $\theta = \xi + \frac{\pi}{2}$ we obtain from (A19) the following expression

$$\begin{aligned} M_R \int_{C_1} \left|e^{st}ds\right| &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{Rt\cos\left(\xi+\frac{\pi}{2}\right)}d\xi \\ &= 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\sin\xi}d\xi \end{aligned} \tag{A20}$$

Plotting the graphs of $y = \sin\xi$ and a straight line, i.e. $y = \frac{2}{\pi}\xi$ it is observed that in the region $0 \le \xi \le \frac{\pi}{2}$, we see $\sin\xi \ge \frac{2}{\pi}\xi$. With this observation we write the following from (A20)

$$\begin{aligned} M_R \int_{C_1} \left|e^{st}ds\right| &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\sin\xi}d\xi \\ &\le 2M_R Re^{\sigma t} \int_0^{\pi/2} e^{-Rt\left(\frac{2}{\pi}\xi\right)}d\xi = \frac{M_R \pi e^{\sigma t}}{t}\left(1 - e^{-Rt}\right) \end{aligned} \tag{A21}$$

Therefore, for any $t > 0$ as $R \uparrow \infty$, we have $M_R \downarrow 0$, the above quantity in (A21) tends to zero. This proves our case i.e. $\lim_{R\uparrow\infty} \int_{C_1} e^{st}X(s)ds = 0$, for $t > 0$.

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