Application of Formula ‘q (t) = c * v’ for Charge and ‘φ (t) = l * i’ for Magnetic Flux to described Memory Based Transmission Line Equation

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Abstract
In this presentation note, we apply the newly developed charge storage expression $q(t)=c(t)*v(t)$ and magnetic flux expression $\varphi(t)=l(t)*i(t)$ as a function of time i.e. via convolution operation for capacitor and inductor respectively. We apply this new formula to various types of configuration for linear dispersion less transmission line modeling. We arrive at diffusion equations, rather Time Fractional Diffusion Wave Equations (TFDWE). We deliberate this for case of fractional capacitor, fractional inductor as well as classical ideal capacitor and classical ideal inductor. The fractional capacitor and inductors that we take are systems with memory. This new formula is different to usual and conventional way of writing i.e. via multiplication $q(t)=c(t)v(t)$ and $\varphi(t)=l(t)*i(t)$ for capacitor and inductor respectively. This new deliberation with convolution operation works well for classical ideal loss less capacitors and loss less ideal inductor where one says that is a constant capacity and inductance, and also for a time varying capacity function and inductance function given by power-law: that gives the formation of fractional capacitor and fractional inductor. This presentation note gives validity of usage of this new formula-for case of linear dispersion less transmission line; that gives Time Fractional Diffusion Wave Equations (TFDWE). This is deliberation is in continuation to our earlier discussion regarding verification of usage of convolution operation instead of multiplication operation in these formulas.

Keywords
Time varying Capacity Function, Time varying Inductance Function, Fractional Capacitor, Fractional Inductor, Ideal Capacitor, Ideal Inductor, Convolution Operation, Causality Principle, Fractional Derivative, Time Fractional Diffusion-Wave Equation (TFDWE)

1. Introduction
This is deliberation is in continuation to our earlier discussion regarding verification of usage of convolution operation instead of multiplication operation in these formulas. The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality we write as follows

$$q(t)=c(t)*v(t)=\int_{-\infty}^{t}c(t-\tau)v(\tau)\mathrm{d}\tau$$

where $v(t)$ is the input impressed voltage, across capacitor device. We call $q(t)$ as effect to cause which is $v(t)$. This is contrary to usual formula i.e. of $q(t)=c(t)v(t)$ i.e. the product of the two. This formulation is deliberated and derived in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t)=C_{\alpha}t^{-\alpha}$; $0<\alpha<1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t) = C\delta(t)$; [1]. In this presentation note we will always take the power-exponent of power-law of decaying capacity function i.e. α or β as between zero and one. This power-law decay function is in singular at origin and in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t) \sim t^{-\alpha}$, $0 < \alpha < 1$ when uncharged system (or even system at rest) of dielectric is stressed by a constant voltage; a step voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36].

On similar lines we formulate 'time varying inductance function' $l(t)$ which is response function of a conductor producing an effect that is magnetic field $\varphi(t)$ caused by a current $i(t)$ carrying in a conductor. We call $\varphi(t)$ is the effect of cause i.e. $i(t)$ given by following convolution operation

$$\varphi(t) = \int_{-\infty}^{t} l(t-t')i(t')\mathrm{d}t' = l(t) * i(t)$$

Where the function $l(t)$ is time varying inductance function, with $i(t)$ the current flowing through the conductor giving the effect of magnetic field $\varphi(t)$.

We note a priori that the constant C_α is proportionality constant of the relation of time varying capacity function i.e. $c(t) \sim t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\mathrm{Farad}/\mathrm{sec}^{1-\alpha}$. The Fractional capacitor is given by following

$$i(t) = C_{F-\alpha}\frac{\mathrm{d}^\alpha v(t)}{\mathrm{d}t^\alpha}; \quad 0 < \alpha < 1, \quad c(t) = C_\alpha t^{-\alpha}; \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$$

The fractional derivative operator $\mathrm{d}^\alpha / \mathrm{d}t^\alpha$ is Riemann-Liouville (detailed in [1]). The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. The Fractional Inductor with time varying inductance function $l(t) = L_\beta t^{-\beta}$, $0 < \beta < 1$ is similarly can be casted as follows

$$v(t) = L_{F-\beta}\frac{\mathrm{d}^\beta i(t)}{\mathrm{d}t^\beta}; \quad L_{F-\beta} = L_\beta \Gamma(1-\beta)$$

The units of fractional capacity $C_{F-\alpha}$ is $\mathrm{Farad}/\mathrm{sec}^{1-\alpha}$ and unit of fractional inductor $L_{F-\beta}$ is $\mathrm{Henry}/\mathrm{sec}^{1-\beta}$. The Time Fractional Diffusion Wave Equation that we obtain is of the following form

$$\frac{\partial^\upsilon}{\partial t^\upsilon} f(x,t) = a\frac{\partial^2}{\partial x^2} f(x,t); \quad 1 < \upsilon \le 2$$

This above TFDWE we will not solve. The [13] gives detailed treatment of this type of TFDWE equation, based on development by Prof Mainardi and Prof Caputo by using Fourier-Laplace tricks.

Causality principle applied for charge storage in a capacitor and magnetic flux in an inductor leading to convolution expression

Consider the capacitor device, which is supplied with an input function i.e. time varying voltage $v(t)$ and as a result the capacitor stores charge $q(t)$. Assume the input voltage $v(t')$ at a time t' is sustained for a short infinitesimal period call it $\mathrm{d}t'$ and we say output i.e. charge stored in capacitor at some later time $t > t'$ is proportional to input, i.e. in the following form

$$\mathrm{d}q(t) = \left(c'(t,t')\right)\left(v(t')\mathrm{d}t'\right)$$

with proportionality term as function $c'(t,t')$. Hence, the function $c'(t,t')$ describes the operation of charge storage in a capacitor. Assuming this operation has no explicit dependence (i.e. no in-built clock that changes its behavior) then the relation between the input voltage and the charge storage will only depend on the time interval i.e. $t-t'$ and not on the absolute time i.e. t. Therefore, we may replace the function $c'(t,t')$ with a function of single variable i.e. $c(t-t')$. This may be called response function of the capacitor, or fundamental impulse response of the device. In this note, we call this $c(t-t')$ as 'capacity function'.

The assumption of 'causality', namely that cause precedes the effect, implies that output at any time t is obtained only due to input at or before t. That is simply, you hear the bell sound (the effect), only after you strike the bell (cause). Hence the expression $\mathrm{d}q(t) = \left(c(t-t')\right)\left(v(t')\mathrm{d}t'\right)$ applies only for $t \ge t'$, or equivalently $c'(t,t') = 0$ for $t' > t$ or we say $c(t-t') = 0$ for $t' > t$. We get the total charge stored at time t by integration of $\mathrm{d}q(t) = \left(c'(t,t')\right)\left(v(t')\mathrm{d}t'\right)$ from $t' = -\infty$ to $t' = t$ (since, $c'(t,t') = 0$ when $t' > t$: i.e. 'because cause cannot be preceding the effect'). We express as follows the causality principle

$$q(t) = \int_{-\infty}^{t} c(t-t')v(t')\mathrm{d}t'$$

The above is convolution operation of charge function and voltage function i.e. $q(t) = c(t) * v(t)$. Where in convolution operation is denoted as $(*)$ and the convolution of two functions $f_1(t)$ and $f_2(t)$ is described as $f_1(t) * f_2(t)$ is

$$f_1(t) * f_2(t) = \int_{-\infty}^{t} \left(f_1(t-t')\right)\left(f_2(t')\right)\mathrm{d}t' = \int_{-\infty}^{t} \left(f_1(t')\right)\left(f_2(t-t')\right)\mathrm{d}t'$$

We note that in frequency-transformed domain we have

$$\mathcal{L}\left\{f_1(t) * f_2(t)\right\} = \mathcal{L}\left\{f_1(t)\right\}\mathcal{L}\left\{f_2(t)\right\}$$

If the application of voltage is at time $t' = 0$ then in equation $q(t) = \int_{-\infty}^{t} c(t-t')v(t')\mathrm{d}t'$ we have charge storage at time after $t > t'$ as

$$q(t) = \int_{0}^{t} c(t-t')v(t')\mathrm{d}t'$$

This is different than the usual conventional way of writing the charge i.e. $q(t) = c(t)v(t)$; which many text books and researchers use. This argument we will explained in [1] in detail.

With $c(t) = C\delta(t)$, that is for ideal loss less capacitor with capacity function as delta function [1], and $v(t) = V_m u(t)$ a step input at $t = 0$ we get the following

$$q(t)=\int_{-\infty}^{t}C\delta(t-t')v(t')\mathrm{d}t'=Cv(t)$$
$$=CV_m;\qquad t\geq 0$$

Inductance is a response function $l(t)$ to a cause of current flow i.e. $i(t)$ in a conductor to effect the formation of magnetic field $\varphi(t)$. An electric current through any conductor creates a magnetic field around the conductor. We may write thus magnetic field $\varphi(t)$ is effect that is caused by current $i(t)$ flowing through conductor, is responded by response function the property of the conductor call it $l(t)$, that we write as

$$\varphi(t)=\int_{-\infty}^{t}l(t-t')i(t')\mathrm{d}t'$$

A changing current creates a changing magnetic field. From Faraday's law of induction any change in magnetic flux through a circuit induces an electromotive force voltage, i.e. voltage across inductor is rate of change of magnetic flux that we get as effect described above.

The classical case we say a memory less system where $l(t)=L\delta(t)$ act as soon as current $i(t)=I_m u(t)$ acts in conductor say at time $t=0$, we have

$$\varphi(t)=\int_{0}^{t}L\delta(t-\tau)i(\tau)\mathrm{d}\tau=Li(t)$$
$$=LI_m;\quad t\geq 0$$

System with memory when we have power law

We know that time rate of change of $\varphi(t)$ is the induced voltage across inductor. So we write the following

$$v(t)=\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}l(t-t')i(t')\mathrm{d}t'$$
$$=\int_{-\infty}^{t}l(t-t')\left(\frac{\mathrm{d}i(t')}{\mathrm{d}t'}\right)\mathrm{d}t'$$

Consider at $t=0$ an uncharged inductor is excited by a current step, $i(t)=I_m u(t)$ then we write classically the following

$$v(t)=L\frac{\mathrm{d}i(t)}{\mathrm{d}t}=LI_m\delta(t);\quad \frac{\mathrm{d}u(t)}{\mathrm{d}t}=\delta(t)$$

This classical equation holds if and only if we write the time varying inductance function as $l(t)=L\delta(t)$, as

$$v(t)=\int_{-\infty}^{t}l(t-t')\left(\frac{\mathrm{d}i(t')}{\mathrm{d}t'}\right)\mathrm{d}t'$$
$$=\int_{-\infty}^{t}L\delta(t-t')\left(\frac{\mathrm{d}i(t')}{\mathrm{d}t'}\right)\mathrm{d}t'=L\frac{\mathrm{d}i(t)}{\mathrm{d}t}=LI_m\delta(t)$$

The above discussion says that the system responds only at $t=0$ and thereafter it forgets that cause has happened at $t=0$. This is memory less response, and valid for classical inductor case. That is the effect $v(t)$ is only at $t=0$, and vanishes after cause's application.

Now we say the response lingers after the cause has appeared at $t=0$, say given by a singular power law i.e. $l(t)=L_\beta t^{-\beta}$, with power law exponent $0<\beta<1$. Then we do following steps by applying integration by parts as

$$\int_0^t \left(f_1(x)\right)\left(f_2(x)\right)\mathrm{d}x=\left[f_1(x)\int f_2(x)\mathrm{d}x\right]_{x=0}^{x=t}-\int_0^t\left(\left(f_1^{(1)}(x)\right)\int_0^t\left(f_2(x)\right)\mathrm{d}x\right)\mathrm{d}x$$

We get

$$\begin{aligned}
v(t)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t l(t-t')i(t')\mathrm{d}t'=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t L_\beta(t-t')^{-\beta}i(t')\mathrm{d}t'\\
&=\frac{\mathrm{d}}{\mathrm{d}t}L_\beta\int_0^t(t-x)^{-\beta}i(x)\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}L_\beta\int_0^t\frac{i(x)\mathrm{d}x}{(t-x)^\beta},\quad f_1(x)=i(x),\quad f_2(x)=(t-x)^{-\beta}\\
&=L_\beta\frac{\mathrm{d}}{\mathrm{d}t}\left(\left[i(x)\int\frac{\mathrm{d}x}{(t-x)^\beta}\right]_{x=0}^{x=t}-\int_0^t\left(i^{(1)}(x)\int\frac{\mathrm{d}x}{(t\text{ - }x)^\beta}\right)\mathrm{d}x\right)\qquad i^{(1)}(x)\equiv\frac{\mathrm{d}i(x)}{\mathrm{d}x}\\
&=L_\beta\frac{\mathrm{d}}{\mathrm{d}t}\left(i(x)\left(-\frac{(t-x)^{1-\beta}}{1-\beta}\right)\Bigg|_{x=0}^{x=t}-\int_0^t i^{(1)}(x)\left(\frac{(-1)(t-x)^{1-\beta}}{1-\beta}\right)\mathrm{d}x\right)\\
&=L_\beta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{i(0)}{1-\beta}t^{1-\beta}+\int_0^t\frac{i^{(1)}(x)}{1-\beta}(t-x)^{1-\beta}\mathrm{d}x\right)\\
&=L_\beta\left(\frac{i(0)}{t^\beta}+\int_0^t(t-x)^{-\beta}i^{(1)}(x)\mathrm{d}x\right)\\
&=L_\beta\frac{i(0)}{t^\beta}+L_\beta\Gamma(1-\beta)\left(\frac{1}{\Gamma(1-\beta)}\int_0^t(t-x)^{-\beta}i^{(1)}(x)\mathrm{d}x\right)\\
&=\frac{L_{F-\beta}}{\Gamma(1-\beta)}t^{-\beta}i(0)+L_{F-\beta}\left({}_0^CD_t^\beta i(t)\right);\quad L_{F-\beta}=L_\beta\,\Gamma(1-\beta)\\
&=L_{F-\beta}\left(\frac{i(0)}{\Gamma(1-\beta)}t^{-\beta}+{}_0^CD_t^\beta i(t)\right),\quad {}_0D_t^\beta i(t)=\frac{i(0)}{\Gamma(1-\beta)}t^{-\beta}+{}_0^CD_t^\beta i(t)\\
&=L_{F-\beta}\left({}_0D_t^\beta i(t)\right)\\
&=L_{F-\beta}\frac{\mathrm{d}^\beta i(t)}{\mathrm{d}t^\beta}
\end{aligned}$$

In above steps, the lower limit of convolution integral $-\infty$ is taken to 0 since, the cause i.e. current is appearing only at $t=0$; and then after.

We note that $\frac{1}{\Gamma(1-\beta)}\int_0^t(t-x)^{-\beta}\frac{\mathrm{d}i(x)}{\mathrm{d}x}\mathrm{d}x\equiv{}_0^CD_t^\beta i(t)$ i.e. the Caputo fractional derivative of order $0<\beta<1$; [12], [13]. For converting it to Riemann-Liouville (RL) fractional derivative we used ${}_0D_t^\beta\left[f(t)\right]={}_0^CD_t^\beta\left[f(t)\right]+\frac{f(0)}{\Gamma(1-\beta)}t^{-\beta}$, $0<\beta<1$ and write a general equation as follows

$$v(t)=L_{F-\beta}\frac{\mathrm{d}^\beta i(t)}{\mathrm{d}t^\beta},\quad 0<\beta<1.\quad L_{F-\beta}=L_\beta\,\Gamma(1-\beta)$$

For uncharged case $i(0)=0$ so above is Caputo as well as RL fractional derivative.

For excitation $i(t) = I_m u(t)$ at $t = 0$ we have the following

$$v(t) = L_{F-\beta}\frac{\mathrm{d}^\beta i(t)}{\mathrm{d}t^\beta} = L_{F-\beta}\frac{\mathrm{d}^\beta I_m}{\mathrm{d}t^\beta}; \quad t \ge 0; \quad \frac{\mathrm{d}^\beta t^m}{\mathrm{d}t^\beta} = \frac{\Gamma(m+1)t^{m-\beta}}{\Gamma(m+1-\beta)}$$

$$= L_{F-\beta} I_m \frac{1}{\Gamma(1-\beta)} t^{-\beta} = L_\beta t^{-\beta}; \quad L_{F-\beta} = L_\beta \Gamma(1-\beta)$$

Therefore, the voltage $v(t)$ the effect lingers even at $t > 0$ that is memorizing the cause, that happened at $t = 0$. The memorized capacity we write as $c(t) = C_\alpha t^{-\alpha}$ with $0 < \alpha < 1$ for which we have equation as

$$i(t) = C_{F-\alpha}\frac{\mathrm{d}^\alpha i(t)}{\mathrm{d}t^\alpha}, \quad 0 < \alpha < 1, \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$$

This derivation of above expression for fractional capacitor we are not repeating. The same steps in details as done for memory based inductor will give the above result, for a memory based capacitor is a fractional capacitor), [1].

Linear and non-dispersive transmission lines with ideal inductance and capacitance a memory less TL

We revise the concept of transmission line (TL), having $l(t)$ as time varying inductance function per unit length, and $c(t)$ as time varying capacitance per init length, as distributed parameters of the transmission line. Let $i(x,t)$ and $v(x,t)$ represents current and voltage variables in x and t . Refer Figure-1, for this distributed TL.

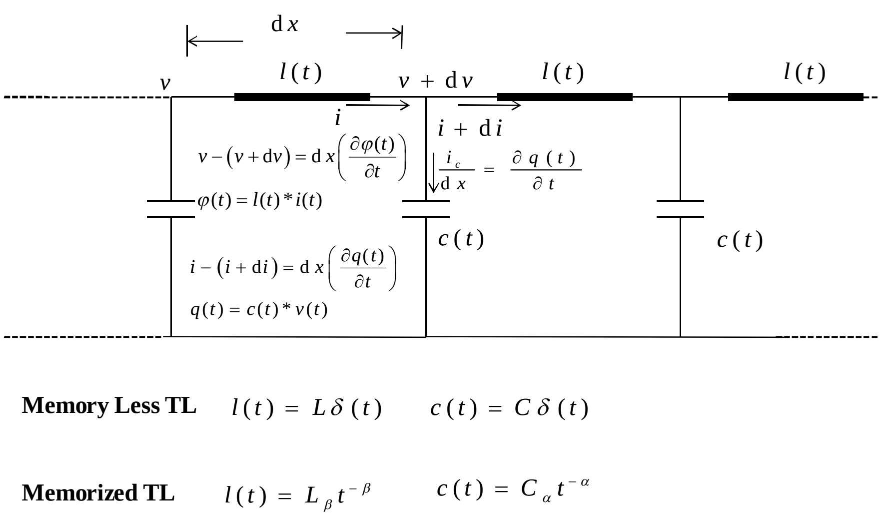

Figure-1: Distributed Transmission Line

For memory less TL take $l(t) = L\delta(t)$ and $c(t) = C\delta(t)$. Then via KVL for element $\mathrm{d}x$ section of TL we write the following equation

$$v-\left(v+\mathrm{d}v\right)=\mathrm{d}\,x\left(\frac{\partial\varphi(t)}{\partial t}\right)$$
$$\varphi(t)=l(t)*i(t)$$
$$l(t)=L\delta(t)$$
$$\varphi(t)=L\delta(t)*i(t)=Li(t)$$

This gives

$$v-\left(v+\mathrm{d}v\right)=\mathrm{d}\,x\left(L\frac{\partial i(t)}{\partial t}\right)$$
$$-\frac{\partial v}{\partial x}=L\frac{\partial i}{\partial t}$$

Similarly using KCL for element unit $\mathrm{d}x$ we write the following

$$i-\left(i+\mathrm{d}i\right)=\mathrm{d}\,x\left(\frac{\partial q(t)}{\partial t}\right)$$
$$q(t)=c(t)*v(t)$$
$$c(t)=C\delta(t)$$
$$q(t)=C\delta(t)*v(t)=Cv(t)$$

This gives

$$-\frac{\partial i}{\partial x}=C\frac{\partial v}{\partial t}$$

Differentiating above wrt t we get and using $(-\partial v/\partial x)=L\left(\partial i/\partial t\right)$ we get the following

$$-\frac{\partial}{\partial t}\frac{\partial i}{\partial x}=C\frac{\partial^2 v}{\partial t^2}$$
$$C\frac{\partial^2 v}{\partial t^2}=-\frac{\partial}{\partial x}\left(\frac{\partial i}{\partial t}\right)$$
$$=-\frac{\partial}{\partial x}\left(-\frac{1}{L}\frac{\partial v}{\partial x}\right)$$

Writing $c_0=\frac{1}{\sqrt{LC}}$ we write the TL equation as

$$\frac{\partial^2 v}{\partial t^2}-c_0^2\frac{\partial^2 v}{\partial x^2}=0$$

The above wave equation gives simple travelling waves in TL moving with velocity c_0.

Linear non-dispersive Transmission line with time varying capacity function and ideal inductance

We take $l(t)=L\delta(t)$, i.e. classical memory less inductor, and $c(t)=C_\alpha t^{-\alpha}$, the fractional capacitor or memory based capacitor. Via KVL for element $\mathrm{d}x$ section of TL we write the following

$$v-\left(v+\mathrm{d}v\right)=\mathrm{d}\,x\left(\frac{\partial\varphi(t)}{\partial t}\right)$$
$$\varphi(t)=l(t)*i(t);$$
$$l(t)=L\delta(t)$$
$$\varphi(t)=L\delta(t)*i(t)=Li(t)$$

This gives

$$v-\left(v+\mathrm{d}v\right)=\mathrm{d}\,x\left(L\frac{\partial i(t)}{\partial t}\right)$$
$$-\frac{\partial v}{\partial x}=L\frac{\partial i}{\partial t}$$

Similarly using KCL for element unit $\mathrm{d}x$ we write the following

$$i-\left(i+\mathrm{d}i\right)=\mathrm{d}\,x\left(\frac{\partial q(t)}{\partial t}\right)\qquad q(t)=c(t)*v(t)$$
$$c(t)=C_\alpha t^{-\alpha};\quad 0<\alpha<1$$
$$q(t)=\int_0^t C_\alpha(t-\tau)^{-\alpha}v(\tau)\mathrm{d}\tau$$

Doing integration by parts and then differentiating we get the following

$$\frac{\partial q(t)}{\partial t}=\frac{\partial}{\partial t}\int_0^t C_\alpha(t-\tau)^{-\alpha}v(\tau)\mathrm{d}\tau$$
$$=C_\alpha v(0)t^{-\alpha}+C_\alpha\Gamma(1-\alpha)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t\mathrm{d}\tau\left((t-\tau)^{-\alpha}\frac{\partial v(\tau)}{\partial\tau}\right)\right)$$

Noting that $\frac{1}{\Gamma(1-\alpha)}\int_0^t\left((t-\tau)^{-\alpha}\frac{\partial v(\tau)}{\partial\tau}\right)\mathrm{d}\tau={}_0^CD_t^\alpha v(t)$ is Caputo fractional derivative formula for order $0<\alpha<1$ we write in terms of RL fractional derivative the following

$$\frac{\partial q(t)}{\partial t}=C_\alpha v(0)t^{-\alpha}+C_\alpha\Gamma(1-\alpha)\left({}_0^CD_t^\alpha v(t)\right);\quad C_{F-\alpha}=C_\alpha\,\Gamma(1-\alpha)$$
$$=C_{F-\alpha}\left(\frac{v(0)}{\Gamma(1-\alpha)}t^{-\alpha}+C_{F-\alpha}\left({}_0^CD_t^\alpha v(t)\right)\right)=C_{F-\alpha}\left({}_0D_t^\alpha v(t)\right)$$
$$=C_{F-\alpha}\frac{\mathrm{d}^\alpha v(t)}{\mathrm{d}t^\alpha}$$

This gives

$$-\frac{\partial i}{\partial x}=C_\alpha\Gamma(1-\alpha)\frac{\partial^\alpha v}{\partial t^\alpha}$$

Differentiating above once wrt t and using $(-\partial v/\partial x)=L\left(\partial i/\partial t\right)$ we get the following

$$-\frac{\partial}{\partial t}\frac{\partial i}{\partial x}=C_\alpha\Gamma(1-\alpha)\frac{\partial}{\partial t}\frac{\partial^\alpha v}{\partial t^\alpha}$$
$$-\frac{\partial}{\partial x}\left(\frac{\partial i}{\partial t}\right)=C_\alpha\Gamma(1-\alpha)\frac{\partial^{\alpha+1}v}{\partial t^{\alpha+1}}$$
$$-\frac{\partial}{\partial x}\left(-\frac{1}{L}\frac{\partial v}{\partial x}\right)=C_\alpha\Gamma(1-\alpha)\frac{\partial^{\alpha+1}v}{\partial t^{\alpha+1}}$$

Writing $c_\alpha=\frac{1}{\sqrt{LC_\alpha\Gamma(1-\alpha)}}=\frac{1}{\sqrt{LC_{F-\alpha}}}$ we write the TL equation as follows

$$\frac{\partial^{\alpha+1}v}{\partial t^{\alpha+1}}-c_\alpha^2\frac{\partial^2 v}{\partial x^2}=0$$

This above is TFDWE of TL with memory element as fractional capacitor.

Linear non-dispersive Transmission line with time varying capacity function and time varying inductance function-doubly memorized TL

Here we take fractional inductor that is memory based, as $l(t)=L_\beta t^{-\beta}$, and fractional capacitor or memory based capacitor as $c(t)=C_\alpha t^{-\alpha}$. Via KVL for element $\mathrm{d}x$ section of TL we write the following

$$v-\left(v+\mathrm{d}v\right)=\mathrm{d}\,x\left(\frac{\partial\varphi(t)}{\partial t}\right)$$
$$\varphi(t)=l(t)*i(t);$$
$$l(t)=L_\beta t^{-\beta};\quad 0<\beta<1$$
$$\frac{\partial\varphi(t)}{\partial t}=\frac{\partial}{\partial t}\left[\left(L_\alpha t^{-\beta}\right)*\left(\left(i(t)\right)\right)\right]$$

Thus we have following steps by using integration by parts formula and then differentiation

$$\frac{\partial\varphi(t)}{\partial t}=\frac{\partial}{\partial t}\left[\left(L_\beta t^{-\beta}\right)*\left(\left(i(t)\right)\right)\right]$$
$$=L_\beta\frac{\partial}{\partial t}\int_0^t(t-\tau)^{-\beta}i(\tau)\mathrm{d}\tau;\quad L_{F-\beta}=L_\beta\Gamma(1-\beta)$$
$$=L_{F-\beta}\left(\frac{i(0)}{\Gamma(1-\beta)}+{}_0^CD_t^\beta i(t)\right);\quad {}_0D_t^\beta\left[f(t)\right]={}_0^CD_t^\beta\left[f(t)\right]+\tfrac{f(0)}{\Gamma(1-\beta)}t^{-\beta}$$
$$=L_{F-\beta}\left({}_0D_t^\beta i(t)\right);\quad 0<\beta<1$$

We get

$$-\frac{\partial v}{\partial x}=L_\beta\Gamma(1-\beta)\frac{\partial^\beta i(t)}{\partial t^\beta};\quad 0<\beta<1$$

Similarly using KCL for unit element $\mathrm{d}x$ we write the following

$$i-\left(i+\mathrm{d}i\right)=\mathrm{d}\,x\left(\frac{\partial q(t)}{\partial t}\right)\qquad q(t)=c(t)*v(t)$$

$$c(t) = C_\alpha t^{-\alpha}; \quad 0 < \alpha < 1$$

$$q(t) = \int_0^t C_\alpha (t-\tau)^{-\alpha} v(\tau) \mathrm{d}\tau$$

Doing integration by parts and then differentiating we write following

$$\begin{aligned}\frac{\partial q(t)}{\partial t} &= \frac{\partial}{\partial t}\int_0^t C_\alpha (t-\tau)^{-\alpha} v(\tau)\mathrm{d}\tau \\ &= C_\alpha v(0) t^{-\alpha} + C_\alpha \Gamma(1-\alpha)\left(\frac{1}{\Gamma(1-\alpha)}\int_0^t \mathrm{d}\tau \left((t-\tau)^{-\alpha}\frac{\partial v(\tau)}{\partial \tau}\right)\right) \\ &= C_{F-\alpha}\left(\frac{v(0)}{\Gamma(1-\alpha)}t^{-\alpha} + {}_0^C D_t^\alpha v(t)\right); \quad 0<\alpha<1, \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)\end{aligned}$$

Noting that ${}_0 D_t^\beta \left[f(t)\right] = {}_0^C D_t^\beta \left[f(t)\right] + \frac{f(0)}{\Gamma(1-\beta)} t^{-\beta}$ is Caputo RL fractional derivative relation formula for order $0<\alpha<1$ we write

$$\frac{\partial q(t)}{\partial t} = C_\alpha \Gamma(1-\alpha)\frac{\partial^\alpha v}{\partial t^\alpha}$$

This gives

$$-\frac{\partial i}{\partial x} = C_\alpha \Gamma(1-\alpha)\frac{\partial^\alpha v}{\partial t^\alpha}$$

Differentiating above once by order β wrt t and using $-\frac{\partial v}{\partial x} = L_\beta \Gamma(1-\beta)\left(\frac{\partial^\beta i(t)}{\partial t^\beta}\right)$

$$-\frac{\partial^\beta}{\partial t^\beta}\frac{\partial i}{\partial x} = C_\alpha \Gamma(1-\alpha)\frac{\partial^{\alpha+\beta} v}{\partial t^{\alpha+\beta}}$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial^\beta i}{\partial t^\beta}\right) = C_\alpha \Gamma(1-\alpha)\frac{\partial^{\alpha+\beta} v}{\partial t^{\alpha+\beta}}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{L_\beta \Gamma(1-\beta)}\frac{\partial v}{\partial x}\right) = C_\alpha \Gamma(1-\alpha)\frac{\partial^{\alpha+\beta} v}{\partial t^{\alpha+\beta}}$$

We used composition and $\frac{\partial^{\alpha+\beta}}{\partial t^{\alpha+\beta}} \equiv \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta}{\partial t^\beta} \equiv \frac{\partial^\beta}{\partial t^\beta}\frac{\partial^\alpha}{\partial t^\alpha}$ are sometimes valid not always for and $\alpha, \beta > 0$ [12], [13]. Writing $c_{\alpha\beta} = \frac{1}{\sqrt{L_\beta C_\alpha \Gamma(1-\alpha)\Gamma(1-\beta)}} = \frac{1}{\sqrt{L_{F-\beta} C_{F-\alpha}}}$ we write the TL equation as

$$\frac{\partial^{\alpha+\beta} v}{\partial t^{\alpha+\beta}} - c_{\alpha\beta}^2 \frac{\partial^2 v}{\partial x^2} = 0$$

The above we get TFDWE for memorized TL, having and $l(t)$ and $c(t)$ as fractional elements, with memory.

Conclusions

We have applied the new formula of charge storage and magnetic flux i.e. via convolution operation, of time varying capacity function and voltage stress for a fractional capacitor; and time varying inductance function and time varying current function for inductor-in a transmission line (TL). We obtained the TL equation for classical memory less case with ideal inductor and capacitive distributed parameters; the equation is simple wave equation. With the fractional capacitor and also with fractional inductor we get Time Fractional Diffusion-Wave equation TFDW for a linear non dispersive TL. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function; for charge in capacitor; and multiplication of inductance and current for magnetic flux in an inductor. Application of these new concepts gets validated here in this application of TL modeling.

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