Verification of formula 'q (t) = c (t) * v (t)' applied to fractional capacitor and classical ideal capacitor in RC- charging circuit

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Abstract

In this presentation note continuation from our earlier deliberation, we apply the newly developed charge storage expression as a function of time i.e. via convolution operation of time varying capacity function and applied voltage function to a capacitor. We apply this new formula to various types of RC circuit as charging the capacitors via constant dc voltage or current sources. This new formula is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor. This new deliberation with convolution operation works well for classical ideal loss less capacitors, where one says that is a constant capacity, and also for a time varying capacity function given by decaying power-law: that gives the formation of fractional capacitor. This presentation note gives validity of usage of this new formula.

Keywords

Mittag-Leffler function, Time varying Capacity Function, Fractional Capacitor, Ideal loss-less Capacitor, Convolution Operation, Laplace Transform, Caputo fractional derivative

Introduction

This is continuation of our earlier deliberations regarding the new formula $q(t)=c(t)*v(t)$ .The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality we write $q(t)=c(t)*v(t)$ where $v(t)$ is the input impressed voltage. This is different to usual formula $q(t)=c(t)v(t)$. This new formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t)\sim t^{-\alpha}$; $0<\alpha<1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t)\sim\delta(t)$; [1]. In this presentation note we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one, i.e. $0<\alpha<1$. This power-law decay function is in singular at origin and in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t)\sim t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11]. The use of non-singular kernel in integration for the formula for fractional derivative and application is developing topic. This concept is used and studied in pioneering works [23]-[36]. Here we are taking singular function $c(t)$ as 'time varying capacity function', as because the same gets derived from basic universal dielectric relaxation law $i(t)\sim t^{-\alpha}$ of Curie-von Schweidler.

In this note we will take capacitor with time varying capacity function $c(t)=C_{\alpha}t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula

$$q(t)=c(t)*v(t)=\int_0^t c(t-\tau)v(\tau)d\tau=\int_0^t c(\tau)v(t-\tau)d\tau$$

and discuss various cases of $q(t)$ for RC charging circuit with ideal capacitor and fractional capacitor. We note a priori that the constant C_{α} is proportionality constant of the relation of time varying capacity function i.e. $c(t)\sim t^{-\alpha}$, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad}/\sec^{1-\alpha}$, and we will use $C_{F-\alpha}=C_{\alpha}\Gamma(1-\alpha)$ to relate the two. The equation of current and voltage, and impedance for fractional capacitor is following, given by fractional derivative $D_t^{\alpha}\equiv d^{\alpha}/dt^{\alpha}$ of Caputo type [6], [7] [8], [12], [13];

$$i(t)=C_{F-\alpha}\frac{d^{\alpha}v(t)}{dt^{\alpha}};\quad Z(s)=\frac{1}{s^{\alpha}C_{F-\alpha}};\quad 0<\alpha<1$$

With $\alpha=1$ we get classical ideal loss less capacitor.

$$i(t)=C\frac{d\ v(t)}{dt};\quad Z(s)=\frac{1}{s\ C}$$

The fractional capacitor appears in studies with super-capacitors and other memory based relaxation phenomena [14]-[22]. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (having output impedance as zero), in the RC charging circuits. We will use Laplace Transform technique in all analysis. In all the cases in subsequent sections, we will apply this new formula and give the validity justification.

Charge in a Capacitor is $q(t)=c(t)*v(t)$, is given via convolution operation and not with the usual way that we write as $q(t)=c(t)v(t)$. Let us have a capacitor with capacity function in time as power-law $c(t)=C_{\alpha}t^{-\alpha}$ ($0<\alpha<1$), that is fractional capacitor, is charged via resistance R. Let a voltage $v_{in}(t)$ or current $i_{in}(t)$ be applied to an uncharged capacitor in the RC circuit at time $t=0$, to this RC charging circuit. Then charge function in time is given as convolution $(*)$ operation as $q(t)=c(t)*v_0(t)$, with $v_0(t)$ is the voltage profile on the capacitor, in the RC circuit. For each case we also study the ideal loss less capacitor given by capacity function as $c(t)=C\delta(t)$, and apply $q(t)=c(t)*v_0(t)$. We will validate verify this new formula.

Charge storage $q(t)$ by step input voltage $v_{in}(t)=V_m u(t)$ excitation to RC circuit with ideal loss less capacitor

In classical circuit theory, if we charge an ideal capacitor, C through a resistor R, via a step input voltage $v_{in}(t)=V_m u(t)$ (Figure-1) we get voltage across capacitor as exponential rise as $v_0(t)=V_m(1-e^{-t/RC})$. In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is ideal capacitor with capacity function as $c(t)=C\delta(t)$. Therefore we have following impedance function

$$Z_2(s)=\frac{1}{s\mathcal{L}\{c(t)\}}=\frac{1}{s\mathcal{L}\{C\delta(t)\}}=\frac{1}{sC} \tag{1}$$

The above way of writing $Z(s)$ for capacitor ideal or fractional we got from application of formula $q(t)=c(t)*v(t)$ in our previous discussion, that we got by differentiating this convolution expression to get $i(t)$ and taking Laplace transform to arrive as $Z(s)=V(s)/I(s)=\left(s\mathcal{L}\{c(t)\}\right)^{-1}$.

We have from circuit theory of Figure-1 the following expressions

$$\begin{aligned} V_0(s)&=\frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\{v_{in}(t)\},\quad \mathcal{L}\{v_{in}(t)\}=\frac{V_m}{s}\\ &=\frac{V_m}{RCs\left(s+\frac{1}{RC}\right)}=V_m\left(\frac{1}{s}-\frac{1}{s+\frac{1}{RC}}\right) \end{aligned} \tag{2}$$

The inverse Laplace Transform of (2) gives following charging equation for capacitor

$$v_0(t)=V_m(1-e^{-t/RC});\quad t\ge 0 \tag{3}$$

The current flowing in the RC circuit at $t\ge 0$ is following

$$\begin{aligned} i(t)&=\mathcal{L}^{-1}\left\{\frac{V_m/s}{R+\frac{1}{Cs}}\right\}=\mathcal{L}^{-1}\left\{\frac{V_m}{R}\left(\frac{1}{s+\frac{1}{RC}}\right)\right\}\\ &=\frac{V_m}{R}e^{-t/RC} \end{aligned} \tag{4}$$

Therefore the charge function $q(t)$ is

$$\begin{aligned} q(t)&=\int_0^t i(\tau)d\tau=\int_0^t \frac{V_m}{R}e^{-\tau/RC}\\ &=V_mC(1-e^{-t/RC});\quad t\ge 0 \end{aligned} \tag{5}$$

We apply the formula $q(t)=c(t)*v(t)$ to ideal capacitor given by $c(t)=C\delta(t)$ across which we are having a voltage profile as $v_0(t)=V_m(1-e^{-t/RC})$, to write following

$$\begin{aligned} Q(s)&=\left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right)\\ &=\left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{V_m(1-e^{-t/RC})\}\right)=C\left(\frac{V_m}{s}-\frac{V_m}{\left(s+\frac{1}{RC}\right)}\right) \end{aligned} \tag{6}$$

The inverse Laplace transform of (6) above gives

$$q(t)=V_mC(1-e^{-t/RC}) \tag{7}$$

This gives validation of formula $q(t)=c(t)*v(t)$ for classical ideal loss less capacitor case.

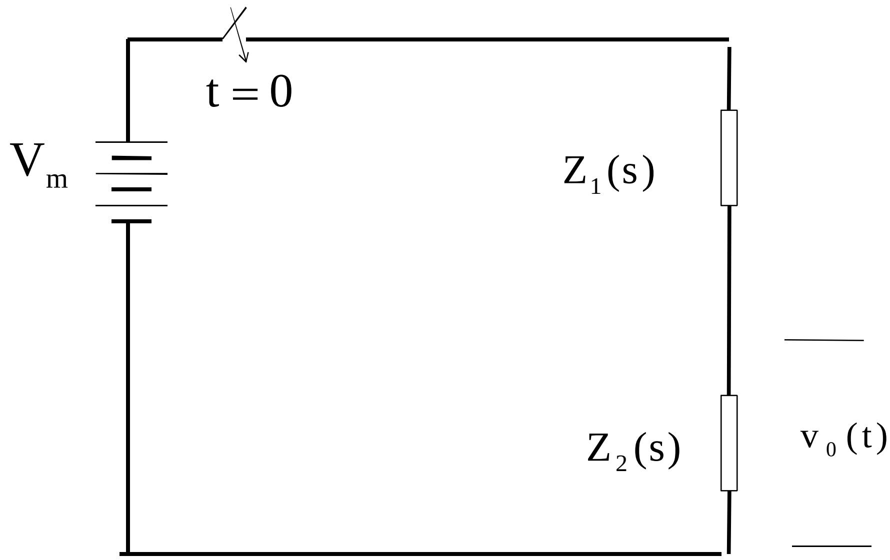

Figure- 1: The constant voltage charging RC circuit

Charge storage $q(t)$ by step input voltage $v_{in}(t)=V_m u(t)$ excitation to RC circuit with fractional capacitor

In Figure-1 consider $Z_1(s)=R$, and $Z_2(s)$ is fractional capacitor with capacity function as $c(t)=C_\alpha t^{-\alpha}$. Therefore we have following impedance function

$$\begin{aligned} Z_2(s) &= \frac{1}{s\mathcal{L}\left\{c(t)\right\}} = \frac{1}{s\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}} = \frac{1}{s\left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)} \\ &= \frac{1}{s^\alpha C_\alpha \Gamma(1-\alpha)} = \frac{1}{s^\alpha C_{F-\alpha}}; \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha) \end{aligned} \tag{8}$$

Here we will use a constant voltage excitation of V_m from time $t=0$, to time $t=T_c$,(as charging phase, through a known resistor R) and thereafter we will switch to discharging phase i.e. voltage source will be made zero (Figure-2). By this we record the charging and discharging profile $v_0(t)$. From the circuit diagram of Figure-1, we write the following [37]

$$\begin{aligned} V_0(s) &= \frac{Z_2(s)}{Z_1(s)+Z_2(s)}\mathcal{L}\left\{v_{in}(t)\right\}, \quad \mathcal{L}\left\{v_{in}(t)\right\} = \frac{V_m}{s} \\ &= \frac{V_m}{RC_{F-\alpha}s\left(s^\alpha + \frac{1}{RC_{F-\alpha}}\right)} = \frac{V_m k s^{-1}}{(s^\alpha + k)}; \quad k = \frac{1}{RC_{F-\alpha}} \end{aligned} \tag{9}$$

We use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\} = \frac{p!s^{\alpha-\beta}}{s^\alpha - a}$ to get $\mathcal{L}^{-1}\left\{\frac{s^{-1}}{s^\alpha - a}\right\} = t^\alpha E_{\alpha,\alpha+1}(at^\alpha)$, by putting $p=0$, $\alpha=\alpha$, $\beta=\alpha+1$. The $E_{\alpha,\beta}(at^\alpha)$ is two parameter Mittag-Leffler function; as defined below;

$$E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)},\qquad E_{\alpha,(\alpha+1)}(-kt^{\alpha})=\sum_{m=0}^{\infty}\frac{(-kt^{\alpha})^m}{\Gamma(m\alpha+\alpha+1)} \tag{10}$$

With this we obtain the following from Laplace inverse of (9)

$$\begin{aligned} v_0(t) &= \mathcal{L}^{-1}\left\{\tfrac{V_m k}{s(s^{\alpha}+k)}\right\}=V_m k t^{\alpha} E_{\alpha,\alpha+1}(-kt^{\alpha}) \\ &= \frac{V_m}{RC_{F-\alpha}}t^{\alpha}E_{\alpha,\alpha+1}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right) \end{aligned} \tag{11}$$

We have alternate derivation via series expansion [13], [37] as follows

$$\begin{aligned} V_0(s) &= \frac{V_m k}{s(s^{\alpha}+k)}=\frac{V_m k}{s^{\alpha+1}\left(1+\frac{k}{s^{\alpha}}\right)} \\ &= \frac{V_m k}{s^{\alpha+1}}\left(1-\frac{k}{s^{\alpha}}+\frac{k^2}{s^{2\alpha}}-\frac{k^3}{s^{3\alpha}}+\ldots\right) \\ &= V_m\left(\frac{k}{s^{\alpha+1}}-\frac{k^2}{s^{2\alpha+1}}+\frac{k^3}{s^{3\alpha+1}}-..\right) \end{aligned} \tag{12}$$

Use Laplace pair $\frac{\Gamma(n+1)}{s^{n+1}}=\mathcal{L}\left\{t^n\right\}$ to invert term by term the above to get following

$$\begin{aligned} v_0(t) &= V_m\left(\frac{kt^{\alpha}}{\Gamma(\alpha+1)}-\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}-\ldots\right) \\ &= V_m\left(1-\left[1-\frac{kt^{\alpha}}{\Gamma(\alpha+1)}+\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right]\right) \\ &= V_m\left(1-\sum_{n=0}^{\infty}\frac{(-kt^{\alpha})^n}{\Gamma(n\alpha+1)}\right)=V_m\left[1-E_{\alpha}(-kt^{\alpha})\right]=V_m\left[1-E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right)\right] \end{aligned} \tag{13}$$

Where, $E_{\alpha}(x)$ is one parameter Mittag-Leffler function used above (13), with $E_1(x)=e^x$. Therefore for classical ideal capacitor with $\alpha=1$, we have normal exponential charging $v_0(t)=V_m(1-e^{-t/RC})$; writing $C_{F-\alpha}\equiv C$. For voltage charging expression for fractional order impedance $Z_2(s)=s^{-\alpha}\,C_{F-\alpha}^{-1}$ we have

$$v_0(t)=V_m\left(1-E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right)\right)=\frac{V_m}{RC_{F-\alpha}}t^{\alpha}E_{\alpha,\alpha+1}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right) \tag{14}$$

For charging current of circuit of Figure-1 with $Z_1=R$ and $Z_2(s)=\frac{1}{s^{\alpha}C_{F-\alpha}}$, we have $Z(s)=Z_1(s)+Z_2(s)$ and write the following

$$I(s)=\frac{1}{Z(s)}\left(\frac{V_m}{s}\right)=\frac{V_m}{s\left(R+\frac{1}{s^{\alpha}C_{F-\alpha}}\right)}=\frac{V_m}{R}\left(\frac{s^{\alpha-1}}{s^{\alpha}+\frac{1}{RC_{F-\alpha}}}\right) \tag{15}$$

Using $\mathcal{L}\left\{E_n(at^n)\right\}=\frac{s^{n-1}}{s^n-a}$, we get inverse of above (15) as

$$i(t)=\frac{V_m}{R}E_{\alpha}\left(-\tfrac{t^{\alpha}}{RC_{F-\alpha}}\right) \tag{16}$$

Clearly for ideal $\alpha = 1$ case we get $i(t) = \frac{V_m}{R} e^{-t/RC}$. Therefore the charge $q(t)$ is from (16) the following

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \tag{17}$$

We apply the formula $q(t) = c(t) * v(t)$ to fractional capacitor given by $c(t) = C_\alpha t^{-\alpha}$ across which we are having a voltage profile as $v_0(t) = V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)$, to write following

$$\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_0(t)\}\right) \\
&= \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1 - E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right)\right\}\right) \\
&= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\tfrac{V_m k}{s(s^\alpha + k)}\right) = \frac{V_m C_{F\text{-}\alpha}\left(\tfrac{1}{RC_{F\text{-}\alpha}}\right)}{s^{2-\alpha}\left(s^\alpha + \tfrac{1}{RC_{F\text{-}\alpha}}\right)}; \quad k = \frac{1}{RC_{F\text{-}\alpha}}, \quad C_{F\text{-}\alpha} = C_\alpha\Gamma(1-\alpha) \qquad (18) \\
&= \left(\frac{V_m}{R}\right)\frac{s^{\alpha-2}}{\left(s^\alpha + \tfrac{1}{RC_{F\text{-}\alpha}}\right)} \qquad\qquad \mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k} \\
&= \left(\frac{V_m}{R}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^\alpha + \tfrac{1}{RC_{F\text{-}\alpha}}\right)}\right)\right) = \left(\frac{V_m}{R}\right)\left(s^{-1}\mathcal{L}\left\{E_\alpha\left(-\tfrac{t^\alpha}{RC_{F\text{-}\alpha}}\right)\right\}\right)
\end{aligned}$$

Taking inverse Laplace transform of (18) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = s^{-1}F(s)$ we write

$$q(t) = \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \tag{19}$$

The same result as we got by using $q(t) = \int_0^t i(\tau)d\tau$ validates the verification of formula $q(t) = c(t) * v(t)$. Put $\alpha = 1$ in (19) and we get ideal loss-less capacitor $C_{F\text{-}\alpha} = C$, and $E_1(x) = e^x$ to write the following case

$$\begin{aligned}
q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau\bigg|_{\alpha=1} = \int_0^t \frac{V_m}{R} e^{-\tau/RC} d\tau \\
&= V_m C\left(1 - e^{-t/RC}\right)
\end{aligned} \tag{20}$$

The above (20) is charge build up relation for ideal-loss less capacitor.

The integration of Mittag-Leffler function is $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ with $E_{\alpha,\beta}(x) = \sum_{m=0}^{\infty} \frac{(x)^m}{\Gamma(\alpha m+\beta)}$. So we have charge build up function on a fractional capacitor in RC charging circuit as follows

$$\begin{aligned}
q(t) &= \int_0^t \frac{V_m}{R} E_\alpha\left(-\tfrac{\tau^\alpha}{RC_{F\text{-}\alpha}}\right)d\tau \\
&= \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha / RC_{F\text{-}\alpha})\right); \quad t \ge 0
\end{aligned} \tag{21}$$

We verify the formula used $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ as in following steps

$$\begin{aligned}
\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau &= \int_0^t\left(1-\frac{k\tau^\alpha}{\Gamma(\alpha+1)}+\frac{k^2\tau^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{k^3\tau^{3\alpha}}{\Gamma(3\alpha+1)}+....\right)d\tau \\
&= t-\frac{kt^{\alpha+1}}{(\alpha+1)\Gamma(\alpha+1)}+\frac{k^2t^{2\alpha+1}}{(2\alpha+1)\Gamma(2\alpha+1)}-\frac{k^3t^{3\alpha+1}}{(3\alpha+1)\Gamma(3\alpha+1)}+... \\
&= t\left(1-\frac{kt^\alpha}{\Gamma(\alpha+2)}+\frac{k^2t^{2\alpha}}{\Gamma(2\alpha+2)}-\frac{k^3t^{3\alpha}}{\Gamma(3\alpha+2)}+...\right), \quad \Gamma(m+1)=m\Gamma(m) \\
&= t\left(E_{\alpha,2}(-kt^\alpha)\right) \quad ; \quad E_{\alpha,\beta}(x)=\sum_{m=0}^{\infty}\frac{(x)^m}{\Gamma(\alpha m+\beta)}
\end{aligned} \tag{22}$$

Let us verify this for $\alpha=1$ from (22)

$$\begin{aligned}
q(t) &= \frac{V_m t}{R}\left(E_{\alpha,2}(-t^\alpha/RC_{F\text{-}\alpha})\right)\Big|_{\alpha=1;C_{F\text{-}\alpha}=C} \quad ; \quad E_{\alpha,2}(-a\,x^\alpha)=\sum_{m=0}^{\infty}\frac{(-1)^m a^m x^{\alpha m}}{\Gamma(\alpha m+2)} \\
&= \frac{V_m}{R}t\left(1-\frac{t}{(RC)\Gamma(3)}+\frac{t^2}{(RC)^2\Gamma(4)}-\frac{t^3}{(RC)^3\Gamma(5)}+...\right) \\
&= \frac{V_m C}{RC}\left(t-\frac{t^2}{(RC)(2)!}+\frac{t^3}{(RC)^2(3!)}-\frac{t^4}{(RC)^3(4)!}+...\right) \\
&= V_m C\left(1-1+\frac{\left(\frac{t}{RC}\right)}{1!}-\frac{\left(\frac{t}{RC}\right)^2}{2!}+\frac{\left(\frac{t}{RC}\right)^3}{3!}-\frac{\left(\frac{t}{RC}\right)^4}{4!}+...\right) \\
&= V_m C\left(1-\left(1-\frac{\left(\frac{t}{RC}\right)}{1!}+\frac{\left(\frac{t}{RC}\right)^2}{2!}-\frac{\left(\frac{t}{RC}\right)^3}{3!}+\frac{\left(\frac{t}{RC}\right)^4}{4!}-...\right)\right) \\
&= V_m C\left(1-e^{-t/RC}\right)
\end{aligned} \tag{23}$$

Thus we have verified the validity of formula $q(t)=c(t)*v(t)$ in RC charging circuit with fractional capacitor

Charging a super-capacitor with ESR via RC charging circuit

The differential equation corresponding to Figure-1 for $\alpha=1$, is ordinary differential equation (ODE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{sC}$ is following

$$RC\frac{dv_0(t)}{dt}+v_0(t)=v_{in}(t) \tag{24}$$

For $\alpha\neq1$ we get fractional differential equation (FDE), with $Z_1(s) = R$ and $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ is following

$$RC_{F\text{-}\alpha}\frac{d^\alpha v_0(t)}{dt^\alpha}+v_0(t)=v_{in}(t) \tag{25}$$

We now consider a lumped ESR R_s for super-capacitor, thus for Figure-1 we have $Z_2(s)=R_s+\frac{1}{s^\alpha C_{F\text{-}\alpha}}=\frac{s^\alpha R_s C_{F\text{-}\alpha}+1}{s^\alpha C_{F\text{-}\alpha}}$ while charging impedance remains at $Z_1(s)=R$. Therefore for any input voltage $V_{in}(s)=\mathcal{L}\{v_{in}(t)\}$, we write the charging current (in Laplace domain) as

$$I_{CH}(s)=\frac{V_{in}(s)}{R+R_s+\frac{1}{s^\alpha C_{F\text{-}\alpha}}}=\frac{s^\alpha C_{F\text{-}\alpha}V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1} \tag{26}$$

Output voltage across $Z_2(s)$ in Laplace domain is therefore is

$$V_0(s)=\left(I_{CH}(s)\right)\left(Z_2(s)\right)=\left(\frac{V_{in}(s)s^\alpha C_{F\text{-}\alpha}}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{s^\alpha R_s C_{F\text{-}\alpha}+1}{s^\alpha C_{F\text{-}\alpha}}\right)$$

$$=\frac{V_{in}(s)+V_{in}s^\alpha R_s C_{F\text{-}\alpha}}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1}=\frac{\frac{V_{in}(s)}{C_{F\text{-}\alpha}(R+R_s)}+\frac{V_{in}(s)s^\alpha R_s}{(R+R_s)}}{s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\qquad \text{put}\qquad V_{in}(s)=\frac{V_m}{s} \tag{27}$$

$$=\left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right)+\left(\frac{V_m R_s}{R+R_s}\right)\left(\frac{s^{\alpha\text{-}1}}{s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}}\right)$$

To get $v_0(t)$ we do inverse Laplace transform of above (27) output voltage expression $V_0(s)$ as

$$v_0(t)=\mathcal{L}^{-1}\{V_0(s)\}=\mathcal{L}^{-1}\left\{\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)s\left(s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\}+\mathcal{L}^{-1}\left\{\frac{V_m R_s s^{\alpha\text{-}1}}{(R+R_s)\left(s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right\} \tag{28}$$

Use $\mathcal{L}\left\{t^{\alpha p+\beta-1}E^{(p)}_{\alpha,\beta}(at^\alpha)\right\}=p!\frac{s^{\alpha-\beta}}{s^\alpha-a}$ with $p=1, \alpha=\alpha, \beta=\alpha+1$ and $p=0, \alpha=\alpha, \beta=1$, to write from (28) the inverse Laplace as

$$v_0(t)=\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^\alpha E_{\alpha,\alpha+1}\left(-\frac{t^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right)+\frac{V_R R_s}{R+R_s}E_{\alpha,1}\left(-\frac{t^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{29}$$

Let us keep the step input from time $t=0$ to $t=T_c$, and then at time $t=T_c$, the output voltage will be

$$v_0(T_c)=\frac{V_R T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\frac{T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right)+\frac{V_R R_s}{R+R_s}E_{\alpha,1}\left(-\frac{T_c^\alpha}{C_{F\text{-}\alpha}(R+R_s)}\right) \tag{30}$$

The charge $q(t)$ will be held only in the element $C_{F\text{-}\alpha}$. We calculate now the voltage profile and then voltage at $t=T_c$, for only fractional impedance part i.e. $\frac{1}{s^\alpha C_{F\text{-}\alpha}}$ of the impedance $Z_2(s)$ comprising of R_s plus this fractional impedance $\frac{1}{s^\alpha C_{F\text{-}\alpha}}$, the voltage is thus

$$V_c(s)=I_{CH}\left(\frac{1}{s^\alpha C_{F\text{-}\alpha}}\right)=\left(\frac{s^\alpha C_{F\text{-}\alpha}V_{in}(s)}{s^\alpha C_{F\text{-}\alpha}(R+R_s)+1}\right)\left(\frac{1}{s^\alpha C_{F\text{-}\alpha}}\right)\qquad \text{put}\qquad V_{in}(s)=\frac{V_m}{s}$$

$$=\left(\frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}\right)\left(\frac{1}{s\left(s^\alpha+\frac{1}{C_{F\text{-}\alpha}(R+R_s)}\right)}\right) \tag{31}$$

Using the previous Laplace identity of Mittag Leffler function $\mathcal{L}\left\{E_n(\mathrm{at}^n)\right\}=\frac{s^{n-1}}{s^n-a}$, we write

$$\begin{aligned} v_c(t) &= \frac{V_m}{C_{F\text{-}\alpha}(R+R_s)}t^{\alpha}E_{\alpha,\alpha+1}\left(-\tfrac{t^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right) \\ v_c(t) &= V_m\left(1-E_{\alpha}\left(-\tfrac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right), \quad 0\le t\le T_c \end{aligned} \tag{32}$$

At $t=T_c$ we thus have the voltage at the fractional impedance

$$v_c(T_c)=\frac{V_mT_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}E_{\alpha,\alpha+1}\left(-\tfrac{T_c^{\alpha}}{C_{F\text{-}\alpha}(R+R_s)}\right)=V_m\left(1-E_{\alpha}\left(-\tfrac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{33}$$

The charge $q(t)$ is $q(t)=c(t)*v_c(t)$ with fractional capacitor with capacity function as $c(t)=C_{\alpha}t^{-\alpha}$ having voltage profile and that is $v_c(t)=V_m\left(1-E_{\alpha}\left(-\tfrac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$ as following

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\left\{c(t)\right\}\right)\left(\mathcal{L}\left\{v_c(t)\right\}\right) \\ &= \left(\mathcal{L}\left\{C_{\alpha}\,t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{V_m\left(1-E_{\alpha}\left(-\tfrac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right\}\right) \\ &= \left(C_{\alpha}\Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{V_m\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s\left(s^{\alpha}+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)=\frac{V_mC_{F\text{-}\alpha}\left(\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}{s^{2-\alpha}\left(s^{\alpha}+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}; \quad k=\frac{1}{(R+R_s)C_{F\text{-}\alpha}}, \quad C_{F\text{-}\alpha}=C_{\alpha}\Gamma(1-\alpha) \\ &= \left(\frac{V_m}{R+R_s}\right)\frac{s^{\alpha-2}}{\left(s^{\alpha}+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)} \qquad \mathcal{L}\left\{E_{\alpha}(-kt^{\alpha})\right\}=\frac{s^{\alpha-1}}{s^{\alpha}+k} \\ &= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\left(\frac{s^{\alpha-1}}{\left(s^{\alpha}+\frac{1}{(R+R_s)C_{F\text{-}\alpha}}\right)}\right)\right) \\ &= \left(\frac{V_m}{R+R_s}\right)\left(s^{-1}\mathcal{L}\left\{E_{\alpha}\left(-\tfrac{t^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right) \end{aligned} \tag{34}$$

Taking inverse Laplace transform of (34) by recognizing $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=s^{-1}F(s)$ we write

$$q(t)=\int_0^t\frac{V_m}{R+R_s}E_{\alpha}\left(-\tfrac{\tau^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau=\frac{V_mt}{R+R_s}\left(E_{\alpha,2}(-t^{\alpha}/(R+R_s)C_{F\text{-}\alpha})\right) \tag{35}$$

At $t=T_c$ we have charge as

$$q(T_c)=\frac{V_mT_c}{R+R_s}E_{\alpha,2}\left(-\tfrac{T_c^{\alpha}}{(R+R_s)C_{F\text{-}\alpha}}\right) \tag{36}$$

For $Z_2(s)=R_s+\frac{1}{sC}$ i.e. an ideal capacitor with ESR, we have the following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right)\\
&= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\left\{V_m\left(1-e^{-\frac{t}{(R+R_s)C}}\right)\right\}\right)\\
&= C\left(\frac{V_m\left(\frac{1}{(R+R_s)C}\right)}{s\left(s+\frac{1}{(R+R_s)C}\right)}\right) = \frac{V_m C\left(\frac{1}{(R+R_s)C}\right)}{s\;\left(s\;+\frac{1}{(R+R_s)C}\right)} = V_m C\left(\frac{1}{s}-\frac{1}{s+\frac{1}{(R+R_s)C}}\right)
\end{aligned}
\tag{37}
$$

$$q(t) = V_m C\left(1-e^{-\frac{t}{(R+R_s)C}}\right)$$

Charge at the end of $t = T_c$ is

$$q(T_c) = V_m C\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{38}$$

The charging current is following from (37)

$$i_{CH}(t) = \frac{dq(t)}{dt} = \frac{V_m e^{-\frac{t}{(R+R_s)C}}}{(R+R_s)},\quad 0 \le t \le T_c \tag{39}$$

The voltage at the end of $t = T_c$ is $v_c(T_c) = V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)$.

After $t = T_c$ we make the voltage $v_{in}(t) = 0$ i.e. we are draining out the stored charge i.e. $q(T_c) = V_m C\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)$ during the discharge phase ($t \ge T_c$); Figure-2. In the discharge phase the voltage $v_c(T_c)$ will decay as $v_c(t') = \left(v_c(T_c)\right)e^{-t'/(R+R_s)C}$, for $t \ge T_c$, writing $t' = t - T_c$. At this point the capacity function $c(t') = C\delta(t')$ will again appear, as there is change in voltage from V_m to 0 at $t' = 0$ (i.e. $t = T_c$). Therefore the discharging $q(t')$ we write as $q(t') = c(t') * v_c(t')$ as

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right),\quad t \ge T_c\\
&= \left(\mathcal{L}\{C\delta(t')\}\right)\left(\mathcal{L}\left\{\left(v_c(T_c)\right)e^{-t'/(R+R_s)C}\right\}\right)\\
&= \left(C\right)\left(\frac{\left(v_c(T_c)\right)}{s+\frac{1}{(R+R_s)C}}\right)
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
q(t') &= Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}};\quad v_c(T_c) = V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)\\
&= V_m C\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)e^{-\frac{t'}{(R+R_s)C}};\quad t' \ge 0;\quad t \ge T_c
\end{aligned}
$$

The discharging current $t \ge T_c$ is as follows

$$
\begin{aligned}
i_{DIS}(t') &= \frac{dq(t')}{dt'} = Cv_c(T_c)\frac{de^{-\frac{t'}{(R+R_s)C}}}{dt'},\quad t \ge T_c\\
&= -\frac{v_c(T_c)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}} = -\frac{V_m\left(1-e^{-\frac{T_c}{(R+R_s)C}}\right)}{(R+R_s)}e^{-\frac{t'}{(R+R_s)C}}
\end{aligned}
\tag{41}
$$

The negative sign indicates that discharge current is opposite to that of charging current. Now we carry on with the above logic for a fractional capacitor with $Z_2(s) = R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}$.

This value $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$ of voltage becomes the initial voltage while we discharge the super-capacitor with time defined as $t' = t - T_c$, for discharge phase where $v_{in}(t') = 0$. Now we see the discharge profile, as the charged fractional capacitor $C_{F\text{-}\alpha}$ with above value $v_c(T_c)$ discharges through R. The discharge current is now for $t' \geq 0$, negative to the charging current is following

$$I_{DIS}(s) = -\frac{v_c(T_c)/s}{R + R_s + \frac{1}{s^\alpha C_{F\text{-}\alpha}}} = -\frac{v_c(T_c)s^{\alpha-1}}{(R+R_s)\left(s^\alpha + \frac{1}{s^\alpha C_{F\text{-}\alpha}(R+R_s)}\right)} \tag{42}$$

The inverse Laplace transform gives discharge current for $t \geq T_c$ is

$$\begin{aligned} i_{DIS}(t') &= \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{s^\alpha C_{F\text{-}\alpha}}}\right\} \\ &= -\frac{v_c(T_c)}{R+R_s}E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right); \quad t \geq T_c, \quad v_c(0) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \end{aligned} \tag{43}$$

For $\alpha = 1$ we have for ideal loss less capacitor $C_{F\text{-}\alpha} = C$ from (43)

$$i_{DIS}(t') = \mathcal{L}^{-1}\left\{-\frac{v_c(T_c)/s}{R+R_s+\frac{1}{sC}}\right\} = -\frac{v_c(T_c)}{R+R_s}e^{-\frac{t'}{(R+R_s)C}}; \quad t \geq T_c, \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) \tag{44}$$

The discharging profile of $q(t')$ with initial charge $q(0) = q(T_c)$ is

$$\begin{aligned} q(t') &= q(0) + \int_0^{t'} -\frac{v_c(T_c)}{R+R_s}e^{-\frac{\tau}{(R+R_s)C}}d\tau = \left[Cv_c(T_c)e^{-\frac{\tau}{(R+R_s)C}}\right]_{\tau=0}^{\tau=t'}; \quad t > T_c \\ &= q(0) + Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}} - Cv_c(T_c) \\ q(T_c) &= q(0) = V_m C\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right) = Cv_c(T_c) \end{aligned} \tag{45}$$

Thus we get $q(t')$ for $t \geq T_c$ with $t' = t - T_c$ as following

$$q(t') = Cv_c(T_c)e^{-\frac{t'}{(R+R_s)C}}; \quad v_c(T_c) = V_m\left(1 - e^{-\frac{T_c}{(R+R_s)C}}\right); \quad t \geq T_c \tag{46}$$

The voltage profile across the fractional capacitor while discharging process is decay, by

$$v_c(t') = v_c(T_c)E_\alpha\left(-\frac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right), \quad t \geq T_c, \quad v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right) \tag{47}$$

The charge profile during the discharge phase is $q(t') = c(t') * v_c(t')$ for $t \geq T_c$ is following

$$
\begin{aligned}
Q(s) &= \left(\mathcal{L}\{c(t')\}\right)\left(\mathcal{L}\{v_c(t')\}\right) \\
&= \left(\mathcal{L}\{C_\alpha (t')^{-\alpha}\}\right)\left(\mathcal{L}\left\{v_c(T_c)E_\alpha\left(-\tfrac{(t')^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right\}\right); \quad C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha) \\
&= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(\frac{v_c(T_c)s^{\alpha-1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}}\right) = C_{F\text{-}\alpha} v_c(T_c) s^{\alpha-1}\frac{s^{\alpha-1}}{s^\alpha + \frac{1}{(R+R_s)C_{F\text{-}\alpha}}} \\
&= C_{F\text{-}\alpha} v_c(T_c)\left(s^{-1}\mathcal{L}\left\{D_{t'}^\alpha E_\alpha(-kt'^\alpha)\right\}\right); \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}}
\end{aligned}
\tag{48}
$$

In above steps, we have used $s^\alpha F(s) \equiv D_t^\alpha f(t)$, for $F(s) = \frac{s^{\alpha-1}}{s^\alpha + k}$, $f(t) = E_\alpha(-kt^\alpha)$. Consider the fractional derivative operator D_t^α as Caputo fractional derivative. We have the Caputo fractional derivative of Mittag-Leffler function $E_\alpha(\lambda x^\alpha)$ as $D_x^\alpha E_\alpha(\lambda x^\alpha) = \lambda E_\alpha(\lambda x^\alpha)$; [13]. Using this we write the following

$$
Q(s) = C_{F\text{-}\alpha} v_c(T_c)\left(s^{-1}\mathcal{L}\left\{-kE_\alpha(-kt'^\alpha)\right\}\right); \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}} \tag{49}
$$

Using inverse Laplace Transform we have

$$
\begin{aligned}
q(t') &= q(0) + \left(-C_{F\text{-}\alpha} v_c(T_c)\int_0^{t'} kE_\alpha(-k\tau^\alpha)d\tau\right); \quad k = \frac{1}{(R+R_s)C_{F\text{-}\alpha}} \\
&= q(0) + \left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'} E_\alpha(-k\tau^\alpha)d\tau\right); \quad t \ge T_c
\end{aligned}
\tag{50}
$$

Where we have $q(0) = q(T_c) = \frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)$ and $v_c(T_c) = V_m\left(1 - E_\alpha\left(-\frac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)$

We use $\int_0^t E_\alpha\left(-k\tau^\alpha\right)d\tau = t\left(E_{\alpha,2}(-kt^\alpha)\right)$ and write the following

$$
\begin{aligned}
q(t') &= q(0) + \left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'} E_\alpha\left(-\tfrac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right); \quad t \ge T_c \\
&= \frac{V_m T_c}{R+R_s}E_{\alpha,2}\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right) - \frac{V_m\left(1-E_\alpha\left(-\tfrac{T_c^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)}{(R+R_s)}\left[t'\left(E_{\alpha,2}\left(-\tfrac{t'^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)\right)\right]
\end{aligned}
\tag{51}
$$

We put $\alpha = 1$ in $q(t') = q(0) + \left(-\frac{v_c(T_c)}{(R+R_s)}\int_0^{t'} E_\alpha\left(-\frac{\tau^\alpha}{(R+R_s)C_{F\text{-}\alpha}}\right)d\tau\right)$; (51) and we get what we got for classical ideal capacitor $C_{F\text{-}\alpha} = C$, i.e. $q(t') = q(0) + \left(-\frac{v_c(T_c)}{R+R_s}\int_0^{t'} e^{-\frac{\tau}{(R+R_s)C}}d\tau\right)$, (45).

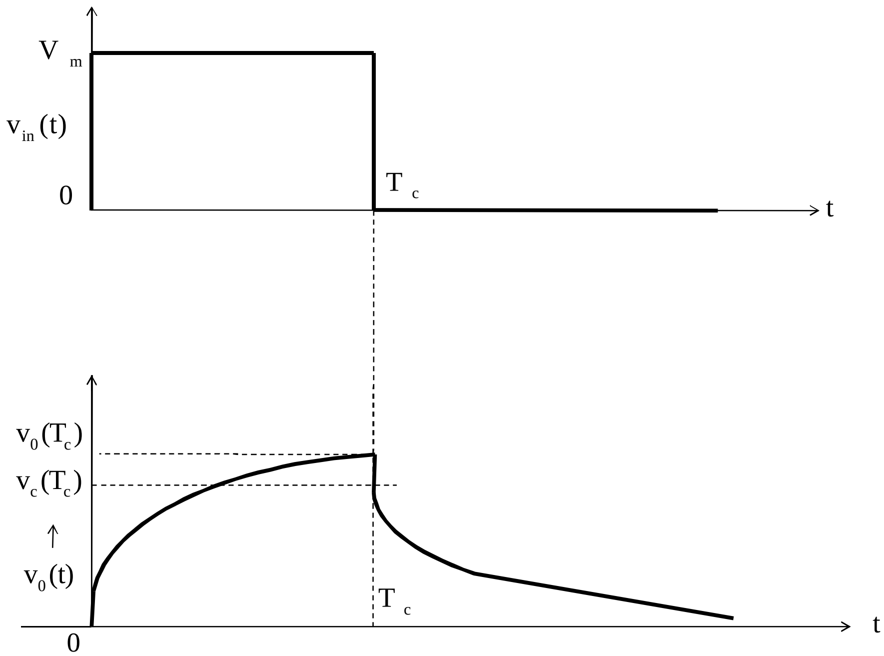

Figure-2: Constant voltage charging and discharging voltage profile at super-capacitor

The Figure-2 displays the curve of voltage profile for a constant voltage charge and discharge case. Here we point out that the charging curve though similar to exponential charging of a text book capacitor $v_0(t) \sim 1 - e^{-t/RC}$, but it is not so, for fractional capacitor described via Mittag-Leffler function. Similarly the discharge profile though similar to exponential decay $v_0(t) \sim e^{-t/RC}$, but is not so for fractional capacitor; here too described by Mittag-Leffler function. All the relations we obtained and also verified our formula $q(t) = c(t) * v(t)$

Charge storage $q(t)$ by step input constant current $i_{in}(t) = I_m u(t)$ excitation to RC circuit with fractional capacitor and ideals capacitor

In the Figure-1 we take $Z_1(s) = R$, $Z_2(s) = \frac{1}{s^\alpha C_{F-\alpha}}$ and instead of $v_{in}(t) = V_m u(t)$, that is voltage source, we take, that as a constant current source i.e. $i_{in}(t) = I_m u(t)$. This constant current charging we apply to initially uncharged fractional capacitor, with capacity function $c(t) = C_\alpha t^{-\alpha}$. The fractional capacitor will develop a voltage across it by law governed by fractional derivative and fractional integral as follows

$$i(t) = C_{F-\alpha} \frac{d^\alpha v(t)}{dt^\alpha}; \quad v(t) = \frac{1}{C_{F-\alpha}} \int_0^t i(\tau)(d\tau)^\alpha = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t); \quad 0 < \alpha < 1 \qquad (52)$$

Therefore, for constant current $i(t) = I_m$ the voltage is fractional integral of a constant I_m

$$v(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} i(t) = \frac{1}{C_{F-\alpha}} D_t^{-\alpha} I_m = \frac{I_m}{C_{F-\alpha}\Gamma(1+\alpha)} t^\alpha \qquad (53)$$

for $t \geq 0$ [12], [13], [37]; we used formula $D_t^{-n} t^m = \frac{\Gamma(m+1)}{\Gamma(m+1+n)} t^{m+n}$ in (53). Therefore the charge function $q(t)$ is $q(t) = c(t) * v(t)$

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\{C_\alpha t^{-\alpha}\}\right)\left(\mathcal{L}\left\{I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} t^\alpha\right\}\right); \quad \tfrac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\{t^n\} \\ &= \left(C_\alpha \Gamma(1-\alpha)s^{\alpha-1}\right)\left(I_m \tfrac{1}{C_{F\text{-}\alpha}\Gamma(1+\alpha)} \tfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}\right); \quad C_\alpha\Gamma(1-\alpha) = C_{F\text{-}\alpha} \\ &= \frac{I_m}{s^2} \end{aligned} \tag{54}$$

Thus we have charge function by taking Laplace inverse of above (54) as

$$q(t) = I_m t; \quad t \geq 0 \tag{55}$$

This is matter of fact as the current flowing through R and $C_{F\text{-}\alpha}$ is $i(t) = I_m$ for $t \geq 0$, and thus the charge will be

$$q(t) = \int_0^t i(\tau)d\tau = \int_0^t I_m d\tau = I_m t; \quad t \geq 0 \tag{56}$$

For an ideal capacitor with $c(t) = C\delta(t)$ the voltage is $v(t) = \frac{1}{C}\int_0^t I_m d\tau = \frac{I_m}{C} t$ so the charge is $q(t) = c(t) * v(t)$ as follows

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\{c(t)\}\right)\left(\mathcal{L}\{v_c(t)\}\right) \\ &= \left(\mathcal{L}\{C\delta(t)\}\right)\left(\mathcal{L}\{I_m \tfrac{1}{C} t\}\right); \quad \tfrac{1}{s^2} = \mathcal{L}\{t\} \\ &= (C)\left(I_m \tfrac{1}{Cs^2}\right) = \frac{I_m}{s^2} \end{aligned} \tag{57}$$

$$q(t) = I_m t; \quad t \geq 0$$

Thus in the case of constant current charging, we verified the validity of $q(t) = c(t) * v(t)$ as for any capacitor fractional or ideal loss less capacitor, the $q(t) = I_m t$; that is always integration of current function.

Let the square pulse of current be described as follows

$$i(t) = I_m u(t) - 2I_m u(t - T_c) + I_m u(t - T_d) \tag{58}$$

Where $u(t - T) = 1$ for $t \geq T$ and $u(t - T) = 0$ for $t < T$, i.e. unit step function at time $t = T$ .Then with identity $\mathcal{L}\{f(t - T)\} = e^{-sT}F(s)$ we write

$$I(s) = \mathcal{L}\{i(t)\} = \frac{I_m}{s} - \frac{2I_m}{s}e^{-sT_c} + \frac{I_m}{s}e^{-sT_d} \tag{59}$$

We have voltage across $Z_2(s) = \frac{1}{s^\alpha C_{F\text{-}\alpha}}$ as follows

$$\begin{aligned} V(s) &= Z_2(s)I(s) \\ &= \left(\frac{1}{C_{F\text{-}\alpha}s^\alpha}\right)\left(\frac{I_m}{s} - \frac{2I_m}{s}e^{-sT_c} + \frac{I_m}{s}e^{-sT_d}\right) = \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}} - \frac{2I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha}s^{\alpha+1}}e^{-sT_d} \end{aligned} \tag{60}$$

Then taking inverse Laplace of (60) we get voltage profile across $C_{F\text{-}\alpha}$ as

$$v(t) = \frac{I_m t^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)} - \frac{2I_m(t - T_c)^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t - T_c) + \frac{I_m(t - T_d)^\alpha}{C_{F\text{-}\alpha}\Gamma(\alpha+1)}u(t - T_d) \tag{61}$$

The charge function is $q(t) = c(t) * v(t)$ as follows, when the voltage profile $v(t)$; (60) is across a fractional capacitor $c(t) = C_\alpha t^{-\alpha}$. This $c(t) = C_\alpha t^{-\alpha}$ gets applied at $t = 0$, $t = T_c$ and $t = T_d$; that is where there is sudden change of state of $v(t)$; that is at points where the differentiability of $v(t)$ is lost.

$$\begin{aligned} Q(s) &= \left(\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{v(t)\right\}\right); \quad \frac{\Gamma(n+1)}{s^{n+1}} = \mathcal{L}\left\{t^n\right\}; \quad C_{F\text{-}\alpha} = C_\alpha \Gamma(1-\alpha) \\ &= C_{F\text{-}\alpha} s^{\alpha-1} \frac{I_m}{C_{F\text{-}\alpha} s^{\alpha+1}} - C_{F\text{-}\alpha} s^{\alpha-1} \frac{2I_m}{C_{F\text{-}\alpha} s^{\alpha+1}} e^{-sT_c} + \frac{I_m}{C_{F\text{-}\alpha} s^{\alpha+1}} e^{-sT_d} \\ &= \frac{I_m}{s^2} - \frac{2I_m}{s^2} e^{-sT_c} + \frac{I_m}{s^2} e^{-sT_d} \\ q(t) &= I_m t - 2I_m (t - T_c) u(t - T_c) + I_m (t - T_d) u(t - T_d) \end{aligned} \tag{62}$$

In similar way we can analyze the ideal loss less capacitor $c(t) = C\delta(t)$, for this wave form of current pulse.

Conclusions

We have applied the new formula of charge storage i.e. via convolution operation $q(t) = c(t) * v(t)$, of time varying capacity function and voltage stress for a fractional capacitor; for verification in RC charging circuit; with dc voltage and current sources. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. This verifies that this new formulation of stored charge via convolution operation is applicable, and can be taken as general formula applicable to fractional capacitor as well as ideal capacitor.

References

[1] Shantanu Das, "**A New Look at Formulation of Charge Storage in Capacitors and Application to Classical Capacitor and Fractional Capacitor Theory";** Asian Journal of Research and Reviews in Physics; 1(3): 1-18, 2018; Article no.AJR2P.43738.

[2] Curie, Jaques, "**Recherches sur le pouvoir inducteur specifique et la conductibilite des corps cristallises**". Annales de Chimie et de Physique. 17 pp 384–434. 1889.

[3] Schweidler, Ergon Ritter von,"**Studien über die Anomalien im Verhalten der Dielektrika (Studies on the anomalous behaviour of dielectrics)**". Annalen der Physik. 329 (14): 711–770, pp 1907.

[4] Jonscher, Andrzej Ka, "**Dielectric Relaxation in Solids**", Chelsea Dielectrics Press Limited 1983.

[5] Jameson, N. Jordan; Azarian, Michael H.; Pecht, Michael, "**Thermal Degradation of Polyimide Insulation and its Effect on Electromagnetic Coil Impedance**". Proceedings of the Society for Machinery Failure Prevention Technology 2017 Annual Conference. 2017.

[6] Svante Westerlund, Lars Ekstam, "**Capacitor Theory**", IEEE Trans on Dielectrics and Insulation, Vol. 1, No. 5, pp826, 1994.

[7] Svante Westerlund, "**Dead Matter has Memory**", Physica Scripta, Vol. 43, pp 174-179, 1991.

[8] Shantanu Das, "**Revisiting the Curie-von Schweidler law for dielectric relaxation and derivation of distribution function for relaxation rates as Zipf's power law and manifestation of fractional differential equation for capacitor**"; Journal of Modern Physics, 2017, 8, 1988-2012.

[9] Shantanu Das NC Pramanik. "**Micro-structural roughness of electrodes manifesting as temporal fractional order differential equation in super-capacitor transfer characteristics**". International Journal of Mathematics and Computation Vol. 20, Issue 3, pp 94-113, 2013.

[10] Shantanu Das, "**Functional Fractional Calculus**", 2nd Edition Springer-Verlag, Germany, 2011.

[11] Shantanu Das, "**Singular vs. Non-Singular Memorized Relaxation for basic Relaxation Current of Capacitor**", Pramana Journal of Physics (in-press 2018).

[12] Oldham K B, Spanier J. "**The Fractional Calculus**", Academic Press 1974.

[13] Shantanu Das, "**Kindergarten of Fractional Calculus",** (Book-under print at Cambridge Scholars Publishers UK- Collection of lecture notes on fractional calculus course at Dept. of Physics Jadavpur University, Phys. Dept. St Xaviers Univ. Kolkata , Dept. of Appl. Mathematics Calcutta University etc.)

[14] Somasri Hazra, Tapati Dutta, Shantanu Das, Sujata Tarafdar, "**Memory of Electric Field in Laponite and How It Affects Crack Formation: Modeling through Generalized Calculus**", Langmuir, DOI:10.1021/acs.langmuir.7b02034, 2017.

[15] Anis Allagui, Di Zhang, Ahmed S. Elwakil; "**Short-term memory in electric double-layer capacitors**", Applied Physics Letters 113, 253901 (2018).

[16] Shantanu Das, N. C. Pramanik, "**Indigenous Development of Carbon Aerogel Farad Super-capacitors and Application in Electronics circuits**" BARC News Letter Issue No. 339 July-August 2014, (2014).

[17] Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, "**Frequency dependent piecewise fractional order modeling of ultra-capacitors using hybrid optimization and fuzzy clustering**", Journal of Power Sources 335, pp 98-104, 2015.

[18] Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, "**Charge discharge energy efficiency analysis of ultra-capacitor with fractional order dynamics using hybrid optimization and its experimental validation**", International Journal of Electronics & Communications (AEU) 78, pp 2714-280 2017.

[19] Elwakil AS, Allagui A, Maundy BJ, Psycchalinos CA, "**A low frequency oscillator using super-capacitor**", AEU –Int. J Electron Commun". 70(7) pp970-3, 2016.

[20] Freebon TJ, Maundy B, Elwakil AS, "**Fractional order models of super-capacitors, batteries, fuel-cell: a survey**". Mater Renew Sustain Energy 4(3) pp 1-7, 2015.

[21] Freebon TJ, Maundy B, Elwakil AS, "**Measurement of super-capacitor fractional order model parameters from voltage excited step response**", IEEE J Emerging Sel Top Circuits Syst. 3(3), pp367-76, 2013.

[22] Geethi Krishnan Shantanu Das Vivek Agarwal, "**A Simple Adaptive Fractional Order Model of Supercapacitor for Pulse Power Applications"**, 2018-IACC-0820, 978-1-5386-4536-9/18/$31.00 © 2018 IEEE

[23] Atangana A, Baleanu D. "**New fractional derivatives with nonlocal and non-singular kernel**": Theory and application to heat transfer model. Thermal Science 2016; 20(2):763-769. DOI: 10.2298/TSCI160111018A.

[24] Caputo M, Fabrizio M. "**A new definition of fractional derivative without singular kernel**". Progr Fract Differ Appl 2015; 1(2):73-85. DOI: 10.12785/pfda/010201.

[25] V.F. Morales-Delgado, J.F. G´omez-Aguilar, M.A. Taneco-Hern´andez and R.F. Escobar-Jim´enez "**A novel fractional derivative with variable- and constant-order applied to a mass-spring-damper system**", Eur. Phys. J. Plus (2018) 133: 78

[26] Trifce Sandev "**Generalized Langevin Equation and the Prabhakar Derivative**" Mathematics **2017**, 5, 66; doi:10.3390/math5040066

[27] Prabhakar T R. "**A singular integral equation with a generalized Mittag-Leffer function in the kernel**", Yokohama Mathematical Journal 1971;19:7

[28] Ortigueira M D, Tenreiro Machado J. "**A critical analysis of the Caputo-Fabrizio operator"**. Communications In Nonlinear Science and Numerical Simulation" 2018;59:608-611. DOI: 10.1016/j.cnsns.2017.12.001

[29] Graham William & David C. Watts, "**Non-Symmetrical Dielectric Relaxation Behaviour Arising from a Simple Empirical Decay Function**", Transactions on Faraday Society, Vol 66, 1970.

[30] Jordan Hristov, "**Electrical circuits of non-integer order: Introduction to an emerging interdisciplinary area with examples**" Springer 2018

[31] Ervin Lenzi K., Tateishi Angel A., Haroldo Ribeiro V., "**The Role of Fractional Time-Derivative Operators on Anomalous Diffusion**" Frontiers in Physics. 5: 1–9 (2017).

[32] Khaled Saad, Abdon Atangana, Dumitru Baleanu, "**New fractional derivatives with non-singular kernel applied to the Burgers equation",** Chaos 28(6) · June 2018 DOI: 10.1063/1.5026284.

[33] Abdon Atangana "**Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties"** Physica A: Statistical Mechanics and its Applications 505 · April 2018 DOI: 10.1016/j.physa.2018.03.056

[34] Andrea Giusti, "**A Comment on some New Definitions of Fractional Derivative**" arXiv:1710.06852v4 [math.CA] 24 Apr 2018

[36] Giusti A, Colombaro I. "**Prabhakar-like fractional viscoelasticity"**. Communications in Nonlinear Science and Numerical Simulation 2018;56:138-143. DOI: 10.1016/j.cnsns.2017.08.002.

[37] Shantanu Das, '**Fractional Order Controllers and Application to Real Life Systems**', (Abandoned Thesis) 2007-2013.