Deliberation on Application of formula of Charge function in time as Convolution of Capacity Function and Applied Voltage in a Capacitor

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Abstract
We have a formula for capacitor i.e. charge function of time, as convolution operation of time varying capacity function and time varying voltage function. This is different to usual and conventional way of writing capacitance multiplied by voltage to get charge stored in a capacitor. This new deliberation with convolution operation works well for classical ideal loss less capacitors, where one says that is a constant capacity, and also for a time varying capacity function given by power-law: that gives the formation of fractional capacitor. In this write-up, we apply the developed charge storage expression via convolution expression and apply to various types of inputs excitation voltage- sinusoidal, step, ramp and analyze the charge stored expressions. With this new formulation, we also evaluate impedance function of a classical capacitor as well as fractional capacitor, and also elaborate on the Nyquist diagram, that is employed to study various dielectric materials. This new approach of charge storage concept is yet to be practically applied. This note gives a validity test of this approach.

Keywords
Capacity Function, Fractional Capacitor, Convolution Operation, Laplace Transform, Memory Effect

Introduction
The voltage change when appears at a capacitor, it reacts or relaxes via relaxation current. The time varying capacity function $c(t)$ is the one that defines the response function; and by principle of causality we write $q(t) = c(t) * v(t)$ where $v(t)$ is the input impressed voltage. This is contrary to usual usage of $q(t) = c(t)v(t)$ i.e. the product of the two. This formulation is deliberated in detail with $c(t)$ as for ideal loss less capacitor case, as well as time varying capacity function (fractional capacitor case) in [1]. The capacity function $c(t)$ is the function which decays with time, and has the form $c(t) \sim t^{-\alpha}$; $0 < \alpha < 1$ and acts only at the time of application of voltage change. For ideal case of loss-less capacitor the capacity function is $c(t) \sim \delta(t)$; [1]. In this note we will always take the power-exponent of power-law of decaying capacity function i.e. α as between zero and one. This power-law decay function is in singular at origin and in tune with singular power law decay relaxation current given by Curie-von Schweidler (universal law) of dielectric relaxation [2]-[5]. In this universal dielectric relaxation law, the relaxing current is a decaying power-law as $i(t) \sim t^{-\alpha}$, when uncharged system of dielectric is stressed by a constant voltage. The use of this universal dielectric relaxation law gives current voltage relation of a capacitor as given by fractional derivative [6]-[10]. The non-singular decaying function gives all together different form of current voltage relations in capacitor is discussed in [11]. In this note we will take capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$ (i.e. a fractional capacitor), and will use the formula $q(t) = c(t) * v(t) = \int_0^t c(t-\tau)v(\tau)\mathrm{d}\tau$ and discuss various cases for sinusoidal voltage

excitation, step voltage excitation, ramp voltage excitation. We note a priori that the constant C_α is proportionality constant, and not Fractional Capacity. The fractional capacity of a fractional capacitor we will represent as $C_{F-\alpha}$ which has units of $\text{Farad}/\sec^{1-\alpha}$. We assume that the fractional capacitor has no resistance, (like ideal capacitor has no resistance) and is excited by ideal voltage sources (having output impedance as zero). We will use Laplace Transform technique in all analysis. In all the cases in subsequent sections, we will apply this new formula and give the validity justification.

Charge storage by Sinusoidal voltage excitation to a fractional capacitor

Charge in a Capacitor is $q(t)=c(t)*v(t)$, is given via convolution operation and not with the usual way that we write as $q(t)=c(t)v(t)$. Let us have a Capacitor with capacity function in time as power-law $c(t)=C_\alpha t^{-\alpha}$ $(0<\alpha<1)$. Let a sinusoidal voltage be applied to an uncharged capacitor $v(t)=V_m\cos\omega_0 t$, at time $t=0$ .Then charge function in time is given as convolution $(*)$ operation as following

$$q(t)=c(t)*v(t)=\left(C_\alpha t^{-\alpha}\right)*\left(V_m\cos\omega_0 t\right)$$

We apply Laplace Transform to the above and write the following

$$\mathcal{L}\left\{q(t)\right\}=\mathcal{L}\left\{c(t)*v(t)\right\}; \qquad Q(s)=C(s)V(s)$$

We have $C(s)=\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}=C_\alpha\Gamma(1-\alpha)s^{-(1-\alpha)}$ and $V(s)=\mathcal{L}\left\{V_m\cos\omega_0 t\right\}=\frac{V_m s}{s^2+\omega_0^2}$. This gives $Q(s)$ as follows

$$\begin{aligned}Q(s)&=\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\times\frac{V_m s}{s^2+\omega_0^2}\\&=V_m C_\alpha\Gamma(1-\alpha)\left(s^\alpha\frac{1}{s^2+\omega_0^2}\right)=\frac{V_m C_\alpha\Gamma(1-\alpha)}{\omega_0}\left(s^\alpha\frac{\omega_0}{s^2+\omega_0^2}\right)\\&=\frac{V_m C_\alpha\Gamma(1-\alpha)}{\omega_0}\left(\mathcal{L}\left\{D_t^\alpha\sin\omega_0 t\right\}\right)\end{aligned}$$

In above we used Laplace Transform of Fractional Derivative as $\mathcal{L}\left\{D_t^\alpha f(t)\right\}=s^\alpha F(s)$ with $f(0)=0$, [10], [12], [13] and $\mathcal{L}\left\{\sin at\right\}=\frac{a}{s^2+a^2}$. Taking the inverse Laplace transform of the above, we get

$$q(t)=\frac{V_m C_\alpha\Gamma(1-\alpha)}{\omega_0}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sin\omega_0 t=\frac{V_m C_{F-\alpha}}{\omega_0}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\sin\omega_0 t$$

Here we introduce a constant $C_{F-\alpha} = C_{\alpha}\Gamma(1-\alpha)$ as fractional capacity in units of $\text{Farad}/\sec^{1-\alpha}$ [1], [6], [7], [8], [9], [11]; this we will elaborate later subsequently. We have fractional derivative of $\sin x$ as following [12], [13]

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\sin x = \sin\left(x + \tfrac{\alpha\pi}{2}\right) + \frac{x^{-1-\alpha}}{\Gamma(-\alpha)} - \frac{x^{-3-\alpha}}{\Gamma(-\alpha-2)} + \ldots$$

We write $x = \omega_0 t$ thus we have $\mathrm{d}x = \omega_0 \mathrm{d}t$, gives $\mathrm{d}t^{\alpha} = \omega_0^{-\alpha}\mathrm{d}x$, with this we write the following

$$\begin{aligned} q(t) &= \frac{V_m C_{F-\alpha}}{\omega_0}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\sin\omega_0 t \\ &= \frac{V_m C_{F-\alpha}}{\omega_0^{1-\alpha}}\left(\sin\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right) + \frac{(\omega_0 t)^{-1-\alpha}}{\Gamma(-\alpha)} - \frac{(\omega_0 t)^{-3-\alpha}}{\Gamma(-\alpha-2)} + \ldots\right) \end{aligned}$$

The transient terms i.e. $t^{-1-\alpha}$, $t^{-3-\alpha}$ … in the above expression decays to zero for large times, thus we write the steady state charge function from above as following

$$q(t) = V_m C_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right)$$

From above steady state current is

$$\begin{aligned} i(t) = \frac{\mathrm{d}}{\mathrm{d}t}q(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(V_m C_{F-\alpha}\omega_0^{\alpha-1}\sin\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right)\right) \\ &= V_m C_{F-\alpha}\omega_0^{\alpha}\cos\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right) \end{aligned}$$

This shows at steady state the current in fractional capacitor leads the voltage by angle $\alpha \times 90^0$. This is true as the way to validate experimentally a frational integrator or differentiator circuit, by sinusoidal input. The leading current is 90^0 to voltage excitation for ideal loss less capacitor where ($\alpha = 1$).

We do the following steps to re-write above for $q(t) = V_m C_{\alpha}\Gamma(1-\alpha)\omega_0^{\alpha-1}\sin\left(\omega_0 t + \tfrac{\alpha\pi}{2}\right)$ as following

$$\begin{aligned} q(t) &= V_m C_{\alpha}\Gamma(1-\alpha)\omega_0^{\alpha-1}\sin\left(\omega_0 t + \tfrac{\alpha\pi}{2} - \tfrac{\pi}{2} + \tfrac{\pi}{2}\right) \\ &= V_m C_{\alpha}\Gamma(1-\alpha)\omega_0^{\alpha-1}\cos\left(\omega_0 t - \tfrac{(\alpha-1)\pi}{2}\right) = Q_p\cos\left(\omega_0 t - \tfrac{(\alpha-1)\pi}{2}\right) \\ Q_p &= V_m C_{\alpha}\Gamma(1-\alpha)\omega_0^{\alpha-1} = V_m C_{F-\alpha}\omega_0^{\alpha-1} \end{aligned}$$

We observe that charge function lags voltage function at steady state by angle $\frac{(\alpha-1)\pi}{2}$. For $\alpha = 1$ these is no phase difference (lag) between charge function and voltage function [1]. This is for ideal loss less capacitor where capacity function is $c(t)=C_1\delta(t)$. With this we get $Q(s)=\mathcal{L}\{C_1\delta(t)\}\times\mathcal{L}\{V_m\cos\omega_0 t\}=V_m C_1\frac{s}{s^2+\omega_0^2}$. We get $q(t)=V_m C_1\cos\omega_0 t=Q_m\cos\omega_0 t$, that is in same phase with voltage function.

We see that for a time varying capacity function $c(t)=C_\alpha t^{-\alpha}$, the charge function $q(t)=Q_p\cos\left(\omega_0 t-\frac{(1-\alpha)\pi}{2}\right)$, the peak value of charge $Q_p=V_m C_{F-\alpha}\omega_0^{\alpha-1}$ varies with operating frequency ω_0 of the excitation voltage $v(t)=V_m\cos\omega_0 t$. With parameters $C_{F-\alpha}$ and α assuming to be constant with varying ω_0, and V_m as maximum rated value of the operational circuit; the peak charge Q_p decreases as ω_0 the input frequency is increased. While for ideal loss less capacitors the peak charge $Q_m=V_m C_1$ remains invariant with frequency of input voltage. Therefore with various input excitation voltages we will be getting different peak charge values, though the excitation voltage is within the capacitor maximum rating V_m. This meaning square wave, triangular wave, trapezoidal wave of voltages will be giving different peak charge stored, as they will be having different fundamental and harmonic frequencies. A square wave voltage of positive and negative cycles, a symmetric triangular wave voltage, and a pure sinusoidal with same period will have different peak charge stored. However the assumption $C_{F-\alpha}$ and α assuming to be constant with varying ω_0, does not hold, that we will explain in Nyquist diagram shortly.

Impedance Expression of fractional capacitor

We have the relation of charge function as $q(t)=c(t)*v(t)$.Differentiation of this [1] we obtain current through capacitor as

$$i(t)=\frac{\mathrm{d}}{\mathrm{d}t}q(t)=c(t)*\frac{\mathrm{d}}{\mathrm{d}t}v(t)=\int_0^t c(t-\tau)v^{(1)}(\tau)\mathrm{d}\tau$$

Taking Laplace Transform of the above we obtain $I(s)=\left(\mathcal{L}\{c(t)\}\right)\left(sV(s)\right)$. The impedance is defined as $Z(s)=V(s)/I(s)$. From this we write for the capacity function of capacitor, the impedance function in Laplace domain as

$$Z(s)=\frac{1}{s\left(\mathcal{L}\{c(t)\}\right)}$$

For ideal loss less capacitor with $c(t)=C_1\delta(t)$ [1] we get the classical impedance formula

$$Z(s)=\frac{1}{s\mathcal{L}\{C_1\delta(t)\}}=\frac{1}{sC_1}$$

With $s=i\omega$ we write

$$Z(\omega)=\frac{1}{i\omega C_1}=-\frac{i}{\omega C_1}$$

For a capacitor having time varying capacity function as $c(t)=C_\alpha t^{-\alpha}$, we get

$$Z(s)=\frac{1}{s\mathcal{L}\left\{C_\alpha t^{-\alpha}\right\}}=\frac{1}{s\left(C_\alpha\Gamma(1-\alpha)s^{\alpha-1}\right)}$$
$$=\frac{1}{s^\alpha C_\alpha\Gamma(1-\alpha)}=\frac{1}{s^\alpha C_{F-\alpha}};\quad C_{F-\alpha}=C_\alpha\Gamma(1-\alpha)$$

From above we get $I(s)=C_\alpha\Gamma(1-\alpha)\left(s^\alpha V(s)\right)$. Using the Laplace transform of fractional derivative, we get

$$i(t)=C_{F-\alpha}\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}v(t);\quad C_{F-\alpha}=C_\alpha\Gamma(1-\alpha)$$

We note that the constant $C_{F-\alpha}$ has unit of $\text{Farad}/\sec^{1-\alpha}$, is unit for Fractional Capacitor.
With $s=i\omega$ we obtain

$$Z(s)=\frac{1}{C_\alpha\Gamma(1-\alpha)}(i\omega)^{-\alpha}=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\left(\cos\tfrac{\alpha\pi}{2}-i\sin\tfrac{\alpha\pi}{2}\right)$$

We have $\operatorname{Re}Z(\omega)=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\cos\frac{\alpha\pi}{2}$ and $-\operatorname{Im}Z(\omega)=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\sin\frac{\alpha\pi}{2}$.

The Nyquist diagram of a fractional capacitor

The impedance spectroscopy gives Nyquist diagram, when $\mathrm{X}=\operatorname{Re}Z(\omega)$ and $\mathrm{Y}=-\operatorname{Im}Z(\omega)$ is plotted with frequency ω varying from 0 to ∞. For ideal loss less capacitor with capacity function as $c(t)=C_1\delta(t)$, the Nyquist diagram is just Y-Axis in units of Ω, with $\mathrm{Y}=(\omega C_1)^{-1}\Omega$ and $\mathrm{X}=0\Omega$. When $\omega\downarrow 0$, the $\mathrm{Y}\uparrow\infty$ and while at very-very high frequency, i.e. $\omega\uparrow\infty$ we have $\mathrm{Y}\downarrow 0$, with $\mathrm{X}=0$ at all frequencies. For ideal loss less capacitors we have equivalent series resistance (ESR) as ZERO at all frequencies.
For a fractional capacitor with time varying capacity function as $c(t)=C_\alpha t^{-\alpha}$, we have $\mathrm{X}=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\cos\frac{\alpha\pi}{2}\Omega$ and $\mathrm{Y}=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\sin\frac{\alpha\pi}{2}\Omega$. We have a slope of the Nyquist diagram as $\tan\frac{\alpha\pi}{2}$. When $\alpha\approx 1$, the angle of slope is tending to 90^0 (i.e. the Nyquist diagram going as vertical line parallel to Y-Axis). When $\alpha\approx 0$, the slope is zero, the angle of the slope is 0^0 and while $\alpha=0.5$, the slope is one, the angle of the slope is 45^0. We remark here that when $\alpha=0.5$ is Warburg Impedance region [10].
The capacitor with time varying capacity function as $c(t)=C_\alpha t^{-\alpha}$, has ESR as $R_s=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\cos\frac{\alpha\pi}{2}\Omega$. This R_s is a function of frequency ω. At very high frequency, this ESR is low value and at very low frequency this ESR is at high value. See Figure-1.
The imaginary part is capacitive impedance in Ω, i.e. $\mathrm{Y}=\frac{1}{C_\alpha\omega^\alpha\Gamma(1-\alpha)}\sin\frac{\alpha\pi}{2}\Omega$. This is also a frequency (ω) dependent. The value $\mathrm{Y}\uparrow\infty$ for very-very low frequency, while $\mathrm{Y}\downarrow 0$ at very-very high frequency. This is shown in Figure-1.
We show in Figure-1 the Warburg region, at around frequency of $\omega=4.0$ Radians/sec. At this point we have ESR as $R_s=0.62\Omega$. At this point of frequency we have $\alpha=0.5$, and $\mathrm{Y}=0.3\Omega$. Therefore at this point we have $C_{F-\alpha}=C_\alpha\Gamma(1-\alpha)=\frac{1}{\mathrm{Y}\omega^\alpha}\sin\frac{\alpha\pi}{2}=\frac{\sin 45^0}{0.3\times(4)^{0.5}}=1.18\text{Farad}/\sec^{0.5}$.
Thus at various points of frequency ω of Nyquist diagram, we get the value of α, ESR (R_s), and Fractional capacity $C_{F-\alpha}$. We observe, that at very low frequency $\omega=0.04$ Radians/sec the

Y value tends towards 1Ω, where we take $\alpha = 0.9$,(that is $\alpha \approx 1.0$ and at this point we get $C_{F-\alpha}\big|_{\alpha\approx 1.0} \approx 25\text{Farad}$, with $R_s = 0.9\Omega$. This impedance Nyquist diagram says that we have equivalent circuit representing capacitor with time varying capacity function $c(t) = C_\alpha t^{-\alpha}$; as series connected ESR R_s with Fractional Capacitor $C_{F-\alpha}$ ($\text{Farad}/\sec^{1-\alpha}$), with $C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$ as dependent on ω.

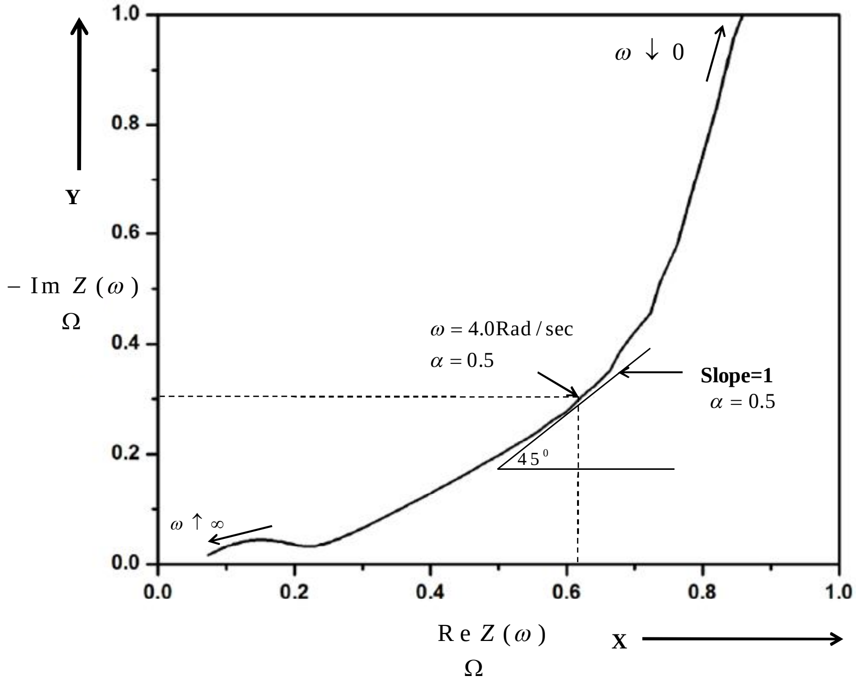

Figure-1: Nyquist Diagram of Super-Capacitor (Courtesy: CMET Thrissur as test result of BRNS Project: Sanction No. 2009/34/31/BRNS dated 22.10.2009)

Charge storage by Step Input Voltage excitation to a fractional capacitor

Let at $t = 0$ $v(t) = V_m$ a step input is given to uncharged capacitor with time varying capacity function as $c(t) = C_\alpha t^{-\alpha}$. Thus we have following from $q(t) = c(t) * v(t)$

$$Q(s) = \mathcal{L}\{q(t)\} = \mathcal{L}\{c(t)\} \times \mathcal{L}\{v(t)\}$$

$$= \frac{C_\alpha \Gamma(1-\alpha)}{s^{1-\alpha}} \times \frac{V_m}{s} = \frac{V_m C_\alpha \Gamma(1-\alpha)}{s^{2-\alpha}}$$

Doing inverse Laplace transform of above, we get the following

$$q(t) = \frac{V_m C_\alpha}{1-\alpha} t^{1-\alpha} = \frac{V_m C_{F-\alpha}}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad C_{F-\alpha} = C_\alpha \Gamma(1-\alpha)$$

The current is

$$i(t) = \frac{\mathrm{d}}{\mathrm{d}t} q(t) = V_m C_\alpha t^{-\alpha}$$

This is Curie-von Schweidler relaxation law for dielectric stressed with constant voltage or Electric field. If the case we take of loss less ideal capacitor given by capacity function $c(t) = C_1 \delta(t)$, then $q(t) = c(t) * v(t)$, with $v(t) = V_m u(t)$, where $u(t)$ is unit step function at $t=0$, is $q(t)=\mathcal{L}^{-1}\{C(s)V(s)\}=\mathcal{L}^{-1}\{C_1V_m/s\}=C_1V_mu(t)$. The current in this ideal case is $i(t)=q^{(1)}(t)=V_mC_1u^{(1)}(t)=V_mC_1\delta(t)$. Both the cases are depicted in Figure-2[1].

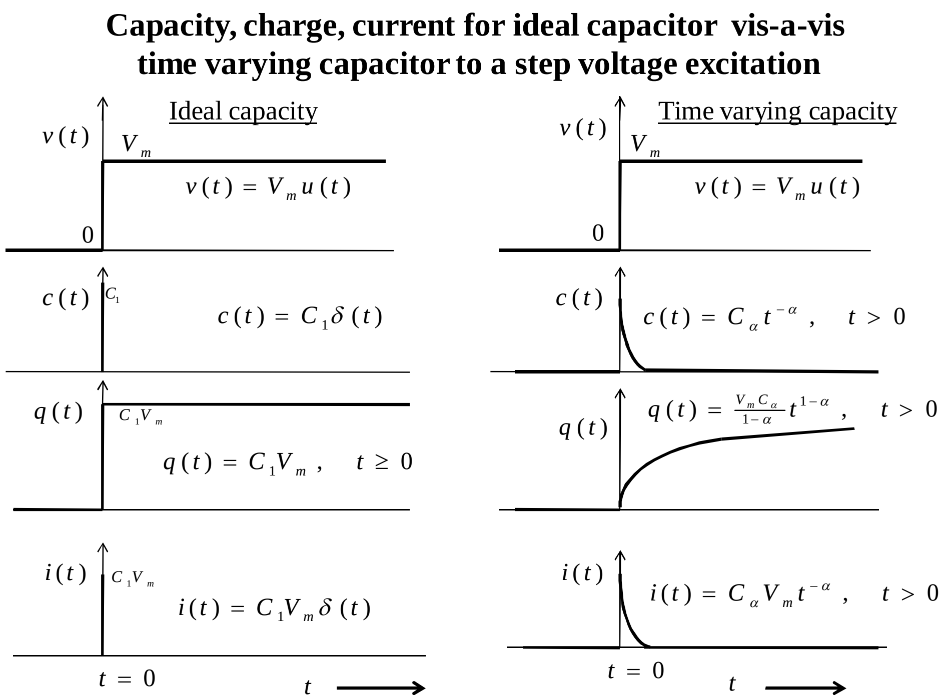

Figure-2: Charge in a ideal capacitor vis-à-vis time varying capacity function

We see when the step input $v(t)$ is kept on at V_m for time T, $2T$, $3T$ we get $q(T)=\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$, $q(2T)=2^{1-\alpha}\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$, $q(3T)=3^{1-\alpha}\frac{V_mC_\alpha}{1-\alpha}T^{1-\alpha}$ respectively. If the $T\uparrow\infty$, we get $q(T)\big|_{T\uparrow\infty}=\infty$. Refer Figure-2. This process leads to a new breakdown mechanism of capacitors noted in [1], [6], [7], called electrostatic breakdown of capacitors.

Memory effect in fractional capacitor

Thus we see if we keep afloat a capacitor (not ideal one), but a capacitor with capacity function $c(t)=C_\alpha t^{-\alpha}$, to a constant voltage V_m first for time T then $2T$ …the charge held will be more in second case, though the terminal voltage that we measure will be same as V_m. After holding for set time, we keep it open circuited (for self discharge case). In all the cases the self discharge decay of voltage we will observe starting from V_m, with a different decay curves. This is because in second and third cases more charge needs to be drained out, in self-discharge case. Thus the capacitor is memorizing its time of charge, i.e. T. This is described in [6], [7]. This explanation is possible only with fractional capacitor and not with ideal capacitors.

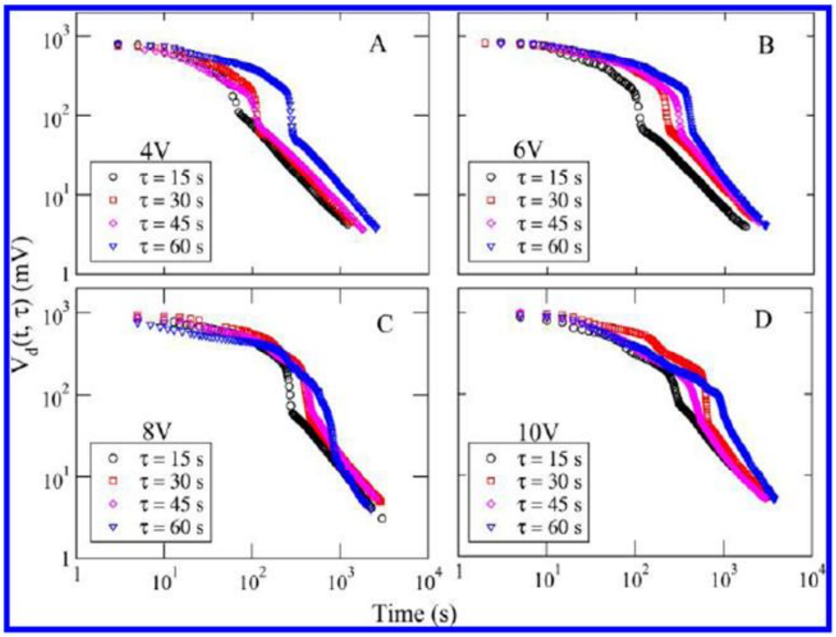

Figure-3: Memorizing charging time of applied electric-field in experiments with Laponite for crack formation studies Courtesy, Condense Matter Physics Research Center (CMPRC), Dept. of Physics, Jadavpur University Kolkata.

The Figure-3 is experimental evidence that a relaxing system (in this case, Laponite stressed with DC-Electric Field/ Voltage) has a memory of being connected to a voltage supply for during time τ and continues as if it is still connected to a non-zero voltage source, after the voltage source is switched-off. This Figure-3 is variation of discharge voltage $V_d(t,\tau)$ with time after the electric field is switched off at time $t=\tau$ for different applied voltages. The self-discharging curve is a function of time (τ); therefore in a way the relaxing system is memorizing its history of charging time [14].

Charge storage by Ramp Input Voltage excitation to fractional capacitor

Say we have a Ramp voltage input as $v(t)=\left(V_m/T\right)t$, i.e. applied at $t=0$ and it linearly rises from zero volts to V_m volts, in time $t=T$. We have $V(s)=\left(V_m/Ts^2\right)$. We apply $q(t)=c(t)*v(t)$; with $c(t)=C_\alpha t^{-\alpha}$, as time varying capacity function. To get the following charge function in Laplace domain

$$Q(s)=\left(\frac{C_\alpha\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(\frac{V_m}{Ts^2}\right)=\frac{V_mC_\alpha\Gamma(1-\alpha)}{Ts^{1+(2-\alpha)}}=\frac{V_mC_{F-\alpha}}{s^{1+(2-\alpha)}T}$$

Doing inverse Laplace transform we obtain the $q(t)$ as follows

$$q(t)=\frac{V_m C_\alpha \Gamma(1-\alpha)}{T\Gamma(3-\alpha)}t^{2-\alpha}=\frac{V_m C_\alpha}{T(1-\alpha)(2-\alpha)}t^{2-\alpha}, \quad 0\le t\le T$$
$$=\frac{V_m C_{F-\alpha}}{T(2-\alpha)\Gamma(2-\alpha)}t^{2-\alpha}$$
$$=\frac{V_m}{T}\frac{C_{F-\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha}$$

The current is following

$$i(t)=\frac{\mathrm{d}}{\mathrm{d}t}q(t)=\frac{V_m C_\alpha}{T(1-\alpha)}t^{1-\alpha}, \quad 0\le t\le T$$

We get the charge at the end of $t=T$ as

$$q(T)=\frac{V_m C_\alpha}{(1-\alpha)(2-\alpha)}T^{1-\alpha}$$

Comparison of charge storage by step and ramp input excitation

For the step input with voltage, held for time $t=T$ we have the charge as $q(T)=\frac{V_m C_\alpha}{1-\alpha}T^{1-\alpha}$, and we have charge at the end of $t=T$ for a ramp input as $q(T)=\frac{V_m C_\alpha}{(1-\alpha)(2-\alpha)}T^{1-\alpha}$. We write the ratio as follows

$$\frac{q(T)\big|_{STEP}}{q(T)\big|_{RAMP}}=\frac{\frac{V_m C_\alpha}{1-\alpha}T^{1-\alpha}}{\frac{V_m C_\alpha}{(1-\alpha)(2-\alpha)}T^{1-\alpha}}=2-\alpha$$

We observe that for time T if we hold the voltage to V_m and charge a capacitor, then we will be holding $(2-\alpha)$ times the charge if we ramp the voltage at rate V_m/T from zero to V_m. Now after this process if we keep the capacitors in self discharge mode, for both the cases the voltage decay will start from V_m, but for step-charging case, since amount of charges held is more, it will take longer time to self discharge as compared to case with ramp-charging. This we expect from the memory effect. The comparison between step input voltage charging and ramp input voltage charging is depicted in Figure-4. This study on similar lines about memory effect from step and ramp charging is also shown in [15].

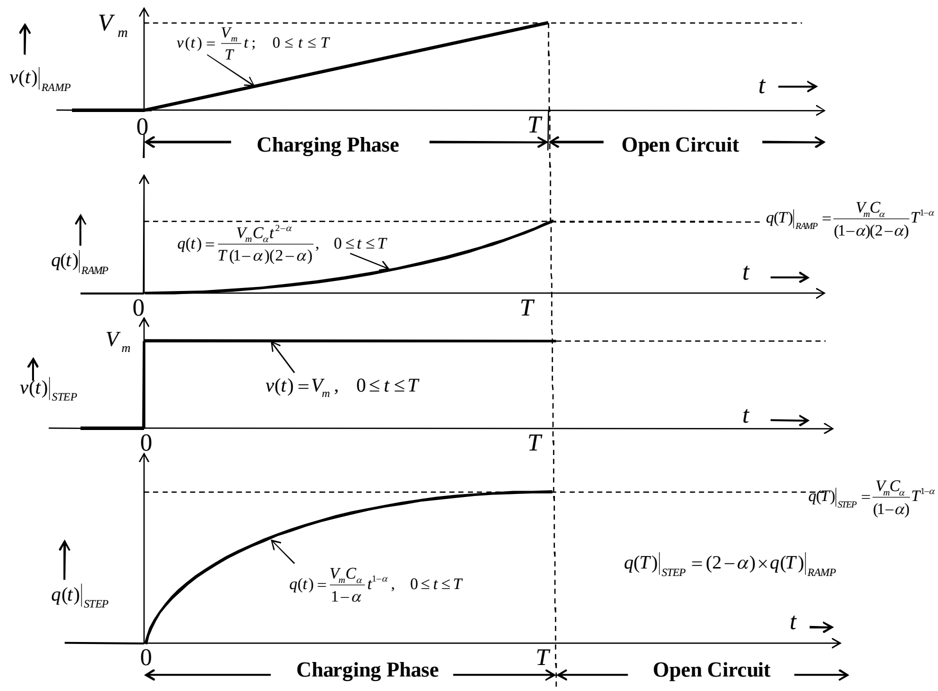

Figure-4: Step input voltage charging and Ramp input voltage charging

Discussions

We have compared the step input and ramp input excitation, by considering when we apply the excitation voltage, the response a capacitor produces to charge itself is by same time varying capacity function, i.e. $c(t)=C_\alpha t^{-\alpha}$. In actual cases the two input functions are having frequency components ω from 0 to infinity (forgetting truncation of the inputs at time $t=T$). As demonstrated in Figure-1, we will be having different values at different frequencies for α and $C_{F-\alpha}$. Therefore; we may ask how far this assumption is valid. A unit step function without truncation has Fourier transform as $\mathcal{F}\left\{u(t)\right\}=\frac{1}{2}\left(\delta(\omega)-\frac{i}{\omega}\right)$ and of unit ramp without truncation is having transform as $\mathcal{F}\left\{r(t)\right\}=-\frac{1}{\omega^2}+i\delta^{(1)}(\omega)$. Both have DC component and frequency components amplitude varying as ω^{-1} and ω^{-2} respectively-but loaded highly towards DC i.e. low frequency. Therefore even if there is difference in α and $C_{F-\alpha}$, that will be small. Moreover these Fourier components of unit step and unit ramp function will be modified by a Fourier of Rectangular Window function centered at time $T/2$, of height unity. Thus one may be justified in selecting same α and C_α for the capacity function $c(t)$ acting for step as well as ramp input, to compare the charge storage function. This analysis is done in [15] but the parameters for ramp and step excitations are not widely varying as shown.

The concept of charge storage as done here $q(t)=c(t)*v(t)$ is very important in dielectric relaxation studies where the capacitor formed as classical electronics capacitor is also having Curie-von Schweidler current relaxation law experimentally verified, as reported in [6], and thus fractional capacitor is reality. The fractional capacitor is more prominent in super capacitor studies, as reported in [15]-[22]. Therefore in this view the concept of charge storage and its formula be revisited.

Conclusions

We have applied the new formula of charge storage i.e. via convolution operation, of time varying capacity function and voltage stress for a fractional capacitor. This new formulation is different to the earlier used formula of multiplication of capacity and voltage function. We have discussed various results obtained for different excitation voltages- sinusoidal, step and ramp; and also revisited the impedance formula, and Nyquist diagrams. We have given interpretations of the various analytical results that were obtained by this new formulation.

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