
Formation of constituent physical laws with of memory kernels of singular and non-singular functions and its analysis

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Abstract

The physical laws are constitutive equations, relating cause and response (effect). The cause is input call it $x(t)$ and its response (effect) is $y(t)$. The system will respond with its characteristic function call it $h(t)$; by universal law as $y(t) = h(t) * x(t)$; i.e. convolution operation. This characteristics function $h(t)$ we term as ‘memory-kernel.’ This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with ‘memory-kernel’, that is $h(t)$. Implying that the value of the process $y(t)$ at present instant is being influenced by all the states of $x(t)$ the system had been through in past. We will study formation of basic constitutive equations with zero-memory case where the memory-kernel is delta-function, and with singular and non-singular memory kernels that decays with time. These decaying functions used as memory kernel gives a reality in which memory fades as time grows. However, the question arises is the memory kernel be of singular or non-singular function? We will see that the zero-memory case that is memory kernel as delta function returns classical constitutive laws that we know but the case with memory kernel as other than delta function returns constitutive equations with fractional derivatives and fractional integrations, different from what we know. We will note that singular function that we use for time-decaying memory kernel gives rise to conjugation to classical constitutive equations where integer-order (classical) derivative is replaced by its fractional counterpart. We will see that non-singular memory kernel gives rise to more complicated constitutive equations as weighted series sum of repeated integrations or weighted series sum of fractional integrations. We will study two types of system with evolution equation defined as $y(t) = h(t) * x(t)$. First considering the cause $x(t)$ is proportional to rate of change of some other physical quantity, i.e. $x(t) \propto f^{(1)}(t)$ and second is a system where response is proportional to rate of change of cause i.e. $y(t) \propto x^{(1)}(t)$. We note the first type of system is like ‘response current’ to a change in applied voltage observed in dielectric relaxations and capacitor. The second one corresponds to population growth or radioactive decay type system. We will study the formation of constituent expressions of physical laws with memory kernel $h(t)$ as delta function, singular power-law decay function, non-singular power law decay function, Mittag-Leffler function, pure exponential function and stretched exponential function. The motivation to have this article is to discuss, issues about using singular and non-singular functions as memory kernels-and its implications to physical laws. We will restrict our analysis to first order constitutive equations of physical laws.

Keywords

Convolution, Memory Kernel, Fractional Derivative, Fractional Integration, power-law, Mittag-Leffler function, stretched exponential function, Hamilton's principle

1. Introduction

The motivation to have this article is to discuss, issues about using singular and non-singular functions in system studies, especially as memory kernel; and get the corresponding constituent equations as physical laws. We observe complications in the physical laws that we derive, by use of non-singular memory kernel in basic evolution equation, as compared to using singular kernels. In this study, we give simple mathematical treatment to derive the relaxation laws (or constitutive equations) of system with several types of memory kernels to a relaxation law that we formulate as convolution integral $y(t) \propto h(t) * x(t)$. The function $h(t)$ is the memory kernel, $x(t)$ is the cause function, and $y(t)$ as effect or response function. This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with 'memory-kernel', that is $h(t)$. This implies the value of the process $y(t)$ at present instant is being influenced by all the states of $x(t)$ the system had been through in past since the application of cause.

Here in this discussion we take the memory kernel $h(t)$ as singular and non-singular functions. The singular functions considered for memory kernel $h(t)$ are delta function $\sim \delta(t)$, power-law decay function $\sim t^{-\alpha}$. The non-singular functions for $h(t)$ are considered as $\sim (1 + \nu t)^{-\alpha}$; (that is non-singular decaying power-law), Mittag-Leffler function $\sim E_{\alpha}(-\lambda t^{\alpha})$ exponential decay function $\sim e^{-\kappa t}$, and stretched exponential function $\delta(t) \sim e^{-(\kappa t)^{\alpha}}$. We arrive at the constitutive relations of $y(t)$ in relation to $x(t)$, with all these memory-kernels. All memory kernels we will use to derive constituent equation from universal evolution equation that is given as convolution, i.e. $y(t) \propto h(t) * x(t)$, are of decaying nature. This is reality, as we all know memory shall be fading away with time.

First we study $y(t) \propto h(t) * x(t)$, with cause as proportional to derivative of some other physical quantity call it $f(t)$, i.e. make $x(t) = f^{(1)}(t)$. In this type of systems, when the memory kernel is delta function, we note we get constituent law as $y(t) \propto f^{(1)}(t)$. This is classical case like for constituent equation for dielectric relaxation current when the dielectric is stressed with a constant voltage stress; that is classical capacitor current equation. This type of classical law is also for case of Newtonian viscous element relating stress to first derivative of strain. We will see by choosing various types of memory kernel this classical law is changed, by use of fractional derivative, and fractional integrals.

The second type of physical laws we will formulate from $y(t) \propto h(t) * x(t)$ by taking $y(t) \propto x^{(1)}(t)$. We will show for a zero-memory case with memory kernel as delta function i.e. $h(t) \propto \delta(t)$ we get classical growth or decay law given by classical constitutive equation i.e. $x^{(1)}(t) \propto x(t)$. The growth or decay is given by classical

exponential function in this case, i.e. $x(t) \sim e^{Ct}$. We will consider other memory kernel as we have done for first case. We will show that a singular power law kernel returns the constitutive equation where first derivative is replaced by fractional derivative i.e. $D^\nu x(t) \propto x(t)$ (or $x^{(\nu)}(t) \propto x(t)$) with $0 < \nu < 1$. We will see that non-singular memory kernels gives rise to more complicated constitutive equations as weighted series sum of repeated integrations or weighted series sum of fractional integrations; for both types of systems considered for evolution equation i.e. $y(t) \propto h(t) * x(t)$.

The article is organized as Sections. The Section-2 is revising the concept of Impulse Response of the system, thereby obtaining general evolution equation. Section-3 gives preliminaries of Fractional Calculus that we use in subsequent sections. Section-4 gives the constitutive equations for physical laws with cause as derivative of some other function, i.e. $x(t) = f^{(1)}(t)$; with all subsections with different types of memory kernels. Section-5 gives formation of constitutive equations with output a derivative of input function i.e. $y(t) = x^{(1)}(t)$. Section-6 gives discussions and observations with summary, followed by conclusion and references.

2. General Evolution Equation for Response (or relaxation) of a system

The output call it $y(t)$ a variable in time ($t \in \mathbb{R}$), of a system represented by function of time variable $h(t)$ to an input variable call it $x(t)$ acting at time $t=0$, is given by evolution equation [5], [6], [25] as follows

$$y(t) = \int_0^t h(t-\tau)x(\tau)d\tau; \quad t \geq 0 \quad (1)$$

If we take Laplace transform, with $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{h(t)\} = H(s)$, we get

$$Y(s) = (H(s))(X(s)) \quad (2)$$

Where variable s is Complex variable, i.e. $s \in \mathbb{C}$. The input $x(t)$ in case is delta-function (an impulse) at $t=0$; we have $y(t) = h(t)$. This $h(t)$ is called ‘impulse response’ of the system [6], [25]. The evolution equation (1) of $y(t)$ in time domain is convolution integral [25]. We will call $h(t)$ as ‘memory-kernel’, and derive constituent expressions with several types of $h(t)$.

2.1. System with cause as derivative of action variable

We have several physical systems that are proportional to rate of change of some other physical quantity that is acting as input. Say we have rate of change of a quantity call it $f(t)$ represented by first time derivative i.e. $f^{(1)}(t)$, then our input variable in (1) is $x(t) = f^{(1)}(t)$; then we have evolution equation in terms of impulse response function of the system, $h(t)$ as

$$y(t) = \int_0^t (h(t-\tau))(f^{(1)}(\tau))d\tau \quad (3)$$

Some physical systems can be casted as (1) and (3). For example, current through a capacitor classically related to voltage given as $i(t) = Cv^{(1)}(t)$; force as rate of change of momentum as $f(t) = p^{(1)}(t)$ and stress related to rate of change in Newtonian viscous

element as $\sigma(t) = \eta \varepsilon^{(1)}(t)$. For these physical systems, as per (3) we have response $y(t)$ as $i(t)$, $f(t)$ and $\sigma(t)$ to excitation $f(t)$ as $v(t)$, $p(t)$ and $\varepsilon(t)$ respectively. These described systems only respond to rate of change; i.e. output effect is proportional to rate of change of input cause.

Looking at the time evolution equation (1), if the input variable $x(t)$ acts only at time $t=0$ thereafter vanishes for $t>0$ and we observe $y(t)$ even at $t>0$ while ($x(t)=0$ for $t>0$); we may term that system is remembering its past input or cause. In that case, we say system relaxes with ‘memory’ [6], [22], [27] [28]. In ideal cases as described by the constitutive equations for capacitor, force function and stress, behave ‘without memory’. It can be seen when the ‘rate terms’ input in the RHS of these constitutive equations vanishes after application at $t=0$, we have no observation of the output at $t>0$. Simply if the rate terms in the RHS of all these constitutive equations is described by delta function, then the output is also delta function at $t=0$.

2.2. System with response as derivative of cause variable

In the evolution equation (1), if we call response $y(t)$ as rate of change of cause i.e. $y(t) = x^{(1)}(t)$, then we have following evolution equation

$$\frac{dx(t)}{dt} = \int_0^t h(t-\tau)x(\tau)d\tau; \quad t \geq 0 \quad (4)$$

The expression (4) denotes physical systems where $x(t)$ grows or decays from an initial value $x(0)$. This is like nuclear reaction growth or radioactive decay type systems, depending on memory-kernel $h(t)$. In subsequent sections we will form constitutive equation from (3) and (4) types of systems, by having various types of decaying memory kernel (singular as well as non-singular in nature).

The convolution integral (1) can in general have lower terminal of integration as $t = -\infty$ or $a > -\infty$ as the case may be for application of input $x(t)$; that we depict as follows

$$y(t) = \int_{-\infty}^t h(t-\tau)x(\tau)d\tau; \quad y(t) = \int_a^t h(t-\tau)x(\tau)d\tau \quad (5)$$

The (5) is the most general representation of the convolution process. In our discussion of forming the constitutive equation, we will consider that the memory kernel acts only when the cause $x(t)$ acts; and we take $t=0$, and thus the response is described for

$t \geq 0$ with convolution process as $y(t) = \int_0^t h(t-\tau)x(\tau)d\tau$.

3. Preliminaries of Fractional Calculus

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration [6], [26], [29] of order $\nu \in \mathbb{R}^+$ is defined as

$${}_0I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau)d\tau \quad (6)$$

Where $\Gamma(\nu)$ is Euler’s Gamma function, is generalization of factorial function [23], [29], we have $\Gamma(\nu) = (\nu-1)!$. The formula (6) is appearing as generalization of Cauchy’s multiple integration formula of m fold integration [6], [26], [29] where $m \in \mathbb{N}$ given as follows

$${}_0I_t^m [f(t)] = \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau; \quad m=1,2,3,\dots \quad (7)$$

The fractional derivative of order ν for $0 < \nu < 1$ by Riemann-Liouville (RL) formula [6], [26], [29] is

$${}_0D_t^\nu [f(t)] = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t (t-\tau)^{-\nu} f(\tau) d\tau; \quad 0 < \nu < 1 \quad (8)$$

The (8) is fractionally integrating the function by order $(1-\nu)$ by formula (6) and then followed by one-whole differentiation. There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative [6], [26], [29] for fractional order $0 < \nu < 1$ is given as

$${}_0^C D_t^\nu [f(t)] = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-\tau)^{-\nu} f^{(1)}(\tau) d\tau; \quad 0 < \nu < 1 \quad (9)$$

Thus for (9) we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order $1-\nu$, by formula (6). The Caputo and Riemann-Liouville (RL) fractional derivative are related [6], [26], [29] by

$${}_0D_t^\nu [f(t)] = {}_0^C D_t^\nu [f(t)] + \frac{f(0)}{\Gamma(1-\nu)} t^{-\nu}; \quad 0 < \nu < 1 \quad (10)$$

We mention that both the fractional derivatives, i.e. RL and Caputo are equal when initial value is zero i.e. $f(0) = 0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^\beta [K] = \frac{K}{\Gamma(1-\beta)} t^{-\beta}$. Whereas, the Caputo fractional derivative of a constant is zero i.e. ${}_0^C D_t^\beta [K] = 0$, [6], [26], [29].

The fractional integration and fractional differentiation of delta function [6], [26], [29] is as follows

$${}_0I_t^\nu \delta(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}; \quad {}_0D_t^\nu \delta(t) = \frac{1}{\Gamma(-\nu)} t^{-\nu-1}, \quad 0 < \nu < 1 \quad (11)$$

Fractional derivative and fractional integration of power function $f(t) = Kt^p$ [6], [26], [29] is

$${}_0I_t^\nu Kt^p = K \frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} t^{p+\nu}, \quad {}_0D_t^\nu Kt^p = K \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} t^{p-\nu}, \quad p > -1 \quad (12)$$

The Laplace transform of fractional integral operation is

$$\mathcal{L} \left\{ {}_0I_t^\nu f(t) \right\} = s^{-\nu} F(s) \quad (13)$$

Laplace transform of Caputo derivative for fractional order $0 < \nu < 1$ is

$$\mathcal{L} \left\{ {}_0^C D_t^\nu f(t) \right\} = s^\nu F(s) - s^{\nu-1} f(0) \quad (14)$$

Like in classical calculus, we have exponential function; similarly, in fractional Calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}; \quad \text{Re}[\alpha, \beta] > 0 \quad (15)$$

For $\beta = 1$ we have $E_{\alpha,1}(z) = E_\alpha(z)$; is called One-Parameter Mittag-Leffler function.

The Laplace transformation of Mittag-Leffler function is

$$\mathcal{L}\{E_\alpha(\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda} \quad (16)$$

We observe that for $E_\alpha(-bt^\alpha)\big|_{\alpha=1} = e^{-bt}$, and $E_\alpha(-at^\alpha)\big|_{\alpha=2} = \cos\sqrt{at}$.

We point here that $f(t) = E_\alpha(\lambda t^\alpha)$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^\alpha f(t) = \lambda f(t)$; and $f(t) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)$ is eigen-function for fractional differential equation with RL fractional derivative, i.e. ${}_0 D_t^\alpha f(t) = \lambda f(t)$ [6], [26], [29]. We will be using these concepts of fractional calculus in our discussion.

4. Forming Constituent expression for $y(t) = h(t) * x(t)$ where $x(t) = f^{(1)}(t)$ with various memory kernels

4.1. Memory Kernel as delta function

We study a simple case, the response $y(t)$ in a system given by (1), when $x(t)$ the input is rate of change of $f(t)$ i.e. $x(t) = f^{(1)}(t)$. That is classically we have relaxation of $y(t)$ as $y(t) \propto f^{(1)}(t)$. The classical case is therefore is following with C as a constant

$$y(t) = C \frac{df(t)}{dt} = C(f^{(1)}(t)) \quad (17)$$

We can modify the above expression i.e. $y(t) = Cf^{(1)}(t)$ or $y(t) = C(D_t^1 f(t))$ and write

$$\begin{aligned} y(t) &= C \int_{-\infty}^t (\delta(t-\tau) f^{(1)}(\tau)) d\tau \\ &= \int_{-\infty}^t (C\delta(t-\tau) f^{(1)}(\tau)) d\tau = (C\delta(t)) * (f^{(1)}(t)) \end{aligned} \quad (18)$$

This comes from property of delta function, i.e. $\int \delta(x-y) f(y) dy = f(x)$ [23]. In above expression (18) for the convolution integral, we have kernel of integration as delta function call it $h(t) = C\delta(t)$. With this we get for $y(t) = Cf^{(1)}(t)$ the following

$$y(t) = (h(t)) * (f^{(1)}(t)), \quad h(t) = C\delta(t), \quad y(t) = Cf^{(1)}(t) \quad (19)$$

This (19) expression we have casted as (1) and (3). The kernel $h(t)$ we will now term as 'memory kernel' of the basic evolution equation i.e. $y(t) = (h(t)) * (f^{(1)}(t))$.

Let us give a unit step input, $f(t) = u(t)$ applied at $t = 0$. This means $f(t) = 1$ for $t \geq 0$ to an initially rest system i.e. $f(t) = 0$ for $t < 0$; then we have $f^{(1)}(t) = \delta(t)$ i.e. differentiation of unit-step input. Placing this value in expression $y(t) = Cf^{(1)}(t)$, we get

$$\begin{aligned} y(t) &= \int_0^t (C\delta(t-\tau) f^{(1)}(\tau)) d\tau = C \int_0^t (\delta(t-\tau) \delta(\tau)) d\tau \\ &= C\delta(t) \end{aligned} \quad (20)$$

This (20) above is direct result of $y(t) = Cf^{(1)}(t)$; as the differentiation of unit step function is delta-function. The Laplace transformed relations of $y(t) = Cf^{(1)}(t)$ is

$$Y(s) = C(sF(s) - f(0)) \quad (21)$$

Doing Laplace transform of $y(t) = (h(t)) * (f^{(1)}(t))$ we get the following

$$\begin{aligned}
 \mathcal{L}\{y(t)\} &= \mathcal{L}\{(h(t)) * (f^{(1)}(t))\} \\
 Y(s) &= \mathcal{L}\{h(t)\} \mathcal{L}\{f^{(1)}(t)\} \\
 &= C(H(s))(sF(s) - f(0)), \quad H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{C\delta(t)\} = C \\
 &= C(sF(s) - f(0))
 \end{aligned} \tag{22}$$

We get the same result as we got earlier (21), i.e. $Y(s) = C(sF(s) - f(0))$.

From the classical theory with Newtonian Calculus as the constitutive equation (17), we get a delta impulse output when system is impressed with a constant step input. This is a singular relaxation current function.

4.1.1. A zero memory case-with memory kernel as delta function

From the classical law we have arrived at the equation, which is following

$$\begin{aligned}
 y(t) &= (h(t)) * (f^{(1)}(t)) \\
 &= \int_{-\infty}^t (h(t - \tau))(f^{(1)}(\tau)) d\tau
 \end{aligned} \tag{23}$$

It so happens that the classical equation i.e. $y(t) = Cf^{(1)}(t)$ is associated with memory kernel $h(t) = C\delta(t)$. This physically implies that the system $y(t) = Cf^{(1)}(t)$ has zero-memory. That is just after the instance of application of input i.e. at $t = 0^+$ the memory kernel vanishes i.e. $h(t) = 0$ for $t > 0$. Whereas $h(t) = \infty$ only at $t = 0$; and is singular function. This is a 'singular memory kernel'. Now we will study the formation of constituent equation relaxation of output to unit step input of for various kernels-singular and non-singular, from the evolution equation $y(t) = h(t) * f^{(1)}(t)$.

4.2. Constitutive equation due to Singular Power Law Memory Kernel

The Nature has many examples of 'power-law' distributions [13]-[16]. Let us have the power law decay memory kernel described as

$$h(t) = Ct^{-\alpha}; \quad 0 < \alpha < 1 \tag{24}$$

In above (24) expression C is a positive constant. The kernel $h(t) = Ct^{-\alpha}$ is singular at origin with its derivative as minus infinity. This means that we have memory kernel $h(t) = \infty$ at $t = 0$ and monotonically decaying after that i.e. $t > 0$, with $h^{(1)}(t)|_{t=0} = -\infty$ this is some way mimicking the actual memory or forgetfulness. That is as the time goes the memory fades away.

With this we have following steps

$$\begin{aligned}
y(t) &= h(t) * f^{(1)}(t) \\
\mathcal{L}\{y(t)\} &= \mathcal{L}\{(h(t)) * (f^{(1)}(t))\} \\
Y(s) &= (\mathcal{L}\{h(t)\})(\mathcal{L}\{f^{(1)}(t)\}), \quad \mathcal{L}\{h(t)\} = \mathcal{L}\{Ct^{-\alpha}\} = C \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}; \quad f(t) = u(t) \\
&= \left(C \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} \right) (sF(s) - f(0)), \quad \mathcal{L}\{f(t)\} = F(s) \\
&= C(\Gamma(1-\alpha))(s^\alpha F(s) - s^{\alpha-1}f(0)), \quad f(0) = 0, \quad F(s) = \frac{1}{s} = \mathcal{L}\{u(t)\} \\
Y(s) &= C(\Gamma(1-\alpha))s^{\alpha-1}
\end{aligned} \tag{25}$$

From above (25) we get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ by using $\mathcal{L}\{t^p\} = \Gamma(p+1)s^{-p-1}$

$$y(t) = Ct^{-\alpha}; \quad 0 < \alpha < 1 \tag{26}$$

Therefore, we are getting a power decay response (output) i.e. $y(t) \sim t^{-\alpha}$ for the memory kernel in evolution equation i.e. $y(t) = h(t) * f^{(1)}(t)$ as a power law $h(t) \sim t^{-\alpha}$; for system unit step input $f(t) = u(t)$.

Now we obtain constitutive relation with memory kernel $h(t) = Ct^{-\alpha}$ that is singular, at the application time of the cause i.e. at $t = 0$ and has no derivative at start point. We write the following steps

$$\begin{aligned}
y(t) &= (h(t)) * (f^{(1)}(t)) \\
&= \int_{-\infty}^t (h(t-\tau))(f^{(1)}(\tau))d\tau; \quad h(t) = Ct^{-\alpha}; \quad t \geq 0 \\
&= \int_0^t (C(t-\tau)^{-\alpha})(f^{(1)}(\tau))d\tau \\
&= C(\Gamma(1-\alpha)) \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-\tau)^{-\alpha})(f^{(1)}(\tau))d\tau \right) \\
&= C(\Gamma(1-\alpha)) \left({}_0^C D_t^\alpha f(t) \right)
\end{aligned} \tag{27}$$

In above (27) steps we have used the definition of Caputo fractional derivative [6], [26], [29] for fractional order $0 < \alpha < 1$ i.e. ${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-\tau)^{-\alpha})(f^{(1)}(\tau))d\tau$.

Thus, our constitutive equation for a system having Singular Power Law Memory kernel is given by fractional differential equation, and is changed from classical with zero-memory case i.e. $y(t) = Cf^{(1)}(t)$ to following case with fractional derivative

$$\begin{aligned}
y(t) &= C(\Gamma(1-\alpha)) \left({}_0^C D_t^\alpha f(t) \right); \quad C_\alpha = C(\Gamma(1-\alpha)) \\
y(t) &= C_\alpha \left({}_0^C D_t^\alpha f(t) \right); \quad 0 < \alpha < 1
\end{aligned} \tag{28}$$

In above (28) putting $\alpha = 1$ we get classical relation i.e. $y(t) = Cf^{(1)}(t)$. The above (28) expression as obtained is used in [5], [7]-[12], [17]-[20], [22], [28].

The Laplace Transform of Caputo Fractional Derivative [6], [26], [29] for fractional order $0 < \alpha < 1$ is $\mathcal{L}\left\{{}^C_0D_t^\alpha f(t)\right\} = s^\alpha F(s) - s^{\alpha-1}f(0)$. Using this we write Laplace Transform of $y(t) = C_\alpha \left({}^C_0D_t^\alpha f(t)\right)$ as following

$$\begin{aligned}\mathcal{L}\{y(t)\} &= C_\alpha \mathcal{L}\left\{{}^C_0D_t^\alpha f(t)\right\}; \quad 0 < \alpha < 1 \\ Y(s) &= C_\alpha \left(s^\alpha F(s) - s^{\alpha-1}f(0)\right)\end{aligned}\tag{29}$$

We note that in above (29) putting $\alpha=1$ we obtain the classical result i.e. $Y(s) = C(sF(s) - f(0))$.

We verify the relaxation expression of $y(t)$ with $F(s) = s^{-1}$ and $f(0) = 0$ i.e. for unit step input $f(t) = u(t)$; $t \geq 0$, applied to system with $f(0) = 0$, $t < 0$ from above

$$\begin{aligned}Y(s) &= C_\alpha \left(s^\alpha F(s) - s^{\alpha-1}f(0)\right); \quad F(s) = \frac{1}{s}, \quad f(0) = 0 \\ Y(s) &= C_\alpha s^{\alpha-1}, \quad \mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{C_\alpha s^{\alpha-1}\} \\ &= \frac{C_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, \quad 0 < \alpha < 1, \quad t \geq 0\end{aligned}\tag{30}$$

With $C_\alpha = C(\Gamma(1-\alpha))$ we get $y(t) = Ct^{-\alpha}$; same as we got earlier (26).

4.2.1. Difference between zero-memory and memory based relaxation cases

We observed that for a classical case the relaxation response i.e. $y(t)$ is delta function at the start of application of input $f(t)$, which is unit step function, i.e. $f(t) = u(t)$. Therefore, as soon as the rate of change i.e. $f^{(1)}(t)$ vanishes at $t > 0$ we have relaxing output $y(t)$ as zero. This is zero memory case with memory kernel as $h(t) \propto \delta(t)$. Where we observe from above (30) with a power-law memory kernel as $h(t) \propto t^{-\alpha}$ ($0 < \alpha < 1$), we have a finite output response $y(t)$ even the rate of change of cause (input) $f^{(1)}(t)$ vanished at $t > 0$. Therefore, the system is memorizing the excitation that once took place as a rate of change in input (cause) and system is relaxing with memory. Well this above case was the case with singular power law memory kernel. Now in subsequent sections we will discuss non-singular memory kernels and see the constitutive relations that we get for these systems.

4.3. Constitutive equation due to Non-Singular power law Memory Kernel

We have seen earlier that the kernel of singular power-law i.e. $h(t) \propto t^{-\alpha}$; $0 < \alpha < 1$ gives a constitutive equation with fractional derivative i.e. $y(t) \propto f^{(\alpha)}(t)$. We modify the power-law to a non-singular type with following type

$$h(t) = C(1+\nu t)^{-\alpha}; \quad 0 < \alpha < 1, \quad \nu > 0, \quad t \geq 0\tag{31}$$

In above (31) C is a positive constant. In above we have $h(0) = C$ and $h^{(1)}(0) = -C\alpha$, unlike the singular kernel, i.e. $h(t) \propto t^{-\alpha}$. With this we do following calculations for obtaining constitutive equation

$$\begin{aligned}
y(t) &= (h(t)) * (f^{(1)}(t)) \\
&= \int_0^t C(1 + \nu(t-\tau)^{-\alpha}) (f^{(1)}(\tau)) d\tau \\
&= C \int_0^t \left(\binom{-\alpha}{0} (\nu(t-\tau))^0 + \binom{-\alpha}{1} (\nu(t-\tau)) + \binom{-\alpha}{2} (\nu(t-\tau))^2 + \dots \right) (f^{(1)}(\tau)) d\tau \\
&= C \int_0^t \left(1 + (-\alpha)(\nu(t-\tau)) + \frac{(-\alpha)(-\alpha-1)}{2!} (\nu(t-\tau))^2 + \dots \right) (f^{(1)}(\tau)) d\tau \\
&= C \left(\int_0^t (f^{(1)}(\tau)) + \frac{(-\alpha)}{1!} \int_0^t (\nu(t-\tau)) (f^{(1)}(\tau)) d\tau \right. \\
&\quad \left. + \frac{(-\alpha)(-\alpha-1)}{2!} \int_0^t (\nu(t-\tau))^2 (f^{(1)}(\tau)) d\tau \right. \\
&\quad \left. + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t (\nu(t-\tau))^3 (f^{(1)}(\tau)) d\tau \dots \right) \tag{32}
\end{aligned}$$

In above (32) we used binomial expansion [23] for series representation of $(1+z)^{-\alpha}$. We use repeated integration formula i.e. ${}_0I_t^m f(t) = \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau$, [6], [26], [29] and get the following expression

$$\begin{aligned}
y(t) &= C \left(\int_0^t (f^{(1)}(\tau)) + \frac{(-\alpha)}{1!} \int_0^t (\nu(t-\tau)) (f^{(1)}(\tau)) d\tau \dots \right. \\
&\quad \left. \dots + \frac{(-\alpha)(-\alpha-1)}{2!} \int_0^t (\nu(t-\tau))^2 (f^{(1)}(\tau)) d\tau + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t (\nu(t-\tau))^3 (f^{(1)}(\tau)) d\tau \dots \right) \\
&= C \left(\int_0^t (f^{(1)}(\tau)) + (-\alpha)\nu \left(\frac{1}{(2-1)!} \int_0^t (t-\tau)^{2-1} (f^{(1)}(\tau)) d\tau \right) + \right. \\
&\quad \left((-\alpha)(-\alpha-1)\nu^2 \left(\frac{1}{(3-1)!} \int_0^t (t-\tau)^{3-1} (f^{(1)}(\tau)) d\tau \right) + \dots \right) \\
&= C \left({}_0I_t^1 f^{(1)}(t) + (-\alpha)\nu ({}_0I_t^2 f^{(1)}(t)) + (-\alpha)(-\alpha-1)\nu^2 ({}_0I_t^3 f^{(1)}(t)) + \dots \right) \tag{33}
\end{aligned}$$

It so happens that for this memory-kernel $h(t) = C(1 + \nu t)^{-\alpha}$ which is non-singular power-law the constitutive equation is for $y(t)$ is weighted sum of integrals (one whole, two whole, three whole; and so on) of $f^{(1)}(t)$; that is rate of change of $f(t)$ (the input).

With excitation $f^{(1)}(t) = \delta(t)$ to initially rest system with $y(0) = 0$, that is applying a unit step input $f(t) = u(t)$ we obtain following from above (33), we write following steps

$$\begin{aligned}
y(t) &= C \left({}_0I_t^1 f^{(1)}(t) + (-\alpha)\nu ({}_0I_t^2 f^{(1)}(t)) + (-\alpha)(-\alpha-1)\nu^2 ({}_0I_t^3 f^{(1)}(t)) + \dots \right) \\
&= C \left({}_0I_t^1 \delta(t) + (-\alpha)\nu ({}_0I_t^2 \delta(t)) + (-\alpha)(-\alpha-1)\nu^2 ({}_0I_t^3 \delta(t)) + \dots \right) \\
&= C \left(1 - \alpha\nu t^2 + \alpha(\alpha+1)\nu^2 \left(\frac{t^2}{2} \right) - \alpha(\alpha+1)(\alpha+2) \left(\frac{t^3}{(3)(2)} \right) + \dots \right) \\
&= C(1 + \nu t)^{-\alpha} \tag{34}
\end{aligned}$$

The above (34) says that the response $y(t)$ lingers in the system while the rate of change of input $f^{(1)}(t)$ vanishes at $t > 0$; giving memorized relaxation response. This was also observed with singular power law memory kernel. However, the constitutive equation for this case is in

$$y(t) = C \left({}_0I_t^1 f^{(1)}(t) + (-\alpha)v \left({}_0I_t^2 f^{(1)}(t) \right) + (-\alpha)(-\alpha-1)v^2 \left({}_0I_t^3 f^{(1)}(t) \right) + \dots \right) \quad (35)$$

This is very different from $y(t) = Cf^{(1)}(t)$ and $y(t) = Cf^{(\alpha)}(t)$; the classical case and case with fractional derivative respectively; that we got for singular memory kernels. Here in (35) above we are getting a series sum of weighted repeated integration

$$y(t) \propto \sum_{n=1}^{\infty} a_n \left({}_0I_t^n f^{(1)}(t) \right) \quad (36)$$

With weights in above as $a_1 = 1$, $a_2 = -\alpha v$, $a_3 = (\alpha)(\alpha+1)v^2 \dots$

4.4. Constitutive equation due to Mittag-Leffler function as Non-Singular Memory Kernel

Here we take Memory Kernel as following for $t \geq 0$, i.e. “One parameter Mittag-Leffler function”

$$h(t) = CE_{\alpha}(-\lambda t^{\alpha}); \quad 0 < \alpha < 1, \quad t \geq 0 \quad (37)$$

In above C and λ are a positive real constants. Where the “One Parameter Mittag-Leffler function” is defined [6], [23], [26], [29] as following

$$E_{\alpha}(-\lambda t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-\lambda t^{\alpha})^n}{\Gamma(\alpha n + 1)}, \quad t \geq 0; \quad \lambda t^{\alpha} \in \mathbb{C}, \quad \alpha \in \mathbb{C}, \quad \text{Re}[\alpha] > 0 \quad (38)$$

The constitutive equation with Memory Kernel $h(t) = CE_{\alpha}(-\lambda t^{\alpha})$ we write the following

$$\begin{aligned} y(t) &= h(t) * f^{(1)}(t) \\ &= C \int_0^t \left(E_{\alpha}(-\lambda(t-\tau)^{\alpha}) \right) \left(f^{(1)}(\tau) \right) d\tau \\ &= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda(t-\tau)^{\alpha})^n}{\Gamma(\alpha n + 1)} \right) \left(f^{(1)}(\tau) \right) d\tau \\ &= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^n}{\Gamma(\alpha n + 1)} \right) \int_0^t (t-\tau)^{\alpha n} f^{(1)}(\tau) d\tau \\ &= C \left(\sum_{n=0}^{\infty} (-1)^n \lambda^n \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} f^{(1)}(\tau) d\tau \right) \\ &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n + 1} \left[f^{(1)}(t) \right] \right) \end{aligned} \quad (39)$$

Where in above (39) we used the operator ${}_0I_t^{\nu}$, $\nu = \alpha n + 1$, which is Riemann-Liouville fractional integration of order ν defined as ${}_0I_t^{\nu} [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$. We write above (39) as series sum of weighted fractional integration

$$\begin{aligned}
y(t) &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n+1} \left[f^{(1)}(t) \right] \right) \\
&= C \left({}_0I_t^1 f^{(1)}(t) - \lambda \left({}_0I_t^{\alpha+1} f^{(1)}(t) \right) + \lambda^2 \left({}_0I_t^{2\alpha+1} f^{(1)}(t) \right) - \lambda^3 \left({}_0I_t^{3\alpha+1} f^{(1)}(t) \right) + \dots \right) \quad (40) \\
&= C \sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+1} f^{(1)}(t) \right)
\end{aligned}$$

With weights in this case as $b_0 = 1$, $b_1 = -\lambda$; ... $b_n = (-1)^n \lambda^n$ We get similar result with $h(t) = CE_{\alpha}(-\lambda t^{\alpha})$ that of with Non-Singular power law memory-kernel i.e. $h(t) = C(1 + \nu t)^{-\alpha}$ and this too is very different from results of singular memory kernels i.e. $h(t) = C\delta(t)$ and $h(t) = Ct^{-\alpha}$ giving $y(t) = Cf^{(1)}(t)$ and $y(t) = Cf^{(\alpha)}(t)$ respectively.

Thus, a Memory Kernel $h(t) = C - \frac{\lambda C t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda^2 C t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots$; $0 < \alpha < 1$ i.e. series-sum of power laws acting on derivative of voltage function i.e. $f^{(1)}(t)$, gives a relaxing response $y(t)$ with series sum of fractional integrations of various orders acting on rate of change of applied excitation i.e. $f^{(1)}(t)$, i.e. $y(t) = C \sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+1} f^{(1)}(t) \right)$. We note that Memory Kernel in this case i.e. $h(t) = CE_{\alpha}(-\lambda t^{\alpha})$ is not singular function at $h(0) = C$, and its derivative is not defined i.e. $h^{(1)}(t) \Big|_{t=0} = -\infty$.

Now we give a unit step input to this system so we have $f(t) = 1$, $t \geq 0$; with $f^{(1)}(t) = \delta(t)$. Placing this in above $y(t) = C \sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+1} f^{(1)}(t) \right)$ (40), we write the following

$$\begin{aligned}
y(t) &= Cf(t) - \lambda C \left({}_0I_t^{\alpha+1} f^{(1)}(t) \right) + \lambda^2 C \left({}_0I_t^{2\alpha+1} f^{(1)}(t) \right) - \lambda^3 C \left({}_0I_t^{3\alpha+1} f^{(1)}(t) \right) + \dots \\
&= C - \lambda C \left({}_0I_t^{\alpha+1} \delta(t) \right) + \lambda^2 C \left({}_0I_t^{2\alpha+1} \delta(t) \right) - \lambda^3 C \left({}_0I_t^{3\alpha+1} \delta(t) \right) + \dots \\
&= C - \lambda C \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} \right) + \lambda^2 C \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) - \lambda^3 C \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right) + \dots \quad (41) \\
&= C \left(1 + \frac{(-\lambda)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(-\lambda)^2(t^{\alpha})^2}{\Gamma(2\alpha+1)} + \frac{(-\lambda)^3(t^{\alpha})^3}{\Gamma(3\alpha+1)} + \dots \right) \\
&= C \left(\sum_{n=0}^{\infty} \frac{(-\lambda t^{\alpha})^n}{\Gamma(\alpha n+1)} \right) = CE_{\alpha}(-\lambda t^{\alpha}), \quad t \geq 0
\end{aligned}$$

In above (41) steps we have used formula for fractional integration of delta function i.e. ${}_0I_t^{\nu} \delta(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$; also we used ${}_0I_t^1 f^{(1)}(t) = f(t)$. What we observe that the relaxation response $y(t)$ to a system excited by unit-step input i.e. $f(t) = u(t)$; relaxes in proportional to the memory kernel function i.e. $y(t) \propto h(t)$ in this case is $h(t) \sim E_{\alpha}(-\lambda t^{\alpha})$. Here in above (41) the current relaxes at $t > 0$ even while the rate of change of voltage has vanished; therefore memorizing the past excitation. We note that by placing $\alpha = 1$ we are not getting classical case i.e. $y(t) = Cf^{(1)}(t)$.

Let us do Laplace Transformation for Mittag-Leffler memory kernel, as depicted as follows

$$\begin{aligned}
y(t) &= h(t) * f^{(1)}(t) \\
\mathcal{L}\{y(t)\} &= \mathcal{L}\{(h(t)) * (f^{(1)}(t))\} \\
Y(s) &= (\mathcal{L}\{h(t)\})(\mathcal{L}\{f^{(1)}(t)\}), \quad \mathcal{L}\{h(t)\} = \mathcal{L}\{CE_\alpha(-\lambda t^\alpha)\} = C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) \\
&= C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) (sF(s) - f(0)) \\
&= C \left(\left(\frac{s^\alpha}{s^\alpha + \lambda} \right) F(s) - \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) f(0) \right), \quad f(0) = 0, \quad F(s) = \frac{1}{s} \\
Y(s) &= C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) \\
y(t) &= C \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + \lambda} \right\} \\
&= CE_\alpha(-\lambda t^\alpha)
\end{aligned} \tag{42}$$

The same that we got earlier in (41).

4.4. Constitutive equation due to exponential function as Non-Singular Memory Kernel

Here we take Memory kernel as

$$h(t) = Ce^{-\kappa t}, \quad t \geq 0, \quad \kappa > 0; \quad C > 0 \tag{43}$$

The constitutive equation with Memory Kernel as $h(t) = Ce^{-\kappa t}$ gives the following

$$\begin{aligned}
y(t) &= h(t) * f^{(1)}(t) \\
&= C \int_0^t (e^{-\kappa(t-\tau)}) (f^{(1)}(\tau)) d\tau \\
&= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\kappa(t-\tau))^n}{n!} \right) (f^{(1)}(\tau)) d\tau \\
&= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n (\kappa)^n}{n!} \right) \int_0^t (t-\tau)^n f^{(1)}(\tau) d\tau \\
&= C \left(\sum_{n=0}^{\infty} (-1)^n \kappa^n \right) \left(\frac{1}{n!} \int_0^t (t-\tau)^n f^{(1)}(\tau) d\tau \right) \\
&= C \sum_{n=0}^{\infty} (-1)^n \kappa^n \left({}_0I_t^{n+1} [f^{(1)}(t)] \right)
\end{aligned} \tag{44}$$

Thus, the memory Kernel which is pure exponential function i.e. $h(t) = Ce^{-\kappa t}$ gives a relaxation response $y(t)$ which is weighted series sum of integer order multiple integration of rate of change of input function $f^{(1)}(t)$. We describe as follows

$$\begin{aligned}
y(t) &= C \sum_{n=0}^{\infty} (-1)^n \kappa^n \left({}_0I_t^{n+1} \left[f^{(1)}(t) \right] \right) \\
&= C \left({}_0I_t^1 f^{(1)}(t) \right) - \kappa C \left({}_0I_t^2 f^{(1)}(t) \right) + \kappa^2 C \left({}_0I_t^3 f^{(1)}(t) \right) - \lambda^3 C \left({}_0I_t^4 f^{(1)}(t) \right) + \dots \quad (45) \\
&= C \sum_{n=0}^{\infty} c_n \left({}_0I_t^{n+1} \left[f^{(1)}(t) \right] \right)
\end{aligned}$$

$$c_0 = 1, \quad c_1 = -\kappa, \quad c_3 = \kappa^2, \quad \dots c_n = (-1)^n \kappa^n$$

In above (39) the constitutive equation with memory kernel as non-singular Mittag-Leffler function, i.e. $y(t) = C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n+1} \left[f^{(1)}(t) \right] \right)$ if we place $\alpha = 1$ we get (45), by calling λ as κ .

We give a step input to system having Memory kernel $h(t) = Ce^{-\kappa t}$ and observe the following

$$\begin{aligned}
y(t) &= C \left({}_0I_t^1 f^{(1)}(t) \right) f(t) - \kappa C \left({}_0I_t^2 f^{(1)}(t) \right) + \kappa^2 C \left({}_0I_t^3 f^{(1)}(t) \right) - \kappa^3 C \left({}_0I_t^4 f^{(1)}(t) \right) + \dots \\
&= C - \kappa C \left({}_0I_t^2 \delta(t) \right) + \kappa^2 C \left({}_0I_t^3 \delta(t) \right) - \kappa^3 C \left({}_0I_t^4 \delta(t) \right) + \dots \\
&= C - \kappa C t + \kappa^2 C \left(\frac{t^2}{2!} \right) - \kappa^3 C \left(\frac{t^3}{3!} \right) + \dots \\
&= C \left(1 + \frac{(-\kappa)t}{1!} + \frac{(-\kappa)^2(t)^2}{2!} + \frac{(-\kappa)^3(t)^3}{3!} + \dots \right) \\
&= C \left(\sum_{n=0}^{\infty} \frac{(-\kappa t)^n}{n!} \right) = Ce^{-\kappa t}, \quad t \geq 0
\end{aligned} \quad (46)$$

In above (46) we have used ${}_0I_t^m \delta(t) = \frac{1}{(m-1)!} t^{m-1}$, $m = 1, 2, 3, \dots$; that is integration of delta-function [23]. In addition, we assumed $f(0) = 0$, thus we wrote ${}_0I_t^1 f^{(1)}(t) = f(t)$. That is in above (46) the relaxation output $y(t)$ to unit step input, $f(t) = u(t)$ to a system having the memory kernel as exponential decay function $h(t) \sim e^{-\kappa t}$ has relaxation as $y(t) \propto h(t)$.

We note that the Memory Kernel $h(t) \sim e^{-\kappa t}$ is non-singular function and has derivative every-where. Let us apply Laplace Transformation as depicted below

$$\begin{aligned}
y(t) &= h(t) * f^{(1)}(t) \\
\mathcal{L}\{y(t)\} &= \mathcal{L}\{(h(t)) * (f^{(1)}(t))\} \\
Y(s) &= (\mathcal{L}\{h(t)\})(\mathcal{L}\{f^{(1)}(t)\}), \quad \mathcal{L}\{h(t)\} = \mathcal{L}\{Ce^{-\kappa t}\} = C\left(\frac{1}{s+\kappa}\right) \\
&= \left(\frac{C}{s+\kappa}\right)(sF(s) - f(0)) \\
&= C\left(\left(\frac{s}{s+\kappa}\right)F(s) - \left(\frac{1}{s+\kappa}\right)f(0)\right), \quad f(0) = 0, \quad F(s) = \frac{1}{s} \\
Y(s) &= C\left(\frac{1}{s+\kappa}\right) \\
y(t) &= C\mathcal{L}^{-1}\left\{\frac{1}{s+\kappa}\right\} \\
&= Ce^{-\kappa t}
\end{aligned} \tag{47}$$

We get same result as earlier in (46).

4.5. Constitutive equation due to stretched exponential Non-Singular Memory Kernel

Here we take Memory Kernel as stretched exponential function

$$h(t) = Ce^{-(\kappa t)^\alpha}, \quad t \geq 0, \quad \kappa > 0; \quad 0 < \alpha < 1; \quad C > 0 \tag{48}$$

With $\alpha = 1$ the situation is same as for the case of pure exponential kernel for memory.

We now proceed in following steps

$$\begin{aligned}
y(t) &= h(t) * f^{(1)}(t) \\
&= C \int_0^t \left(e^{-(\kappa(t-\tau))^\alpha} \right) (f^{(1)}(\tau)) d\tau \\
&= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{((-\kappa(t-\tau))^\alpha)^n}{n!} \right) (f^{(1)}(\tau)) d\tau \\
&= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \kappa^{\alpha n}}{n!} \right) \int_0^t (t-\tau)^{\alpha n} f^{(1)}(\tau) d\tau \\
&= C \left(\sum_{n=0}^{\infty} \Gamma(\alpha n + 1) \left(\frac{(-1)^n \kappa^{\alpha n}}{n!} \right) \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} f^{(1)}(\tau) d\tau \right) \\
&= C \sum_{n=0}^{\infty} (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right) \left({}_0I_t^{\alpha n + 1} [f^{(1)}(t)] \right)
\end{aligned} \tag{49}$$

This gives the constitutive equation for $y(t)$ with series weighted sum of fractional integration of various orders of input $f^{(1)}(t)$; similar to the case with Mittag-Leffler function as Memory kernel, represented as following

$$y(t) = C \sum_{n=0}^{\infty} d_n \left({}_0I_t^{\alpha n + 1} [f^{(1)}(t)] \right); \quad d_n = (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right) \tag{50}$$

We note that with $\alpha=1$ we obtain the exact case for memory kernel with pure-exponential function.

5. Forming Constitutive Equation from evolution equation $y(t) = (h(t)) * (x(t))$ with $y(t) = x^{(1)}(t)$ for various memory-kernels $h(t)$

5.1. Memory kernel as delta function the zero memory case

We have the following evolution equation

$$y(t) = \int_0^t h(t-\tau)x(\tau)d\tau; \quad t \geq 0; \quad y(t) = x^{(1)}(t) \quad (51)$$

$$\frac{dx(t)}{dt} = \int_0^t h(t-\tau)x(\tau)d\tau$$

Take memory-kernel as $h(t) = C\delta(t)$, with C as real constant. Then we have

$$\frac{dx(t)}{dt} = \int_0^t h(t-\tau)x(\tau)d\tau \quad (52)$$

$$= \int_0^t C\delta(t-\tau)x(\tau)d\tau = Cx(t)$$

From above (52) we have a solution

$$x(t) = x(0)e^{Ct}; \quad t \geq 0 \quad (53)$$

We get classical expression of exponential growth for $C > 0$ and case of classical exponential decay with $C < 0$ for the above equation, i.e. $x(t) = x(0)e^{Ct}$. The case with memory kernel as delta function $h(t) = C\delta(t)$ gives classical growth or decay equation given as

$$\frac{dx(t)}{dt} = Cx(t) \quad (54)$$

i.e. rate of change of growing (or decaying) quantity is proportional to quantity itself. This is a classical first order linear homogeneous equation i.e. $x^{(1)}(t) - Cx(t) = 0$; solution of this is $x(t) = x(0)e^{Ct}$. The constituent equation for zero-memory case we got (54) as $x^{(1)}(t) = Cx(t)$, can be re-casted in following way (by integrating both sides)

$$x(t) - x(0) = C \int_0^t x(\tau)d\tau \quad (55)$$

$$x(t) - x(0) = C \left({}_0I_t^1 x(t) \right)$$

5.2. Memory kernel as singular power law function

Take memory kernel as $h(t) = Ct^{-\alpha}$ with $1 < \alpha \leq 2$, and C as real constant. Then we have following steps

$$\frac{dx(t)}{dt} = \int_0^t h(t-\tau)x(\tau)d\tau$$

$$= \int_0^t (C(t-\tau)^{-\alpha})(x(\tau))d\tau \quad (56)$$

$$= C(\Gamma(1-\alpha)) \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-\tau)^{-\alpha})(x(\tau))d\tau \right)$$

$$= C(\Gamma(1-\alpha)) \left({}_0I_t^{(1-\alpha)} x(t) \right)$$

We have used fractional integration formula i.e. ${}_0I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$, in above (56) steps. Integrate (56) once to write the following

$$x(t) - x(0) = C(\Gamma(1-\alpha)) \left({}_0I_t^{(2-\alpha)} x(t) \right) \quad (57)$$

Operate ${}_0D_t^{2-\alpha}$ i.e. fractional derivative operator (RL) on both sides to get the following

$$\begin{aligned} {}_0D_t^{(2-\alpha)} (x(t) - x(0)) &= C(\Gamma(1-\alpha)) \left({}_0D_t^{(2-\alpha)} {}_0I_t^{(2-\alpha)} x(t) \right) \\ {}_0D_t^{(2-\alpha)} x(t) - \frac{1}{\Gamma(\alpha-1)} x(0) &= C(\Gamma(1-\alpha)) x(t) \end{aligned} \quad (58)$$

We have used ${}_0D_t^\nu K = K \frac{t^{-\nu}}{\Gamma(1-\nu)}$ in above (58) steps. We can always write from above (58) i.e. ${}_0D_t^{(2-\alpha)} (x(t) - x(0)) = {}_0^c D_t^{(2-\alpha)} x(t)$ with condition $0 < (2-\alpha) < 1$. That is we used the relation of RL and Caputo derivatives for order less than unity, with non-zero initial value. This gives the condition $1 < \alpha < 2$. Thus for a memory-kernel as $h(t) = Ct^{-\alpha}$ with $1 < \alpha < 2$, we have constituent expression given by Caputo fractional derivative.

$${}_0^c D_t^{(2-\alpha)} x(t) = C(\Gamma(1-\alpha)) x(t) \quad (59)$$

For $0 < (2-\alpha) < 1$ we have solution to above as

$$\begin{aligned} {}_0^c D_t^{(2-\alpha)} x(t) &= C(\Gamma(1-\alpha)) x(t) \\ x(t) &= x(0) E_{2-\alpha} \left(C(\Gamma(1-\alpha)) t^{(2-\alpha)} \right) \end{aligned} \quad (60)$$

In (60) we have used One-parameter Mittag-Leffler function as eigen-function, for Caputo differential operator.

With $C < 0$, call it $C = -\lambda$, we write the law as

$$\begin{aligned} {}_0^c D_t^{(2-\alpha)} x(t) &= -\lambda (\Gamma(1-\alpha)) x(t) \\ x(t) &= x(0) E_{2-\alpha} \left(-\lambda (\Gamma(1-\alpha)) t^{(2-\alpha)} \right) \end{aligned} \quad (61)$$

With $\alpha = 1$ in (61) we get $x(t) \sim e^{-\lambda t}$ that is we have classical relaxation law as exponential decay $x^{(1)}(t) = -\lambda x(t)$ that we described with memory kernel as delta function (54). With $\alpha = 0$ we have memory kernel $h(t) = C$ which is constant function (i.e. memory kernel is constant) and relaxation function is sustained oscillations, i.e. $x(t) \sim \cos \sqrt{\lambda} t$, with constitutive equation as $x^{(2)}(t) = -\lambda x(t)$. Thus, we have a generalization from a zero memory case to a constant memory case; in $h(t) = Ct^{-\alpha}$ for $0 \leq \alpha \leq 2$.

We can write the constituent equation with memory-kernel as $h(t) = Ct^{-\alpha}$ as in terms of fractional integral as we have for zero-memory case i.e. $x(t) - x(0) = C \left({}_0I_t^1 x(t) \right)$ as follows

$$x(t) - x(0) = C(\Gamma(1-\alpha)) \left({}_0I_t^{(2-\alpha)} x(t) \right) \quad (62)$$

Let us take $\nu = 2 - \alpha$ and write $C(\Gamma(1-\alpha)) = C_\alpha$. With this we write Laplace transform of $x(t) - x(0) = C_\alpha \left({}_0I_t^\nu x(t) \right)$ as $X(s) - \frac{x(0)}{s} = C_\alpha \left(\frac{1}{s^\nu} X(s) \right)$. Rearranging this Laplace

transformed equation we get $X(s) = \frac{s^{\nu-1}}{s^{\nu}-C_{\alpha}} x(0)$. Taking inverse Laplace transform we get $x(t) = x(0)E_{\nu}(C_{\alpha}t^{\alpha})$. That is the same we wrote above as solution above (60).

5.3. Non singular power law memory-kernel

We take the case as $h(t) = C(1 + \nu t)^{-\alpha}$, with C as a real constant. That is a non-singular memory kernel. With this we write following steps to get constituent equation

$$\begin{aligned}
 \frac{dx(t)}{dt} &= h(t) * x(t) = \int_0^t h(t-\tau)x(\tau)d\tau \\
 &= \int_0^t C(1 + \nu(t-\tau))^{-\alpha} (x(\tau)) d\tau \\
 &= C \int_0^t \left(\binom{-\alpha}{0} (\nu(t-\tau))^0 + \binom{-\alpha}{1} (\nu(t-\tau)) + \binom{-\alpha}{2} (\nu(t-\tau))^2 + \dots \right) (x(\tau)) d\tau \\
 &= C \int_0^t \left(1 + (-\alpha)(\nu(t-\tau)) + \frac{(-\alpha)(-\alpha-1)}{2!} (\nu(t-\tau))^2 + \dots \right) (x(\tau)) d\tau \tag{63} \\
 &= C \left(\int_0^t (x(\tau)) + \frac{(-\alpha)}{1!} \int_0^t (\nu(t-\tau))(x(\tau)) d\tau \right. \\
 &\quad \left. + \frac{(-\alpha)(-\alpha-1)}{2!} \int_0^t (\nu(t-\tau))^2 (x(\tau)) d\tau \right. \\
 &\quad \left. + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t (\nu(t-\tau))^3 (x(\tau)) d\tau \dots \right) \\
 &= C \left({}_0I_t^1 x(t) + (-\alpha)\nu \left({}_0I_t^2 x(t) \right) + (-\alpha)(-\alpha-1)\nu^2 \left({}_0I_t^3 x(t) \right) + \dots \right)
 \end{aligned}$$

We have used ${}_0I_t^m [f(t)] = \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau$ in above derivation. We integrate once to get the following constituent relation

$$\begin{aligned}
 x(t) - x(0) &= C \left({}_0I_t^2 x(t) + (-\alpha)\nu \left({}_0I_t^3 x(t) \right) + (-\alpha)(-\alpha-1)\nu^2 \left({}_0I_t^4 x(t) \right) + \dots \right) \\
 x(t) - x(0) &= C \left(\sum_{n=0}^{\infty} a_n \left({}_0I_t^{n+2} x(t) \right) \right) \tag{64}
 \end{aligned}$$

$$a_0 = 1, \quad a_1 = -\alpha\nu, \quad a_2 = \alpha(\alpha+1)\nu^2 \dots$$

In contrary to the cases for constituent equations with memory kernel as singular type (55), (62), here in (64) we are having weighted sum series of multiple integrations of orders, 2, 3, 4... We note the expression (64) is difficult to solve for obtaining $x(t)$. Since there is no close form standard Laplace transform formula exists for $h(t) = C(1 + \nu t)^{-\alpha}$ we are unable to proceed with solution using Laplace transform technique to (63). We will use Laplace transform technique for Mittag-Leffler kernel and exponential kernel.

5.4. Memory Kernel non-singular Mittag-Leffler function

We take $h(t) = CE_{\alpha}(-\lambda t^{\alpha})$ with C and λ as positive constants, and write following steps

$$\begin{aligned}
y(t) &= (h(t)) * (x(t)) = \int_0^t h(t-\tau)x(\tau)d\tau, \quad y(t) = x^{(1)}(t) \\
\frac{dx(t)}{dt} &= C \int_0^t (E_\alpha(-\lambda(t-\tau)^\alpha))(x(\tau))d\tau \\
&= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n + 1)} \right) (x(\tau))d\tau \\
&= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^n}{\Gamma(\alpha n + 1)} \right) \int_0^t (t-\tau)^{\alpha n} x(\tau) d\tau \\
&= C \left(\sum_{n=0}^{\infty} (-1)^n \lambda^n \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} x(\tau) d\tau \right) \\
&= C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n + 1} [x(t)] \right)
\end{aligned} \tag{65}$$

We perform one whole integration on both sides to write the following constituent equation

$$x(t) - x(0) = C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n + 2} [x(t)] \right) \tag{66}$$

Here we see the constituent equation (66) is weighted infinite series sum of fractional integration of various orders. The (66) is difficult to solve for $x(t)$. However one can use Laplace transformation to (65) and write following

$$\begin{aligned}
\frac{dx(t)}{dt} &= CE_\alpha(-\lambda t^\alpha) * x(t) \\
sX(s) - x(0) &= C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) X(s) \\
x(t) &= x(0) \mathcal{L}^{-1} \left\{ \frac{s^\alpha + \lambda}{s^{\alpha+1} + \lambda s - C s^{\alpha-1}} \right\}
\end{aligned} \tag{67}$$

The inverse Laplace transform of (67) one can obtain via Contour Integration technique or Berberan Santos method [29], [38], [39]. That will give solutions as integral representation of $x(t)$.

5.5. Memory kernel as pure Exponential function

Take $h(t) = Ce^{-\kappa t}$ with C and κ as positive real constants, and write following steps

$$\begin{aligned}
y(t) &= (h(t)) * (x(t)); \quad h(t) = Ce^{-\kappa t}; \quad y(t) = x^{(1)}(t) \\
\frac{dx(t)}{dt} &= C \int_0^t (e^{-\kappa(t-\tau)}) (x(\tau)) d\tau \\
&= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\kappa(t-\tau))^n}{n!} \right) (x(\tau)) d\tau \\
&= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n (\kappa)^n}{n!} \right) \int_0^t (t-\tau)^n x(\tau) d\tau \\
&= C \left(\sum_{n=0}^{\infty} (-1)^n \kappa^n \right) \left(\frac{1}{n!} \int_0^t (t-\tau)^n x(\tau) d\tau \right) \\
&= C \sum_{n=0}^{\infty} (-1)^n \kappa^n \left({}_0I_t^{n+1} [x(t)] \right)
\end{aligned} \tag{68}$$

In above steps we used ${}_0I_t^m [f(t)] = \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau$. Integrate once both the sides to get constituent equation as

$$x(t) - x(0) = C \sum_{n=0}^{\infty} (-1)^n \kappa^n \left({}_0I_t^{n+2} [x(t)] \right) \tag{69}$$

The constituent equation (69) is difficult to solve where the formation is infinite series sum of multiple integer order integrals. However, one can apply Laplace transform to (68) and write solution as

$$\begin{aligned}
\frac{dx(t)}{dt} &= Ce^{-\kappa t} * x(t) \\
sX(s) - x(0) &= C \left(\frac{1}{s + \kappa} \right) X(s) \\
x(t) &= x(0) \mathcal{L}^{-1} \left\{ \frac{s + \kappa}{s^2 + \kappa s - C} \right\}
\end{aligned} \tag{70}$$

The (70) can be solved by usual partial fraction methods and with roots of $s^2 + \kappa s - C$. For example taking $C = -20$, $\kappa = 12$, doing partial fractions of (7) and then taking inverse Laplace transformation we get, $x(t) = x(0) \left(\frac{5}{4} e^{-2t} - \frac{1}{4} e^{-10t} \right)$.

5.6. Stretched Exponential function as memory-kernel

Take $h(t) = Ce^{-(\kappa t)^\alpha}$, $0 < \alpha < 1$ and with C , κ as positive constants, and write following steps

$$\begin{aligned}
y(t) &= (h(t)) * (x(t)), \quad h(t) = Ce^{-(\kappa t)^\alpha}, \quad y(t) = x^{(1)}(t) \\
\frac{dx(t)}{dt} &= C \int_0^t \left(e^{-(\kappa(t-\tau)^\alpha} \right) (x(\tau)) d\tau \\
&= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{\left((-\kappa(t-\tau)^\alpha)^n \right)}{n!} \right) (x(\tau)) d\tau \\
&= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \kappa^{\alpha n}}{n!} \right) \int_0^t (t-\tau)^{\alpha n} x(\tau) d\tau \\
&= C \left(\sum_{n=0}^{\infty} \Gamma(\alpha n + 1) \left(\frac{(-1)^n \kappa^{\alpha n}}{n!} \right) \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} x(\tau) d\tau \right) \\
&= C \sum_{n=0}^{\infty} (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right) \left({}_0 I_t^{\alpha n + 1} [x(t)] \right)
\end{aligned} \tag{71}$$

In above (71) we used the formula ${}_0 I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$, the RL formula of fractional integration. Integrating (71) once, we write the constitutive expression

$$x(t) - x(0) = C \sum_{n=0}^{\infty} (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right) \left({}_0 I_t^{\alpha n + 2} [x(t)] \right) \tag{72}$$

The (72) is difficult to solve for $x(t)$, we are getting constitutive equation as infinite series sum of weighted fractional integrations of various orders.

6.0. Discussions and Observations

6.1. Classical genesis of constituent equations for physical laws

We have discussed two types of laws, classically described by first order differentials. For zero memory case i.e. with memory kernel (singular) delta function, i.e. $h(t) = \delta(t)$ we get from the evolution equation i.e. $y(t) = h(t) * x(t)$ with a) $x(t) = f^{(1)}(t)$ b) $y(t) = x^{(1)}(t)$ the following constituent equations

$$\begin{aligned}
y &= C \frac{df}{dt} \\
\frac{dx}{dt} &= Cx
\end{aligned} \tag{73}$$

There is genesis to these fundamental equations (73) of physical laws-as described above.

The first one is Newton's second law of motion, relating force acting on a particle is proportional to rate of change of velocity i.e. $f = mv^{(1)}$. This is similar to our first constituent equation, i.e. $y = Cf^{(1)}$. These laws as $y = Cf^{(1)}$ are fundamental ones, derived from Hamilton's principle. Since Newton's time, classical mechanics has been elegantly reformulated as a single unifying principle known as Hamilton's principle. Following Feynman [30] and Landau [31], Hamilton's principle is frequently called "the principle of

least action." The action is defined as $S \equiv \int_{t_1}^{t_2} (\text{KE} - \text{PE})dt$. Where KE is Kinetic Energy and PE is potential energy. The action between time t_1 and t_2 is viewed as position of particle at time t_1 call it x_1 and position of particle at time t_2 call it x_2 . According to the least action formulation of classical mechanics, a particle moves along the path for which the action S is a minimum. The law $f = mv^{(1)}$ is got by minimizing the action S in a linearly varying potential field, with $f = -\frac{d(\text{PE})}{dx}$. This is second law of Newton's law of motion. The first law says that velocity is constant when potential field is constant or zero, and in that case, action is a straight line. We stress here the classical laws $y = Cf^{(1)}$ for zero-memory case is derived from Hamilton's principle of least action. The case of capacitor current proportional to rate of change of applied voltage, or inductor voltage proportional to rate of change of current through inductor or stress of a Newtonian viscous element proportional to rate of change of strain etc all are derived from Hamilton's principle of least action in a potential field. We are not discussing the variation calculus and Hamilton's principle. But what we observe that we get the constituent equation $y = Cf^{(1)}$ for a zero-memory case where memory kernel is $h(t) = C\delta(t)$ from evolution equation i.e. $y(t) = h(t)*x(t)$ with $x(t) = f^{(1)}(t)$.

Most of the physical laws extracted from experimental observations and are expressed mathematically in terms of differential equations. A simple example is when an observable y is a function of a single variable t and the experimental observations indicate a relationship between y and its first and/or higher differentials with respect to t . Such a relationship is a differential equation, and the solution of the equation gives the function y and determines how it varies with t . Suppose there is a fixed amount of radioactive isotope, which decays to a neighboring stable nucleus. The radioactive material is divided into batches of different mass and the number of decays from each batch measured over the same time interval. The data show that the numbers of decays are proportional to the mass of material used, whatever the time interval over which the measurements on the different samples were made, and show that as the time interval is varied the numbers of decays are proportional to the size of that interval. These observations suggest that the number of decays dN in an infinitesimal time interval dt is proportional to the number of radioactive atoms present and to the interval dt . That is $dN = -\lambda Ndt$ or $dN/N = -\lambda dt$ where λ is the proportionality constant. The minus sign is present because the decays dN represent a reduction in the number N of nuclei. This is the classical law $x^{(1)}(t) = Cx(t)$, that is zero-memory case with memory kernel as $h(t) = \delta(t)$ in our evolution equation i.e. $x^{(1)}(t) = h(t)*x(t)$ that we have dealt in general terms and analyzed with various memory kernels singular and non-singular thereafter.

6.2. Summary of analysis

The Table-1 gives the summary of the evolution equation $y(t) = h(t)*x(t)$ with $x(t) = f^{(1)}(t)$, studied for various memory kernels. Table-2 gives summary of the

evolution equation $y(t) = h(t) * x(t)$ with $y(t) = x^{(1)}(t)$ studied for various memory kernels.

S.No.	Function of Memory Kernel	Type	Memory Kernel Function $h(t)$	Constitutive Equation of $y(t) = h(t) * f^{(1)}(t)$ $n = 0, 1, 2, \dots; 0 < \alpha < 1$	Relaxation of $y(t)$ for unit step input of $f(t)$
1	Delta Function	Singular	$C\delta(t)$	$y(t) = Cf^{(1)}(t)$	$C\delta(t)$
2	Power Law	Singular	$Ct^{-\alpha}$ $0 < \alpha < 1$	$y(t) = Cf^{(\alpha)}(t)$	$Ct^{-\alpha}$
3	Non-singular Power Law	Non-Singular	$C(1 + vt)^{-\alpha}$	$y(t) = C \sum_{n=1}^{\infty} a_n ({}_0I_t^n f^{(1)}(t))$ $a_1 = 1, a_2 = -\alpha v,$ $a_3 = (\alpha)(\alpha + 1)v^2 \dots$	$C(1 + vt)^{-\alpha}$
4	Mittag-Leffler	Non-Singular	$CE_{\alpha}(-\lambda t^{\alpha})$	$y(t) = C \sum_{n=0}^{\infty} b_n ({}_0I_t^{\alpha n + 1} f^{(1)}(t))$ $b_0 = 1, b_1 = -\lambda; \dots$ $b_n = (-1)^n \lambda^n \dots$	$CE_{\alpha}(-\lambda t^{\alpha})$
5	Exponential	Non-Singular	$Ce^{-\kappa t}$	$y(t) = C \sum_{n=0}^{\infty} c_n ({}_0I_t^{n+1} [f^{(1)}(t)])$ $c_0 = 1, c_1 = -\kappa, c_3 = \kappa^2$ $\dots c_n = (-1)^n \kappa^n$	$Ce^{-\kappa t}$
6	Stretched-Exponential	Non-Singular	$Ce^{-(\kappa t)^{\alpha}}$	$y(t) = C \sum_{n=0}^{\infty} d_n ({}_0I_t^{\alpha n + 1} [f^{(1)}(t)])$ $d_n = (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right)$	$Ce^{-(\kappa t)^{\alpha}}$

Table-1: Summary of results of various singular and Non-singular Memory Kernels for evolution equation $y(t) = (h(t)) * (f^{(1)}(t))$

S.No.	Function of Memory Kernel	Type	Memory Kernel Function $h(t)$	Constitutive Equation of $x^{(1)}(t) = h(t) * x(t)$	Relaxation of $x(t)$ from initial value $x(0)$
1	Delta Function	Singular	$C\delta(t)$	$x^{(1)}(t) = Cx(t)$ $x(t) - x(0) = C({}_0I_t^1 x(t))$	$x(0)e^{Ct}$
2	Power Law	Singular	$Ct^{-\alpha}$ $1 < \alpha \leq 2$	${}_0^C D_t^\nu x(t) = C_\alpha x(t)$ $x(t) - x(0) = C_\alpha ({}_0I_t^\nu x(t))$ $C_\alpha = C(\Gamma(1-\alpha)), \quad \nu = 2 - \alpha$	$x(0)E_\nu(C_\alpha t^\nu)$
3	Non-singular Power Law	Non-Singular	$C(1+\nu t)^{-\alpha}$	$x(t) - x(0) = C \sum_{n=0}^{\infty} a_n ({}_0I_t^{n+2} x(t))$ $a_0 = 1, \quad a_1 = -\alpha\nu$ $a_2 = \alpha(\alpha+1)\nu^2 \dots$	X
4	Mittag-Leffler	Non-Singular	$CE_\alpha(-\lambda t^\alpha)$	$x(t) - x(0) = C \sum_{n=0}^{\infty} b_n ({}_0I_t^{\alpha n+2} [x(t)])$ $b_n = (-1)^n \lambda^n$	$x(0)\mathcal{L}^{-1}\left\{\frac{s^\alpha + \lambda}{s^{\alpha+1} + \lambda s - C s^{\alpha-1}}\right\}$
5	Exponential	Non-Singular	$Ce^{-\kappa t}$	$x(t) - x(0) = C \sum_{n=0}^{\infty} c_n ({}_0I_t^{n+2} [x(t)])$ $c_n = (-1)^n \kappa^n$	$x(0)\mathcal{L}^{-1}\left\{\frac{s+\kappa}{s^2 + \kappa s - C}\right\}$
6	Stretched-Exponential	Non-Singular	$Ce^{-(\kappa t)^\alpha}$	$x(t) - x(0) = C \sum_{n=0}^{\infty} d_n ({}_0I_t^{\alpha n+2} [x(t)])$ $d_n = (-1)^n \left(\frac{\kappa^{\alpha n} \Gamma(\alpha n+1)}{n!}\right)$	X

Table-2: Summary of results of various singular and Non-singular Memory Kernels for the evolution equation $x^{(1)}(t) = h(t) * x(t)$

6.3. Few experimental evidences

We have given few examples of Memory Kernel that gives constitutive expression for relaxation quantity $y(t)$, for evolution equation $y(t) = h(t) * f^{(1)}(t)$. The memory-less relaxation (zero-memory) is via Memory Kernel with delta function $h(t) = \delta(t)$, gives a classical constitutive formula, i.e. $y(t) \propto f^{(1)}(t)$. The memory kernel if formulated via a singular power-law kernel then we have a fractional derivative of Caputo type relating relaxation quantity $y(t)$ and impressed excitation $f(t)$, as $y(t) \propto f^{(\alpha)}(t)$. This power law memory kernel is singular in nature and non-differentiable at start. We make modification, and try to write a non-singular power-law memory kernel and derive its constitutive equation for relaxation quantity $y(t)$. We observe that here we get weighted sum of integrations of input excitation i.e. the rate of change of excitation quantity as $y(t) \propto \sum_{n=1}^{\infty} a_n ({}_0I_t^n f^{(1)}(t))$. This we get all together different from the singular kernels

results, for classical as well as fractional cases i.e. $y(t) \propto f^{(1)}(t)$ and $y(t) \propto f^{(\alpha)}(t)$ respectively. We extend this analysis with memory-kernel, which is Mittag-Leffler function. This kernel is non-singular at origin but the derivative at origin does not exist. With this, we get the constitutive equation as $y(t) \propto \sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+1} f^{(1)}(t) \right)$. We note that the structure of this expression is much away from that $y(t) \propto f^{(1)}(t)$ the classical and the $y(t) \propto f^{(\alpha)}(t)$ the fractional one. This comprises of series of fractional integration, may thus be mathematically fine but we may not be getting physical sense. Thereafter we take the memory kernel as pure exponential decay function, which is non-singular and everywhere differentiable function. With this, we construct a constitutive equation given as $y(t) \propto \sum_{n=0}^{\infty} c_n \left({}_0I_t^{\alpha n+1} \left[f^{(1)}(t) \right] \right)$ that is series sum of integer order repeated integrations- and is very much off from the capacitor dynamics classical case or fractional case i.e. $y(t) \propto f^{(1)}(t)$ or $y(t) \propto f^{(\alpha)}(t)$ respectively. We modify this non-singular memory kernel to a stretched exponential function. This function is non-singular. We get constitutive equation here that is weighted sum-series of the fractional integrations of input function i.e. rate of change of applied voltage given as $y(t) \propto \sum_{n=0}^{\infty} d_n \left({}_0I_t^{\alpha n+1} \left[f^{(1)}(t) \right] \right)$. The constitutive equation in stretched exponential case is similar to that with memory kernel as decaying Mittag-Leffler function.

However, mathematically it is fine, to have non-singular memory kernels yet physical applicability of the constitutive expressions obtained is questionable; presently because we are used to the classical law $y(t) \propto f^{(1)}(t)$ and fractional law $y(t) \propto f^{(\alpha)}(t)$ for some real physical systems. Presently we are unable to give interpretation to the weighted sum series of integrations and fractional integrations of rate of change of voltage that appears for capacitor dynamics when we take non-singular memory kernel.

We note that though classical textbook capacitors currents $i(t)$ to applied voltage $v(t)$ is expressed as in $i(t) \propto v^{(1)}(t)$, yet in reality they have power-law decay current, when excited by a step-voltage for an uncharged capacitor. This is well established by Curie-von Schweidler law the current relaxation is $i(t) \sim t^{-\alpha}$; $0 < \alpha < 1$ [1]-[4], [28], when a constant step voltage is applied to an uncharged capacitor. Therefore, the memory-kernel associated with relaxation dynamics is $h(t) \sim t^{-\alpha}$; that is singular power-law function, is applicable. Here the fractional derivative appears in constitutive expression i.e. $i(t) \propto v^{(\alpha)}(t)$; $0 < \alpha < 1$, [5]-[12], [17]-[20] [22], [28].

In Rheology studies, there is a visco-elastic element, described as $\sigma \propto \varepsilon^{(\alpha)}$, where σ is stress function of time and ε as strain function with α as fractional order of derivative (Caputo type) $0 < \alpha < 1$. This fractional dynamic is similar to the fractional capacitor case as described above, i.e. $i \propto v^{(\alpha)}$. From linear viscoelasticity theory, the relaxation modulus $g(t)$ is obtained by property $\tilde{\sigma}(s) = sG(s)\tilde{\varepsilon}(s)$. This expression is written in Laplace transformed domain. Where $G(s) = \mathcal{L}\{g(t)\}$, $\tilde{\sigma}(s) = \mathcal{L}\{\sigma(t)\}$ and $\varepsilon(s) = \mathcal{L}\{\tilde{\varepsilon}(t)\}$. This is Scott-Blair model, and gives a ‘relaxation modulus’ $g(t) \propto t^{-\alpha}$, that is the stress function to unit step strain function [32], [33]. This constitutive equation $\sigma(t) \propto \varepsilon^{(\alpha)}(t)$ i.e. Scott-Blair model also thus fits into the evolution equation,

$y(t) = h(t) * f^{(1)}(t)$ with $y(t) \equiv \sigma(t)$ and $f(t) \equiv \varepsilon(t)$, having singular power-law memory kernel; i.e. $h(t) \propto t^{-\alpha}$. The classical Newtonian viscous element is $\sigma \propto \varepsilon^{(1)}$ and pure spring element is $\sigma \propto \varepsilon$. Thus, Scott-Blair model is in-between pure-spring and pure viscous element. The relaxation modulus of pure Newtonian viscous element is $g(t) \propto \delta(t)$ i.e. delta function. The pure Newtonian element is zero memory case with $h(t) \propto \delta(t)$. The use of fractional derivative in Rheological studies for non-Newtonian fluids is found in, [34]-[37].

The second physical phenomenon that is of growth and decay was using evolution equation $x^{(1)}(t) = h(t) * x(t)$, with various memory kernels, (summarized in Table-2). This gives very interesting observations like, use of singular kernels like zero-memory case gives exponential relaxation of $x(t)$ observed for pure radioactive decay or pure neutron population growth, that is $x(t) \sim e^{Ct}$. The singular power law kernel gives a constituent law i.e. $x^{(\alpha)}(t) \sim x(t)$ with relaxation function as $x(t) = E_{2-\alpha}(C_\alpha t^{2-\alpha})$. We note that both the singular memory-kernels give a relaxation function $x(t)$ that is non-singular. The observation is similar to the earlier case when the memory kernel is non-singular; giving constituent equation for decay or growth phenomena as weighted sum series of integer order or fractional order integrals. These constituent equations are difficult to solve to get relaxation function $x(t)$. The classical growth or decay is exponential function, we say that is Debye type, the anomalous relaxation is non-Debye type can be modeled by various non-exponential relaxation functions [38]-[41].

Conclusions

We address a very relevant question that is ‘if we have in reality a singular memory kernel or a non-singular memory kernel’, for system dynamics for process evolution that is governed by convolution integral. This study shows if we are having a singular memory kernel, then we observe the reality better. That is we obtain constituent expressions that are either expressed with classical derivative, or via use of fractional derivative. Though mathematically non-singular memory kernels are possible, yet the constitutive equation for system relaxation dynamics does not give the useful information, results in constituent equations that are weighted infinite series sums of multiple integer order integrals or fractional order integrals. This maybe because we are unable presently assigns physical sense to mathematically correct constitutive equations for system relaxation, due to non-singular memory kernel. Therefore, may we not ask that is singular memory kernel be the order of Nature? If so then presently we are unable to give physical insight into reasons of singularity of memory kernels that we showed to have given reality in constituent equations for physical laws that we described.

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