Singular vs. Non-Singular Memorized Relaxation for basic Relaxation Current of Capacitor

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Abstract

We have constructed the basic capacitor expression for relaxing current and applied voltage as convolution operation of the chosen memory kernel and rate of change of applied voltage. We have studied types of memory kernels singular and non-singular. With these, we form constitutive equations for capacitor dynamics. We conclude that though mathematically we can use non-singular kernel yet this does not give presently much useful practical or physically realizable results and interpretations. May be we are not able to interpret these constitutive expressions of capacitor relaxation with non-singular memory kernel. Therefore, we have a question, does natural relaxation dynamics for dielectrics have a singular memory kernel, and the relaxation current function is singular in nature? Is it the singular relaxation function for capacitor dynamics with singular memory kernel remains universal law for dielectric relaxation? However, we are not questioning researchers modeling relaxation of dielectric via non-singular functions, yet we are hinting about complexity of basic constituent equation of capacitor dynamics thus obtained via considering non-singular relaxations.

Keywords

Convolution, Memory Kernel, Fractional Derivative, Fractional Integration, power-law, Mittag-Leffler function, stretched exponential function

Introduction

In this paper, we give simple mathematical treatment to derive the dielectric relaxation laws (or constitutive equations) of capacitor with several types of memory kernels to a relaxation law that we formulate as convolution integral $\mathrm{i}(t) \propto k(t) * \mathrm{v}^{(1)}(t)$. The function $k(t)$ is the memory kernel, $\mathrm{v}(t)$ is the applied voltage stress function, with $\mathrm{v}^{(1)}(t)$ as its first derivative and $\mathrm{i}(t)$ as relaxing current. Here in this discussion we take the memory kernel $k(t)$ as singular and non-singular functions like delta function, power-law decay function, Mittag-Leffler function and exponential decay function, and arrive at the constitutive relations of voltage and currents of a capacitor. The empirically and experimentally derived law called 'universal dielectric relaxation law' also called as

Curie-von Schweidler law, is observed since late 19th century; [1]-[4], [28]. This is classical power law for current decay i.e. $\mathrm{i}(t)\propto t^{-\alpha}$; $0<\alpha<1$. Here relaxation of current is inverse of power of time for a constant step-voltage excitation to uncharged capacitor. This universal power law relaxation is described by a singular function. This empirical Curie-von Schewidler relaxation law is used to derive fractional differential equations describing constituent expression for capacitor i.e. $\mathrm{i}(t)\propto {}_0D_t^{\alpha}\mathrm{v}(t)$, that is 'fractional capacitor', [5]-[12], [17]-[20]. Where ${}_0D_t^{\alpha}$ is fractional differentiation operation. With $\alpha=1$, we have classical capacitor i.e. $\mathrm{i}(t)\propto D_t^1\mathrm{v}(t)$ or $\mathrm{i}(t)\propto \mathrm{v}^{(1)}(t)$, described with classical one-whole differentiation i.e. D_t^1. Here we will show for classical case, the memory kernel is delta-function, or the relaxing system is with 'no-memory'. We will derive this law i.e. $\mathrm{i}(t)\propto {}_0D_t^{\alpha}\mathrm{v}(t)$ with memory kernel as singular kernel of power-law type, i.e. $k(t)\propto t^{-\alpha}$ .We will also show that if the memory kernel were of nonsingular functions then the constitutive equations of current and voltage of those capacitors are too complicated and does not give physical sense of interpretability, though mathematically doable. With this described method, we can derive various constitutive equations for various other types of memory kernels.

Impulse Response of a system-a review

The output call it $\mathrm{y}(t)$ a variable in time ($t\in\mathbb{R}$), of a system represented by function of time variable $\mathrm{h}(t)$ to an input variable call it $\mathrm{x}(t)$ acting at time $t=0$, is given by evolution equation [5], [6], [25] as follows

$$\mathrm{y}(t)=\int_0^t \mathrm{h}(t-\tau)\mathrm{x}(\tau)\mathrm{d}\tau;\quad t\geq 0 \tag{1}$$

If we take Laplace transform, with $\mathcal{L}\{\mathrm{x}(t)\}=\mathrm{X}(s)$, $\mathcal{L}\{\mathrm{y}(t)\}=\mathrm{Y}(s)$ and $\mathcal{L}\{\mathrm{h}(t)\}=\mathrm{H}(s)$, we get

$$\mathrm{Y}(s)=\left(\mathrm{H}(s)\right)\left(\mathrm{X}(s)\right) \tag{2}$$

Where variable s is Complex variable, i.e. $s\in\mathbb{C}$. The input $\mathrm{x}(t)$ in case is delta-function (an impulse) at $t=0$; we have $\mathrm{y}(t)=\mathrm{h}(t)$. This $\mathrm{h}(t)$ is called 'impulse response' of the system [6], [25]. The evolution equation (1) of $\mathrm{y}(t)$ in time domain is convolution integral.

We have several physical systems that are proportional to rate of change of some other physical quantity that is acting as input. Say we have rate of change of a quantity call it $\mathrm{g}(t)$ represented by first time derivative i.e. $\mathrm{g}^{(1)}(t)$, then our input variable in (1) is $\mathrm{x}(t)=\mathrm{g}^{(1)}(t)$; then we have evolution equation in terms of impulse response function of the system, $\mathrm{h}(t)$ as

$$\mathrm{y}(t)=\int_0^t \left(\mathrm{h}(t-\tau)\right)\left(\mathrm{g}^{(1)}(\tau)\right)\mathrm{d}\tau \tag{3}$$

Some physical systems can be casted as (1) and (3). For example, current through a capacitor classically related to voltage given as $\mathrm{i}(t)=\mathrm{C}\mathrm{v}^{(1)}(t)$; velocity function to rate of displacement as $\mathrm{v}(t)=\mathrm{x}^{(1)}(t)$, force on a mass to rate of change of velocity as

$f(t) = mv^{(1)}(t)$ and stress related to rate of change in Newtonian viscous element as $\sigma(t) = \eta\varepsilon^{(1)}(t)$.

Looking at the time evolution equation (1), if the input variable $x(t)$ acts only at time $t = 0$ thereafter vanishes at $t > 0$ and we observe $y(t)$ even at $t > 0$ while ($x(t) = 0$ for $t > 0$); we may term that system is remembering its past input. In that case, we say system relaxes with 'memory' [6], [22], [27]. In ideal cases as described by the constitutive equations for capacitor $i(t)$, velocity function $v(t)$, force function $f(t)$ and stress $\sigma(t)$ behave 'without memory'. It can be seen when the 'rate terms' input in the RHS of these constitutive equations vanishes after application at $t = 0$, we have no observation of the output at $t > 0$. Simply if the rate terms in the RHS of all these constitutive equations is described by delta function, then the output is also delta function at $t = 0$.

The convolution integral (1) can in general have lower terminal of integration as $t = -\infty$ or $a > -\infty$ as the case may be for application of input $x(t)$; that we depict as follows

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)\,d\tau; \quad y(t) = \int_{a}^{t} h(t-\tau)x(\tau)\,d\tau \tag{4}$$

Preliminaries of Fractional Calculus

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration [6], [26], [29] of order $\upsilon \in \mathbb{R}^{+}$ is defined as

$$ {}_0I_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)\,d\tau \tag{5}$$

Where $\Gamma(\upsilon)$ is Euler's Gamma function, is generalization of factorial function [23], [29], we have $\Gamma(\upsilon) = (\upsilon - 1)!$. The formula (5) is appearing as generalization of Cauchy's multiple integration formula of m fold integration [6], [26], [29] where $m \in \mathbb{N}$ given as follows

$$ {}_0I_t^{m}\left[f(t)\right] = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)\,d\tau; \quad m = 1, 2, 3, ... \tag{6}$$

The fractional derivative of order β for $0 < \beta < 1$ by Riemann-Liouville (RL) formula [6], [26], [29] is

$$ {}_0D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta} f(\tau)\,d\tau; \quad 0 < \beta < 1 \tag{7}$$

The (7) is fractionally integrating the function by order $(1-\beta)$ by formula (5) and then followed by one-whole differentiation. There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative [6], [26], [29] for fractional order $0 < \beta < 1$ is given as

$$ {}_0^{C}D_t^{\beta}\left[f(t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta} f^{(1)}(\tau)\,d\tau; \quad 0 < \beta < 1 \tag{8}$$

Thus for (8) we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order β, by formula (5). The Caputo and Riemann-Liouville (RL) fractional derivative are related [6], [26], [29] by

$$ {}_0D_t^{\beta}\left[f(t)\right] = {}_0^CD_t^{\beta}\left[f(t)\right] + \frac{f(0)}{\Gamma(1-\beta)}t^{-\beta}; \quad 0<\beta<1 \tag{9}$$

We mention that both the fractional derivatives are equal when initial value is zero i.e. $f(0)=0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^{\beta}\left[\mathrm{K}\right] = \frac{\mathrm{K}}{\Gamma(1-\beta)}t^{-\beta}$. Whereas the Caputo's fractional derivative of a constant is zero, i.e. ${}_0^CD_t^{\beta}\left[\mathrm{K}\right]=0$, [6], [26], [29].

The fractional integration and fractional differentiation of delta function [6], [26], [29] is as follows

$$ {}_0I_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}; \quad {}_0D_t^{\upsilon}\delta(t) = \frac{1}{\Gamma(-\upsilon)}t^{-\upsilon-1}, \qquad 0<\upsilon<1 \tag{10}$$

Fractional derivative and fractional integration of power function $f(t)=\mathrm{K}t^p$ [6], [26], [29] is

$$ {}_0I_t^{\upsilon}\mathrm{K}t^p = \mathrm{K}\frac{\Gamma(p+1)}{\Gamma(p+\upsilon+1)}t^{p+\upsilon}, \qquad {}_0D_t^{\upsilon}\mathrm{K}t^p = \mathrm{K}\frac{\Gamma(p+1)}{\Gamma(p-\upsilon+1)}t^{p-\upsilon}, \quad p>-1 \tag{11}$$

Classical Dielectric Relaxation case of Capacitor

The response current flows through a capacitor if and only if there is rate of change in the applied voltage across it. That is classically we have relaxation current $\mathrm{i}(t)$ as $\mathrm{i}(t)\propto \mathrm{v}^{(1)}(t)$. The classical Capacitor expression relating time function of current through capacitor to voltage stress applied is following

$$ \mathrm{i}(t) = \mathrm{C}\frac{\mathrm{dv}(t)}{\mathrm{d}t} = \mathrm{C}\left(\mathrm{v}^{(1)}(t)\right) \tag{12}$$

We can modify the above expression i.e. $\mathrm{i}(t)=\mathrm{C}\left(\mathrm{v}^{(1)}(t)\right)$ or $\mathrm{i}(t)=\mathrm{C}\left(D_t^1\mathrm{v}(t)\right)$ and write

$$ \begin{aligned} \mathrm{i}(t) &= \mathrm{C}\int_{-\infty}^{t}\left(\delta(t-\tau)\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau \\ &= \int_{-\infty}^{t}\left(\mathrm{C}\delta(t-\tau)\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau = \left(\mathrm{C}\delta(t)\right)*\left(\mathrm{v}^{(1)}(t)\right) \end{aligned} \tag{13}$$

This comes from property of delta function, i.e. $\int\delta(x-y)f(y)\mathrm{d}y = f(x)$ [23]. In (13) for the convolution integral, we have kernel of integration as delta function call it $k(t)=\mathrm{C}\delta(t)$. With this we get

$$ \mathrm{i}(t) = \left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right) \tag{14}$$

The expression (14) we have casted as (1) and (3). The kernel $k(t)$ we will now term as Memory Kernel.

Let us give a unit voltage step input, $\mathrm{v}(t)=u(t)$ applied at $t=0$. This means $\mathrm{v}(t)=1$ for $t\geq 0$ to an uncharged capacitor i.e. $\mathrm{v}(t)=0$ for $t<0$; then we have $v^{(1)}(t)=\delta(t)$ i.e. differentiation of unit-step input. Placing this value in (13), we get

$$
\begin{aligned}
\mathrm{i}(t) &= \int_0^t \left(\mathrm{C}\delta(t-\tau)\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau = \mathrm{C}\int_0^t \left(\delta(t-\tau)\delta(\tau)\right)\mathrm{d}\tau \\
&= \mathrm{C}\delta(t)
\end{aligned}
\tag{15}
$$

This (15) is direct result of (12); as the differentiation of unit step function is delta-function. The Laplace transformed relations of (12) is

$$
\mathrm{I}(s) = \mathrm{C}\left(s\mathrm{V}(s) - \mathrm{v}(0)\right) \tag{16}
$$

Doing Laplace transform of (14), we get

$$
\begin{aligned}
\mathcal{L}\left\{\mathrm{i}(t)\right\} &= \mathcal{L}\left\{\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\right\} \\
\mathrm{I}(s) &= \mathcal{L}\left\{k(t)\right\}\mathcal{L}\left\{\mathrm{v}^{(1)}(t)\right\} \\
&= \mathrm{C}\left(K(s)\right)\left(s\mathrm{V}(s)-\mathrm{v}(0)\right), \quad K(s) = \mathcal{L}\left\{k(t)\right\} = \mathcal{L}\left\{\mathrm{C}\delta(t)\right\} = \mathrm{C} \\
&= \mathrm{C}\left(s\mathrm{V}(s)-\mathrm{v}(0)\right)
\end{aligned}
\tag{17}
$$

We get the same result as (16).

From the classical theory with Newtonian Calculus as the constitutive equation of capacitor (12) we get a delta impulse current uncharged capacitor is impressed with a constant step voltage. This is a singular relaxation current function.

Constitutive equation of Classical capacitor with Memory Kernel-a zero memory case

From the classical law we have arrived at the equation, which is following

$$
\begin{aligned}
\mathrm{i}(t) &= \left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right) \\
&= \int_{-\infty}^{t} \left(k(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau
\end{aligned}
\tag{18}
$$

It so happens that the classical capacitor equation (12) is associated with Memory Kernel $k(t) = \mathrm{C}\delta(t)$.

This physically implies that the system (12) has zero-memory. That is just after the instance of application of voltage stress i.e. at $t = 0^{+}$ the memory kernel vanishes i.e. $k(t) = 0$ for $t > 0$. Whereas at $k(t) = \infty$ only at $t = 0$; and is singular function. This is a 'singular memory kernel'. Now we will study relaxation of currents to unit step input of capacitors for various kernels-singular and non-singular.

Constitutive equation of Capacitor due to Singular Power Law Memory Kernel

Let us have the power law decay kernel described as

$$
k(t) = \mathrm{A}t^{-\alpha}; \quad 0 < \alpha < 1 \tag{19}
$$

In (19) A is a positive constant. The (19) is singular at origin with its derivative as minus infinity. This means that we have memory kernel $k(t) = \infty$ at $t = 0$ and monotonically decaying after that i.e. $t > 0$, with $k^{(1)}(t)\Big|_{t=0} = -\infty$ .However, we say that this kernel (19) is a singular kernel. This is some way mimicking the actual memory or forgetfulness. That is as the time goes the memory fades away.

With this we have following steps

$$\mathrm{i}(t)=\Big(\big(k(t)\big)*\big(\mathrm{v}^{(1)}(t)\big)\Big)$$

$$\mathcal{L}\{\mathrm{i}(t)\}=\mathcal{L}\Big\{\Big(\big(k(t)\big)*\big(\mathrm{v}^{(1)}(t)\big)\Big)\Big\}$$

$$\begin{aligned}\mathrm{I}(s)&=\mathrm{C}\big(\mathcal{L}\{k(t)\}\big)\big(\mathcal{L}\{\mathrm{v}^{(1)}(t)\}\big),\quad \mathcal{L}\{k(t)\}=\mathcal{L}\{\mathrm{A}t^{-\alpha}\}=\mathrm{A}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\\&=\left(\mathrm{A}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\big(s\mathrm{V}(s)-\mathrm{v}(0)\big)\\&=\mathrm{A}\big(\Gamma(1-\alpha)\big)\big(s^{\alpha}\mathrm{V}(s)-s^{\alpha-1}\mathrm{v}(0)\big),\quad \mathrm{v}(0)=0,\qquad \mathrm{V}(s)=\frac{1}{s}\\\mathrm{I}(s)&=\mathrm{A}\left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\end{aligned}\tag{20}$$

From above (20) we get $\mathrm{i}(t)=\mathcal{L}^{-1}\{\mathrm{I}(s)\}$ as

$$\mathrm{i}(t)=\mathrm{A}\big(\Gamma(1-\alpha)\big)t^{-\alpha};\quad 0<\alpha<1\tag{21}$$

Therefore, we are getting a power law current decay $\mathrm{i}(t)\sim t^{-\alpha}$ for the memory kernel in constitutive equation as a power law $k(t)\sim t^{-\alpha}$; for an uncharged capacitor stressed with unit step voltage. This (21) law i.e. singular function, is also called Universal Dielectric relaxation law, observed since late 19th century [1]-[4], [28]. This is termed as power-law [13]-[16], [27], [28].

Now we obtain constitutive relation for capacitor with memory kernel that is singular and has no derivative at start point i.e. (19). We write

$$\begin{aligned}\mathrm{i}(t)&=\Big(\big(k(t)\big)*\big(\mathrm{v}^{(1)}(t)\big)\Big)\\&=\int_{-\infty}^{t}\big(k(t-\tau)\big)\big(\mathrm{v}^{(1)}(\tau)\big)\mathrm{d}\tau;\quad k(t)=\mathrm{A}t^{-\alpha};\quad t\ge 0\\&=\int_{0}^{t}\big(\mathrm{A}(t-\tau)^{-\alpha}\big)\big(\mathrm{v}^{(1)}(\tau)\big)\mathrm{d}\tau\\&=\mathrm{A}\big(\Gamma(1-\alpha)\big)\left(\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\big((t-\tau)^{-\alpha}\big)\big(\mathrm{v}^{(1)}(\tau)\big)\mathrm{d}\tau\right)\\&=\mathrm{A}\big(\Gamma(1-\alpha)\big)\big({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\big)\end{aligned}\tag{22}$$

In (22) we have used the definition of Caputo fractional derivative [6], [26], [29] for fractional order $0<\alpha<1$ i.e. ${}_{0}^{C}D_{t}^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\big((t-\tau)^{-\alpha}\big)\big(f^{(1)}(\tau)\big)\mathrm{d}\tau$ (8).

Thus, our constitutive equation for a capacitor having Singular Power Law Memory kernel is given by fractional differential equation (22), and is changed from classical capacitor case (12), i.e.

$$\begin{aligned}\mathrm{i}(t)&=\mathrm{A}\big(\Gamma(1-\alpha)\big)\big({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\big);\quad \mathrm{C}_{\alpha}=\mathrm{A}\big(\Gamma(1-\alpha)\big)\\\mathrm{i}(t)&=\mathrm{C}_{\alpha}\big({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\big);\quad 0<\alpha<1\end{aligned}\tag{23}$$

In (23) putting $\alpha=1$ we get classical relation (12). The expression (23) is obtained and is used in [5], [7]-[12], [17]-[20].

The Laplace Transform of Caputo Fractional Derivative [6], [26], [29] for fractional order $0<\alpha<1$ is $\mathcal{L}\left\{{}_0^CD_t^\alpha f(t)\right\}=s^\alpha F(s)-s^{\alpha-1}f(0)$. Using this we write Laplace Transform of (23) as

$$\begin{aligned}&\mathcal{L}\left\{\mathrm{i}(t)\right\}=\mathrm{C}_\alpha\mathcal{L}\left\{{}_0^CD_t^\alpha \mathrm{v}(t)\right\};\quad 0<\alpha<1\\&\mathrm{I}(s)=\mathrm{C}_\alpha\left(s^\alpha\mathrm{V}(s)-s^{\alpha-1}\mathrm{v}(0)\right)\end{aligned}\tag{24}$$

We note that in (24) putting $\alpha=1$ we obtain the classical result i.e. (17).

We verify the relaxation current with $\mathrm{V}(s)=s^{-1}$ and $\mathrm{v}(0)=0$ i.e. for unit step input $\mathrm{v}(t)=u(t);\quad t\geq 0$, applied to initially uncharged capacitor with $\mathrm{v}(0)=0,\quad t<0$ from (24)

$$\begin{aligned}&\mathrm{I}(s)=\mathrm{C}_\alpha\left(s^\alpha\mathrm{V}(s)-s^{\alpha-1}\mathrm{v}(0)\right);\quad \mathrm{V}(s)=\frac{1}{s},\quad \mathrm{v}(0)=0\\&\mathrm{I}(s)=\mathrm{C}_\alpha s^{\alpha-1},\quad \mathcal{L}\left\{t^p\right\}=\frac{\Gamma(p+1)}{s^{p+1}}\\&\mathrm{i}(t)=\mathcal{L}^{-1}\left\{\mathrm{I}(s)\right\}=\mathcal{L}^{-1}\left\{\mathrm{C}_\alpha s^{\alpha-1}\right\}\\&\qquad=\frac{\mathrm{C}_\alpha}{\Gamma(1-\alpha)}t^{-\alpha},\quad 0<\alpha<1,\qquad t\geq 0\end{aligned}\tag{25}$$

With $\mathrm{C}_\alpha=\mathrm{A}\left(\Gamma(1-\alpha)\right)$ (23), we get from (25) $\mathrm{i}(t)=\mathrm{CA}\left(\Gamma(1-\alpha)\right)t^{-\alpha}$; same as we got in (21).

Difference between zero-memory and memory based relaxation cases of Capacitor

We observed that for a classical case the relaxation current $\mathrm{i}(t)$ is delta function at the start of application of voltage, which is unit step function. Therefore, as soon as the rate of change of voltage vanishes at $t>0$ we have relaxing current as zero. This is zero memory case with memory kernel as $k(t)\propto\delta(t)$. Where we observe from (25) with a power-law memory kernel as $k(t)\propto t^{-\alpha}$ ($0<\alpha<1$), we have a finite current even the rate of change of voltage vanished at $t>0$. Therefore, the capacitor is memorizing the excitation that once took place as a rate of change in voltage and capacitor is relaxing with memory. Well this (25) was the case with singular power law memory kernel. Now in subsequent sections we will discuss non-singular memory kernels and see the constitutive relations that we get for capacitor expression.

Constitutive equation for Capacitor due to Non-Singular power law Memory Kernel

We have seen earlier that the kernel of singular power-law i.e. $k(t)\propto t^{-\alpha}$; $0<\alpha<1$ gives a constitutive equation with fractional derivative i.e. $\mathrm{i}(t)\propto\mathrm{v}^{(\alpha)}(t)$. We modify the power-law to a non-singular type with following type

$$k(t)=\mathrm{P}(1+\upsilon t)^{-\alpha};\quad 0<\alpha<1,\quad \upsilon>0\tag{26}$$

In (26) P is a positive constant. In (26) we have $k(0)=\mathrm{P}$ and $k^{(1)}(0)=-\mathrm{P}\alpha$, unlike singular kernel (19). With this we do following calculations for obtaining constitutive equation

$$
\begin{aligned}
\mathrm{i}(t) &= \left(\left(k(t)\right) * \left(\mathrm{v}^{(1)}(t)\right)\right) \\
&= \int_0^t \left(\mathrm{P}(1+\upsilon(t-\tau))\right)^{-\alpha} \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \\
&= \mathrm{P}\int_0^t \left(\binom{-\alpha}{0} \left(\upsilon(t-\tau)\right)^0 + \binom{-\alpha}{1} \left(\upsilon(t-\tau)\right) + \binom{-\alpha}{2} \left(\upsilon(t-\tau)\right)^2 + \ldots \right) \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \\
&= \mathrm{P}\int_0^t \left(1 + (-\alpha)\left(\upsilon(t-\tau)\right) + \tfrac{(-\alpha)(-\alpha-1)}{2!}\left(\upsilon(t-\tau)\right)^2 + \ldots\right) \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \\
&= \mathrm{P}\left(\begin{array}{l} \int_0^t \left(\mathrm{v}^{(1)}(\tau)\right) + \tfrac{(-\alpha)}{1!} \int_0^t \left(\upsilon(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \\ \quad + \tfrac{(-\alpha)(-\alpha-1)}{2!} \int_0^t \left(\upsilon(t-\tau)\right)^2 \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \\ \quad + \tfrac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t \left(\upsilon(t-\tau)\right)^3 \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \ldots\ldots\ldots\ldots\ldots \end{array} \right)
\end{aligned}
\tag{27}
$$

In (27) we used binomial expansion [23] for $(1+z)^{-\alpha}$. We use repeated integration formula (6) i.e. ${}_0I_t^m f(t) = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1} f(\tau)\mathrm{d}\tau$, [6], [26], [29] and get the following expression

$$
\begin{aligned}
\mathrm{i}(t) &= \mathrm{P}\left(\begin{array}{l} \int_0^t \left(\mathrm{v}^{(1)}(\tau)\right) + \tfrac{(-\alpha)}{1!} \int_0^t \left(\upsilon(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \ldots \\ \ldots. + \tfrac{(-\alpha)(-\alpha-1)}{2!} \int_0^t \left(\upsilon(t-\tau)\right)^2 \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau + \tfrac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t \left(\upsilon(t-\tau)\right)^3 \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau .. \end{array} \right) \\
&= \mathrm{P}\left(\begin{array}{l} \int_0^t \left(\mathrm{v}^{(1)}(\tau)\right) + (-\alpha)\upsilon \left(\tfrac{1}{(2-1)!} \int_0^t \left(t-\tau\right)^{2-1} \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \right) + \\ (-\alpha)(-\alpha-1)\upsilon^2 \left(\tfrac{1}{(3-1)!} \int_0^t \left(t-\tau\right)^{3-1} \left(\mathrm{v}^{(1)}(\tau)\right) \mathrm{d}\tau \right) + \ldots \end{array} \right) \\
&= \mathrm{P}\left({}_0I_t^1 \mathrm{v}^{(1)}(t) + (-\alpha)\upsilon \left({}_0I_t^2 \mathrm{v}^{(1)}(t) \right) + (-\alpha)(-\alpha-1)\upsilon^2 \left({}_0I_t^3 \mathrm{v}^{(1)}(t) \right) + \ldots\ldots\ldots. \right)
\end{aligned}
\tag{28}
$$

It so happens that for this kernel (26), which is non-singular power-law the constitutive equation is for $\mathrm{i}(t)$ is weighted sum of integrals (one whole, two whole, three whole; and so on) of $\mathrm{v}^{(1)}(t)$.

With excitation $\mathrm{v}^{(1)}(t) = \delta(t)$ to an uncharged capacitor, that is applying a unit step input we obtain following from (28)

$$
\begin{aligned}
\mathrm{i}(t) &= \mathrm{P}\left({}_0I_t^1 \mathrm{v}^{(1)}(t) + (-\alpha)\upsilon \left({}_0I_t^2 \mathrm{v}^{(1)}(t) \right) + (-\alpha)(-\alpha-1)\upsilon^2 \left({}_0I_t^3 \mathrm{v}^{(1)}(t) \right) + \ldots\ldots\ldots. \right) \\
&= \mathrm{P}\left({}_0I_t^1 \delta(t) + (-\alpha)\upsilon \left({}_0I_t^2 \delta(t) \right) + (-\alpha)(-\alpha-1)\upsilon^2 \left({}_0I_t^3 \delta(t) \right) + \ldots\ldots\ldots. \right) \\
&= \mathrm{P}\left(1 - \alpha\upsilon t^2 + \alpha(\alpha+1)\upsilon^2 \left(\tfrac{t^2}{2}\right) - \alpha(\alpha+1)(\alpha+2)\left(\tfrac{t^3}{(3)(2)}\right) + \ldots\ldots \right) \\
&= \mathrm{P}(1+\upsilon t)^{-\alpha}
\end{aligned}
\tag{29}
$$

The (29) says that the current lingers in a capacitor while the rate of change of voltage vanishes at $t > 0$; giving memorized relaxation of current. This was also observed with singular power law memory kernel. However, the constitutive equation for capacitor in this case is

$$\mathrm{i}(t) = \mathrm{P}\left({}_0I_t^1 \mathrm{v}^{(1)}(t) + (-\alpha)\upsilon \left({}_0I_t^2 \mathrm{v}^{(1)}(t) \right) + (-\alpha)(-\alpha-1)\upsilon^2 \left({}_0I_t^3 \mathrm{v}^{(1)}(t) \right) + ... \right) \tag{30}$$

This is very different from (12) and (23); the classical case and case with fractional derivative respectively. Here in (30) we are getting a series sum of weighted repeated integration

$$\mathrm{i}(t) \propto \sum_{n=1}^{\infty} a_n \left({}_0I_t^n \mathrm{v}^{(1)}(t) \right) \tag{31}$$

With weights in (31) as $a_1 = 1$, $a_2 = -\alpha\upsilon$, $a_3 = (\alpha)(\alpha+1)\upsilon^2 \ldots$

Constitutive equation of Capacitor due to Mittag-Leffler function as Non-Singular Memory Kernel

Here we take Memory Kernel as following for $t \geq 0$

$$k(t) = \mathrm{B}E_\alpha(-\lambda t^\alpha); \quad 0 < \alpha < 1 \tag{32}$$

In (32) B and λ are a positive real constants. Where the Mittag-Leffler function is defined [6], [23], [26], [29] as following

$$E_\alpha(-\lambda t^\alpha) = \sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+1)}, \quad t \geq 0; \quad \lambda t^\alpha \in \mathbb{C}, \quad \alpha \in \mathbb{C}, \quad \mathrm{Re}\left[\alpha\right] > 0 \tag{33}$$

The constitutive equation with Memory Kernel (32) we write the following

$$\begin{aligned}
\mathrm{i}(t) &= \left(\left(k(t)\right) * \left(\mathrm{v}^{(1)}(t)\right) \right) \\
&= \mathrm{B}\int_0^t \left(E_\alpha(-\lambda(t-\tau)^\alpha) \right) \left(\mathrm{v}^{(1)}(\tau) \right) \mathrm{d}\tau \\
&= \mathrm{B}\int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n+1)} \right) \left(\mathrm{v}^{(1)}(\tau) \right) \mathrm{d}\tau \\
&= \mathrm{B}\sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^n}{\Gamma(\alpha n+1)} \right) \int_0^t (t-\tau)^{\alpha n} \mathrm{v}^{(1)}(\tau)\, \mathrm{d}\tau \\
&= \mathrm{B}\left(\sum_{n=0}^{\infty} (-1)^n \lambda^n \right) \left(\frac{1}{\Gamma(\alpha n+1)} \int_0^t (t-\tau)^{\alpha n} \mathrm{v}^{(1)}(\tau)\, \mathrm{d}\tau \right) \\
&= \mathrm{B}\sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n+1} \left[\mathrm{v}^{(1)}(t) \right] \right)
\end{aligned} \tag{34}$$

Where in (34) we used the operator ${}_0I_t^\upsilon, \upsilon = \alpha n + 1$, which is Riemann-Liouville fractional integration (5) of order υ defined as ${}_0I_t^\upsilon\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1} f(\tau)\mathrm{d}\tau$. We write (34) as series sum of weighted fractional integration

$$\begin{aligned}
\mathrm{i}(t) &= \mathrm{B}\sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n+1} \left[\mathrm{v}^{(1)}(t) \right] \right) \\
&= \mathrm{B}\left({}_0I_t^1 \mathrm{v}^{(1)}(t) - \lambda \left({}_0I_t^{\alpha+1} \mathrm{v}^{(1)}(t) \right) + \lambda^2 \left({}_0I_t^{2\alpha+1} \mathrm{v}^{(1)}(t) \right) - \lambda^3 \left({}_0I_t^{3\alpha+1} \mathrm{v}^{(1)}(t) \right) + \ldots \right) \\
&= \mathrm{B}\sum_{n=0}^{\infty} b_n \left({}_0I_t^{\alpha n+1} \mathrm{v}^{(1)}(t) \right)
\end{aligned} \tag{35}$$

With weights in this case as $b_0 = 1$, $b_1 = -\lambda$;… $b_n = (-1)^n \lambda^n$ …..We get similar result that of (31) and this too is very different from singular kernels of (12) and (23).
Thus, a Memory Kernel (32) $k(t) = \mathrm{B} - \frac{\lambda \mathrm{B} t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda^2 \mathrm{B} t^{2\alpha}}{\Gamma(2\alpha+1)} - ...;\quad 0 < \alpha < 1$ i.e. series-sum of power laws acting on derivative of voltage function $\mathrm{v}^{(1)}(t)$, gives a relaxing current $\mathrm{i}(t)$ with series sum of fractional integrations of various orders acting on rate of change of voltage (35).
We note that Memory Kernel in this case (32) is not singular function at $k(0) = \mathrm{B}$, and its derivative is not defined i.e. $\left. k^{(1)}(t) \right|_{t=0} = -\infty$.
Now we give a unit step input to this system so we have $\mathrm{v}(t) = 1,\quad t \geq 0$; with $\mathrm{v}^{(1)}(t) = \delta(t)$. Placing this in (35), we write the following

$$\begin{aligned}
\mathrm{i}(t) &= \mathrm{B}\mathrm{v}(t) - \lambda\,\mathrm{B}\left({}_0 I_t^{\alpha+1} \mathrm{v}^{(1)}(t)\right) + \lambda^2 \mathrm{B}\left({}_0 I_t^{2\alpha+1} \mathrm{v}^{(1)}(t)\right) - \lambda^3 \mathrm{B}\left({}_0 I_t^{3\alpha+1} \mathrm{v}^{(1)}(t)\right) + ... \\
&= \mathrm{B} - \lambda\,\mathrm{B}\left({}_0 I_t^{\alpha+1} \delta(t)\right) + \lambda^2 \mathrm{B}\left({}_0 I_t^{2\alpha+1} \delta(t)\right) - \lambda^3 \mathrm{B}\left({}_0 I_t^{3\alpha+1} \delta(t)\right) + ... \\
&= \mathrm{B} - \lambda \mathrm{B}\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) + \lambda^2 \mathrm{B}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) - \lambda^3 \mathrm{B}\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right) + ... \qquad (36) \\
&= \mathrm{B}\left(1 + \frac{(-\lambda) t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(-\lambda)^2 (t^{\alpha})^2}{\Gamma(2\alpha+1)} + \frac{(-\lambda)^3 (t^{\alpha})^3}{\Gamma(3\alpha+1)} + ...\right) \\
&= \mathrm{B}\left(\sum_{n=0}^{\infty} \frac{(-\lambda t^{\alpha})^n}{\Gamma(\alpha n+1)}\right) = \mathrm{B} E_{\alpha}(-\lambda t^{\alpha}), \quad t \geq 0
\end{aligned}$$

In (36) we have used formula for fractional integration of delta function i.e. ${}_0 I_t^{\upsilon} \delta(t) = \frac{1}{\Gamma(\upsilon)} t^{\upsilon-1}$ (11). What we observe that the relaxation current $\mathrm{i}(t)$ to uncharged capacitor excited by unit-step voltage input; relaxes in proportional to the memory kernel function i.e. $\mathrm{i}(t) \propto k(t)$ in this case $k(t) \sim E_{\alpha}(-\lambda t^{\alpha})$. Here in (36) the current relaxes at $t > 0$ even while the rate of change of voltage has vanished; therefore memorizing the past excitation.We note that by placing $\alpha = 1$ we are not getting classical case (1).
Let us do Laplace Transformation for Mittag-Leffler memory kernel, as depicted as follows

$$
\begin{aligned}
&\mathrm{i}(t)=\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&\mathcal{L}\left\{\mathrm{i}(t)\right\}=\mathcal{L}\left\{\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\right\}\\
&\mathrm{I}(s)=\left(\mathcal{L}\left\{k(t)\right\}\right)\left(\mathcal{L}\left\{\mathrm{v}^{(1)}(t)\right\}\right),\quad \mathcal{L}\left\{k(t)\right\}=\mathcal{L}\left\{\mathrm{B}E_\alpha(-\lambda t^\alpha)\right\}=\mathrm{B}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\\
&\qquad=\mathrm{B}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\left(s\mathrm{V}(s)-\mathrm{v}(0)\right)\\
&\qquad=\mathrm{B}\left(\left(\tfrac{s^\alpha}{s^\alpha+\lambda}\right)\mathrm{V}(s)-\left(\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\mathrm{v}(0)\right),\quad \mathrm{v}(0)=0,\qquad \mathrm{V}(s)=\frac{1}{s}\\
&\mathrm{I}(s)=\mathrm{B}\left(\frac{s^{\alpha-1}}{s^\alpha+\lambda}\right)\\
&\mathrm{i}(t)=\mathrm{B}\mathcal{L}^{-1}\left\{\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\right\}\\
&\qquad=\mathrm{B}E_\alpha(-\lambda t^\alpha)
\end{aligned}
\tag{37}
$$

The same that we got in (36).

Constitutive equation of Capacitor due to exponential function as Non-Singular Memory Kernel

Here we take Memory kernel as

$$
k(t)=\mathrm{M}\mathrm{e}^{-\kappa t},\quad t\ge 0,\quad \kappa>0;\quad \mathrm{M}>0 \tag{38}
$$

The constitutive equation with Memory Kernel as (37) gives the following

$$
\begin{aligned}
\mathrm{i}(t)&=\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&=\mathrm{M}\int_0^t\left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{M}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{M}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t(t-\tau)^n\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\\
&=\mathrm{M}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t(t-\tau)^n\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\
&=\mathrm{M}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)
\end{aligned}
\tag{39}
$$

Thus, the memory Kernel which is pure exponential function (37) gives a relaxation current which is weighted series sum of integer order multiple integration of voltage function; like in (31) we write the following

$$\begin{aligned}
\mathrm{i}(t) &= \mathrm{M}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right) \\
&= \mathrm{M}\left({}_0I_t^1\mathrm{v}^{(1)}(t)\right)-\kappa\,\mathrm{M}\left({}_0I_t^2\mathrm{v}^{(1)}(t)\right)+\kappa^2\mathrm{CM}\left({}_0I_t^3\mathrm{v}^{(1)}(t)\right)-\lambda^3\mathrm{CM}\left({}_0I_t^4\mathrm{v}^{(1)}(t)\right)+\ldots \\
&= \mathrm{M}\sum\nolimits_{n=0}^{\infty}c_n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)
\end{aligned} \tag{40}$$

$c_0 = 1, \quad c_1 = -\kappa, \quad c_3 = \kappa^2, \quad \ldots c_n = (-1)^n\kappa^n$

In (36) the constitutive equation with memory kernel as non-singular Mittag-Leffler function, if we place $\alpha = 1$ we get (24).

We give a step input to system having Memory kernel (38) and observe the following

$$\begin{aligned}
\mathrm{i}(t) &= \mathrm{Mv}(t)-\kappa\,\mathrm{M}\left({}_0I_t^2\mathrm{v}^{(1)}(t)\right)+\kappa^2\mathrm{M}\left({}_0I_t^3\mathrm{v}^{(1)}(t)\right)-\kappa^3\mathrm{M}\left({}_0I_t^4\mathrm{v}^{(1)}(t)\right)+\ldots \\
&= \mathrm{M}-\kappa\,\mathrm{M}\left({}_0I_t^2\delta(t)\right)+\kappa^2\mathrm{M}\left({}_0I_t^3\delta(t)\right)-\kappa^3\mathrm{M}\left({}_0I_t^4\delta(t)\right)+\ldots \\
&= \mathrm{M}-\kappa\mathrm{M}t+\kappa^2\mathrm{M}\left(\frac{t^2}{2!}\right)-\kappa^3\mathrm{M}\left(\frac{t^3}{3!}\right)+\ldots \\
&= \mathrm{M}\left(1+\frac{(-\kappa)t}{1!}+\frac{(-\kappa)^2(t)^2}{2!}+\frac{(-\kappa)^3(t)^3}{3!}+\ldots.\right) \\
&= \mathrm{M}\left(\sum_{n=0}^{\infty}\frac{(-\kappa t)^n}{n!}\right) = \mathrm{Me}^{-\kappa t}, \quad t \geq 0
\end{aligned} \tag{41}$$

In (41) we have used ${}_0I_t^m\delta(t) = \frac{1}{(m-1)!}t^{m-1}, \quad m = 1,2,3,\ldots.$; that is integration of delta-function [23]. In addition, we assumed $\mathrm{v}(0) = 0$, that is voltage stress applied to uncharged capacitor, thus we wrote ${}_0I_t^1\,\mathrm{v}^{(1)}(\mathrm{t}) = \mathrm{v}(t)$ in the (41).

That is in (41) the relaxation current to unit step voltage input to an uncharged capacitor having the memory kernel as exponential decay function (38), $k(t) \sim \mathrm{e}^{-\kappa t}$ has relaxation current $\mathrm{i}(t) \propto k(t)$.

We note that the Memory Kernel (38) is non-singular function and has derivative everywhere. Let us apply Laplace Transformation as depicted below

$$
\begin{aligned}
&\mathrm{i}(t)=\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&\mathcal{L}\left\{\mathrm{i}(t)\right\}=\mathcal{L}\left\{\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\right\}\\
&\mathrm{I}(s)=\left(\mathcal{L}\left\{k(t)\right\}\right)\left(\mathcal{L}\left\{\mathrm{v}^{(1)}(t)\right\}\right),\quad \mathcal{L}\left\{k(t)\right\}=\mathcal{L}\left\{\mathrm{M}\mathrm{e}^{-\kappa t}\right\}=\mathrm{M}\left(\frac{1}{s+\kappa}\right)\\
&\qquad=\left(\frac{\mathrm{M}}{s+\kappa}\right)\left(s\mathrm{V}(s)-\mathrm{v}(0)\right)\\
&\qquad=\mathrm{M}\left(\left(\tfrac{s}{s+\kappa}\right)\mathrm{V}(s)-\left(\tfrac{1}{s+\kappa}\right)\mathrm{v}(0)\right),\quad \mathrm{v}(0)=0,\qquad \mathrm{V}(s)=\frac{1}{s}\\
&\mathrm{I}(s)=\mathrm{M}\left(\frac{1}{s+\kappa}\right)\\
&\mathrm{i}(t)=\mathrm{M}\mathcal{L}^{-1}\left\{\tfrac{1}{s+\kappa}\right\}\\
&\qquad=\mathrm{M}\mathrm{e}^{-\kappa t}
\end{aligned}
\tag{42}
$$

We get same result of (41).

Constitutive equation of Capacitor due to stretched exponential Non-Singular Memory Kernel

Here we take Memory Kernel as stretched exponential function

$$
k(t)=\mathrm{N}\mathrm{e}^{-(\omega t)^{\alpha}}\ ,\quad t\geq 0,\quad \omega>0;\quad 0<\alpha<1;\quad \mathrm{N}>0 \tag{43}
$$

With $\alpha=1$ the situation is same as for the case of pure exponential kernel for memory. We now proceed in following steps

$$
\begin{aligned}
\mathrm{i}(t)&=\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&=\mathrm{N}\int_0^t\left(\mathrm{e}^{-(\omega(t-\tau))^{\alpha}}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{N}\int_0^t\left(\sum_{n=0}^{\infty}\frac{\left((-\omega(t-\tau))^{\alpha}\right)^n}{n!}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{N}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\omega^{\alpha n}}{n!}\right)\int_0^t(t-\tau)^{\alpha n}\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\\
&=\mathrm{N}\left(\sum_{n=0}^{\infty}\Gamma(\alpha n+1)\left(\frac{(-1)^n\omega^{\alpha n}}{n!}\right)\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t(t-\tau)^{\alpha n}\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\
&=\mathrm{N}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\omega^{\alpha n}\Gamma(\alpha n+1)}{n!}\right)\left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)
\end{aligned}
\tag{44}
$$

This gives the constitutive equation for $\mathrm{i}(t)$ with series weighted sum of fractional integration of various orders of input $\mathrm{v}^{(1)}(t)$; similar to the case with Mittag-Leffler function as Memory kernel.

$$\mathrm{i}(t)=\mathrm{N}\sum_{n=0}^{\infty}d_n\left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right);\quad d_n=(-1)^n\left(\tfrac{\omega^{\alpha n}\Gamma(\alpha n+1)}{n!}\right) \tag{45}$$

We note that with $\alpha=1$ we obtain the exact case for memory kernel with exponential function (41).

Summary

We summarize the results of our study in Table-1

S.No.	Function of Memory Kernel	Type	Memory Kernel Function $k(t)$	Constitutive Equation of Capacitor	Relaxation current to unit step voltage $\mathrm{i}(t)$
1	Delta Function	Singular	$\mathrm{C}\delta(t)$	$\mathrm{i}(t)=\mathrm{Cv}^{(1)}(t)$	$\mathrm{C}\delta(t)$
2	Power Law	Singular	$\mathrm{C}t^{-\alpha}$ $0<\alpha<1$	$\mathrm{i}(t)=\mathrm{Cv}^{(\alpha)}(t)$	$\mathrm{C}t^{-\alpha}$
3	Non-singular Power Law	Non-Singular	$\mathrm{C}(1+\upsilon t)^{-\alpha}$	$\mathrm{i}(t)=\mathrm{C}\sum_{n=1}^{\infty}a_n\left({}_0I_t^{n}\mathrm{v}^{(1)}(t)\right)$	$\mathrm{C}(1+\upsilon t)^{-\alpha}$
4	Mittag-Leffler	Non-Singular	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$	$\mathrm{i}(t)=\mathrm{C}\sum_{n=0}^{\infty}b_n\left({}_0I_t^{\alpha n+1}\mathrm{v}^{(1)}(t)\right)$	$\mathrm{C}E_\alpha(-\lambda t^\alpha)$
5	Exponential	Non-Singular	$\mathrm{Ce}^{-\kappa t}$	$\mathrm{i}(t)=\mathrm{C}\sum_{n=0}^{\infty}c_n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)$	$\mathrm{Ce}^{-\kappa t}$
6	Stretched-Exponential	Non-Singular	$\mathrm{Ce}^{-(\omega t)^\alpha}$	$\mathrm{i}(t)=\mathrm{C}\sum_{n=0}^{\infty}d_n\left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)$	$\mathrm{Ce}^{-(\omega t)^\alpha}$

Table-1: Summary of results of various singular and Non-singular Memory Kernels

Discussions and Observations

We have given few examples of Memory Kernel that gives constitutive expression for relaxation current to capacitor. The memory-less relaxation is via Memory Kernel with delta function, gives a classical constitutive formula of capacitor, i.e. (12). The memory kernel if formulated via a singular power-law kernel (19), then we have a fractional derivative of Caputo type relating relaxation current and impressed voltage (23) for a capacitor. This power law memory kernel is singular in nature and non-differentiable at start. We make modification, and try to write a non-singular power-law memory kernel, (26), and derive its constitutive equation for capacitor current. We observe that here we get weighted sum of integrations of input excitation i.e. the rate of change of voltage. This we get all together different from the singular kernels results, for classical as well as fractional cases (12) and (23) respectively. We extend this analysis with memory-kernel, which is Mittag-Leffler function (32). This kernel (32) is non-singular at origin but the derivative at origin does not exist. With this, we get the constitutive equation as depicted in (35). We note that the structure of this expression is much away from that (12) the classical and the (23) the fractional one. This comprises of series of fractional integration, may thus be mathematically fine but we may not be getting physical sense. Thereafter we take the memory kernel as pure exponential decay function (38), which is non-singular and everywhere differentiable function. With this, we construct a constitutive equation for capacitor (40) that is series sum of integer order repeated integrations-and is very much off from the capacitor dynamics classical case or fractional case i.e. (12) or (23) respectively. We modify this non-singular memory kernel to a stretched exponential function (43). This function is non-singular and everywhere differentiable. We get constitutive equation here that is weighted sum-series of the fractional integrations of input function i.e. rate of change of applied voltage (44). The constitutive equation in stretched exponential case is similar to that with memory kernel as decaying Mittag-Leffler function (32).

However, mathematically it is fine, to have non-singular memory kernels yet physical applicability of the constitutive expressions obtained is questionable; presently because we are used to the classical law (12) and fractional law (23) for a capacitor dynamics. Presently we are unable to give interpretation to the weighted sum series of integrations and fractional integrations of rate of change of voltage that appears for capacitor dynamics when we take non-singular memory kernel.

We note that though classical textbook capacitors are expressed as in (12), yet in reality they have power-law decay current, when excited by a step-voltage for an uncharged capacitor. This is well established by Curie-von Schweidler law the current relaxation is $\mathrm{i}(t)\sim t^{-\alpha};\quad 0<\alpha<1$ [1]-[4], [28]. Therefore, the memory-kernel associated with relaxation dynamics is $k(t)\sim t^{-\alpha}$; that is singular power-law function. Here the fractional derivative appears in constitutive expression i.e. $\mathrm{i}(t)\propto \mathrm{v}^{(\alpha)}(t);\quad 0<\alpha<1$. In this short analysis, we observe that whatever be the nature of Memory-Kernel, (say delta-function, singular power-law function, non-singular power law function, Mittag-Leffler function, classical exponential function, or stretched exponential function)-the same is the nature of relaxation current for a step input voltage excitation to an uncharged capacitor.

Conclusions

A very relevant question that is: if we have in reality a singular memory kernel or a non-singular memory kernel, for capacitor relaxation dynamics. This study shows if we are having a singular memory kernel, then we observe the reality better. With singular kernels, we see that the constituent expression of capacitor for current is first derivative of voltage or fractional derivative of voltage applied. Though mathematically non-singular memory kernel is possible, yet the constitutive equation for capacitor relaxation dynamics does not give the useful information. The universal dielectric relaxation law of curie-von Schweidler still holds with singular power-law memory kernel and not via non-singular memory kernels. This maybe because we are unable presently assigns physical sense to mathematically correct constitutive equations for capacitor relaxation, due to non-singular memory kernel. Here we are not questioning researchers modeling relaxation of dielectric via non-singular functions, yet we are hinting about validity of basic constituent equation of capacitor dynamics thus obtained via considering non-singular relaxations, that is weighted series sum of several orders (fractional or integer order) integrals operating on rate of change of voltage.

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