About Singular and Non-Singular Memory Kernel in Relaxation Dynamics of Capacitors

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Abstract

We have constructed the basic capacitor expression for relaxing current and applied voltage as convolution operation of the chosen memory kernel and rate of change of applied voltage. We have studied four types of memory kernels singular and non-singular. With these, we form constitutive equations for capacitor. We conclude that though mathematically we can use non-singular kernel yet this does not give much useful practical or physically realizable results and interpretations. Therefore we have a question, does natural relaxation dynamics for dielectrics have singular memory kernel-is it universal.

Keywords

Convolution, Memory Kernel, Fractional Derivative, Fractional Integration, power-law, Mittag-Leffler function

Introduction

In this paper, we give simple mathematical treatment to derive the dielectric relaxation laws of capacitor with several types of memory kernels to a relaxation law that we have formulated as convolution integral $\mathrm{i}(t) \propto k(t) * \mathrm{v}^{(1)}(t)$. The function $k(t)$ is the memory kernel, $\mathrm{v}(t)$ is the applied voltage stress function, with $\mathrm{v}^{(1)}(t)$ as its first derivative and $\mathrm{i}(t)$ as relaxing current. Here in this discussion we take the memory kernel $k(t)$ as singular and non-singular functions like delta function, power-law decay function, Mittag-Leffler function and exponential decay function, and arrive at the constitutive relations of voltage and currents of a capacitor. The empirically and experimentally derived law called 'universal dielectric relaxation law' also called as Curie-von Schweidler law, is observed since late 19th century. This is classical power law for current decay i.e. $\mathrm{i}(t) \propto t^{-\alpha}; \quad 0 < \alpha < 1$. Here relaxation of current is inverse of power of time for a constant step-voltage excitation to uncharged capacitor. This empirical Curie-von Schewidler relaxation law is used to derive fractional differential equations describing constituent expression for capacitor i.e. $\mathrm{i}(t) \propto {}_0^C D_t^{\alpha} \mathrm{v}(t)$, that is 'fractional capacitor'. Where ${}_0^C D_t^{\alpha}$ is fractional differentiation operation. With $\alpha = 1$, we have classical capacitor i.e. $\mathrm{i}(t) \propto D_t^1 \mathrm{v}(t)$ or $\mathrm{i}(t) \propto \mathrm{v}^{(1)}(t)$, described with classical one-whole differentiation i.e. D_t^1. Here we will show for classical case, the memory kernel is delta-function, or the relaxing system is with no-memory. We will derive this law i.e. $\mathrm{i}(t) \propto {}_0^C D_t^{\alpha} \mathrm{v}(t)$ with memory kernel as singular kernel of power-law type. We will also show that if the memory kernel were of nonsingular functions then the constitutive equations of current and voltage of those capacitors are too complicated and does not give physical sense of interpretability, though mathematically doable. With this described method, we can derive various constitutive equations for various other types of memory kernels.

Classical Dielectric Relaxation case

The classical Capacitor expression relating time function of current through capacitor to voltage stress applied is following

$$\mathrm{i}(t) = \mathrm{C}\frac{\mathrm{dv}(t)}{\mathrm{d}t} = \mathrm{C}\left(\mathrm{v}^{(1)}(t)\right) \tag{1}$$

We can modify the above expression i.e. $\mathrm{i}(t) = \mathrm{C}\left(\mathrm{v}^{(1)}(t)\right)$ or $\mathrm{i}(t) = \mathrm{C}\left(D_t^1 \mathrm{v}(t)\right)$ and write

$$\mathrm{i}(t) = \mathrm{C}\int_{-\infty}^{t}\left(\delta(t-\tau)\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau \tag{2}$$

This comes from property of delta function, i.e. $\int \delta(x-y) f(y)\mathrm{d}y = f(x)$. In (2) for the convolution integral, we have kernel of integration as delta function call it $k(t) = \delta(t)$. With this we get

$$\mathrm{i}(t) = \mathrm{C}\left(\left(k(t)\right) * \left(\mathrm{v}^{(1)}(t)\right)\right) \tag{3}$$

Let us give a unit voltage step input, $\mathrm{v}(t) = u(t)$ applied at $t = 0$. This means $\mathrm{v}(t) = 1$ for $t \geq 0$ to an uncharged capacitor i.e. $\mathrm{v}(t) = 0$ for $t < 0$; then we have $v^{(1)}(t) = \delta(t)$ i.e. differentiation of unit-step input. Placing this value in (2), we get

$$\begin{aligned} \mathrm{i}(t) &= \mathrm{C}\int_{0}^{t}\left(\delta(t-\tau)\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau = \mathrm{C}\int_{0}^{t}\left(\delta(t-\tau)\delta(\tau)\right)\mathrm{d}\tau \\ &= \mathrm{C}\delta(t) \end{aligned} \tag{4}$$

This (4) is direct result of (1). The Laplace transformed relations of (1) is

$$\mathrm{I}(s) = \mathrm{C}\left(s\mathrm{V}(s) - \mathrm{v}(0)\right) \tag{5}$$

Doing Laplace transform of (3), we get

$$\begin{aligned} \mathcal{L}\left\{\mathrm{i}(t)\right\} &= \mathcal{L}\left\{\mathrm{C}\left(\left(k(t)\right) * \left(\mathrm{v}^{(1)}(t)\right)\right)\right\} \\ \mathrm{I}(s) &= \mathrm{C}\mathcal{L}\left\{k(t)\right\}\mathcal{L}\left\{\mathrm{v}^{(1)}(t)\right\} \\ &= \mathrm{C}\left(K(s)\right)\left(s\mathrm{V}(s) - \mathrm{v}(0)\right), \quad K(s) = \mathcal{L}\left\{k(t)\right\} = \mathcal{L}\left\{\delta(t)\right\} = 1 \\ &= \mathrm{C}\left(s\mathrm{V}(s) - \mathrm{v}(0)\right) \end{aligned} \tag{6}$$

We get the same result as (5).

From the classical theory with Newtonian Calculus as the constitutive equation of capacitor (1) we get a delta impulse current uncharged capacitor is impressed with a constant step voltage.

Constitutive equation of Classical capacitor with Memory Kernel-a zero memory case

From the classical law we have arrived at the equation, which is following

$$\begin{aligned} \mathrm{i}(t) &= \mathrm{C}\left(\left(k(t)\right) * \left(\mathrm{v}^{(1)}(t)\right)\right) \\ &= \mathrm{C}\int_{-\infty}^{t}\left(k(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{7}$$

It so happens that the classical capacitor equation (1) is associated with Memory Kernel $k(t) = \delta(t)$.

This physically implies that the system (1) has zero-memory. That is just after the instance of application of voltage stress i.e. at $t = 0^{+}$ the memory kernel vanishes i.e. $k(t) = 0$ for $t > 0$. Whereas at $k(t) = \infty$ only at $t = 0$. This is a singular kernel. Now we will study relaxation of currents to unit step input of capacitors for various kernels-singular and non-singular kernels.

Constitutive equation due to Power Law Decay Memory Kernel

Let us have the power law decay kernel described as

$$k(t) = \mathrm{A}t^{-\alpha}; \quad 0 < \alpha < 1 \tag{8}$$

In (8) A is a constant. The (8) is singular at origin with its derivative as minus infinity. This means that we have memory kernel $k(t) = \infty$ at $t = 0$ and monotonically decaying after that i.e. $t > 0$. However, we say that this kernel (8) is also a singular kernel. This is some way mimicking the actual memory or forgetfulness. That is as the time goes the memory fades away. With this we have following

$$\begin{aligned}
&\mathrm{i}(t) = \mathrm{C}\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&\mathcal{L}\left\{\mathrm{i}(t)\right\} = \mathcal{L}\left\{\mathrm{C}\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\right\}\\
&\mathrm{I}(s) = \mathrm{C}\left(\mathcal{L}\left\{k(t)\right\}\right)\left(\mathcal{L}\left\{\mathrm{v}^{(1)}(t)\right\}\right), \quad \mathcal{L}\left\{k(t)\right\} = \mathcal{L}\left\{\mathrm{A}t^{-\alpha}\right\} = \mathrm{A}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\\
&\qquad = \mathrm{C}\left(\mathrm{A}\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\right)\left(s\mathrm{V}(s) - \mathrm{v}(0)\right)\\
&\qquad = \mathrm{CA}\left(\Gamma(1-\alpha)\right)\left(s^{\alpha}\mathrm{V}(s) - s^{\alpha-1}\mathrm{v}(0)\right), \quad \mathrm{v}(0) = 0, \qquad \mathrm{V}(s) = \frac{1}{s}\\
&\mathrm{I}(s) = \mathrm{CA}\left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}\right)
\end{aligned} \tag{9}$$

From above (9) we get $\mathrm{i}(t) = \mathcal{L}^{-1}\left\{\mathrm{I}(s)\right\}$ as

$$\mathrm{i}(t) = \mathrm{CA}\left(\Gamma(1-\alpha)\right)t^{-\alpha}; \quad 0 < \alpha < 1 \tag{10}$$

Therefore, we are getting a power law current decay $\mathrm{i}(t) \sim t^{-\alpha}$ for the memory kernel in constitutive equation as a power law $k(t) \sim t^{-\alpha}$. Now we obtain constitutive relation for capacitor with memory kernel that is singular and has no derivative at start point i.e. (8). We write

$$\begin{aligned}
\mathrm{i}(t) &= \mathrm{C}\left(\left(k(t)\right)*\left(\mathrm{v}^{(1)}(t)\right)\right)\\
&= \mathrm{C}\int_{-\infty}^{t}\left(k(t-\tau)\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau; \quad k(t) = \mathrm{A}t^{-\alpha}; \quad t \ge 0\\
&= \mathrm{C}\int_{0}^{t}\left(\mathrm{A}(t-\tau)^{-\alpha}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&= \mathrm{CA}\left(\Gamma(1-\alpha)\right)\left(\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\left((t-\tau)^{-\alpha}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\right)\\
&= \mathrm{CA}\left(\Gamma(1-\alpha)\right)\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\right)
\end{aligned} \tag{11}$$

In (11) we have used the definition of Caputo fractional derivative for fractional order $0<\alpha<1$ i.e. for a differentiable function $f(t)$ we have following

$$ {}_{0}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left((t-\tau)^{-\alpha}\right)\left(f^{(1)}(\tau)\right) \mathrm{d}\tau \tag{12} $$

Thus, our constitutive equation for a capacitor having Power Law Memory kernel is given by fractional differential equation (13), and is changed from (1)

$$ \begin{aligned} &\mathrm{i}(t) = \mathrm{CA}\left(\Gamma(1-\alpha)\right)\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\right); \quad \mathrm{C}_{\alpha} = \mathrm{CA}\left(\Gamma(1-\alpha)\right) \\ &\mathrm{i}(t) = \mathrm{C}_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\right); \quad 0<\alpha<1 \end{aligned} \tag{13} $$

In (13) putting $\alpha = 1$ we get classical relation (1).

The Laplace Transform of Caputo Fractional Derivative for fractional order $0<\alpha<1$ is $\mathcal{L}\left\{{}_{0}^{C}D_{t}^{\alpha} f(t)\right\} = s^{\alpha}F(s) - s^{\alpha-1}f(0)$, using this we write Laplace Transform of (13) as

$$ \begin{aligned} &\mathcal{L}\left\{\mathrm{i}(t)\right\} = \mathrm{C}_{\alpha}\mathcal{L}\left\{{}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\right\}; \quad 0<\alpha<1 \\ &\mathrm{I}(s) = \mathrm{C}_{\alpha}\left(s^{\alpha}\mathrm{V}(s) - s^{\alpha-1}\mathrm{v}(0)\right) \end{aligned} \tag{14} $$

We note that in (14) putting $\alpha = 1$ we obtain the classical result i.e. (5).

We verify the relaxation current with $\mathrm{V}(s) = s^{-1}$ and $\mathrm{v}(0) = 0$ i.e. for unit step input $\mathrm{v}(t) = u(t); \quad t \ge 0$, applied to initially uncharged capacitor $\mathrm{v}(0) = 0, \quad t<0$ from (13)

$$ \begin{aligned} &\mathrm{I}(s) = \mathrm{C}_{\alpha}\left(s^{\alpha}\mathrm{V}(s) - s^{\alpha-1}\mathrm{v}(0)\right); \quad \mathrm{V}(s) = \frac{1}{s}, \quad \mathrm{v}(0) = 0 \\ &\mathrm{I}(s) = \mathrm{C}_{\alpha}s^{\alpha-1}, \quad \mathcal{L}\left\{t^{p}\right\} = \frac{\Gamma(p+1)}{s^{p+1}} \\ &\mathrm{i}(t) = \mathcal{L}^{-1}\left\{\mathrm{I}(s)\right\} = \mathcal{L}^{-1}\left\{\mathrm{C}_{\alpha}s^{\alpha-1}\right\} \\ &\qquad = \frac{\mathrm{C}_{\alpha}}{\Gamma(1-\alpha)}t^{-\alpha}, \quad 0<\alpha<1, \qquad t \ge 0 \end{aligned} \tag{15} $$

With $\mathrm{C}_{\alpha} = \mathrm{CA}\left(\Gamma(1-\alpha)\right)$ (13), we get from (15) $\mathrm{i}(t) = \mathrm{CA}\left(\Gamma(1-\alpha)\right)t^{-\alpha}$; same as we got in (10).

Constitutinve equation due to Mittag-Leffler function as Memory Kernel

Here we take Memory Kernel as following for $t \ge 0$

$$ k(t) = \mathrm{B}E_{\alpha}(-\lambda t^{\alpha}); \quad 0<\alpha<1 \tag{16} $$

In (16) B and λ are a positive real constants. Where the Mittag-Leffler function is defined as following

$$ E_{\alpha}(-\lambda t^{\alpha}) = \sum_{n=0}^{\infty} \frac{(-\lambda t^{\alpha})^{n}}{\Gamma(\alpha n+1)}, \quad t \ge 0; \quad \lambda t^{\alpha} \in \mathbb{C}, \quad \alpha \in \mathbb{C}, \quad \mathrm{Re}\left[\alpha\right] > 0 \tag{17} $$

The constitutive equation with Memory Kernel (16) we write the following

$$
\begin{aligned}
\mathrm{i}(t) &= \mathrm{C}\Big(\big(k(t)\big)*\big(\mathrm{v}^{(1)}(t)\big)\Big) \\
&= \mathrm{CB}\int_0^t \Big(E_\alpha(-\lambda(t-\tau)^\alpha)\Big)\Big(\mathrm{v}^{(1)}(\tau)\Big)\mathrm{d}\tau \\
&= \mathrm{CB}\int_0^t \left(\sum_{n=0}^{\infty}\frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n+1)}\right)\Big(\mathrm{v}^{(1)}(\tau)\Big)\mathrm{d}\tau \\
&= \mathrm{CB}\sum_{n=0}^{\infty}\left(\frac{(-1)^n\lambda^n}{\Gamma(\alpha n+1)}\right)\int_0^t (t-\tau)^{\alpha n}\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau \\
&= \mathrm{CB}\left(\sum_{n=0}^{\infty}(-1)^n\lambda^n\right)\left(\frac{1}{\Gamma(\alpha n+1)}\int_0^t (t-\tau)^{\alpha n}\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\right) \\
&= \mathrm{CB}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)
\end{aligned}
\tag{18}
$$

Where in (18) we used the operator ${}_0I_t^{\upsilon}$, $\upsilon=\alpha n+1$, which is Riemann-Liouville fractional integration of order υ defined as

$$
{}_0I_t^{\upsilon}\left[f(t)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^t (t-\tau)^{\upsilon-1}f(\tau)\mathrm{d}\tau \tag{19}
$$

Considering $\mathrm{v}(0)=0$ we can write ${}_0I_t^1\mathrm{v}^{(1)}(t)=\mathrm{v}(t)$ and expand (18) to write

$$
\begin{aligned}
\mathrm{i}(t) &= \mathrm{CB}\sum_{n=0}^{\infty}(-1)^n\lambda^n\left({}_0I_t^{\alpha n+1}\left[\mathrm{v}^{(1)}(t)\right]\right) \\
&= \mathrm{CBv}(t)-\lambda\,\mathrm{CB}\left({}_0I_t^{\alpha+1}\mathrm{v}^{(1)}(t)\right)+\lambda^2\mathrm{CB}\left({}_0I_t^{2\alpha+1}\mathrm{v}^{(1)}(t)\right)-\lambda^3\mathrm{CB}\left({}_0I_t^{3\alpha+1}\mathrm{v}^{(1)}(t)\right)+\ldots
\end{aligned}
\tag{20}
$$

Thus, a Memory Kernel (16) $k(t)=\mathrm{B}-\frac{\lambda \mathrm{B}t^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda^2\mathrm{B}t^{2\alpha}}{\Gamma(2\alpha+1)}-\ldots;\quad 0<\alpha<1$ i.e. series-sum of power laws acting on derivative of voltage function $\mathrm{v}^{(1)}(t)$, gives a relaxing current $\mathrm{i}(t)$ with series sum of fractional integrations of various orders acting on rate of change of voltage (20).

We note that Memory Kernel in this case (16) is not singular function at $k(0)=\mathrm{B}$, as $k^{(1)}(0)=\mathrm{B}$ though its derivative is not defined i.e. $k^{(1)}(t)\big|_{t=0}=-\infty$.

Now we give a unit step input to this system so we have $\mathrm{v}(t)=1,\quad t\geq 0$; with $\mathrm{v}^{(1)}(t)=\delta(t)$. Placing this in (20), we write the following

$$
\begin{aligned}
\mathrm{i}(t) &= \mathrm{CBv}(t)-\lambda\,\mathrm{CB}\left({}_0I_t^{\alpha+1}\mathrm{v}^{(1)}(t)\right)+\lambda^2\mathrm{CB}\left({}_0I_t^{2\alpha+1}\mathrm{v}^{(1)}(t)\right)-\lambda^3\mathrm{CB}\left({}_0I_t^{3\alpha+1}\mathrm{v}^{(1)}(t)\right)+\ldots \\
&= \mathrm{CB}-\lambda\,\mathrm{CB}\left({}_0I_t^{\alpha+1}\delta(t)\right)+\lambda^2\mathrm{CB}\left({}_0I_t^{2\alpha+1}\delta(t)\right)-\lambda^3\mathrm{CB}\left({}_0I_t^{3\alpha+1}\delta(t)\right)+\ldots \\
&= \mathrm{CB}-\lambda\mathrm{CB}\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)+\lambda^2\mathrm{CB}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right)-\lambda^3\mathrm{CB}\left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right)+\ldots \\
&= \mathrm{CB}\left(1+\frac{(-\lambda)t^\alpha}{\Gamma(\alpha+1)}+\frac{(-\lambda)^2(t^\alpha)^2}{\Gamma(2\alpha+1)}+\frac{(-\lambda)^3(t^\alpha)^3}{\Gamma(3\alpha+1)}+\ldots.\right) \\
&= \mathrm{CB}\left(\sum_{n=0}^{\infty}\frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+1)}\right)=\mathrm{CB}E_\alpha(-\lambda t^\alpha),\quad t\geq 0
\end{aligned}
\tag{21}
$$

In (21) we have used formula for fractional integration of delta function i.e. ${}_0I_t^{\upsilon}\delta(t)=\frac{1}{\Gamma(\upsilon)}t^{\upsilon-1}$. What we observe that the relaxation current $\mathrm{i}(t)$ to uncharged capacitor excited by unit-step voltage input; relaxes in proportional to the memory kernel function i.e. $\mathrm{i}(t)\propto k(t)$ in this case $k(t)\sim E_{\alpha}(-\lambda t^{\alpha})$.

We note that by placing $\alpha=1$ we are not getting classical case (1).

Constitutive equation due to exponential function as Memory Kernel

Here we take Memory kernel as

$$k(t)=\mathrm{M}\mathrm{e}^{-\kappa t}\,,\quad t\ge 0\,,\quad \kappa>0 \tag{22}$$

The constitutive equation with Memory Kernel as (22) gives the following

$$\begin{aligned}
\mathrm{i}(t)&=\mathrm{C}\Big(\big(k(t)\big)*\big(\mathrm{v}^{(1)}(t)\big)\Big)\\
&=\mathrm{CM}\int_0^t\left(\mathrm{e}^{-\kappa(t-\tau)}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{CM}\int_0^t\left(\sum_{n=0}^{\infty}\frac{(-\kappa(t-\tau))^n}{n!}\right)\left(\mathrm{v}^{(1)}(\tau)\right)\mathrm{d}\tau\\
&=\mathrm{CM}\sum_{n=0}^{\infty}\left(\frac{(-1)^n(\kappa)^n}{n!}\right)\int_0^t(t-\tau)^n\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\\
&=\mathrm{CM}\left(\sum_{n=0}^{\infty}(-1)^n\kappa^n\right)\left(\frac{1}{n!}\int_0^t(t-\tau)^n\mathrm{v}^{(1)}(\tau)\,\mathrm{d}\tau\right)\\
&=\mathrm{CM}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)
\end{aligned} \tag{23}$$

Thus the memory Kernel which is pure exponential function (22) gives a relaxation current which is series sum of integer order multiple integration of voltage function; like in (20) we write the following

$$\begin{aligned}
\mathrm{i}(t)&=\mathrm{CM}\sum_{n=0}^{\infty}(-1)^n\kappa^n\left({}_0I_t^{n+1}\left[\mathrm{v}^{(1)}(t)\right]\right)\\
&=\mathrm{CMv}(t)-\kappa\,\mathrm{CM}\left({}_0I_t^2\mathrm{v}^{(1)}(t)\right)+\kappa^2\mathrm{CM}\left({}_0I_t^3\mathrm{v}^{(1)}(t)\right)-\lambda^3\mathrm{CM}\left({}_0I_t^4\mathrm{v}^{(1)}(t)\right)+\dots
\end{aligned} \tag{24}$$

We give a step input to system having Memory kernel (22) and observe the following

$$\begin{aligned}
\mathrm{i}(t)&=\mathrm{CMv}(t)-\kappa\,\mathrm{CM}\left({}_0I_t^2\mathrm{v}^{(1)}(t)\right)+\kappa^2\mathrm{CM}\left({}_0I_t^3\mathrm{v}^{(1)}(t)\right)-\kappa^3\mathrm{CM}\left({}_0I_t^4\mathrm{v}^{(1)}(t)\right)+\dots\\
&=\mathrm{CM}-\kappa\,\mathrm{CM}\left({}_0I_t^2\delta(t)\right)+\kappa^2\mathrm{CM}\left({}_0I_t^3\delta(t)\right)-\kappa^3\mathrm{CM}\left({}_0I_t^4\delta(t)\right)+\dots;\quad {}_0I_t^m\delta(t)=\tfrac{1}{(m-1)!}t^{m-1}\\
&=\mathrm{CM}-\kappa\mathrm{CM}t+\kappa^2\mathrm{CM}\left(\frac{t^2}{2!}\right)-\kappa^3\mathrm{CM}\left(\frac{t^3}{3!}\right)+\dots\\
&=\mathrm{CM}\left(1+\frac{(-\kappa)t}{1!}+\frac{(-\kappa)^2(t)^2}{2!}+\frac{(-\kappa)^3(t)^3}{3!}+\dots.\right)\\
&=\mathrm{CM}\left(\sum_{n=0}^{\infty}\frac{(-\kappa t)^n}{n!}\right)=\mathrm{CMe}^{-\kappa t}\,,\quad t\ge 0
\end{aligned} \tag{25}$$

That is in (25) the relaxation current to unit step voltage input to an uncharged capacitor having the memory kernel as exponential decay function (22), $k(t) \sim \mathrm{e}^{-\kappa t}$ has relaxation current $\mathrm{i}(t) \propto k(t)$.
We note that the Memory Kernel (22) is non-singular function and has derivative every-where.
Let us apply Laplace Transformation as depicted below

$$
\begin{aligned}
&\mathrm{i}(t) = \mathrm{C}\Big(\big(k(t)\big) * \big(\mathrm{v}^{(1)}(t)\big)\Big) \\
&\mathcal{L}\{\mathrm{i}(t)\} = \mathcal{L}\Big\{\mathrm{C}\Big(\big(k(t)\big) * \big(\mathrm{v}^{(1)}(t)\big)\Big)\Big\} \\
&\mathrm{I}(s) = \mathrm{C}\big(\mathcal{L}\{k(t)\}\big)\big(\mathcal{L}\{\mathrm{v}^{(1)}(t)\}\big), \quad \mathcal{L}\{k(t)\} = \mathcal{L}\{\mathrm{Me}^{-\kappa t}\} = \mathrm{M}\left(\frac{1}{s+\kappa}\right) \\
&\qquad = \mathrm{C}\left(\frac{\mathrm{M}}{s+\kappa}\right)\big(s\mathrm{V}(s) - \mathrm{v}(0)\big) \\
&\qquad = \mathrm{CM}\Big(\big(\tfrac{s}{s+\kappa}\big)\mathrm{V}(s) - \big(\tfrac{1}{s+\kappa}\big)\mathrm{v}(0)\Big), \quad \mathrm{v}(0) = 0, \quad \mathrm{V}(s) = \frac{1}{s} \\
&\mathrm{I}(s) = \mathrm{CM}\left(\frac{1}{s+\kappa}\right) \\
&\mathrm{i}(t) = \mathrm{CM}\mathcal{L}^{-1}\left\{\tfrac{1}{s+\kappa}\right\} \\
&\qquad = \mathrm{CMe}^{-\kappa t}
\end{aligned} \tag{26}
$$

We get same result of (25).
Let us do Laplace Transformation for Mittag-Leffler memory kernel, as depicted as follows

$$
\begin{aligned}
&\mathrm{i}(t) = \mathrm{C}\Big(\big(k(t)\big) * \big(\mathrm{v}^{(1)}(t)\big)\Big) \\
&\mathcal{L}\{\mathrm{i}(t)\} = \mathcal{L}\Big\{\mathrm{C}\Big(\big(k(t)\big) * \big(\mathrm{v}^{(1)}(t)\big)\Big)\Big\} \\
&\mathrm{I}(s) = \mathrm{C}\big(\mathcal{L}\{k(t)\}\big)\big(\mathcal{L}\{\mathrm{v}^{(1)}(t)\}\big), \quad \mathcal{L}\{k(t)\} = \mathcal{L}\{\mathrm{B}E_\alpha(-\lambda t^\alpha)\} = \mathrm{B}\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right) \\
&\qquad = \mathrm{CB}\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right)\big(s\mathrm{V}(s) - \mathrm{v}(0)\big) \\
&\qquad = \mathrm{CB}\Big(\big(\tfrac{s^\alpha}{s^\alpha+\lambda}\big)\mathrm{V}(s) - \big(\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\big)\mathrm{v}(0)\Big), \quad \mathrm{v}(0) = 0, \quad \mathrm{V}(s) = \frac{1}{s} \\
&\mathrm{I}(s) = \mathrm{CB}\left(\frac{s^{\alpha-1}}{s^\alpha + \lambda}\right) \\
&\mathrm{i}(t) = \mathrm{CB}\mathcal{L}^{-1}\left\{\tfrac{s^{\alpha-1}}{s^\alpha+\lambda}\right\} \\
&\qquad = \mathrm{CB}E_\alpha(-\lambda t^\alpha)
\end{aligned} \tag{27}
$$

The same that we got in (21). In (27) placing $\alpha = 1$ we get (26) but not (6), the classical case.

Discussions and Observations

We have given few examples of Memory Kernel that gives constitutive expression for relaxation current to capacitor. The memory-less relaxation is via Memory Kernel with unit delta function, gives a classical constitutive formula of capacitor, i.e. (1). The memory kernel if formulated via a

singular power-law kernel (8), then we have a fractional derivative of Caputo type relating relaxation current and impressed voltage (11) and (13); for a capacitor. This power law memory kernel is singular in nature and non-differentiable at start. We extend this analysis with memory-kernel, which is Mittag-Leffler type (16). This kernel (16) is non-singular at origin but the derivative at origin does not exist. With this we get the constitutive equation as depicted in (20). We note that the structure of this expression is much away from that (1) the classical one and the (13) the fractional one. This comprises of series of fractional integration, may thus be mathematically fine but we may not be getting physical sense. Lastly we get the constitutive equation with non-singular and everywhere differentiable memory kernel i.e. (22) ; which is exponential function. With this, we construct a constitutive function that is series sum of integer order repeated integrations-and is very much off from the capacitor dynamics classical case or fractional case i.e. (1) or (13) respectively. Though mathematically fine, yet physical applicability is questionable.
We note that though classical textbook capacitors are expressed as in (1), yet in reality they have power-law decay current, when excited by a step-voltage for an uncharged capacitor. This is well established by Curie-von Schweidler law the current relaxation is $\mathrm{i}(t)\sim t^{-\alpha};\quad 0<\alpha<1$. Therefore, the memory-kernel associated with relaxation dynamics is $k(t)\sim t^{-\alpha}$; that is power-law singular function. Here the fractional derivative appears in constitutive expression i.e. $\mathrm{i}(t)=\mathrm{C}_{\alpha}\left({}_{0}^{C}D_{t}^{\alpha}\mathrm{v}(t)\right);\quad 0<\alpha<1$. In this short analysis, we observe that whatever be the nature of Memory-Kernel, (say delta-function, singular power-law function, Mittag-Leffler function or classical exponential function)-the same is the nature of relaxation current for a step input voltage excitation to an uncharged capacitor. We can have several other type of memory kernel like-stretched exponential, Two-parameter-Mittag leffler, Three parameter Mittag-Leffler, Hyperbolic, etc and derive the constitutive relations for each of them.

Conclusions

A very relevant question that is: if we have in reality a singular memory kernel or a non-singular memory kernel, for capacitor relaxation dynamics. This study shows if we are having a singular memory kernel, then we observe the reality better. Though mathematically non-singular memory kernel is possible, yet the constitutive equation for capacitor relaxation dynamics does not give the useful information. The universal dielectric relaxation law of curie-von Schweidler still holds with singular power law memory kernel and not via non-singular memory kernels.

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