How disordered (Non-Debye) decaying relaxation manifests ordered relaxation rate distribution histograms

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Abstract

We observe relaxations that is non-Debye in nature that is deviated from standard (ideal text book) Debye type decay (that are pure exponential decay)-in various systems e.g. Newton's law of cooling in liquids, dielectric relaxations, in capacitors super-capacitors discharge-charge, responses in visco-elastic systems etc. These are the systems following Fractional Dynamics described with fractional calculus. We call these are disordered systems, and classically call the observation is due to non-linearity. This non-Debye relaxation may be interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of time constants or relaxation rates varying from zero to infinity, and the mathematics gives an order to these infinite relaxation rates-i.e. we get a ordered histogram for a particular disordered decay. Primarily we see that Debye relaxation that is a pure exponential decay curve, that has a unique relaxation rate and that gets spread or stretched when the relaxation is via Mittag-Leffler function-or any other non-Debye decay function. That is the delta distributed histogram of relaxation rate for (ideal) Debye case gets changed to various forms (exponential distribution, Gaussian distribution etc) for various non-Debye cases that we see for various orders of Mittag-Leffler function. The Mittag-Leffler function appears as solution to 'linear' Fractional Differential equation. We study the decay curves which are monotonically decaying given by Mittag-Leffler function with its order varying from zero to one, and for oscillatory decay function with Mittag-Leffler function with order varying from one to two. Thus this Mittag-Leffler function maps from order zero i.e. a hyperbolic decay to order one- i.e. pure exponential decay (Debye-type) and then to order equals two a pure sustained oscillatory relaxation. In order to get the histogram function for distribution of relaxation rates for these non-Debye relaxations, one has to get inverse Laplace transform of 'time' decay functions-that we will derive. We will describe various analytical ways to get inverse Laplace transform via integration on Bromwich path i.e. via modern method of Berberan Santo and also via classical Contour integration on Henkel's path using Residue Calculus. For this we consider time variable of decay function as complex quantity and get the decay function inverse Laplace transformed to obtain ordered histogram of relaxation rates. We will apply these methods for Mittag-Leffler function of order zero to two, thus describing the histogram of relaxation rates-describing non-Debye relaxation. We will also mention other type of non-Debye relaxations like power law decay observed since late 19th century for dielectric relaxations called Currie-Von Schweidler law, as well as Kohlraush's Stretched Exponential decay, Becquerel's compressed hyperbola decay and derive their relaxation rate distribution histograms-via the described methodologies. At the end as example we will use these results to obtain relaxation rate histograms for a disordered relaxation of temperature of a cooling body, where the cooling law is governed by Caputo or Riemann-Liouville fractional derivative, instead of classical integer order derivative-in Newton's law of cooling. Thus our discussion is as how the analytical mathematical procedures allow us to order the disordered non-Debye relaxations-that is observed in real physical systems.

Keywords

Residue Calculus, Bromwich Integration, Hankel Path, Berberan-Santo method, Laplace integral

Introduction

In this deliberation we are not discussing Fractional Calculus, However dynamic systems that behave via non-Newtonian calculus; or system having dynamics described by Fractional Differential Equations, manifest relaxation functions in form of non-Debye relaxation functions. Debye relaxation is by pure exponential decay. The non-Debye one is via Mittag-Leffler, power law function etc.-and we term them as disordered (or non-disciplined relaxation). We will see techniques of having these disordered relaxing functions analyzed by how their undisciplined relaxing rates are distributed-via extracting the rate relaxation histogram function. We will discuss Residue Calculus, Bromwich integration, Hankel path integration-for obtaining the histogram function i.e. by inverse Laplace transform done on 'time function', taking 'time as complex variable'. We will also obtain new method of getting inverse Laplace transform with-out Residue Calculus or Contour- Integration-i.e. Berberan-Santo method. We use these techniques to write rate relaxation histogram function for Newton's Law of cooling-formulated by non-Newtonian calculus of Caputo or Riemann-Liouville fractional derivative, and other types of non-Debye decay. We will see these techniques nicely orders the disordered relaxations via some pattern of histogram function obtained for relaxation rates.

The Curie-von Schweidler law relates to the relaxation current in dielectric when a step DC voltage is applied and is given by power law relaxation function $f(t) \sim t^{-\alpha}$ where $t > 0; \quad 0 < \alpha < 1$ and the power (exponent) α is called relaxation constant or decay constant. We note that α the exponent is non-integer. Note the usual Debye relaxation is $f(t) \sim \mathrm{e}^{-\lambda_0 t}$ The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments. For a step voltage $\mathrm{V_{BB}}$ impressed at $t = 0$ to uncharged capacitor the current is $\mathrm{i}(t) = K_\alpha \mathrm{V_{BB}} t^{-\alpha}; \quad 0 < \alpha < 1; \quad t > 0$. This is empirically & experimentally derived. If step input voltage is $\mathrm{v}(t) = \mathrm{V_{BB}} u(t)$ where $u(t)$ is unit step, then this non-Debye relaxation current is following fractional differential equation i.e.

$$\mathrm{i}(t) = K_\alpha \left(\Gamma(1-\alpha) \right) \frac{\mathrm{d}^\alpha \mathrm{v}(t)}{\mathrm{d}t^\alpha}$$

$$= C_\alpha \frac{\mathrm{d}^\alpha \mathrm{v}(t)}{\mathrm{d}t^\alpha}$$

The above example of a non-Debye relaxation via power law-shows that the constituent equation needs be fractional differential equation; however in this deliberation we are not discussing fractional calculus.

Revising basics of Laplace Transforms

For a function $x(t)$ which is zero for $t<0$, the Laplace transform is following Laplace integral

$$X(s)=\int_0^\infty \mathrm{e}^{-st}x(t)\mathrm{d}t$$

Here we have complex variable $s=\sigma+i\omega$, so we have

$$X(s)=\int_0^\infty \mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t$$

The inverse problem is given $X(s)$ how to get $x(t)$. Whereas the Fourier transform of $x(t)$ is $\hat{x}(\omega)$ defined as $\hat{x}(\omega)=\int_{-\infty}^{\infty}\mathrm{e}^{-i\omega t}x(t)\mathrm{d}t$ and inverse Fourier transform is the following integral

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{i\omega t}\hat{x}(\omega)\mathrm{d}\omega$$

In the Laplace transform expression $X(s)=\int_0^\infty \mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t$ as obtained above, let us take

$$\begin{aligned}\phi(t)&=\mathrm{e}^{-\sigma t}x(t); \quad t\ge 0\\ &=0 \qquad\qquad\quad t<0\end{aligned}$$

Where σ is a constant. Taking the Fourier transform of $\phi(t)$ we write

$$\begin{aligned}\hat{\phi}(\omega)&=\int_0^\infty \mathrm{e}^{-i\omega t}\phi(t)\mathrm{d}t\\ &=\int_0^\infty \mathrm{e}^{-i\omega t}\left(\mathrm{e}^{-\sigma t}x(t)\right)\mathrm{d}t\\ &=\int_0^\infty \mathrm{e}^{-(\sigma+i\omega)t}x(t)\mathrm{d}t=X(\sigma+i\omega)\end{aligned}$$

Now we take inverse Fourier transform of $X(\sigma+i\omega)$ as following expression

$$\phi(t)=\mathrm{e}^{-\sigma t}x(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{i\omega t}X(\sigma+i\omega)\mathrm{d}\omega$$

Therefore

$$\begin{aligned}x(t)&=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{\sigma t}\left(\mathrm{e}^{i\omega t}X(\sigma+i\omega)\right)\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty}\mathrm{e}^{(\sigma+i\omega)t}X(\sigma+i\omega)\mathrm{d}\omega\end{aligned}$$

Letting $s=\sigma+i\omega$ we have $\mathrm{d}s=i\mathrm{d}\omega$, we get the following from above

$$\begin{aligned}x(t)&=\frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega}\mathrm{e}^{st}X(s)\mathrm{d}s; \quad t\ge 0\\ &=0; \quad t<0\end{aligned}$$

The inverse Laplace transform via contour integration

The formula that we derived that is

$$x(t)=\frac{1}{2\pi i}\int_{s=\sigma-i\omega}^{s=\sigma+i\omega}\mathrm{e}^{st}X(s)\mathrm{d}s$$

is the integral that gives inverse Laplace transform. Now we ask the following
1). How we choose the real part of s i.e. σ
2). How do we calculate/evaluate the above integral in complex domain.
We already know that $x(t)=0$ for $t<0$; that will help to answer (1).

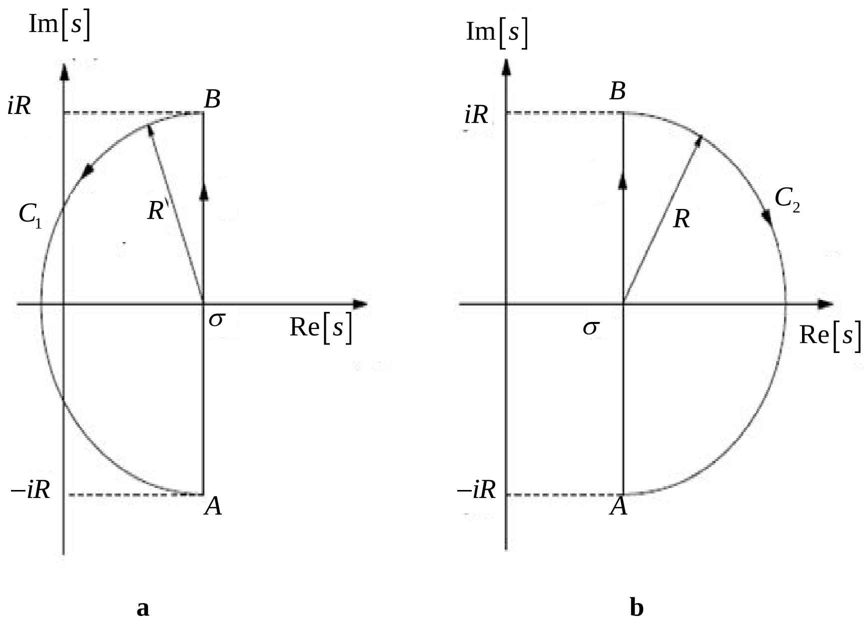


Figure-1 Contour in complex-plane to evaluate inverse Laplace transforms

Consider Figure-1a the closed contour is $A\to B\to C_1$, we write the contour integration as following statement coming from residue calculus

$$\begin{aligned}\int_{A\to B\to C_1} \mathrm{e}^{st}X(s)\mathrm{d}s &= \int_{A\to B} \mathrm{e}^{st}X(s)\mathrm{d}s + \int_{C_1} \mathrm{e}^{st}X(s)\mathrm{d}s \\ &= \int_{\sigma-iR}^{\sigma+iR} \mathrm{e}^{st}X(s)\mathrm{d}s + \int_{C_1} \mathrm{e}^{st}X(s)\mathrm{d}s \\ &= 2\pi i\sum_{\text{poles}} \text{Residues}\left[\mathrm{e}^{st}X(s)\right]\end{aligned}$$

We stress that residues are at the poles inside the closed contour $A\to B\to C_1$ of Figure-1a. Now as $R\uparrow\infty$ the integral $\int_{\sigma-iR}^{\sigma+iR}\mathrm{e}^{st}X(s)\mathrm{d}s$ is the integral of interest that we require to evaluate inverse Laplace transform. We note that the integral on the line $A\to B$ is the Bromwich integral, and this is for finding the inverse Laplace transform. We will use Jordan lemma (described shortly) which says for $t>0$, $\lim_{R\uparrow\infty}\int_{C_1}\mathrm{e}^{st}X(s)\mathrm{d}s=0$.

Therefore for $t>0$ we write the following (as $R\uparrow\infty$)

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}X(s)\mathrm{d}s = \sum_{\text{poles left of }\sigma} \text{Residues}\left[\mathrm{e}^{st}X(s)\right]$$

Consider Figure-1b the closed contour is $A\to B\to C_2$, we write the contour integration as following

$$\int_{A\to B\to C_2} \mathrm{e}^{st}X(s)\mathrm{d}s = \int_{A\to B} \mathrm{e}^{st}X(s)\mathrm{d}s + \int_{C_2} \mathrm{e}^{st}X(s)\mathrm{d}s$$
$$= \int_{\sigma-iR}^{\sigma+iR} \mathrm{e}^{st}X(s)\mathrm{d}s + \int_{C_2} \mathrm{e}^{st}X(s)\mathrm{d}s$$
$$= -2\pi i \sum_{\text{poles}} \text{Residues}\left[\mathrm{e}^{st}X(s)\right]$$

We stress that residues are at the poles inside the closed contour $A\to B\to C_2$ of Figure-1b; and the negative sign indicate that contour is taken in clock-wise direction. We will use Jordan lemma (described shortly) which says for $t<0$, $\lim_{R\uparrow\infty}\int_{C_2} \mathrm{e}^{st}X(s)\mathrm{d}s = 0$, thus we write for $t<0$ the following

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{st}X(s)\mathrm{d}s = -\sum_{\text{poles right of }\sigma} \text{Residues}\left[\mathrm{e}^{st}X(s)\right]$$

We know that this above integral must be zero, since for $t<0$, we have $x(t)=0$, as $\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{st}X(s)\mathrm{d}s = x(t)$; that is from definition of inverse Laplace transform. This means LHS of above is zero thus making $\sum_{\text{poles right of }\sigma}\text{Residues}\left[\mathrm{e}^{st}X(s)\right]=0$, implying that the contour of Figure-1b does not contain any poles.

Therefore the $\mathrm{Re}[s]=\sigma$, the line AB (Figure-1), or Bromwich line, must be chosen such that contour of Figure-1b i.e. $A\to B\to C_2$ does not contain any poles of $\mathrm{e}^{st}X(s)$ as $R\uparrow\infty$; and thus the contour of Figure-1a, i.e. $A\to B\to C_1$ must have all poles of $\mathrm{e}^{st}X(s)$. This gives answer to point (1) above.

Also we note that since e^{st} is analytic everywhere (i.e. it has no poles in entire complex-s plane), the poles of $\mathrm{e}^{st}X(s)$ are same as of $X(s)$. This gives reply to point (2) as posed above, thus we apply residue calculus of complex analysis and write

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{e}^{st}X(s)\mathrm{d}s = \sum_{\text{all poles of } X(s)} \text{Residues}\left[\mathrm{e}^{st}X(s)\right]$$

This is how we need evaluate the integral for obtaining inverse Laplace transform.

The Jordan Lemma

While we discussed the inverse Laplace transform using contour integration and Residue calculus, we very well stated that we need a condition that is (arc in Figure-1a)

$$\lim_{R\uparrow\infty}\int_{C_1} \mathrm{e}^{st}X(s)\mathrm{d}s = 0 \qquad t>0$$

While for $t>0$ the points on this arc C_1 is given as

$$s = \sigma + R\mathrm{e}^{i\theta}; \quad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

Examining standard Laplace transform tables we observe most functions satisfy $\lim_{|s|\uparrow\infty} X(s) = 0$; for example $X(s)=\frac{1}{s}$, $X(s)=\frac{1}{s+a}$, $X(s)=\frac{a}{s^2+a^2}$ etc. Therefore in those cases as $R\uparrow\infty$, $X(s)\downarrow 0$. This means that for any $M_R>0$ a radius R can be found such that $\left|X(s)\right| = \left|X(\sigma + R\mathrm{e}^{i\theta})\right| < M_R$. By using the inequality i.e. $\left|\int_C f(s)\mathrm{d}s\right| \le \int_C \left|f(s)\right|\mathrm{d}s$ for this R we have the following,

$$\left|\int_{C_1} \mathrm{e}^{st} X(s)\mathrm{d}s\right| \le \int_{C_1} \left|\mathrm{e}^{st} X(s)\right| \mathrm{d}s$$

$$\left|\int_{C_1} \mathrm{e}^{st} X(s)\mathrm{d}s\right| \le M_R \int_{C_1} \left|\mathrm{e}^{st}\right|\mathrm{d}s$$

On the arc C_1 for $t > 0$, $s = \sigma + R\mathrm{e}^{i\theta}$; $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, $\mathrm{d}s = iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta$, we write the following

$$\begin{aligned}\left|\mathrm{e}^{st}\right| = \left|\mathrm{e}^{(\sigma + R\mathrm{e}^{i\theta})t}\right| &= \left|\mathrm{e}^{(\sigma + R\cos\theta + iR\sin\theta)t}\right| \\ &= \left|\mathrm{e}^{(\sigma + R\cos\theta)t}\mathrm{e}^{iRt\sin\theta}\right|, \quad \left|\mathrm{e}^{i(Rt\sin\theta)}\right| = 1 \\ &= \left|\mathrm{e}^{(\sigma + R\cos\theta)t}\right|, \quad \mathrm{e}^{(\sigma t + Rt\cos\theta)} > 0 \\ &= \mathrm{e}^{\sigma t}\mathrm{e}^{Rt\cos\theta}\end{aligned}$$

Further we write the following steps

$$\begin{aligned}M_R \int_{C_1} \left|\mathrm{e}^{st}\mathrm{d}s\right| &= M_R \int_{\pi/2}^{3\pi/2} \left|\mathrm{e}^{\sigma t}\mathrm{e}^{Rt\cos\theta} iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta\right| \\ &\le M_R R\mathrm{e}^{\sigma t} \int_{\pi/2}^{3\pi/2} \mathrm{e}^{Rt\cos\theta}\mathrm{d}\theta = 2M_R R\mathrm{e}^{\sigma t} \int_{\pi/2}^{\pi} \mathrm{e}^{Rt\cos\theta}\mathrm{d}\theta\end{aligned}$$

Since the function i.e. $\mathrm{e}^{Rt\cos\theta}$ is even function we wrote the last step in above. By changing variable $\theta = \phi + \frac{\pi}{2}$ we thus obtain the following

$$\begin{aligned}M_R \int_{C_1} \left|\mathrm{e}^{st}\mathrm{d}s\right| &\le 2M_R R\mathrm{e}^{\sigma t} \int_0^{\pi/2} \mathrm{e}^{Rt\cos\left(\phi + \frac{\pi}{2}\right)}\mathrm{d}\phi \\ &= 2M_R R\mathrm{e}^{\sigma t} \int_0^{\pi/2} \mathrm{e}^{-Rt\sin\phi}\mathrm{d}\phi\end{aligned}$$

In the above equation, plotting the graphs of $y = \sin\phi$ and a straight line $y = \frac{2}{\pi}\phi$ says that in the region $0 \le \phi \le \frac{\pi}{2}$, we see $\sin\phi \ge \frac{2}{\pi}\phi$; with this we write the following

$$\begin{aligned}M_R \int_{C_1} \left|\mathrm{e}^{st}\mathrm{d}s\right| &\le 2M_R R\mathrm{e}^{\sigma t} \int_0^{\pi/2} \mathrm{e}^{-Rt\sin\phi}\mathrm{d}\phi \\ &\le 2M_R R\mathrm{e}^{\sigma t} \int_0^{\pi/2} \mathrm{e}^{-Rt\left(\frac{2}{\pi}\phi\right)}\mathrm{d}\phi = \frac{M_R \pi \mathrm{e}^{\sigma t}}{t}\left(1 - \mathrm{e}^{-Rt}\right)\end{aligned}$$

Therefore, for any $t > 0$ as $R \uparrow \infty$, we have $M_R \downarrow 0$, the above quantity tends to zero. This proves our case $\lim_{R\uparrow\infty} \int_{C_1} \mathrm{e}^{st} X(s)\mathrm{d}s = 0$.

Application of Residue Calculus in obtaining inverse Laplace transform via contour integration

We would like to evaluate inverse Laplace transform of $X(s) = \frac{2\mathrm{e}^{-2s}}{s^2+4}$, the Bromwich path integral for inverse Laplace transform is

$$x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{d}s$$
$$=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}\mathrm{d}s$$

We observe that the given function has two simple poles at $s=2i$ and $s=-2i$, both have $\mathrm{Re}[s]=0$. In Figure-1a, thus $\sigma=0$. We thus take an arbitrary (positive) small σ. We can distinguish the two cases (i) $t<2$ and (ii) $t>2$.

For $t<2$ the exponent $s(t-2)$ has negative real part if $\mathrm{Re}[s]>0$. We note that $\mathrm{e}^{s(t-2)}=\mathrm{e}^{(\mathrm{Re}[s]+i\,\mathrm{Im}[s])(t-2)}=\mathrm{e}^{(t-2)\mathrm{Re}[s]}\mathrm{e}^{i(t-2)\mathrm{Im}[s]}$, therefore the part $\mathrm{e}^{(t-2)\mathrm{Re}[s]}$ determines the function at infinity, since $\left|\mathrm{e}^{i(t-2)\mathrm{Im}[s]}\right|=1$. Therefore as $\mathrm{Re}[s]\uparrow\infty$ the function $\mathrm{e}^{s(t-2)}$ goes to zero. At the same time the denominator s^2+4 diverges as $\mathrm{Re}[s]\uparrow\infty$; and means the term $\frac{1}{s^2+4}$ that multiplies $\mathrm{e}^{s(t-2)}$ along the path C_2 (Figure-1b), for $R\uparrow\infty$. Therefore $\lim_{R\uparrow\infty}\int_{C_2}\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\mathrm{d}s=0$, i.e. integral on curve C_2 as $R\uparrow\infty$. We can calculate Bromwich path integral by considering $A\to B\to C_2$, but since this closed contour does not have any poles we say Residues are zero, resulting $x(t)$ as zero.

For $t>2$ the function $\mathrm{e}^{s(t-2)}$ goes to zero as $\mathrm{Re}[s]\downarrow-\infty$. That means the $\lim_{R\uparrow\infty}\int_{C_1}\frac{2\mathrm{e}^{(s-2)}}{s^2+4}\mathrm{d}s=0$ along the curve C_1 (Figure 1a) for $R\uparrow\infty$. For the residue theorem, this integral on the closed path $A\to B\to C_1$ is given by sum of residues of the function $\mathrm{e}^{st}X(s)=\frac{2\mathrm{e}^{(s-2)}}{s^2+4}$ at all the poles namely;

$$x(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\mathrm{d}s=\mathrm{Residue}_{\text{at }s=2i}\left(\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\right)+\mathrm{Residue}_{\text{at }s=-2i}\left(\frac{2\mathrm{e}^{-2s}}{s^2+4}\mathrm{e}^{st}\right)$$
$$=\lim_{s=2i}(s-2i)\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}+\lim_{s=-2i}(s+2i)\frac{2\mathrm{e}^{s(t-2)}}{s^2+4}$$
$$=\lim_{s=2i}\frac{2\mathrm{e}^{s(t-2)}}{s+2i}+\lim_{s=-2i}\frac{2\mathrm{e}^{s(t-2)}}{s-2i}=\frac{\mathrm{e}^{2i(t-2)}}{2i}-\frac{\mathrm{e}^{-2i(t-2)}}{2i}$$
$$=\sin\big(2(t-2)\big)$$

Thus we write

$$x(t)=\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2+4}\right\}=\begin{cases}\sin\big(2(t-2)\big) & t>2\\ 0 & t<2\end{cases}$$

Branch cut in complex plane for multi-valued function and multiple Riemann-sheets

We take $f(z)=\ln z$ where z is complex-variable. This function has singularity at $z=0$. Near The point $z=0$ while we encircle this point we get multiple values. That is by making $z=r\mathrm{e}^{i(\theta+2\pi n)}$ we get $f(z)=\ln r+i(2\pi n);\quad \theta\sim 0$. The number n is called winding number, and $n=0$ corresponds to the Principal Value. Thus the function has different values as we go around zero-and we call this as Branch Point (instead of pole). Introduce Branch-Cut which forms barriers through which z cannot go and thus $f(z)$ remains single valued. We show primary and Secondary Riemann-sheets in Figure-2.

For multi-valued function say $\sqrt{z-a}$; $\ln z$; $\frac{1}{z^{\alpha}+a}$,.... we cut the complex plane and take only primary Riemann-Sheet into consideration.

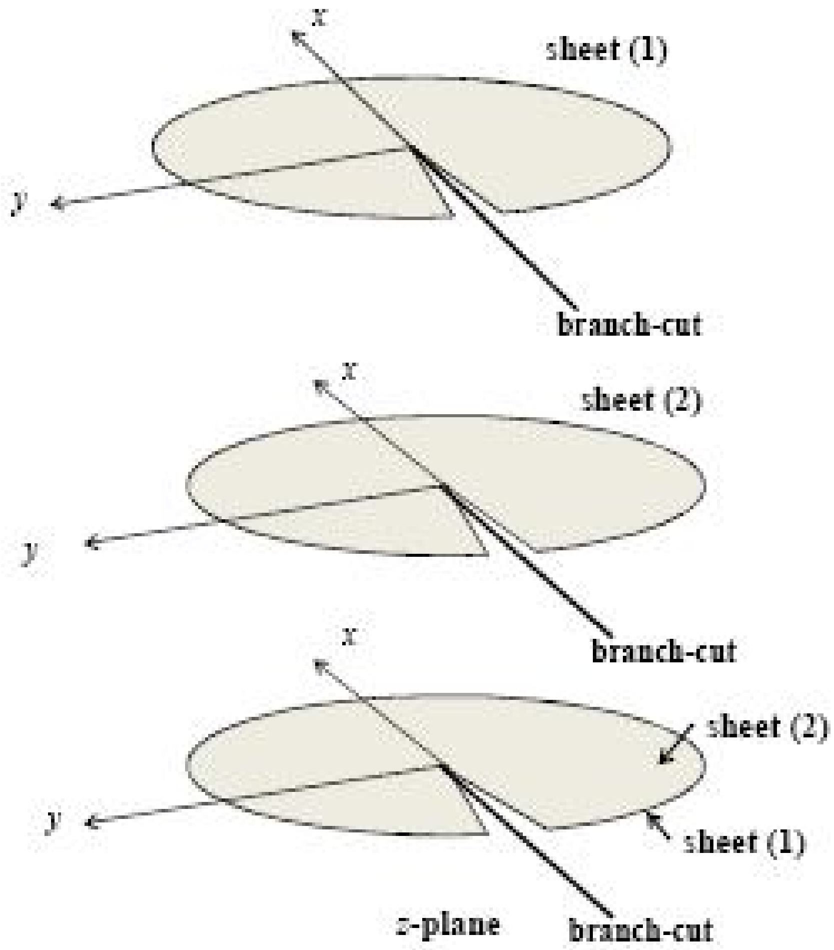


Figure-2: Primary and Secondary Riemann-Sheet and Branch Cut for multi-valued functions in complex variable

Application of Residue Calculus for Multi-valued function with branch cut on the complex plane

We take a function $X(s)=\sqrt{s-a}$, with $a\in\mathbb{R}$ .The inverse Laplace transform is following

$$x(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}\sqrt{s-a}\,\mathrm{d}s$$

The function $\mathrm{e}^{st}\sqrt{s-a}$ has no poles but the function $\sqrt{z}$ (taking $z=s-a$ is a multi-valued function in complex plane. Therefore we see a branch point at $z=0$, namely at $s=a$. This is the only singularity of our function $X(s)\mathrm{e}^{st}$ and therefore to evaluate Bromwich path integral, we have to take σ larger than a. Thus integral will be

$$x(t)=\mathcal{L}^{-1}\left\{\sqrt{s-a}\right\}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}\sqrt{s-a}\,\mathrm{d}s;\quad s-a=z$$

$$=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\mathrm{e}^{(a+z)t}\sqrt{z}\,\mathrm{d}z$$

$$=\frac{\mathrm{e}^{at}}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\mathrm{e}^{zt}\sqrt{z}\,\mathrm{d}z$$

In this case the branch point is zero, therefore $\mathrm{Re}\left[z\right]=\lambda$ can be arbitrarily small (but always larger than zero). Since $z=0$ is a branch point of the function to integrate we have to introduce a branch cut to evaluate the integral. Although the positive real axis also branch cut, we say that this choice is arbitrary and to make the function $\sqrt{z}$ single value it is enough that closed curves are not allowed enclosing the origin. We can therefore take a branch cut as the negative real axis. Figure-3 we indicate the contour used to integrate the given function. Since the closed contour $ABCDEFGA$ does not enclose any singularity, its integral is zero. To evaluate Bromwich path integral, (namely along AB) we have to calculate the integral along the arcs C,G along the straight lines D,F and along the circumference on the small circle E (Figure-3).

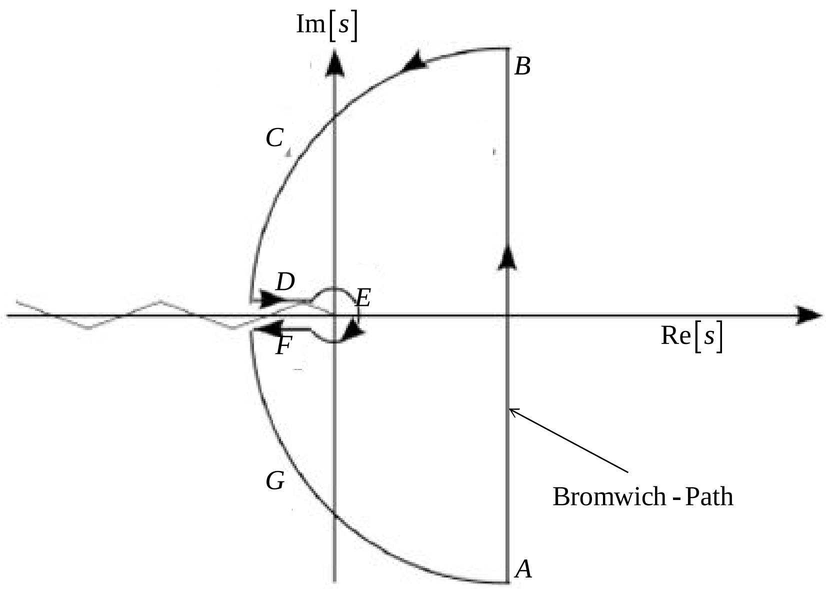


Figure-3: Contour of integration and Bromwich path on a branch cut complex plane

The function $\mathrm{e}^{zt}\sqrt{z}$ goes to zero, for $\mathrm{Re}\left[z\right]\downarrow-\infty$ (the term $\sqrt{z}$ cannot match the exponential decay of e^{zt} for $t>0$ as always); thus the integral along the arcs C,G is zero that disappears as the radius of these arcs grows. To evaluate the integral along small circle E, we take $z=\in\mathrm{e}^{i\theta}$, with θ in the interval π to $-\pi$ and we take the limit $\in\downarrow 0$. With this we have

$$\int_E \mathrm{e}^{zt}\sqrt{z}\mathrm{d}z = \int_{\pi}^{-\pi} \mathrm{e}^{\in t\mathrm{e}^{i\theta}} \sqrt{\in}\mathrm{e}^{i\left(\frac{\theta}{2}\right)} i \in \mathrm{e}^{i\theta}\mathrm{d}\theta$$

The above integrating function clearly tends to zero for $\in \downarrow 0$, thus there is no contribution from the integration over the circumference of E.

Along the straight lines D, F we can assume that the arguments of the complex variables lying on them are π (along D) and $-\pi$ (along F) and that their imaginary parts are close to zero. Therefore we have $z = r\mathrm{e}^{i\pi}$ (for D) and $z = r\mathrm{e}^{-i\pi}$ (for F). Consequently $\mathrm{d}z = \mathrm{e}^{i\pi}\mathrm{d}r$ (for D) and $\mathrm{d}z = \mathrm{e}^{-i\pi}\mathrm{d}r$ (for F). We note $\mathrm{e}^{i\pi} = \mathrm{e}^{-i\pi} = -1$.

The parameter r runs between $+\infty$ and 0 for line D and between 0 and $+\infty$ for line F. The integrals are given as following

$$\int_D \sqrt{z}\mathrm{e}^{zt}\mathrm{d}z = \int_{\infty}^{0} \sqrt{r}\mathrm{e}^{i\left(\frac{\pi}{2}\right)}\mathrm{e}^{tr\mathrm{e}^{i\pi}}\,\mathrm{e}^{i\pi}\mathrm{d}r$$

$$= \int_{\infty}^{0} \sqrt{r}(i)(\mathrm{e}^{-rt})(-1)\mathrm{d}r = i\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$

$$\int_F \sqrt{z}\mathrm{e}^{zt}\mathrm{d}z = \int_0^{\infty} \sqrt{r}\mathrm{e}^{i\left(-\frac{\pi}{2}\right)}\mathrm{e}^{tr\mathrm{e}^{-i\pi}}\,\mathrm{e}^{-i\pi}\mathrm{d}r$$

$$= \int_0^{\infty} \sqrt{r}(-i)(\mathrm{e}^{-rt})(-1)\mathrm{d}r = i\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$

From the above calculations of contour integrations on C, D, E, F, G with residue theorem, we get the following (the closed contour Figure-2, A, B, C, D, E, F, G, A does not enclose any poles, so residue is zero)

$$x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$$

$$= \mathcal{L}^{-1}\left\{\sqrt{s-a}\right\}$$

$$= \frac{1}{2\pi i}\int_{A\to B}\mathrm{e}^{st}\sqrt{s-a}\mathrm{d}s = -\frac{\mathrm{e}^{at}}{2\pi i}\int_{D+E}\sqrt{z}\mathrm{e}^{zt}\mathrm{d}z$$

$$= -\frac{\mathrm{e}^{at}}{\pi}\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$$

In order to evaluate $\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r$ put $rt = \tau^2$ and $\mathrm{d}r = \frac{2\tau\mathrm{d}\tau}{t}$, and write

$$\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r = \frac{1}{t^{3/2}}\int_0^{\infty}\tau\mathrm{e}^{-\tau^2}\,2\tau\mathrm{d}\tau$$

We observe that $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{e}^{-\tau^2} = -2\tau\mathrm{e}^{-\tau^2}$, and thus we do integration by parts for above expression and write

$$\int_0^{\infty}\sqrt{r}\mathrm{e}^{-rt}\mathrm{d}r = -\frac{1}{t^{3/2}}\left(\left[\tau e^{-\tau^2}\right]_0^{\infty} - \int_0^{\infty}\mathrm{e}^{-\tau^2}\mathrm{d}\tau\right)$$

$$= \frac{\sqrt{\pi}}{2t^{3/2}}$$

We used known result $\int_0^{\infty}\mathrm{e}^{-\tau^2}\mathrm{d}\tau = \frac{\sqrt{\pi}}{2}$ in above derivation. Now compactly we write the result

$$\mathcal{L}^{-1}\left\{\sqrt{s-a}\right\} = -\frac{\mathrm{e}^{at}}{2\sqrt{\pi t^3}}$$

We can write form above derivation $\mathcal{L}^{-1}\left\{-\sqrt{s-a}\right\}=\frac{\mathrm{e}^{at}}{2\sqrt{\pi t^3}}$, and thus we have following useful Laplace transform identity

$$\mathcal{L}^{-1}\left\{\sqrt{s-a}-\sqrt{s-b}\right\}=\frac{\mathrm{e}^{bt}-\mathrm{e}^{at}}{2\sqrt{\pi t^3}}$$

Bromwich Path and Hankel path integration for inverse Laplace transformation for multi-valued function

In Figure-3, we have obtained the inverse Laplace transformation by performing the contour integration. We have closed contour via branch-point, branch cut as A,B,C,D,E,F,G,A. Residue Calculus says

$$\int_{A,B,C,D,E,F,G,A} F(s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi i\sum \text{Residues of poles}$$

Jordan Lemma gives

$$t>0 \quad : \lim_{R\uparrow\infty}\int_{C\text{ and }G}\mathrm{e}^{st}F(s)\mathrm{d}s=0$$

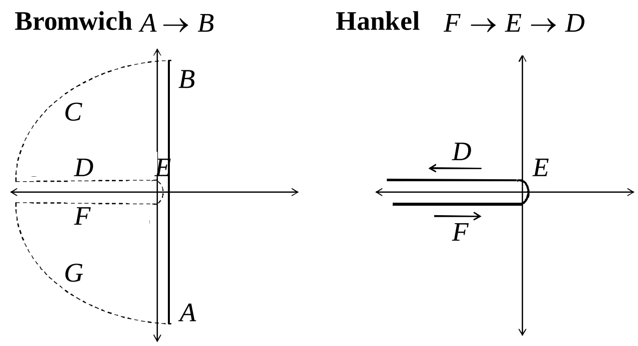
$$\int_{A,B}F(s)\,\mathrm{e}^{st}\mathrm{d}s=-\int_{C}F(s)\,\mathrm{e}^{st}\mathrm{d}s-\int_{D}F(s)\,\mathrm{e}^{st}\mathrm{d}s-\int_{E}F(s)\,\mathrm{e}^{st}\mathrm{d}s-\int_{F}F(s)\,\mathrm{e}^{st}\mathrm{d}s-\int_{G}F(s)\,\mathrm{e}^{st}\mathrm{d}s$$
$$+2\pi i\sum\text{Residues}$$

$$\begin{aligned}f(t)=\mathcal{L}^{-1}\left\{F(s)\right\}&=\lim_{R\uparrow\infty}\frac{1}{2\pi i}\int_{\text{Bromwich}}F(s)\mathrm{e}^{st}\mathrm{d}s\\&=-\int_{D+E+F}F(s)\mathrm{e}^{st}\mathrm{d}s+\sum\text{Residue}\\&=\int_{\text{Hankel}}F(s)\mathrm{e}^{st}\mathrm{d}s+\sum\text{Residue}\end{aligned}$$

Thus we have inverse Laplace transform as following

$$\begin{aligned}f(t)=\mathcal{L}^{-1}\left\{F(s)\right\}&=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}F(s)\mathrm{e}^{st}\mathrm{d}s\\&=\lim_{R\uparrow\infty}\frac{1}{2\pi i}\int_{\text{Bromwich}}F(s)\mathrm{e}^{st}\mathrm{d}s=\int_{\text{Hankel}}F(s)\mathrm{e}^{st}\mathrm{d}s+\sum\text{Residue}\end{aligned}$$

This is depicted in Figure-4, where we deform the Bromwich path to Hankel path covering Branch Point and Branch cut line-for obtaining inverse Laplace Transform. We stress here only for case of multi-valued function, where the poles of the function are spread onto multiple Riemann sheets, the Bromwich line gets deformed as Hankel contour. This is because the isolation of primary-Riemann sheet is required which is obtained via standard Branch cut line, as shown in Figure-3. For a function which is single valued, having poles only on the primary Riemann-sheet , the contour integration is performed on contours of Figure-1, where for inverse Laplace transformation one need not deform the Bromwich path.



$$\lim_{R\uparrow\infty} \frac{1}{2\pi i}\int_{\text{Bromwich}} F(s)\mathrm{e}^{st}\mathrm{d}s = \int_{\text{Hankel}} F(s)\mathrm{e}^{st}\mathrm{d}s + \sum \text{Residue}$$

Figure-4: Bromwich and Hankel contour

Inverse Laplace transformation without contour integration the Berberan-Santo Method

The $x(t)$ inverse Laplace Transform of $X(s)$ is obtained by following formula

$$x(t) = \frac{\mathrm{e}^{\sigma t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left[X(\sigma + i\omega)\right]\cos(\omega t) - \mathrm{Im}\left[X(\sigma + i\omega)\right]\sin(\omega t)\right)\mathrm{d}\omega$$

Here $s = \sigma + i\omega$ where $\mathrm{Re}[s] = \sigma$ is the line, where right of that line there is no singularity. This above formula is Berberan-Santo formula. Here there is no requirement of contour integration, but comes directly from Bromwich integral i.e. integration on the line *AB* (Figure-1a);

$$x(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \mathrm{e}^{st}X(s)\mathrm{d}s, \quad s = \sigma + i\omega, \quad \mathrm{d}s = i\mathrm{d}\omega$$

$$= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t + i\sin\omega t\right)\left(X(\sigma + i\omega)\right)\mathrm{d}\omega; \quad X = \mathrm{Re}[X] + i\,\mathrm{Im}[X]$$

$$x(t) = \mathrm{Re}\left[\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t + i\sin\omega t\right)\left(\mathrm{Re}[X] + i\,\mathrm{Im}[X]\right)\mathrm{d}\omega\right]$$

$$= \frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}[X]\cos\omega t - \mathrm{Im}[X]\sin\omega t\right)\mathrm{d}\omega$$

Since $x(t)$ is real function we have only extracted the real part and say

$$\mathrm{Im}\left[\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}(\cos\omega t+i\sin\omega t)\big(\mathrm{Re}[X]+i\,\mathrm{Im}[X]\big)\mathrm{d}\omega\right]=0$$

$$\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\big(\mathrm{Re}[X]\sin\omega t+\mathrm{Im}[X]\cos\omega t\big)\mathrm{d}\omega=0$$

But we have from definition of Laplace transform i.e. $X(s)=\int_0^{\infty}\big(x(t)\big)\mathrm{e}^{-st}\mathrm{d}t$ and by putting $s=\sigma+i\omega$ we get following

$$X(\sigma+i\omega)=\int_0^{\infty}\big(x(t)\big)\mathrm{e}^{-t(\sigma_0+i\omega)}\mathrm{d}t$$

$$=\int_0^{\infty}\mathrm{e}^{-\sigma_0 t}\big(x(t)\big)\cos(\omega t)\mathrm{d}t-i\int_0^{\infty}\mathrm{e}^{-\sigma_0 t}\big(x(t)\big)\sin(\omega t)\mathrm{d}t$$

This gives following

$$\mathrm{Re}[X]=\int_0^{\infty}\mathrm{e}^{-\sigma_0 t}\big(x(t)\big)\cos(\omega t)\mathrm{d}t$$

$$\mathrm{Im}[X]=-\int_0^{\infty}\mathrm{e}^{-\sigma_0 t}\big(x(t)\big)\sin(\omega t)\mathrm{d}t$$

We find that function $\mathrm{Re}[X]$ is even function, call it $e(\omega)$ in variable ω and the function $\mathrm{Im}[X]$ is odd function in variable ω, call it $o(\omega)$ .We get

$$\big(\mathrm{Re}[X]\cos\omega t-\mathrm{Im}[X]\sin\omega t\big)=\big(e(\omega)\big)\cos\omega t-\big(o(\omega)\big)\sin\omega t$$

as 'even function'. That is even function $e(\omega)$ multiplied by even function i.e. $\cos\omega t$ gives even function, and odd function $o(\omega)$ multiplied by odd function i.e. $\sin\omega t$ gives even function. Therefore for overall even function integrand we have the integral as

$$x(t)=\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\big(\mathrm{Re}[X]\cos\omega t-\mathrm{Im}[X]\sin\omega t\big)\mathrm{d}\omega$$

$$=\frac{\mathrm{e}^{\sigma t}}{\pi}\int_{0}^{\infty}\big(\mathrm{Re}[X]\cos\omega t-\mathrm{Im}[X]\sin\omega t\big)\mathrm{d}\omega$$

Mittag-Leffler function used to formulate non-Debye relaxation-as generalized disordered relaxation

We will discuss the non-Debye type relaxation

$$f(t)\sim E_{\alpha}(-kt^{\alpha}),\quad t>0;\quad k>0$$

i.e. via Mittag-Leffler function, for cases where $0<\alpha<1$ i.e. the decay function is monotonically decaying; also for cases $1<\alpha<2$ with oscillatory decay. While Debye case is $f(t)\sim\mathrm{e}^{-\lambda t}$ which is also 'ordered relaxation'.

The Mittag-Leffler function is defined as following in series representation, where $E_{\alpha}(x)$ is one-parameter and $E_{\alpha,\beta}(x)$ is two-parameter Mittag-Leffler function

$$E_\alpha(-z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\alpha n+1)},\quad E_{\alpha,\beta}(-z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\alpha n+\beta)},\quad E_{\alpha,1}(-z)=E_\alpha(-z)$$

$$E_0(-z)=\frac{1}{1+z}$$

$$E_1(-z)=\mathrm{e}^{-z}$$

$$E_2(-z)=\cosh\sqrt{-z}=\cos\sqrt{z}$$

For the order of Mittag-Leffler function α between zero and one- we see it as monotonically decaying curve, thus the order between zero and one Mittag-Leffler interpolates decaying relaxation function of pure hyperbolic decay to a pure exponential decay (Debye Type) see Figure-5.

For order of Mittag-Leffler function α between one and two we sees oscillatory decaying curve, thus the order between one and two the Mittag-Leffler interpolates decaying relaxation function of pure exponential decay (Debye) to a pure oscillatory decay, refer Figure-6.

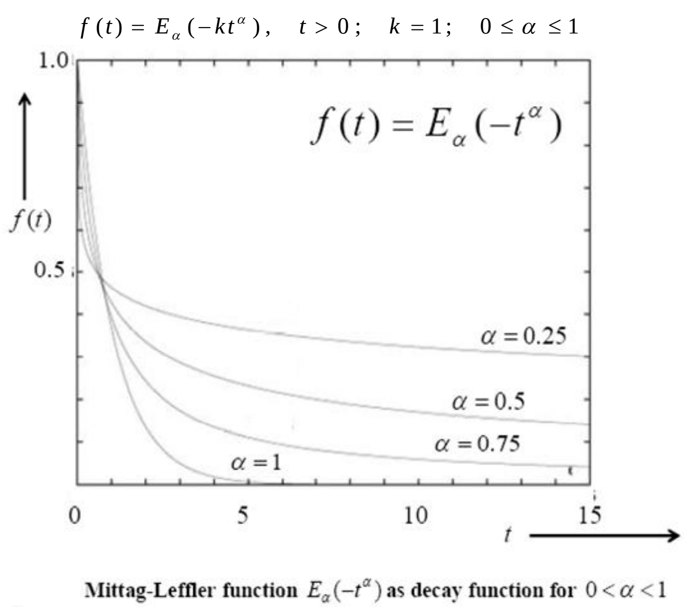
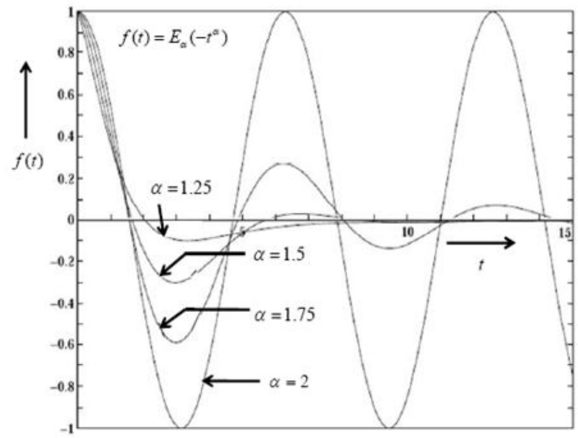


Figure-5: Mittag-Leffler function interpolating non-Debye relaxations by various orders from zero to one

$$f(t) = E_{\alpha}(-kt^{\alpha}), \quad t > 0; \quad k = 1; \quad 1 < \alpha \le 2$$



: **Relaxation decay with oscillation with Mittag-Leffler function** $f(t) = E_{\alpha}(-t^{\alpha})$ **for** $1 < \alpha < 2$

Figure-6: Oscillatory decaying relaxation via Mittag-Leffler function by various orders between one and two

Disordered and Ordered Relaxation

The non-Debye relaxation has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of relaxation rates call as λ, varying zero to infinity. Figure-7 gives picture of several bodies relaxing or decaying at various rates. The Debye relaxation has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning all bodies with relaxation rates same λ -having same rate of decay for all bodies. Thus all bodies are orderly decaying with discipline. Figure-8 gives the picture of several bodies relaxing in disciplined way with same rate of decay. In Figure-9, we see the rates for Debye relaxation is concentrated at one point i.e. at $\lambda = 100$ and for disciplined relaxation we do not have any spread in the rate of relaxation i.e. we have

$$f(t) \sim \mathrm{e}^{-100t} + \mathrm{e}^{-100t} + \mathrm{e}^{-100t} + \ldots\ldots$$

$$\lambda_{\max} - \lambda_{\min} = 0$$

For disordered relaxation (non-Debye) we have the various rates as depicted in the Figure-9, i.e.

$$f(t) = \mathrm{e}^{-0,0005t} + \mathrm{e}^{-0.01t} + \mathrm{e}^{-t} + \mathrm{e}^{-100t} + \mathrm{e}^{-5000t} + \mathrm{e}^{-10^{8}t} \ldots\ldots$$

$$\lambda_{\max} - \lambda_{\min} \sim \infty$$

Where the spread of relaxation rate between various multi-bodies are almost infinity.

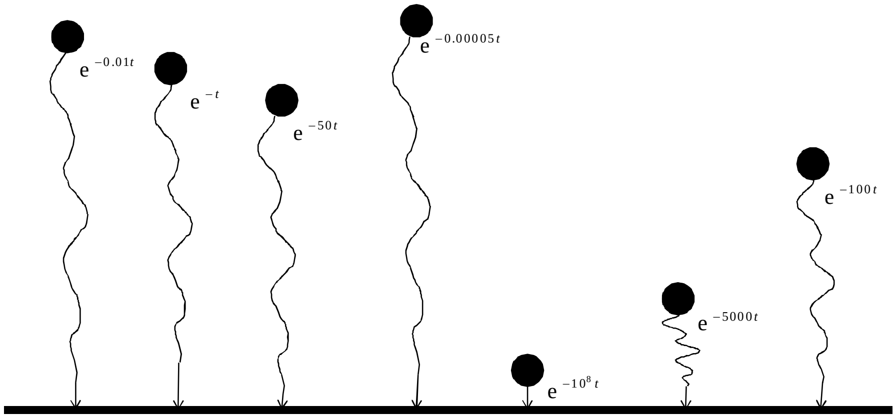


Figure-7: Many bodies relaxing simultaneously all at different rates of decay-Disordered (Non-Debye) relaxation

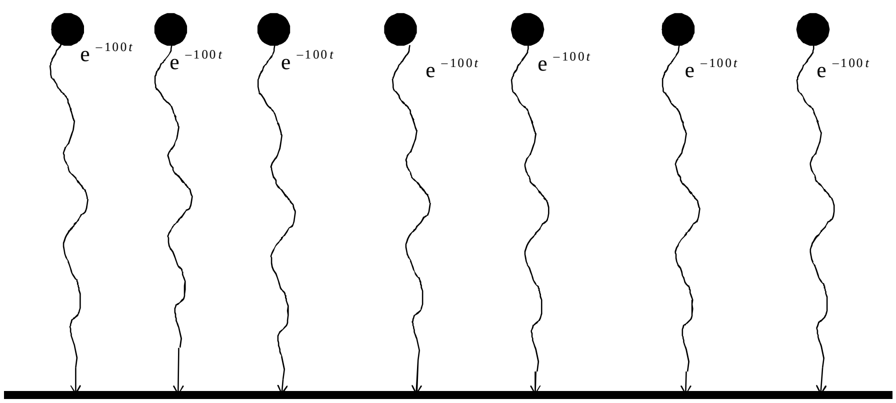


Figure-8: Many bodies relaxing simultaneously with same rate of decay-Ordered Debye relaxation

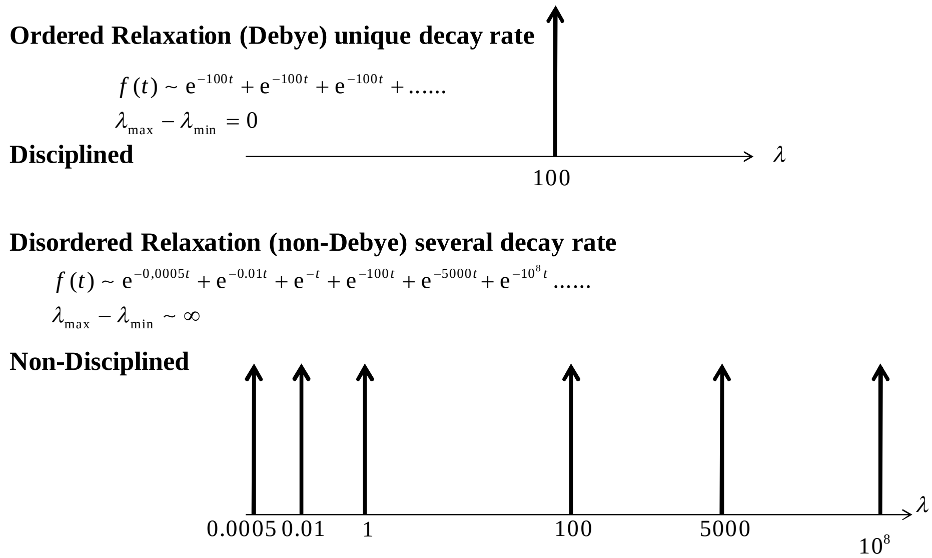


Figure-9: Ordered and disordered relaxation

Formulation of disordered relaxation

The complex decay is expressed as following with several Debye rate constants $\lambda_1, \lambda_2, \lambda_3, \ldots$ with weights $a_1, a_2, a_3, \ldots$. We write following composite relaxation expression as sum of several 'discrete' relaxations of Debye type i.e.

$$f(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + \ldots$$
$$= \sum a_j e^{-\lambda_j t}$$
$$f(0) = a_1 + a_2 + a_3 + \ldots$$

In continuum limit we may write the above as

$$f(t) = \int_0^\infty \left(H(\lambda) \right) e^{-\lambda t} d\lambda$$

The function i.e. $H(\lambda)$ is the distribution-function of the rate of the relaxation λ of the process, or we may call it as histogram of relaxation rates. We note here the weights a_j can be positive or negative. Also $H(\lambda)$ can be too having positive or negative values.

While for the case with discrete set of relaxation rates i.e. $\lambda_j = k_1, k_2, k_3, \ldots$ the rate distribution function would be having discrete delta functions $\delta(\lambda - \lambda_j)$, $j = 1, 2, 3, \ldots$ at points $k_1, k_2, k_3, \ldots$ which we express as by using the property of delta function i.e. $\int \left(\delta(x - x_0) \right) g(x) dx = g(x_0)$.

$$f(t)=a_1\mathrm{e}^{-k_1t}+a_2\mathrm{e}^{-k_2t}+a_3\mathrm{e}^{-k_3t}+...=\int_0^\infty\left(H(\lambda)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$$
$$=\int_0^\infty a_1\delta(\lambda-k_1)\,\mathrm{e}^{-\lambda t}\,\mathrm{d}\lambda+\int_0^\infty a_2\delta(\lambda-k_2)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda+.....$$
$$\int\left(\delta(x-x_0)\right)g(x)\mathrm{d}x=g(x_0)$$
$$H(\lambda)=a_1\delta(\lambda-k_1)+a_2\delta(\lambda-k_2)+a_3\delta(\lambda-k_3)+...$$
$$=\sum a_j\delta(\lambda-\lambda_j);\qquad \lambda_j\Big|_{j=1,2,3,...}=k_1,k_2,k_3,....$$

From above formulation if we have only one single Debye relaxation i.e. having only one rate constant $\lambda=k_0$ then $H(\lambda)=a_0\delta(\lambda-k_0)$ and we have following

$$f(t)=\int_0^\infty\left((H(\lambda))\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$$
$$=\int_0^\infty\left(a_0\delta(\lambda-k_0)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda=a_0\mathrm{e}^{-k_0t}$$

Thus Non-Debye or disordered relaxation has Histogram of relaxation rates (discrete or continuous). We depict this in Figure-10, where composition of non-Debye relaxation function with several relaxation functions of Debye type in discrete and continuum realization gives Histogram. The shape/type of Histogram function gives ordering of relaxation rates for disordered relaxation

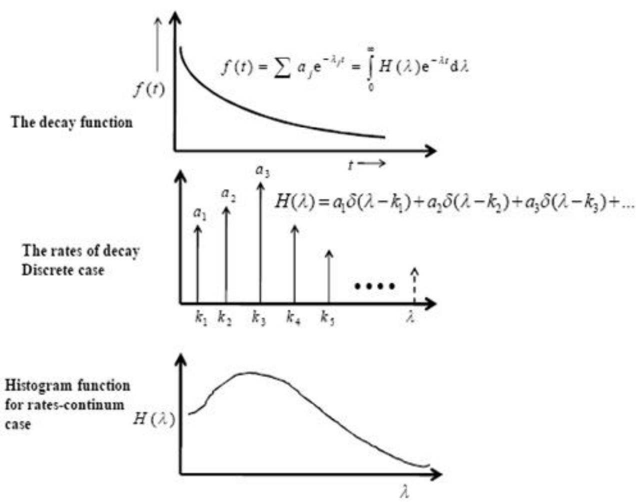


Figure-10: Composition of non-Debye relaxation function and histogram of relaxation rates

Extraction of relaxation rate distribution function-Histogram

Following integral transform relation i.e. called Laplace integral

$$F(s) \overset{\text{def}}{=} \int_0^{\infty} \left(f(t) \right) \mathrm{e}^{-st} \mathrm{d}t, \quad t > 0, \quad s = \mathrm{Re}[s] + i\omega; \quad i = \sqrt{-1}$$

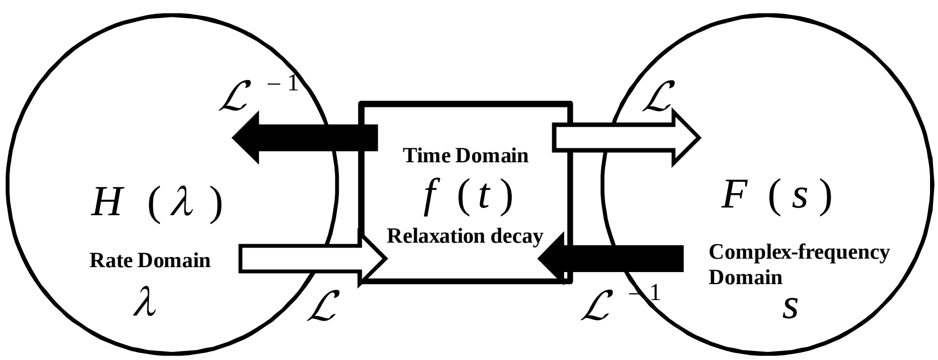
$$F(s) = \mathcal{L}\{ f(t) \} \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(F(s) \right) \mathrm{e}^{st} \mathrm{d}s$$

We have derived/formulated non-Debye relaxation as $f(t) = \int_0^{\infty} \left(H(\lambda) \right) \mathrm{e}^{-t\lambda} \mathrm{d}\lambda$ .Therefore in order to get the rate distribution-function $H(\lambda)$ from the decay curve or relaxation-function $f(t)$ we need to perform inverse Laplace Transform of the time function. The definition of inverse Laplace Transform is described as following integral expressions on Bromwich path-on complex 'time'-plane $t = \mathrm{Re}[t] + i\,\mathrm{Im}[t] = x_0 + iy$

$$H(\lambda) = \mathcal{L}^{-1}\{ f(t) \} = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \left(f(t) \right) \mathrm{e}^{t\lambda} \mathrm{d}t$$

Figure-11, gives the extraction process via Laplace transformation formula



$$H(\lambda) = \mathcal{L}^{-1}\{ f(t) \} = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} f(t) \mathrm{e}^{\lambda t} \mathrm{d}t$$

$$f(t) = \mathcal{L}\{H(\lambda)\} = \int_0^{\infty} H(\lambda) \mathrm{e}^{-t\lambda} \mathrm{d}\lambda$$

$$t = \mathrm{Re}[t] + i\,\mathrm{Im}[t] = x_0 + iy$$

Here time is complex variable

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) \mathrm{e}^{st} \mathrm{d}s$$

$$F(s) = \mathcal{L}\{ f(t) \} = \int_0^{\infty} f(t) \mathrm{e}^{-st} \mathrm{d}t$$

$$s = \mathrm{Re}[s] + i\,\mathrm{Im}[s] = \sigma + i\omega$$

Here frequency is complex variabl

Figure-11: For extracting histogram function do inverse Laplace transform on time function unlike usual done on complex function of frequency

Thus for a Universal dielectric relaxation as we mentioned as a case of non-Debye response, with $f(t) = t^{-\alpha}; \quad 0 < \alpha < 1$, with this extraction method we obtain $H(\lambda) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha - 1}, \quad \lambda > 0$ as ordered histogram function in form of Zipf's power law distribution. With this method we can write from standard Laplace transform tables few histogram functions for corresponding decay functions, as mentioned in Table-1.

S.No.	$f(t), \quad t \geq 0$	$H(\lambda); \quad \lambda \geq 0$
1	e^{-t}	$\delta(\lambda - 1)$
2	$\frac{1}{t+1}$	$e^{-\lambda}$
3	$\frac{1}{t}$	1
4	$\frac{1}{t^2}$	λ
5	$\frac{1}{t^n}, \quad n = 1,2,3...$	$\frac{1}{(n-1)!}\lambda^{n-1}$
6	$\frac{1}{t^\alpha}; \quad 0 < \alpha < 1$	$\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$
7	$\frac{1}{t^2+1}$	$\sin\lambda$
8	$\frac{t}{t^2+1}$	$\cos\lambda$

Table-1: Some relaxation decay functions of non-oscillatory type and their rate relaxation histograms

Relaxation function as Mittag-Leffler function (order $0 < \alpha < 1$)

We have given the gist in previously about Mittag-leffler function that can interpolate various disordered relaxation or decay curves, while its order varies from $0 < \alpha < 1$ as monotonically decaying relaxation and order $1 < \alpha < 2$ as oscillatory decaying function. We start our study with following Laplace identity for one parameter Mittag-Leffler function i.e.

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k}; \quad k > 0$$

This is verified as follows by series definition of Mittag-Leffler function

$$E_\alpha(-kt^\alpha) = 1 - \frac{kt^\alpha}{\Gamma(1+\alpha)} + \frac{k^2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{k^3t^{3\alpha}}{\Gamma(1+3\alpha)} + ...$$

and then taking term by term Laplace transform, with known Laplace identity $\mathcal{L}\left\{t^n\right\} = \frac{\Gamma(1+n)}{s^{1+n}}$ as

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \mathcal{L}\left\{1\right\} - \mathcal{L}\left\{\tfrac{kt^\alpha}{\Gamma(1+\alpha)}\right\} + \mathcal{L}\left\{\tfrac{k^2t^{2\alpha}}{\Gamma(1+2\alpha)}\right\} - \mathcal{L}\left\{\tfrac{k^3t^{3\alpha}}{\Gamma(1+3\alpha)}\right\} + ...$$

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \tfrac{1}{s} - \tfrac{k}{s^{1+\alpha}} + \tfrac{k^2}{s^{1+2\alpha}} - \tfrac{k^3}{s^{1+3\alpha}} + ...$$

$$= \tfrac{1}{s}\left(1 - \tfrac{k}{s^\alpha} + \left(\tfrac{k}{s^\alpha}\right)^2 - \left(\tfrac{k}{s^\alpha}\right)^3 + ...\right)$$

$$= \tfrac{1}{s}\left(\tfrac{1}{1+(k/s^\alpha)}\right) \quad \text{using} \quad \tfrac{1}{1+x} = 1 - x + x^2 - x^3 .. \qquad |x| < 1$$

$$= \tfrac{s^{\alpha-1}}{s^\alpha + k}; \qquad \left|\tfrac{k}{s^\alpha}\right| < 1$$

Thus we get $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha+k}$ with $\mathrm{Re}[s] > k^{1/\alpha}$

Now we are expressing Mittag-Leffler function in integral on Hankel path. In terms of inverse Laplace integral, we write the following

$$E_{\alpha}(-kt^{\alpha}) = \mathcal{L}^{-1}\left\{\tfrac{s^{\alpha}}{s^{\alpha}+k}\right\}; \quad k>0$$

$$E_{\alpha}(-kt^{\alpha}) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s + \sum \text{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\right]$$

This we get from Figure-4. We bend the Bromwich path of integration into Hankel path , a loop which starts from $-\infty$ along the lower side of negative real axis encircles the circular disc $|s| = \in$ in positive (anticlockwise sense) and ends at $-\infty$ along the upper side of negative real axis, (Refer Figure-4). So we write

$$\begin{aligned} f(t) = E_{\alpha}(-kt^{\alpha}) &= \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s + \sum \text{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\right] \\ &= f_1(t) + f_2(t) \end{aligned}$$

For the function $\frac{s^{\alpha-1}}{s^{\alpha}+k}$ the poles are at $s = |k|^{1/\alpha}\exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right); \quad m = 0, \pm 1, \pm 2, \ldots$

For $0<\alpha<1$ we have $\left|(2m+1)\frac{\pi}{\alpha}\right| > \pi$ meaning that no poles are in $-\pi < \arg[s] < \pi$ (i.e. in the Primary Riemann Sheet). Therefore there are no poles in the 'primary Riemann-sheet'. Thus for $0<\alpha<1$

$$f_2(t) = \sum \text{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)\right] = 0$$

and we have

$$E_{\alpha}(-kt^{\alpha}) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = f_1(t); \quad 0<\alpha<1$$

There are three contributions on the Hankel path (Refer Figure-4). The one (1) is on the circle $s = \in \mathrm{e}^{i\theta}$ as limit $\in \downarrow 0$ that is

$$\begin{aligned} \frac{1}{2\pi i}\int_{s=\in \mathrm{e}^{i\theta}} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s &= \frac{1}{2\pi i}\lim_{\in\downarrow 0}\int_{s=\in \mathrm{e}^{i\theta}} \mathrm{e}^{\in t\exp(i\theta)}\left(\frac{\in^{\alpha-1}\mathrm{e}^{i(\alpha-1)\theta}}{\in^{\alpha}\mathrm{e}^{i\alpha\theta}+k}\right)\left(\in i\mathrm{e}^{i\theta}\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in \mathrm{e}^{i\theta}} (1)\left(\frac{\in^{\alpha}\mathrm{e}^{i(\alpha-1)\theta}}{\in^{\alpha}\mathrm{e}^{i\alpha\theta}+k}\right)\left(\mathrm{e}^{i\theta}\right)\mathrm{d}\theta; \quad \alpha>0 \\ &= \frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in \mathrm{e}^{i\theta}} \frac{\in^{\alpha}\mathrm{e}^{i\alpha\theta}}{k}\mathrm{d}\theta = 0 \end{aligned}$$

The second (2) contribution is from line below negative real axis we call $s = r\mathrm{e}^{-i\pi}$ with r varying from ∞ to 0 that gives

$$\begin{aligned} \frac{1}{2\pi i}\int_{s=r\mathrm{e}^{-i\pi}} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s &= \frac{1}{2\pi i}\int_{\infty}^{0} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=r\mathrm{e}^{-i\pi}}(-\mathrm{d}r) \\ &= \frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=r\mathrm{e}^{-i\pi}}(\mathrm{d}r) \end{aligned}$$

The third (3) contribution is from line above negative real axis we call $s = r\mathrm{e}^{i\pi}$ with r varying from 0 to ∞ that gives

$$\frac{1}{2\pi i}\int_{s=re^{i\pi}} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)ds = \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{i\pi}}(-dr)$$

Thus total contributions from Hankel path is sum of the three components is following

$$E_{\alpha}(-kt^{\alpha}) = \frac{1}{2\pi i}\int_{Ha} e^{st}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)ds$$

$$= -\frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{i\pi}} dr + \frac{1}{2\pi i}\int_0^{\infty} e^{-rt}\left(\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}} dr$$

Since $E_{\alpha}(-kt^{\alpha})$ is a real function, we can write the above as by only writing the real part i.e.

$$E_{\alpha}(-kt^{\alpha}) = -\frac{1}{2\pi}\int_0^{\infty} e^{-rt}\,\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}} dr + \frac{1}{2\pi}\int_0^{\infty} e^{-rt}\,\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{-i\pi}} dr$$

$$= \int_0^{\infty} e^{-rt}\left(-\tfrac{1}{\pi}\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}}\right)dr = f_1(t); \quad 0<\alpha<1$$

We compare with the formula of Laplace transform i.e. $F(s)=\mathcal{L}\{f(t)\}=\int_0^{\infty} f(t)e^{-st}dt$ and we write the following

$$\mathcal{L}\left(-\tfrac{1}{\pi}\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}}\right) = E_{\alpha}(-kt^{\alpha}); \quad 0<\alpha<1; \quad k>0$$

$$\mathcal{L}^{-1}\left\{E_{\alpha}(-kt^{\alpha})\right\} = -\tfrac{1}{\pi}\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}}$$

$$= \frac{1}{\pi}\left(\frac{kr^{\alpha-1}\sin\alpha\pi}{r^{2\alpha}+2kr^{\alpha}\cos\alpha\pi+k^2}\right), \quad 0<\alpha<1$$

Thus we have obtained $\mathcal{L}^{-1}\left\{E_{\alpha}(-kt^{\alpha})\right\}; \quad 0<\alpha<1$ as a function of r varying from 0 to ∞.

Relaxation rate histogram function for relaxation as Mittag-Leffler function of order $0<\alpha<1$

We compare with the formula derived i.e. $f(t)=\int_0^{\infty}\left(H(\lambda)\right)e^{-t\lambda}d\lambda$ we may write the derived expression as in previous section, i.e. $\mathcal{L}\left(-\tfrac{1}{\pi}\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}}\right) = E_{\alpha}(-kt^{\alpha}); \quad 0<\alpha<1$ by changing variable r to λ as following

$$E_{\alpha}(-kt^{\alpha}) = \int_0^{\infty} e^{-\lambda t}\left(-\tfrac{1}{\pi}\mathrm{Im}\left[\tfrac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=\lambda e^{i\pi}}\right)d\lambda = \int_0^{\infty} e^{-t\lambda}\left(H_{\alpha}(\lambda)\right)d\lambda; \quad 0<\alpha<1; \quad k>0$$

That gives the histogram of relaxation rates $H_{\alpha}(\lambda)$ as follows for $f(t)=E_{\alpha}(-kt^{\alpha}), \quad 0<\alpha<1$

$$f(t)=E_{\alpha}(-kt^{\alpha}), \quad 0<\alpha<1; \quad t>0, \quad k=1$$

$$H_{\alpha}(\lambda) = -\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+1}\right]_{s=\lambda e^{i\pi}} = \frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^{\alpha}\cos\alpha\pi+1}\right) = \mathcal{L}^{-1}\left\{f_1(t)\right\}; \quad 0<\alpha<1$$

We observe that the histogram function is positive for all $\lambda>0$ for $0<\alpha<1$ and the histogram function is negative for all λ for $1<\alpha<2$. The denominator of histogram function is $>(\lambda^{\alpha}-1)^2\geq 0$. Figure-12 gives the plots of $H_{\alpha}(\lambda)$ for various values $0<\alpha<1$.

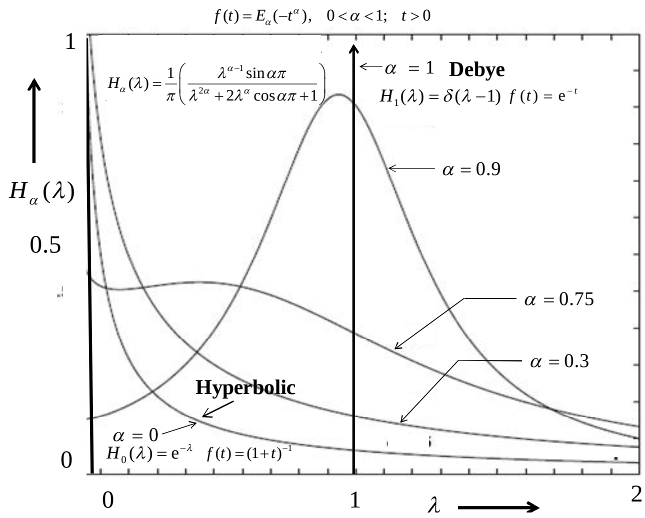


Figure-12: Histogram showing various non-Debye relaxations for monotonically decaying Mittag-Leffler function as the order varies from zero to one.

We observe that for $\alpha=0$ we have exponential distribution as $H_0(\lambda)=e^{-\lambda}$, $f(t)=(1+t)^{-1}$, that is for pure hyperbolic decay. Thereafter $H_\alpha(\lambda)$ it is given by the formula $H_\alpha(\lambda)=\frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right)$ for $f(t)=E_\alpha(-t^\alpha),\quad 0<\alpha<1$, for various disordered relaxation. At $\alpha=1$ we get disciplined ordered as the histogram of relaxation rate is $H_1(\lambda)=\delta(\lambda-1),\quad f(t)=E_1(-t)=e^{-t}$; that is pure Debye relaxation. The extreme cases $\alpha=0$ and $\alpha=1$ are obtained via Table-1, from standard Laplace transformation formulas.

Thus we write inverse Laplace transform of Mittag-Leffler function of one-parameter as following ways

$$\mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{1}{\pi}\left(\frac{k\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2k\lambda^\alpha\cos\alpha\pi+k^2}\right);\quad 0<\alpha<1:\quad t\mapsto\lambda$$

$$\mathcal{L}^{-1}\left\{E_\alpha(-ks^\alpha)\right\}=\frac{1}{\pi}\left(\frac{kt^{\alpha-1}\sin\alpha\pi}{t^{2\alpha}+2kt^\alpha\cos\alpha\pi+k^2}\right);\quad 0<\alpha<1:\quad s\mapsto t$$

In the case of one-parameter Mittag-Leffler function, one gets a compact representation as above due to the fact that we have Laplace pair i.e. $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{s^{\alpha-1}}{s^\alpha+k}$. If we do not have this type of compact representation, then this described method is difficult to apply. For example two parameter Mittag-Leffler function i.e. $E_{\alpha,\beta}(-kt^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n k^n t^{\alpha n}}{\Gamma(\alpha n+\beta)}$ does not have any such Laplace transform as we have for one-parameter Mittag-Leffler function.

We have Laplace transform pair i.e.

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha+k}$$

For this type of relaxation, we can use the above described technique of integration on Hankel path and get the following relaxation rate distribution histogram

$$H_{\alpha,\beta}(\lambda)=\mathcal{L}^{-1}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^{\alpha})\right\}=-\frac{1}{\pi}\operatorname{Im}\left[\frac{s^{\alpha-\beta}}{s^{\alpha}+k}\right]_{s=\lambda e^{i\pi}}\,,\quad \alpha,\beta>0;\quad 0<\alpha<1$$

$$=\frac{1}{\pi}\left(\lambda^{\alpha-\beta}\,\frac{k\sin(\beta-\alpha)\pi+\lambda^{\alpha}\sin\beta\pi}{\lambda^{2\alpha}+2k\lambda^{\alpha}\cos\alpha\pi+k^{2}}\right)$$

Mittag-Leffler function as Laplace transform of M-Wright function to get histogram of relaxing rates for non-Debye Mittag-Leffler decay for order $0<\alpha<1$

The Histogram function can also be analyzed by the identity

$$\mathcal{L}\left\{M_{\alpha}(t)\right\}=E_{\alpha}(-s)$$

for $0<\alpha<1$ where $M_{\alpha}(t)$ is M-Wright Function (M-for Minardi). The series definition is

$$M_{\alpha}(z)=\sum_{n=0}^{\infty}\left(\frac{(-1)^{n}z^{n}}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\right)\qquad 0<\alpha<1$$

We have some following-Wright functions

$$M_{0}(z)=e^{-z}\qquad M_{\frac{1}{2}}(z)=\frac{1}{\sqrt{\pi}}e^{\left(-z^{2}/4\right)}\qquad M_{1}(z)=\delta(t-1)$$

With this M-Wright function we have Histogram function for Mittag-Leffler decay curve, as

$$f(t)=E_{\alpha}(-kt^{\alpha});\quad kt^{\alpha}=\overline{t}\,,\qquad H_{\alpha}(\lambda)=\mathcal{L}^{-1}\left\{E_{\alpha}(-\overline{t})\right\}=M_{\alpha}(\lambda)$$

For $\alpha=\frac{1}{2}$ the distribution of relaxing rate or histogram is Gaussian function with peak at zero, i.e. $H_{\frac{1}{2}}(\lambda)=M_{\frac{1}{2}}(\lambda)=\frac{1}{\sqrt{\pi}}e^{-\lambda^{2}/4}$, and for $\alpha=0$ we have exponential distribution histogram, $H_{0}=e^{-\lambda}$ and for order one (Debye relaxation), we have delta distributed $H_{1}(\lambda)=\delta(\lambda-1)$. In this deliberation we are not going into details of M-Wright function.

Relaxation function as Mittag-Leffler function of order $1<\alpha<2$ **oscillatory decay and its rate relaxation distribution**

For, the case $1<\alpha<2$ the function $\frac{s^{\alpha-1}}{s^{\alpha}+k}$ the poles are at $s=|k|^{\frac{1}{\alpha}}\exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right)$. At $m=0$ we have pole at $s_{1}=|k|^{\frac{1}{\alpha}}\exp\left(i\frac{\pi}{\alpha}\right)$ and for $m=-1$ the pole is at $s_{2}=|k|^{\frac{1}{\alpha}}\exp\left(-i\frac{\pi}{\alpha}\right)$ these two are complex conjugate and remain in the primary Riemann sheet- responsible for response. The rest infinite poles are at other Riemann sheets, does not influence the primary response. We represent them as $s_{1,2}=\sigma_{0}\pm i\omega_{0}$ where $\sigma_{0}=|k|^{\frac{1}{\alpha}}\cos\left(\frac{\pi}{\alpha}\right)$ and $\omega_{0}=|k|^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)$. Thus, for $1<\alpha<2$ case, we have $f_{2}(t)=\sum\text{Residues}\left[e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\right]\neq 0$. We recall we have split $f(t)=E_{\alpha}(-kt^{\alpha})$ into two functions $f_{1}(t)$, which comes from Hankel's path, and $f_{2}(t)$ due to residues of poles in primary Riemann-sheet-left of Bromwich line. We thus have following

$$E_\alpha(-kt^\alpha) = f_1(t) + f_2(t) = \mathcal{L}^{-1}\left\{\tfrac{s^{\alpha-1}}{s^\alpha+k}\right\} = \tfrac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s \qquad 1<\alpha<2$$

$$= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s + \sum \text{Residues}\left[\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s_1,s_2}$$

$$= \int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\,\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=\lambda \mathrm{e}^{i\pi}}\right)\mathrm{d}\lambda + \sum \text{Residues}\left[\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s_1,s_2}$$

$$s_1 = \sigma_0 + i\omega_0 = |k|^{1/\alpha}\,\mathrm{e}^{i\pi/\alpha}, \quad s_2 = \sigma_0 - i\omega_0 = |k|^{1/\alpha}\,\mathrm{e}^{-i\pi/\alpha} \qquad \sigma_0 = |k|^{1/\alpha}\cos\tfrac{\pi}{\alpha}, \quad \omega_0 = |k|^{1/\alpha}\sin\tfrac{\pi}{\alpha}$$

The residue calculation is following for $\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}$ at poles in primary Riemann-sheets those as $s_{1,2} = \sigma_0 \pm i\omega_0$, for the case $1<\alpha<2$.

$$\sum \text{Residues}\left[\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s_1,s_2} = \sum \text{Residues}\left[\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{(s-s_1)(s-s_2)}\right]_{s_1,s_2}$$

$$= \lim_{s\to s_1}(s-s_1)\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{(s-s_1)(s-s_2)} + \lim_{s\to s_2}(s-s_2)\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{(s-s_1)(s-s_2)}$$

$$= \mathrm{e}^{s_1 t}\,\frac{s_1^{\alpha-1}}{s_1-s_2} + \mathrm{e}^{s_2 t}\,\frac{s_2^{\alpha-1}}{s_2-s_1}$$

$$= \frac{\mathrm{e}^{\sigma_0 t}\mathrm{e}^{i\omega_0 t}\left(|k|^{1/\alpha}\,\mathrm{e}^{i\pi/\alpha}\right)^{\alpha-1}}{2i\omega_0} - \frac{\mathrm{e}^{\sigma_0 t}\mathrm{e}^{-i\omega_0 t}\left(|k|^{1/\alpha}\,\mathrm{e}^{-i\pi/\alpha}\right)^{\alpha-1}}{2i\omega_0}$$

$$= |k|^{\frac{\alpha-1}{\alpha}}\,\frac{\mathrm{e}^{\sigma_0 t}}{2i|k|^{1/\alpha}\sin\frac{\pi}{\alpha}}\left(\mathrm{e}^{i\omega_0 t}\mathrm{e}^{i\pi\left(\frac{\alpha-1}{\alpha}\right)} - \mathrm{e}^{-i\omega_0 t}\mathrm{e}^{-i\pi\left(\frac{\alpha-1}{\alpha}\right)}\right)$$

$$= |k|^{\left(1-\frac{2}{\alpha}\right)}\,\frac{\mathrm{e}^{\sigma_0 t}}{\sin\frac{\pi}{\alpha}}\left(\frac{-\mathrm{e}^{i\left(\omega_0 t-\frac{\pi}{\alpha}\right)} + \mathrm{e}^{-i\left(\omega_0 t-\frac{\pi}{\alpha}\right)}}{2i}\right)$$

$$= \frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}}\,\mathrm{e}^{\sigma_0 t}\sin\left(\tfrac{\pi}{\alpha}-\omega_0 t\right)$$

$$= \left(\frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}}\right)\mathrm{e}^{\left(|k|^{1/\alpha}\cos\frac{\pi}{\alpha}\right)t}\sin\left(\tfrac{\pi}{\alpha}-\left(|k|^{1/\alpha}\sin\tfrac{\pi}{\alpha}\right)t\right)$$

For $k=1$ we have for $1<\alpha<2$ the decay function is $f(t) = E_\alpha(-t^\alpha) = f_1(t) + f_2(t)$ with

$$f_2(t) = \sum \text{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+1}\right)\right]_{s_1,s_2} = \left(\tfrac{1}{\sin\frac{\pi}{\alpha}}\right)\mathrm{e}^{\left(\cos\frac{\pi}{\alpha}\right)t}\sin\left(\tfrac{\pi}{\alpha}-\left(\sin\tfrac{\pi}{\alpha}\right)t\right)$$

The part i.e. $f_2(t)$ is oscillatory decaying part, given as $f_2(t) = -\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\omega_0 t - \frac{\pi}{\alpha}\right)$ where $\sigma_0 = \cos\frac{\pi}{\alpha} < 0$ and for $1<\alpha<2$. This factor gives exponential decay of amplitude $f_2(0) = -\mathrm{A}\sin\left(-\frac{\pi}{\alpha}\right) = \left(\frac{1}{\sin\frac{\pi}{\alpha}}\right)\left(\sin\left(-\frac{\pi}{\alpha}\right)\right) = 1$ at $t=0$. The oscillatory part is $\sin\left(\omega_0 t - \frac{\pi}{\alpha}\right)$ with $\omega_0 = \sin\frac{\pi}{\alpha}$.

We observe that for $\alpha = 2$ we have $f(t) = E_2(-t^2) = \cos t$ as $\sin\left(\frac{\pi}{2} - \sin\frac{\pi}{2}t\right) = \cos t$, $\mathrm{A} = 1$ and $\sigma_0 = 0$. Also we note that for $\alpha = 2$ the term $\mathcal{L}^{-1}\left\{f_1(t)\right\} = \frac{1}{\pi}\left(\frac{r^{\alpha-1}\sin\alpha\pi}{r^{2\alpha}+2r^{\alpha}\cos\alpha\pi+1}\right) = 0$ i.e. contribution from Hankel path is zero. We note that $\alpha = 2$ gives function as $\frac{s}{s^2+1}$ with poles at $s_{1,2} = \pm i$; that gives inverse Laplace transform as $\cos t = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$, the standard Laplace identity.

Histogram function of decay rates for oscillatory decay of Mittag-Leffler function of order $1 < \alpha < 2$

The histogram function for $f(t) = E_\alpha(-t^\alpha) = f_1(t) + f_2(t)$, for $1 < \alpha < 2$ is following where $\omega_0 = \sin\frac{\pi}{\alpha}$, $\sigma_0 = \cos\frac{\pi}{\alpha}$ $\sigma_0 < 0$ $\mathrm{A} = \frac{1}{\sin\frac{\pi}{\alpha}}$

$$f(t) = E_\alpha(-t^\alpha); \quad 1 < \alpha < 2$$

$$\begin{aligned}
E_\alpha(-t^\alpha) &= f_1(t) + f_2(t) \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda + \left(\frac{1}{\sin\frac{\pi}{\alpha}}\right)\mathrm{e}^{\left(\cos\frac{\pi}{\alpha}\right)t}\sin\left(\frac{\pi}{\alpha} - \left(\sin\frac{\pi}{\alpha}\right)t\right) \\
&= \int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda - \mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t - \frac{\pi}{\alpha}\right)
\end{aligned}$$

$$\begin{aligned}
H_\alpha(\lambda) &= \mathcal{L}^{-1}\left\{f_1(t)\right\} + \mathcal{L}^{-1}\left\{f_2(t)\right\} \\
&= \frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right) + \mathcal{L}^{-1}\left\{-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t - \frac{\pi}{\alpha}\right)\right\}
\end{aligned}$$

The first term of $H_\alpha(\lambda)$ for $1 < \alpha < 2$ call it $H_{\alpha-1}(\lambda)$ comes from contribution from Hankel's path integration. We see $H_{\alpha-1}(\lambda)$ is negative for all $0 < \lambda < \infty$. The second term is $\mathcal{L}^{-1}\left\{-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t - \frac{\pi}{\alpha}\right)\right\}$ call it $H_{\alpha-2}(\lambda)$ comes from Residues of poles in Primary Riemann sheet, $H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\left\{f_2(t)\right\}$; that we will derive via Berberan Santos formula.Figure-13, depicts $H_{\alpha-1}(\lambda)$.

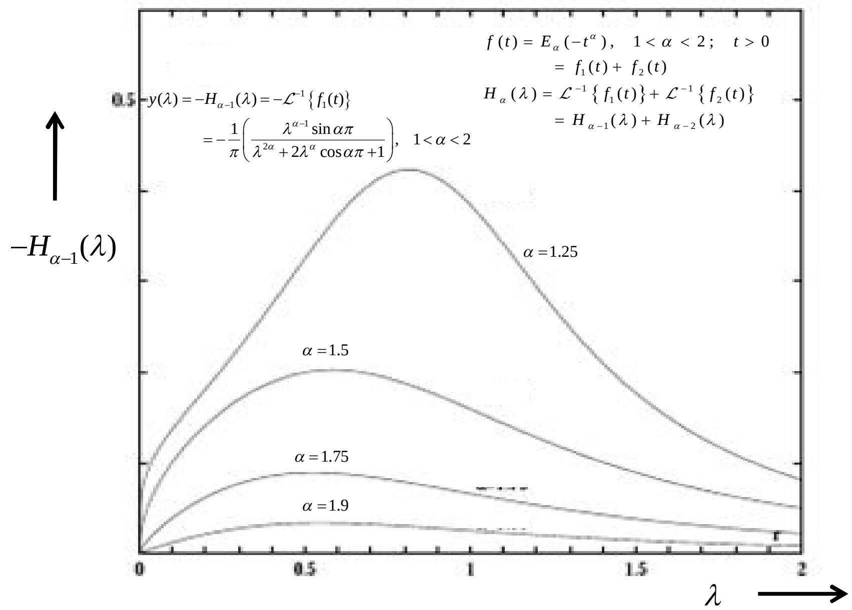


Figure-13: Histogram function for oscillatory decaying Mittag-Leffler function from Hankel's path contribution

We note that as order tends to 2, the $-H_{\alpha-1}(\lambda)$ tends to zero; i.e. for pure cosine decay.

In the derivation of Berberan Santo formula we have derived the formula of inverse Laplace transform from complex frequency domain to time domain. We apply the same to invert time domain function to get histogram of rate relaxation

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\left\{-\mathrm{A}e^{\cos\frac{\pi}{\alpha}t}\sin\left(\sin\tfrac{\pi}{\alpha}t - \tfrac{\pi}{\alpha}\right)\right\} = \mathcal{L}^{-1}\left\{\mathrm{B}e^{-k_0 t}\sin\omega_0 t\right\}$$

$$\mathrm{B} = -\mathrm{A} = \tfrac{1}{\sin\frac{\pi}{\alpha}}; \quad k_0 = -\cos\tfrac{\pi}{\alpha}, \quad \omega_0 = \sin\tfrac{\pi}{\alpha}t - \tfrac{\pi}{\alpha}; \quad 1 < \alpha < 2$$

Now we apply Berberan-Santos formula in following steps

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\{f(t)\} = \frac{e^{c_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[f(c_0 + iy)\right]\cos\lambda y - \mathrm{Im}\left[f(c_0 + iy)\right]\sin\lambda y\right)\mathrm{d}y$$

$$f(t) = \mathrm{B}e^{-k_0 t}\sin\omega_0 t, \quad t = x + iy; \quad \text{choose} \quad x = c_0 = 0$$

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\left\{\mathrm{B}e^{-k_0 t}\sin\omega_0 t\right\} = \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left[f\right]\cos\lambda y - \mathrm{Im}\left[f\right]\sin\lambda y\right)\mathrm{d}y$$

$$f(iy) = \mathrm{B}e^{-k_0(iy)}\sin(i\omega_0 y) = \mathrm{B}\left(\cos k_0 y - i\sin k_0 y\right)\left(i\sinh\omega_0 y\right)$$

$$\mathrm{Re}\left[f\right] = \mathrm{B}\sin k_0 y\sinh\omega_0 y \qquad \mathrm{Im}\left[f\right] = \mathrm{B}\cos k_0 y\sinh\omega_0 y$$

We obtain

$$H_{\alpha-2}(\lambda) = \frac{\mathrm{B}}{\pi}\int_0^\infty \left(\sinh\omega_0 y\right)\left(\sin\left((k_0 - \lambda)y\right)\right)\mathrm{d}y$$

Interesting results in usual inverse Laplace transform convention they are

$$\mathcal{L}^{-1}\left\{\mathrm{e}^{as}\sin bs\right\} = -\tfrac{1}{\pi}\int_0^\infty \left(\sinh b\omega\right)\left(\sin\left((a+t)\omega\right)\right)\mathrm{d}\omega$$

$$\mathcal{L}^{-1}\left\{\sin bs\right\} = -\tfrac{1}{\pi}\int_0^\infty \left(\sinh b\omega\right)\left(\sin \omega t\right)\mathrm{d}\omega$$

Some observations regarding pole movement and Hankel's path contribution vis-à-vis order of Mittag-Leffler function

We have analyzed the behavior of $f(t) = E_\alpha(-t^\alpha) = f_1(t) + f_2(t)$ from known Laplace transform identity of Mittag-Leffler function as $\frac{s^{\alpha-1}}{s^\alpha+1}$, for $0<\alpha<1$ and $1<\alpha<2$. We have seen that $f_1(t)$ part is due to contribution of Hankel's path and $f_2(t)$ is due to residues of poles, $s_{1,2} = \sigma_0 \pm i\omega_0$.

We have observed that for $0<\alpha<1$, the function $\frac{s^{\alpha-1}}{s^\alpha+1}$ has no poles in the primary-Riemann sheet, ($-\pi < \arg[s] < \pi$) however it has on secondary Riemann-sheets, as poles are given by the expression $s = \exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right)$. Since in this case we do not have poles in primary-Riemann sheet, we say $f_2(t) = 0$. Also when $0<\alpha<1$, the function $\frac{s^{\alpha-1}}{s^\alpha+1}$ is multi-valued too. Thus we require Branch-cut-depicted in Figure-3, and in this values of order, $0<\alpha<1$ non-integer, we have $f_1(t)$ the contribution of Hankel's path.

For integer case $\alpha=1$ we note that the function $\frac{s^{\alpha-1}}{s^\alpha+1}$ is $\frac{1}{s+1}$ single valued with pole at $s=-1$, for this case there is no branch cut required, and no contribution of Hankel's path, and we have $f_1(t)=0$ with residue at $s=-k$ gives $f_2(t) = \mathrm{e}^{-t}$.

For the case $1<\alpha<2$, (non-integer) we have both f_1 and f_2, with poles $s_{1,2} = \sigma_0 \pm i\omega_0$, where $\sigma_0 < 0$, that gives oscillatory decaying $f_2(t)$ part.

Again when $\alpha=2$ we have $f_1(t)=0$, that is no contribution of the Hankel's path, as no branch cut is required for function $\frac{s}{s^2+1}$. We note that poles in this case $\alpha=2$ has moved from left half of complex plane to exactly on the imaginary axis, with $\sigma_0=0$, $s_{1,2}=\pm i$. The residue calculation gives for $f_2(t)=\cos t$.

While we go further say $2<\alpha<3$ we will have same observation, but here the poles $s_{1,2}$ will have real part $\sigma_0 = \cos\frac{\pi}{\alpha} > 0$.Thus we see that the poles have moved towards right half of the complex plane for $2<\alpha<3$ and thus we have a growing oscillatory response. This is unstable response. Thus we restrict ourselves to stable region of operation $0<\alpha<2$.

We have observed that Branch cut or Hankel path appears when α is non-integer, with no poles in the case of $0<\alpha<1$, giving $f_2(t)=0$ and poles in left-half plane for $1<\alpha<2$ giving $f(t) = f_1(t) + f_2(t)$. At integer points $\alpha=1$ and $\alpha=2$ we do not have any Branch cut, and response is only by $f_2(t)$, with $f_1(t)=0$.

Berberan-Santo formula for getting histogram function for some non-Debye decay function

We have used contour integration method with residue calculus in earlier sections. Now we apply method of using Berberan-Santo formula i.e. without using the contour integration to have inverse Laplace transformation for some non-Debye decay functions. First we take Mittag-Leffler function $f(t)=E_\alpha(-kt^\alpha)$, and have following steps

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\left\{f(t)\right\}=\frac{e^{c_0\lambda}}{\pi}\int_0^\infty\Big(\mathrm{Re}\big[f(c_0+iy)\big]\cos\lambda y-\mathrm{Im}\big[f(c_0+iy)\big]\sin\lambda y\Big)dy$$

$$f(t)=E_\alpha(-kt^\alpha),\quad kt^\alpha\equiv\overline{t}=x+iy;\quad \text{choose}\quad x=c_0=0$$

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\}=\frac{1}{\pi}\int_0^\infty\Big(\mathrm{Re}\big[E_\alpha(iy)\big]\cos\lambda y-\mathrm{Im}\big[E_\alpha(iy)\big]\sin\lambda y\Big)dy$$

$$E_\alpha(-\overline{t})=1-\frac{\overline{t}}{\Gamma(1+\alpha)}+\frac{\overline{t}^2}{\Gamma(1+2\alpha)}-\frac{\overline{t}^3}{\Gamma(1+3\alpha)}+\ldots$$

$$E_\alpha(iy)=1-\frac{iy}{\Gamma(1+\alpha)}+\frac{i^2y^2}{\Gamma(1+2\alpha)}-\frac{i^3y^3}{\Gamma(1+3\alpha)}+\ldots$$

We get the following

$$\mathrm{Re}\big[E_\alpha(-\overline{t})\big]=1-\frac{y^2}{\Gamma(1+2\alpha)}+\frac{y^4}{\Gamma(1+4\alpha)}-\frac{y^6}{\Gamma(1+6\alpha)}+\ldots$$

$$\mathrm{Im}\big[E_\alpha(-\overline{t})\big]=-\left(\frac{y}{\Gamma(1+\alpha)}-\frac{y^3}{\Gamma(1+3\alpha)}+\frac{y^5}{\Gamma(1+5\alpha)}-\ldots\right)$$

Thus we can calculate the following

$$\begin{aligned}H_\alpha(\lambda)&=\mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\}=\frac{1}{\pi}\int_0^\infty\Big(\mathrm{Re}\big[E_\alpha(iy)\big]\cos\lambda y-\mathrm{Im}\big[E_\alpha(iy)\big]\sin\lambda y\Big)dy\\&=\frac{1}{\pi}\int_0^\infty\left(1-\frac{y^2}{\Gamma(1+2\alpha)}+\frac{y^4}{\Gamma(1+4\alpha)}-\frac{y^6}{\Gamma(1+6\alpha)}+\ldots\right)\cos(\lambda y)dy\\&\quad+\frac{1}{\pi}\int_0^\infty\left(\frac{y}{\Gamma(1+\alpha)}-\frac{y^3}{\Gamma(1+3\alpha)}+\frac{y^5}{\Gamma(1+5\alpha)}-\ldots\right)\sin(\lambda y)dy\end{aligned}$$

For Kohlraush's Stretched exponential decay we apply Berberan Santo method we get following

$$f(t)=e^{-(t/\tau_0)^\alpha}$$

$$H(\lambda)=\frac{1}{\pi}\int_0^\infty dy\left(e^{-(y/\tau_0)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}\right)\cos\left(\lambda y-\left(\frac{y}{\tau_0}\right)^\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)$$

For Becquerel's Compressed hyperbolic decay we apply Berberan Santo method and we get the following

$$f(t)=\frac{1}{\left(1+\frac{(1-\alpha)t}{\tau_0}\right)^{1/(1-\alpha)}};\quad 0<\alpha<1$$

$$H(\lambda)=\frac{1}{\pi}\int_0^\infty dy\left(1+\left(\frac{(1-\alpha)y}{\tau_0}\right)^2\right)^{-\frac{1}{2(1-\alpha)}}\cos\left(\lambda y-\frac{\tan^{-1}\left(\frac{(1-\alpha)y}{\tau_0}\right)}{1-\alpha}\right)$$

For Asymptotic power law decay we apply Berberan-Santo formula to get the following

$$f(t)=\frac{1}{1+\left(\frac{t}{\tau_0}\right)^{\alpha}},\quad 0<\alpha<1$$

$$H(\lambda)=\tau_0^{\alpha}\lambda^{\alpha-1}E_{\alpha,\alpha}\left(-(\tau_0\lambda)^{\alpha}\right)$$

$$\alpha=1$$

$$f(t)=\frac{1}{1+\left(\frac{t}{\tau_0}\right)};\quad H(\lambda)=\tau_0 E_{1,1}\left(-(\tau_0\lambda)\right)=\tau_0\mathrm{e}^{-\tau_0\lambda}$$

Table-2 gives integral representation obtained via Berberan Santos Method

S. No.	$G(s)$ **The transfer function as a function of complex frequency** $s=\sigma_0+i\omega$	$g(t)=\mathcal{L}^{-1}\{G(s)\}$ **In integral representation of function in time domain by inverse Laplace transform by Berberan-Santos method**
1	$\frac{1}{s+a}$	$\frac{1}{\pi}\int_0^{\infty}\frac{a\cos(\omega t)+\omega\sin(\omega t)}{\omega^2+a^2}\mathrm{d}\omega$
2	$\frac{s}{s^2+1}$	$\frac{\mathrm{e}^t}{\pi}\int_0^{\infty}\frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega$
3	e^{-sT_d}	$\frac{1}{\pi}\int_0^{\infty}\cos\left(\omega(t-T_d)\right)\mathrm{d}\omega$
4	1	$\frac{1}{\pi}\int_0^{\infty}\cos(\omega t)\mathrm{d}\omega$
5	$\frac{1}{s}$	$\frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega t}{\omega}\mathrm{d}\omega$
6	$s^{-\alpha}$	$\frac{1}{\pi}\int_0^{\infty}\frac{\cos\left(\omega t-\frac{\alpha\pi}{2}\right)}{\omega^{\alpha}}\mathrm{d}\omega$
7	$\left(1+(1-\beta)\left(\frac{s}{a}\right)\right)^{-\frac{1}{(1-\beta)}}$	$\frac{a}{\pi(1-\beta)}\int_0^{\infty}\left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{aut-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u;\quad u=\frac{(1-\beta)\omega}{a}$
8	$\mathrm{e}^{-(s/a)^{\beta}}$	$\frac{a}{\pi}\int_0^{\infty}\left(\mathrm{e}^{-u^{\beta}\cos(\beta\pi/2)}\right)\cos\left(atu-u^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u;\quad u=\frac{\omega}{a}$
9	$\frac{k}{k+s^{\alpha}}$	$\frac{\sqrt[\alpha]{k}}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u;\quad u=\frac{\omega}{\sqrt[\alpha]{k}}$
10	$\frac{s^{\alpha}}{s(1+s^{\alpha})}$	$\frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega$
11	$\frac{k}{s(s^{\alpha}+k)}$	$\frac{k}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega$

Table-2: List of inverse Laplace transforms in integral representation using Berberan-Santos formulation

From the Table-2 we see $\mathcal{L}^{-1}\left\{\mathrm{e}^{-sT_d}\right\}=\frac{1}{\pi}\int_0^{\infty}\cos\left(\omega(t-T_d)\right)\mathrm{d}\omega$ obtained via Berberan-Santos formula whereas we know that from standard Laplace transform tables, that $\mathcal{L}^{-1}\left\{e^{-sT_d}\right\}=\delta(t-T_d)$. Thus one of integral representation of Delta function is $\delta(t-T_d)=\frac{1}{\pi}\int_0^{\infty}\cos\left(\omega(t-T_d)\right)\mathrm{d}\omega$.

By changing the variables $s\mapsto t$ in the Table-2 for functions $G(s\mapsto t)\equiv f(t)$ the time decay function. The $\mathcal{L}^{-1}\left\{f(t)\right\}=H(\lambda)$, and from Table-2 we get $H(\lambda)$ by changing variable of $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$, and $g(t\mapsto\lambda)\equiv H(\lambda)$. Thus for $f(t)=e^{-t}$ we get $H(\lambda)=\frac{1}{\pi}\int_0^{\infty}\cos\left(y(\lambda-1)\right)\mathrm{d}y$; which is integral representation of $H(\lambda)=\delta(\lambda-1)$. Similarly entries S.No. 1 to 6 can be equated to the functions that are listed in standard Laplace transform tables.

Newton's law of cooling-is it via Newtonian Calculus?

Figure-14 gives the basic experiment on cooling of water from near boiling point $T_0=90^0\mathrm{C}=T_{init}$ to ambient temperature $T_a=T_{amb}=27.6^0\mathrm{C}$. The theoretical and experimental curves are shown in the same Figure-14. The classical Newton's law of cooling gives the relaxation function as

$$T(t)=T_{amb}+\left(T_{init}-T_{amb}\right)e^{-at}$$

The differential equation describing the above classical cooling is

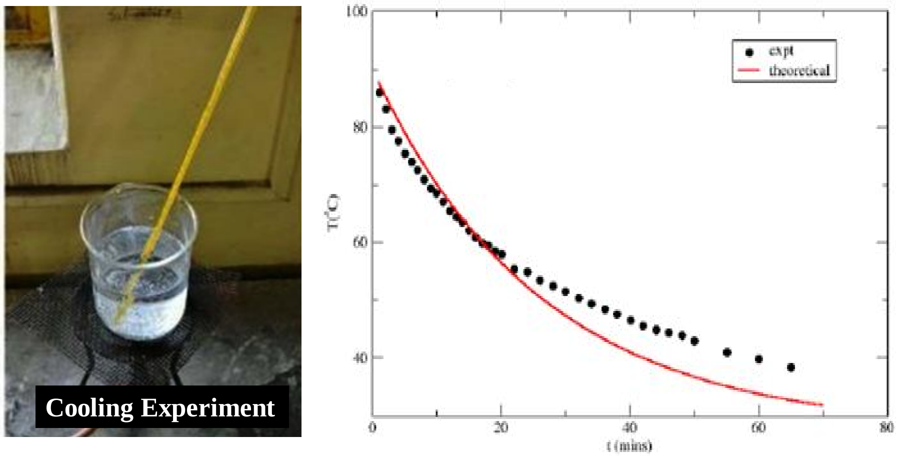
$$\frac{\mathrm{d}T(t)}{\mathrm{d}t}=-a\left(T(t)-T_{amb}\right),\quad a>0,\quad T(0)=T_{init}$$

The rate distribution histogram is by taking inverse Laplace transform, gives

$$H_1(\lambda)=\mathcal{L}^{-1}\left\{T_{amb}\right\}+\left(T_{init}-T_{amb}\right)\mathcal{L}^{-1}\left\{e^{-at}\right\}$$

$$H_1(\lambda)=T_{amb}\delta(t)+\left(T_{init}-T_{amb}\right)\left(\delta(\lambda-a)\right)$$

The experiment suggests that there is difference, so we ask if the Newtonian Calculus suffice or shall we have different approach-i.e. with non-Newtonian calculus or Fractional Calculus. Well if the cooling law is via Fractional Calculus, then which derivative shall define the fractional rate of change of temperature? We note that experiments with various liquids confirm that, cooling law does follow fractional differential equation (Table-3)



Figur-14: Comparison of experimental cooling curve for water with theoretical solution using classical calculus

If the cooling law is composed with Caputo fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}\right|^{C}=-b\left(T(t)-T_{amb}\right),\quad 0<\alpha<1;\quad b>0;\quad T(0)=T_{init}$$

The solution is

$$\left.T(t)\right|_{C}=T_{amb}+\left(T_{init}-T_{amb}\right)E_{\alpha}(-bt^{\alpha})$$

The rate distribution function histogram is

$$H_2(\lambda)=\mathcal{L}^{-1}\left\{T_{amb}\right\}+\left(T_{init}-T_{amb}\right)\mathcal{L}^{-1}\left\{E_{\alpha}(-bt^{\alpha})\right\}$$

$$\left.H_2(\lambda)\right|_{\text{Contour-Integration}}=T_{amb}\delta(t)+\frac{\left(T_{init}-T_{amb}\right)}{\pi}\left(\frac{b\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2b\lambda^{\alpha}\cos\alpha\pi+b^{2}}\right),\quad 0<\alpha<1$$

$$\left.H_2(\lambda)\right|_{\text{Berberan Santo}}=T_{amb}\delta(t)+\frac{(T_{init}-T_{amb})}{\pi}\int_0^{\infty}\left(1-\frac{y^2}{\Gamma(1+2\alpha)}+\frac{y^4}{\Gamma(1+4\alpha)}-\frac{y^6}{\Gamma(1+6\alpha)}+\dots\right)\cos\lambda y\mathrm{d}y$$
$$+\frac{(T_{init}-T_{amb})}{\pi}\int_0^{\infty}\left(\frac{y}{\Gamma(1+\alpha)}-\frac{y^3}{\Gamma(1+3\alpha)}+\frac{y^5}{\Gamma(1+5\alpha)}-\dots\right)\sin\lambda y\mathrm{d}y$$

In the histogram function derivation, we have used inverse Laplace transformation via the two techniques as discussed in the earlier sections.
If the cooling law is composed with Riemann-Liouville fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}\right|^{RL} = -c\left(T(t)-T_{amb}\right), \quad 0<\alpha<1; \quad c>0; \quad T(0)=T_{init}$$

The solution is

$$T(t)\big|_{RL} = T_{amb}\left(\Gamma(\alpha)\right)E_{\alpha,\alpha}(-ct^{\alpha}) + T_{init}\left(1-E_{\alpha}(-ct^{\alpha})\right)$$

The histogram of rate distribution is thus

$$H_3(\lambda) = T_{amb}\left(\Gamma(\alpha)\right)\mathcal{L}^{-1}\left\{E_{\alpha,\alpha}(-ct^{\alpha})\right\} + \mathcal{L}^{-1}\left\{T_{init}\right\} - \mathcal{L}^{-1}\left\{E_{\alpha}\left\{-ct^{\alpha}\right\}\right\}$$

$$H_3(\lambda) = \frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{1}{\Gamma(\alpha)} - \frac{y^2}{\Gamma(3\alpha)} + \frac{y^4}{\Gamma(5\alpha)} - \frac{y^6}{\Gamma(7\alpha)} + \ldots\right)\cos\lambda y\,\mathrm{d}y$$

$$+\frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{y}{\Gamma(2\alpha)} - \frac{y^3}{\Gamma(4\alpha)} + \frac{y^5}{\Gamma(6\alpha)} - \ldots\right)\sin\lambda y\,\mathrm{d}y$$

$$+T_{init}\delta(t) - \frac{T_{init}}{\pi}\left(\frac{c\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha} + 2c\lambda^{\alpha}\cos\alpha\pi + c^2}\right)$$

Here we note that since no direct compact representation of two parameter Mittag-Leffler function in Laplace transform relation, we have used Berberan-Santos formula to write expression for $\mathcal{L}^{-1}\left\{E_{\alpha,\alpha}(-ct^{\alpha})\right\}$.

In Table-3 we give the experimentally extracted parameters: the fractional order of the derivative, the values of proportionality constants for various liquids.

Why Fractional Calculus in Newton's Law of Cooling?

We note here the classical Newton's law of cooling only holds for solids. The reason that we have to have fractional rate is due to mixed modes of heat transfer at different time scales happens in liquids-i.e. heterogeneous dynamics, whereas in solids the heat transfer is via homogeneous time scales with one mode. Thus the elemental infinitesimal unit in case of liquids is $(\Delta t)^{\alpha} > \Delta t$ $0<\alpha<1$, gives comfortable time slice to view the heterogeneous process-and this gives fractional rate of change with limit $\Delta t \downarrow 0$.The use of fractional differentials in case of non-homogeneous dynamics gives fractional derivatives and fractional integrals and thus non-Newtonian calculus

$$\frac{\Delta T}{(\Delta t)^{\alpha}} \propto \left(T(t)-T_{amb}\right)$$

$$\frac{\mathrm{d}^{\alpha}T}{\mathrm{d}t^{\alpha}} = K\left(T(t)-T_{amb}\right)$$

Now what we studied if α the fractional order of derivative with Riemann-Liouville definition as well as Caputo definition, and evaluated experimentally these values, and corresponding proportionality constants.

Liquid	Volume (ml)	$T_0(°C)$	$T_a(°C)$	λ	κ	Area (cm^2)	α
Water	40	92.5	26.1	-0.114	-0.075	11.345	0.79
Water	80	100	23.5	-0.104	-0.070	14.527	0.79
Water	300	100	23.5	-0.070	-0.048	60.845	0.79
Mustard Oil	X	105.0	23.4	-0.115	-0.075	14.527	0.88
Mercury	X	105.0	23.2	-0.32	-0.19	14.527	0.92

Constants and parameters for experiments with various liquids

$$\left.\frac{\mathrm{d}^\alpha T(t)}{\mathrm{d}t^\alpha}\right|^C = \lambda\left(T(t)-T_a\right) \qquad \lambda<0 \qquad T(t)\big|^C = T_a + \left(T_0 - T_a\right)E_{\alpha,1}\left(\lambda t^\alpha\right)$$

$$\left.\frac{\mathrm{d}^\alpha T(t)}{\mathrm{d}t^\alpha}\right|^{RL} = \kappa\left(T(t)-T_a\right) \qquad \kappa<0 \qquad T(t)\big|^{RL} = T_a\left(\Gamma(\alpha)\right)E_{\alpha,\alpha}\left(\kappa t^\alpha\right)+T_0\left(1-E_{\alpha,1}\left(\kappa t^\alpha\right)\right)$$

Table-3: Summary of fractional Newton's law of cooling with various liquids

Conclusions

These mathematical techniques discussed to extract ordered relaxation rate distribution function or histogram functions-for disordered relaxation observed in many systems, are very important for applied mathematics physics and engineering. Though various reaction curves (responses) are obtained in various system studies where we have given reasons for the disordered anomalous reactions observed – to Fractional Differential Equations; yet these discussed mathematical techniques based on theories of complex analysis are yet to be applied to get the spectra or ordered histograms of these experimental records. Especially the crisp physical interpretation is missing for the negative histogram values appearing for oscillatory decaying Mittag-Leffler function-and other non-Debye decays. Therefore it leads scope for extension of these techniques for various physical system dynamics.

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