How disordered (Non-Debye) decaying relaxation manifests ordered relaxation rate distribution histograms

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Dedicating this deliberation to Prof. Sujata Tarafdar Professor, Dept. of Phys, Jadavpur University Kolkata

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Prof. SujataTarafdar was the first one to encourage this subject of fractional calculus way back in 2008. She invited me to present a talk on this topic at Jadavpur Univ in 2009 for Lecture organized by Indian Society for Non-Linear Analysts (ISNA), for seminar called "Theoretical Techniques in Disordered Systems".

The Lecture-Topic was : "Ordering Disordered System by Fractional Calculus" 18-12-2009

Elated to be here again from where my journey on this topic (seriously) started way back more than a decade ago.

Thank you Sujata-di for involving this subject in Condensed Matter Physics Research
-the subject has grown since then

Contents of this deliberation

The topic is : "How disordered (Non-Debye) decaying relaxation manifests ordered relaxation rate distribution histograms"

In this deliberation we are not discussing Fractional Calculus However dynamic systems that behave via non-Newtonian calculus; or system having dynamics described by Fractional Differential Equations, manifest relaxation functions in form of non-Debye relaxation functions. Debye relaxation is by pure exponential decay. The non-Debye one are via Mittag-Leffler, power law function etc.-and we term them as disordered (or non-disciplined relaxation). We will see techniques of having these disordered relaxing functions analyzed by how their undisciplined relaxing rates are distributed-via extracting the rate relaxation histogram function. We will enjoy Residue Calculus, Bromwich integration, Hankel path integration-for obtaining the histogram function i.e. by inverse Laplace transform done on 'time function', taking 'time as complex variable'. We will also obtain new method of getting inverse Laplace transform with-out Residue Calculus or Contour-Integration. We use these technique to write rate relaxation histogram function for Newton's Law of cooling-formulated by non-Newtonian calculus of Caputo or Riemann-Liouville fractional derivative, and various other non-Debye decay.

We will see these techniques nicely orders the disordered relaxations via some pattern of histogram function obtained for relaxation rates

Universal law for dielectric relaxations-non-Debye one

The Curie-von Schweidler law relates to the relaxation current in dielectric when a step DC voltage is applied and is given by power law relaxation function

$$f(t) \sim t^{-\alpha} \quad \textit{where} \quad t > 0; \quad 0 < \alpha < 1$$

and the power (exponent) α ***is called relaxation constant or decay constant with*** $0 < \alpha < 1$ ***. We note that*** α ***the exponent is non-integer***

Note the usual Debye relaxation is $f(t) \sim e^{-\lambda_0 t}$ ***pure exponential decay***

The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments

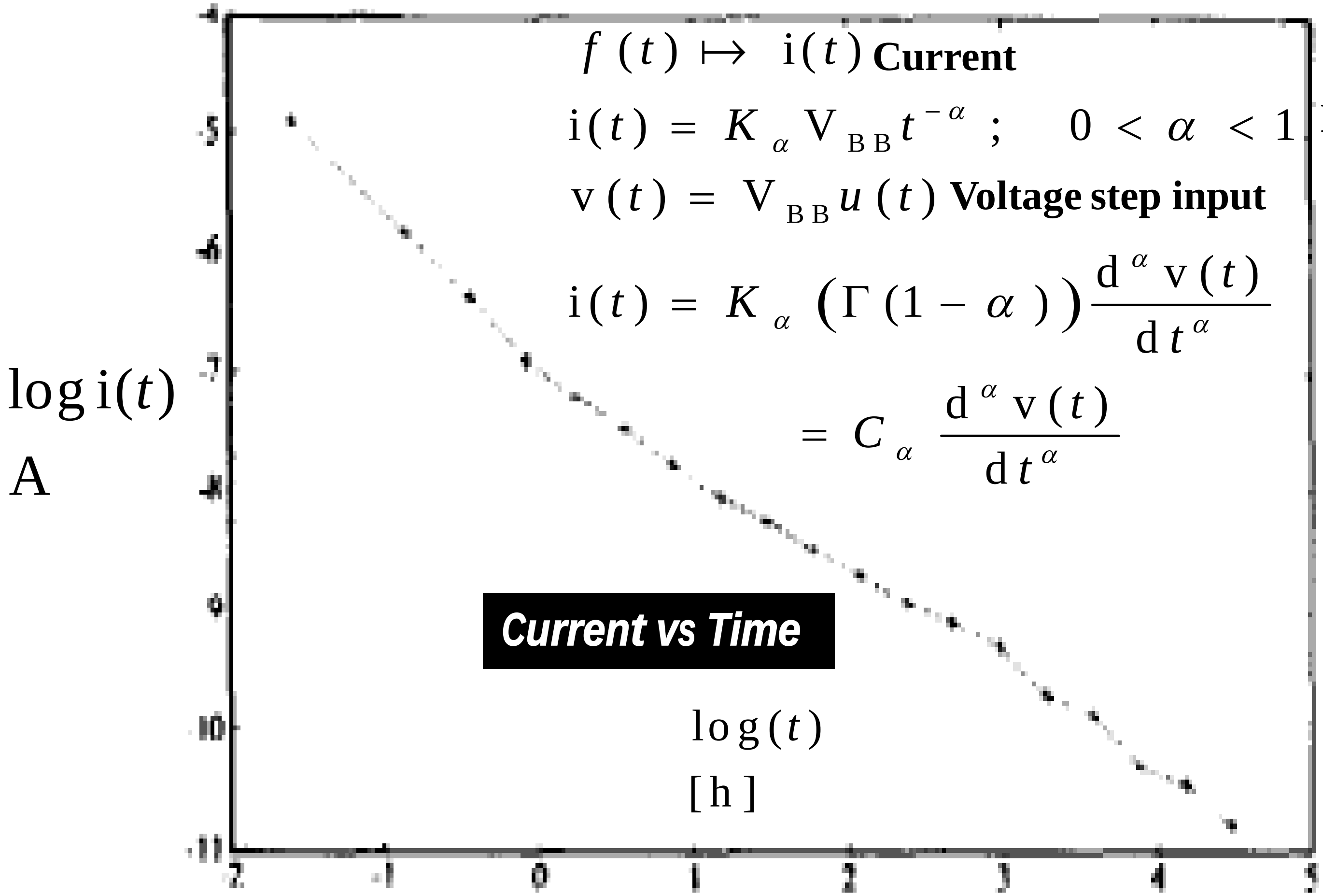
For a step voltage, V_{BB} ***impressed at*** $t=0$ ***to uncharged capacitor the current is***

$$i(t) = K_\alpha \frac{V_{BB}}{t^\alpha}; \quad 0 < \alpha < 1; \quad t > 0$$

This is empirically & experimentally derived

Jaques, C. (1889) Annales de Chimie et de Physique , 17, 384-434.
Schweidler, E.R. (1907) Annalen der Physik, 329, 711-770.
Jonscher, A.K. (1983) Dielectric Relaxation in Solids. Chelsea Dielectrics Press Limited.

Experimental evidence of universal law, for dielectric Relaxations-manifests as Fractional Differential Equation



$$f(t) \mapsto i(t)$$ **Current**

$$i(t) = K_{\alpha} V_{BB} t^{-\alpha} ; \quad 0 < \alpha < 1$$ **Power Law decay**

$$v(t) = V_{BB} u(t)$$ **Voltage step input**

$$i(t) = K_{\alpha} \left(\Gamma(1-\alpha)\right) \frac{d^{\alpha} v(t)}{dt^{\alpha}} = C_{\alpha} \frac{d^{\alpha} v(t)}{dt^{\alpha}}$$

Fractional Differential Equation

At time zero a voltage of 100V is connected to a 0.47uF metalized paper dielectric capacitor; in log-log scales average slope is -0.86. Thus exponent of relaxation current is non-integer

IEEE Trans on Dielectrics and Insulation, 1,826. (1994), S. Westerlund, L. Ekstam

Mittag-Leffler function may use to formulate non-Debye relaxation-as Generalized disordered relaxation

We will discuss the non-Debye type relaxation $f(t) \sim E_\alpha(-kt^\alpha), \quad t > 0; \quad k > 0$

i.e. via Mittag-Leffler function, for cases where $0 < \alpha < 1$

i.e. the decay function is monotonically decaying; also for cases $1 < \alpha < 2$ ***with oscillatory decay***

While Debye case is $f(t) \sim e^{-\lambda t}$ ***which is also 'ordered relaxation'***

The Mittag-Leffler function (of one parameter) is

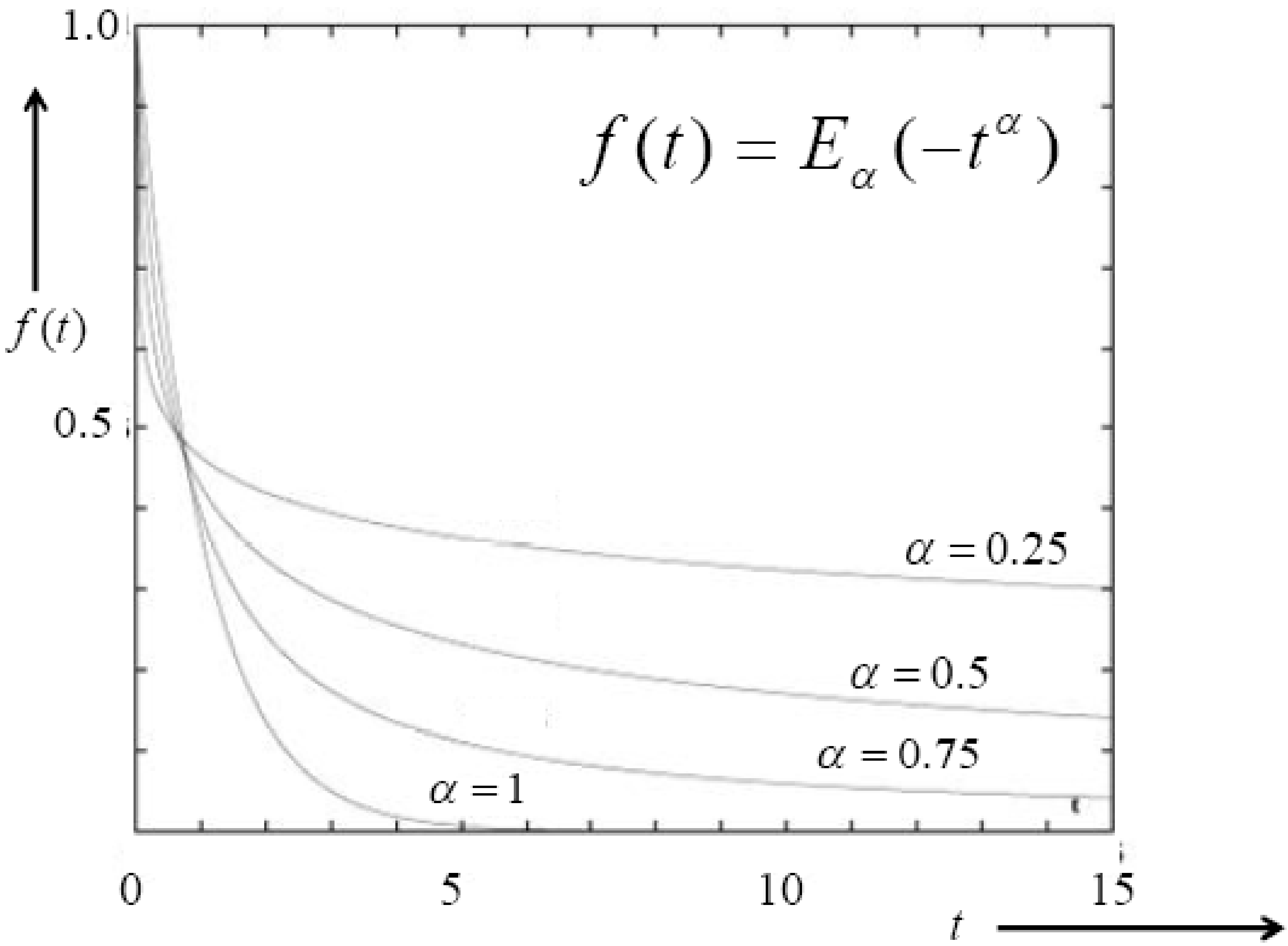
$$E_\alpha(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + 1)}, \quad E_0(-z) = \frac{1}{1+z}, \quad E_1(-z) = e^{-z}, \quad E_2(-z) = \cosh\sqrt{-z} = \cos\sqrt{z}$$

For the order of Mittag-Leffler function α ***between zero and one- we sees monotonically decaying curve, thus the order between zero and one Mittag-Leffler interpolates decaying relaxation function of pure hyperbolic decay to a pure exponential decay (Debye Type)***

For order of Mittag-Leffler function α ***between one and two we sees oscillatory decaying curve, thus the order between one and two the Mittag-Leffler interpolates decaying relaxation function of pure exponential decay (Debye) to a pure oscillatory decay***

Monotonically decaying Mittag-Leffler function

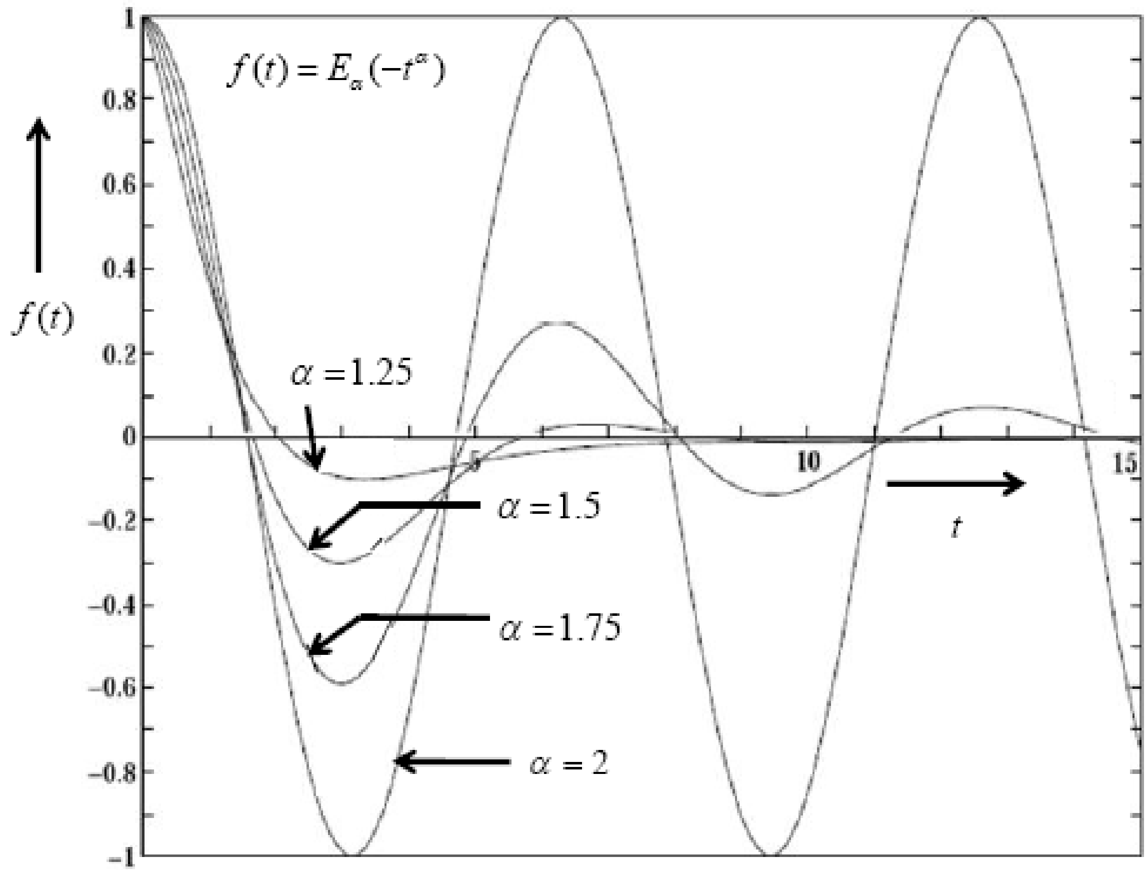
$$f(t) = E_\alpha(-kt^\alpha), \quad t > 0; \quad k = 1; \quad 0 \le \alpha \le 1$$



Mittag-Leffler function $E_\alpha(-t^\alpha)$ **as decay function for** $0 < \alpha < 1$

Oscillatory decaying Mittag-Leffler function

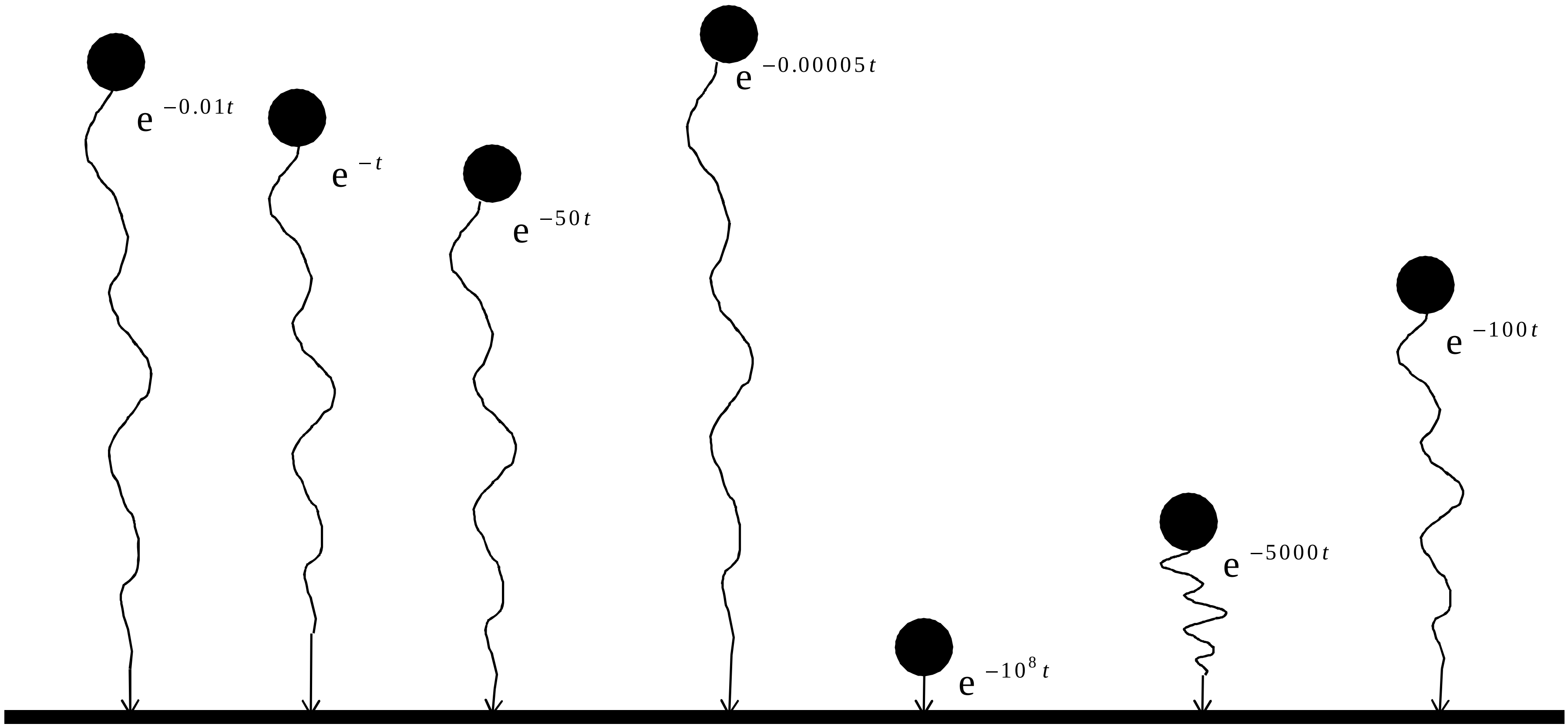
$$f(t) = E_\alpha(-kt^\alpha), \quad t > 0; \quad k = 1; \quad 1 < \alpha \le 2$$



: **Relaxation decay with oscillation with Mittag-Leffler function** $f(t) = E_\alpha(-t^\alpha)$ **for** $1 < \alpha < 2$

Disordered Relaxation

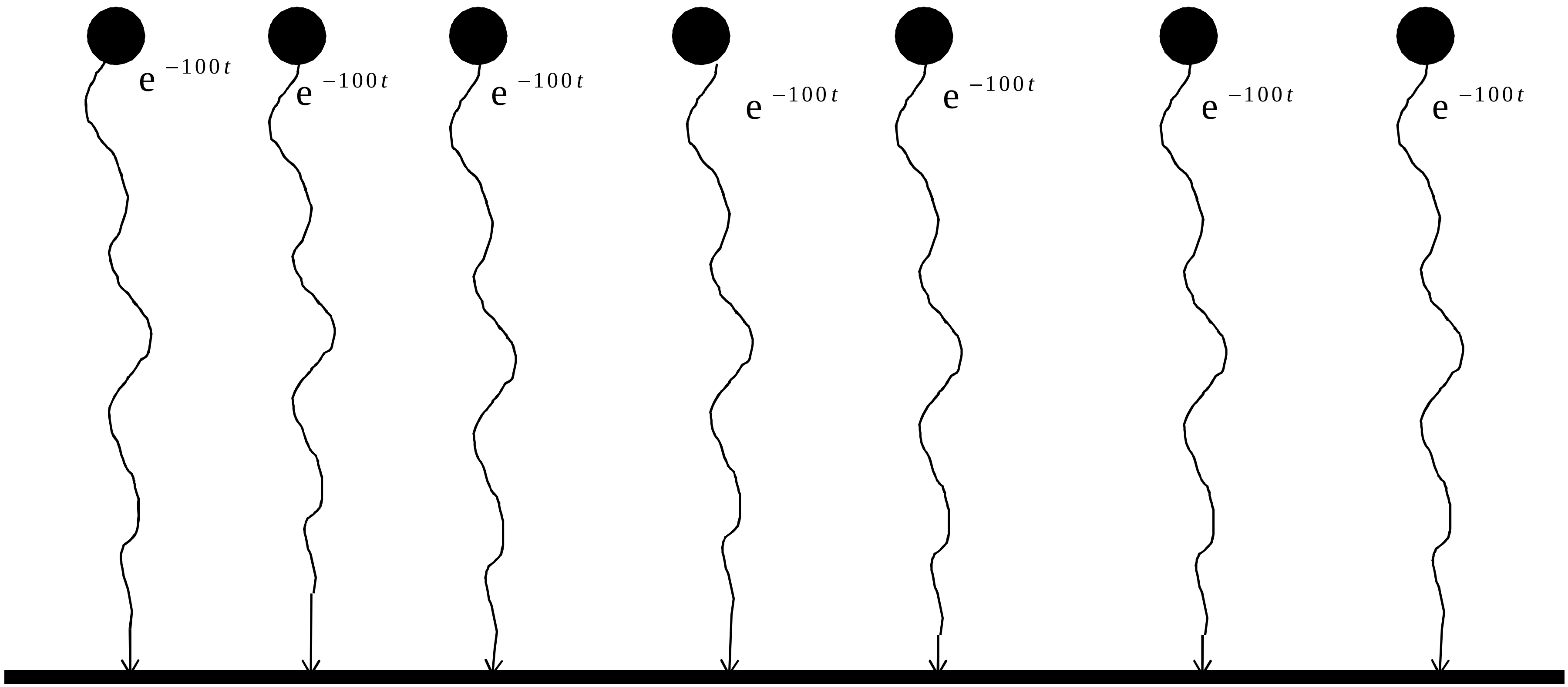
The non-Debye relaxation has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of relaxation rates λ varying from zero to infinity



Indiscipline relaxation in multi-body system

Ordered relaxation

The Debye relaxation has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning all bodies with relaxation rates same λ -having same rate of decay for all bodies



Disciplined relaxation in multi-body system

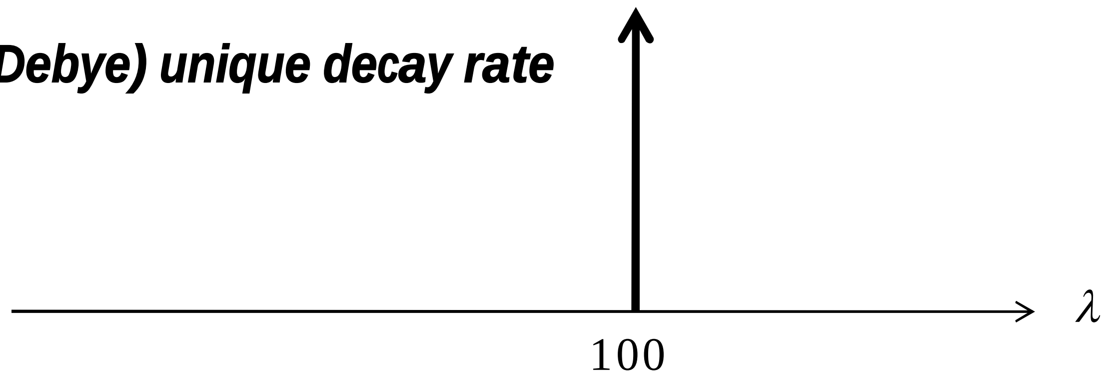
Some indication of ordered to disordered relaxation

Ordered Relaxation (Debye) unique decay rate

$$f(t) \sim e^{-100t}$$

$$\lambda_{max} - \lambda_{min} = 0$$

Disciplined

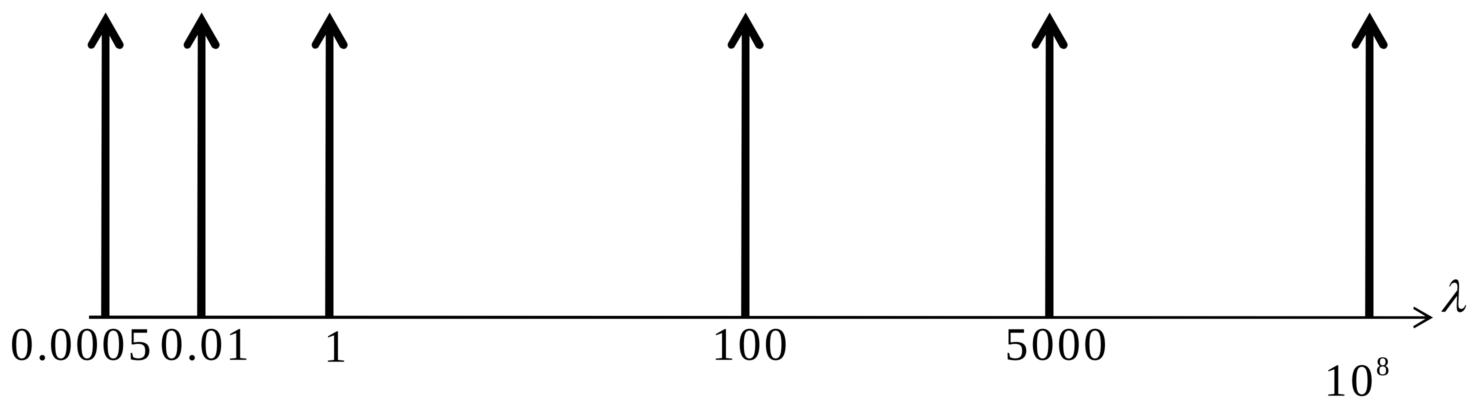


Disordered Relaxation (non-Debye) several decay rate

$$f(t) \sim e^{-0,0005t} + e^{-0.01t} + e^{-t} + e^{-100t} + e^{-5000t} + e^{-10^8 t} \ldots\ldots$$

$$\lambda_{max} - \lambda_{min} \sim \infty$$

Non-Disciplined



Formulation of disordered relaxation

The complex decay is expressed as following with several Debye rate constants

$$\lambda_1, \lambda_2, \lambda_3, \ldots$$

with weights $a_1, a_2, a_3, \ldots$ ***We write following composite relaxation expression as sum of several 'discrete' relaxations of Debye type i.e.***

$$f(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + \ldots$$

$$= \sum a_j e^{-\lambda_j t}$$

$$f(0) = a_1 + a_2 + a_3 + \ldots$$

In continuum limit we may write the above as $f(t) = \int_0^\infty \left(H(\lambda)\right) e^{-\lambda t} d\lambda$

The function i.e. $H(\lambda)$ ***is the distribution-function of the rate of the relaxation*** λ ***of the process, or we may call it as histogram of relaxation rates.***

We note here the weights a_j ***can be positive or negative. Also*** $H(\lambda)$ ***can be too have positive or negative values***

Disordered relaxation in discrete sense

While for the case with discrete set of relaxation rates i.e. $\lambda_j = k_1, k_2, k_3, \ldots$

the rate distribution function would be having discrete delta functions

$\delta(\lambda - \lambda_j), \quad j = 1, 2, 3, \ldots$ ***at points*** $k_1, k_2, k_3, \ldots$ ***which we express as***

$$f(t) = a_1 e^{-k_1 t} + a_2 e^{-k_2 t} + a_3 e^{-k_3 t} + \ldots = \int_0^\infty \left(H(\lambda)\right) e^{-\lambda t} \mathrm{d}\lambda$$

$$= \int_0^\infty a_1 \delta(\lambda - k_1) e^{-\lambda t} \mathrm{d}\lambda + \int_0^\infty a_2 \delta(\lambda - k_2) e^{-\lambda t} \mathrm{d}\lambda + \ldots..$$

Property of Delta function is used above i.e. $\int \left(\delta(x - x_0)\right) g(x) \mathrm{d}x = g(x_0)$

$$H(\lambda) = a_1 \delta(\lambda - k_1) + a_2 \delta(\lambda - k_2) + a_3 \delta(\lambda - k_3) + \ldots$$

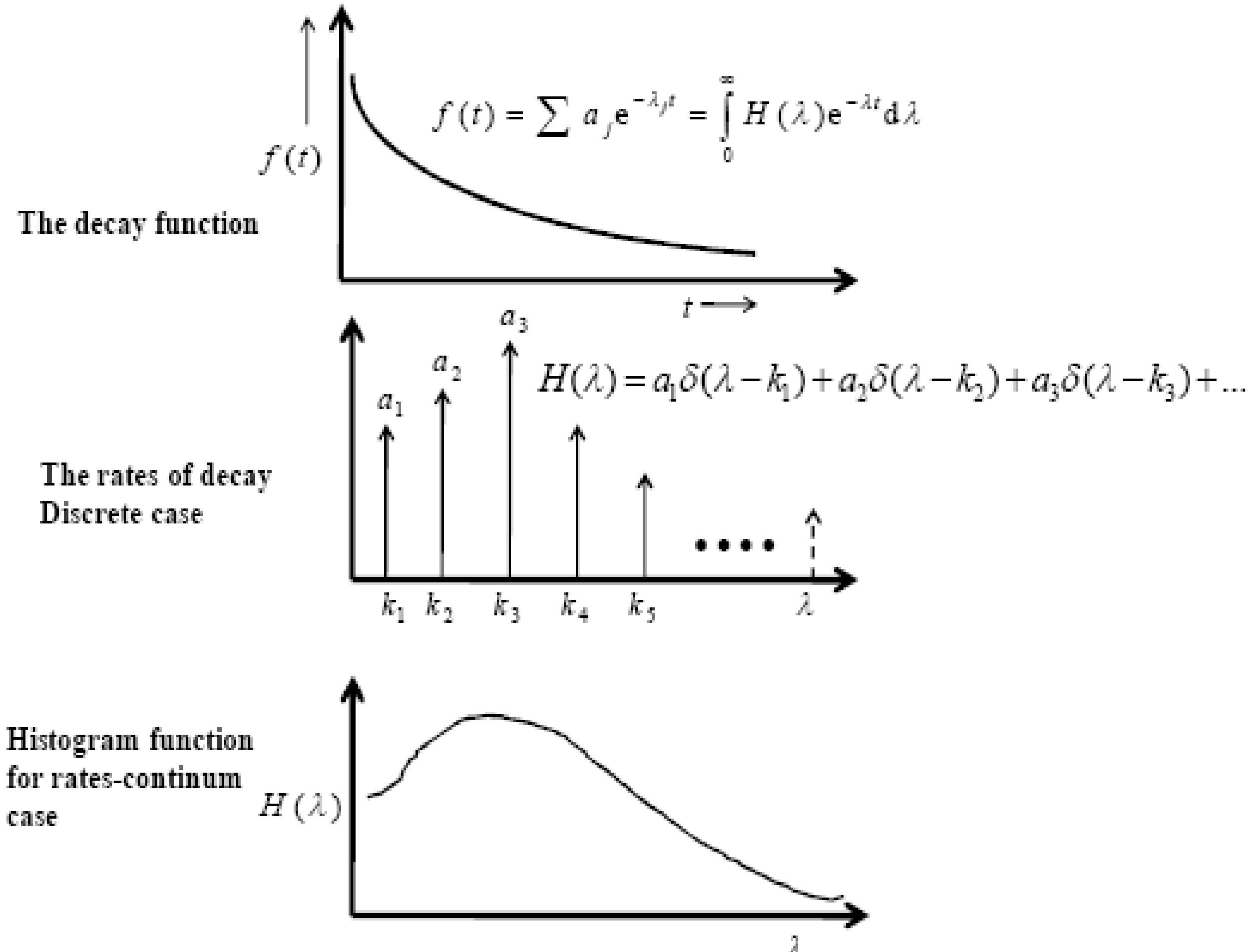
$$= \sum a_j \delta(\lambda - \lambda_j); \qquad \lambda_j \Big|_{j=1,2,3,\ldots} = k_1, k_2, k_3, \ldots.$$

From above formulation if we have only one single Debye relaxation i.e. having only one rate constant $\lambda = k_0$ ***then*** $H(\lambda) = a_0 \delta(\lambda - k_0)$

$$f(t) = \int_0^\infty \left((H(\lambda))\right) e^{-\lambda t} \mathrm{d}\lambda$$

$$= \int_0^\infty \left(a_0 \delta(\lambda - k_0)\right) e^{-\lambda t} \mathrm{d}\lambda = a_0 e^{-k_0 t}$$

Non-Debye or disordered relaxation has Histogram of relaxation rates



Composition of non-Debye relaxation function with several relaxation functions of Debye type in discrete and continuum realization gives Histogram. The shape/type of histogram function gives ordering of relaxation rates for disordered relaxation

Extraction of relaxation rate distribution function -Histogram

Following integral transform relation i.e. called Laplace integral

$$F(s) \overset{\text{def}}{=} \int_0^{\infty} \left(f(t) \right) e^{-st} dt, \quad t > 0, \quad s = \text{Re}[s] + i\omega; \quad i = \sqrt{-1}$$

$$F(s) = \mathcal{L}\{f(t)\} \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left(F(s) \right) e^{st} ds$$

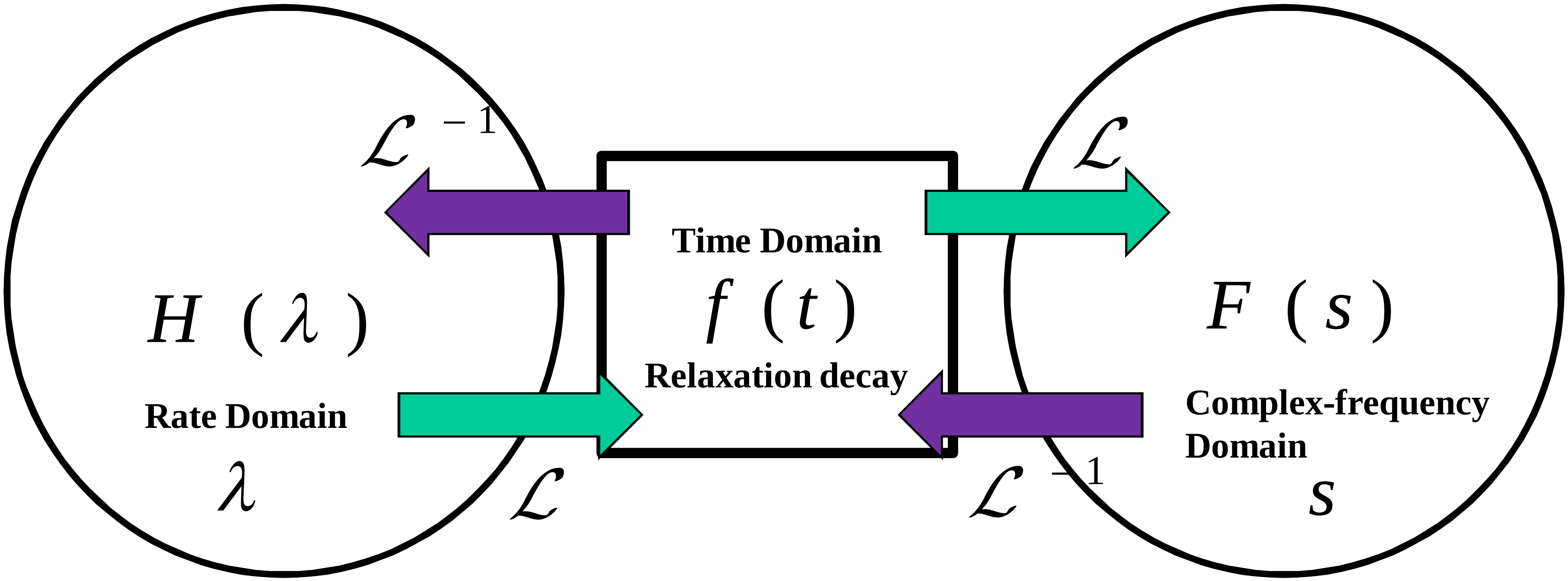
We have derived/formulated non-Debye relaxation as $f(t) = \int_0^{\infty} \left(H(\lambda) \right) e^{-t\lambda} d\lambda$

Therefore in order to get the rate distribution-function $H(\lambda)$ ***from the decay curve or relaxation-function*** $f(t)$ ***we need to perform inverse Laplace Transform of the time function*** $f(t)$

The definition of inverse Laplace Transform is described as following integral expressions on Bromwich path-on complex 'time'-plane $t = \text{Re}[t] + i\,\text{Im}[t] = x_0 + iy$

$$H(\lambda) = \mathcal{L}^{-1}\{f(t)\} = \frac{1}{2\pi i} \int_{x_0 - i\infty}^{x_0 + i\infty} \left(f(t) \right) e^{t\lambda} dt$$

For extracting histogram function do inverse Laplace transform on time function unlike usual done on complex function of frequency



$$H(\lambda) = \mathcal{L}^{-1}\{f(t)\} = \frac{1}{2\pi i}\int_{x_0 - i\infty}^{x_0 + i\infty} f(t)\mathrm{e}^{\lambda t}\mathrm{d}t$$

$$f(t) = \mathcal{L}\{H(\lambda)\} = \int_0^\infty H(\lambda)\mathrm{e}^{-t\lambda}\mathrm{d}\lambda$$

$$t = \mathrm{Re}[t] + i\,\mathrm{Im}[t] = x_0 + iy$$

Here time is complex variable

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} F(s)\mathrm{e}^{st}\mathrm{d}s$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t$$

$$s = \mathrm{Re}[s] + i\,\mathrm{Im}[s] = \sigma + i\omega$$

Here frequency is complex variable

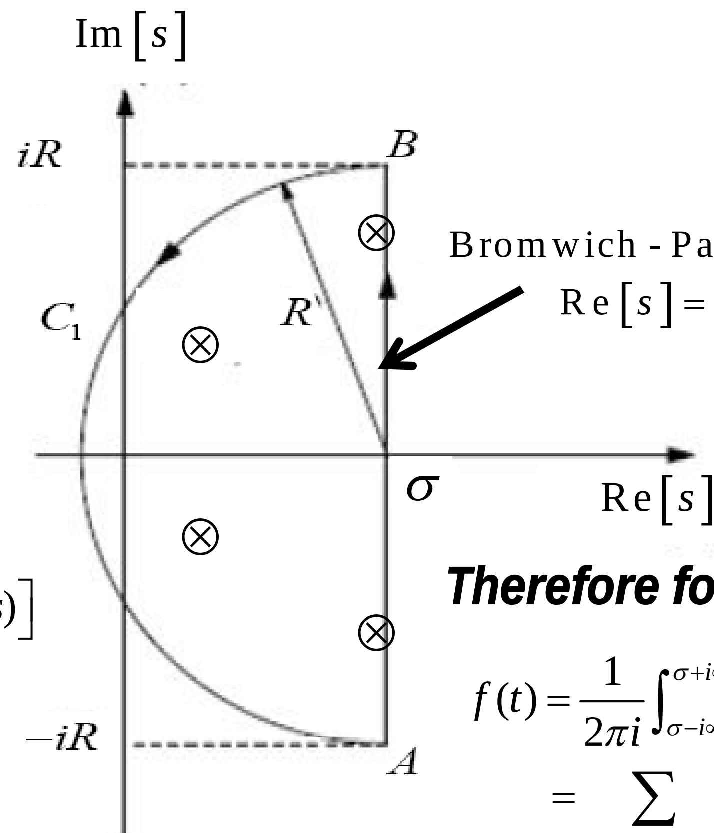
Inverse Laplace transform via Residue Calculus & Contour Integration

In the expression of Bromwich integration for inverse Laplace transform i.e.

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(F(s)\right)e^{ts}ds$$

σ ***being such that*** $F(s)$ ***has some form of singularity on the real line*** $\mathrm{Re}[s] = \sigma$ ***but is analytic in the complex plane to the right of that line,***

i.e. for $\mathrm{Re}[s] > \sigma$



Bromwich - Path: $A \rightarrow B$

$\mathrm{Re}[s] = \sigma$

Consider the closed contour

$A \rightarrow B \rightarrow C_1$ ***as*** $R \uparrow \infty$

Residue calculus says

$$\int_{A\to B\to C_1} e^{st}F(s)ds = \int_{A\to B} e^{st}F(s)ds + \int_{C_1} e^{st}F(s)ds$$

$$= \int_{\sigma-iR}^{\sigma+iR} e^{st}F(s)ds + \int_{C_1} e^{st}F(s)ds$$

$$= 2\pi i\sum_{\text{poles}}\text{Residues}\left[e^{st}F(s)\right]$$

Jordan Lemma gives

$$t > 0 \quad :\lim_{R\uparrow\infty}\int_{C_1} e^{st}F(s)ds = 0$$

Therefore for $t > 0$

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}F(s)ds$$

$$= \sum_{\text{poles left of }\sigma}\text{Residues}\left[e^{st}F(s)\right]$$

Multi-valued function treated via Branch cut

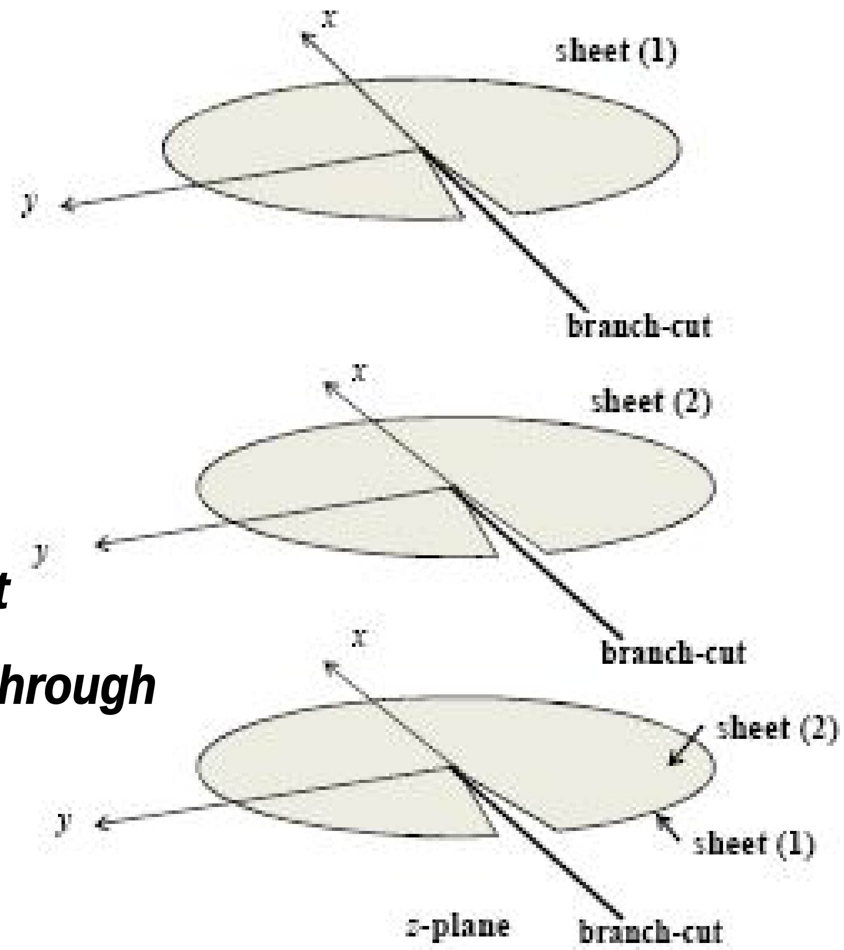
Made use for complex analysis by Arfken 1985 and Kahan 1987

We take $f(z) = \ln z$ ***has singularity at*** $z = 0$ ***Near*** $z = 0$ ***while we encircle this point we get multiple values*** $z = re^{i(\theta + 2\pi n)}$

$$f(z) = \ln r + i(2\pi n); \quad \theta \sim 0$$

Thus has different values as we go around zero-and we call this as Branch Point

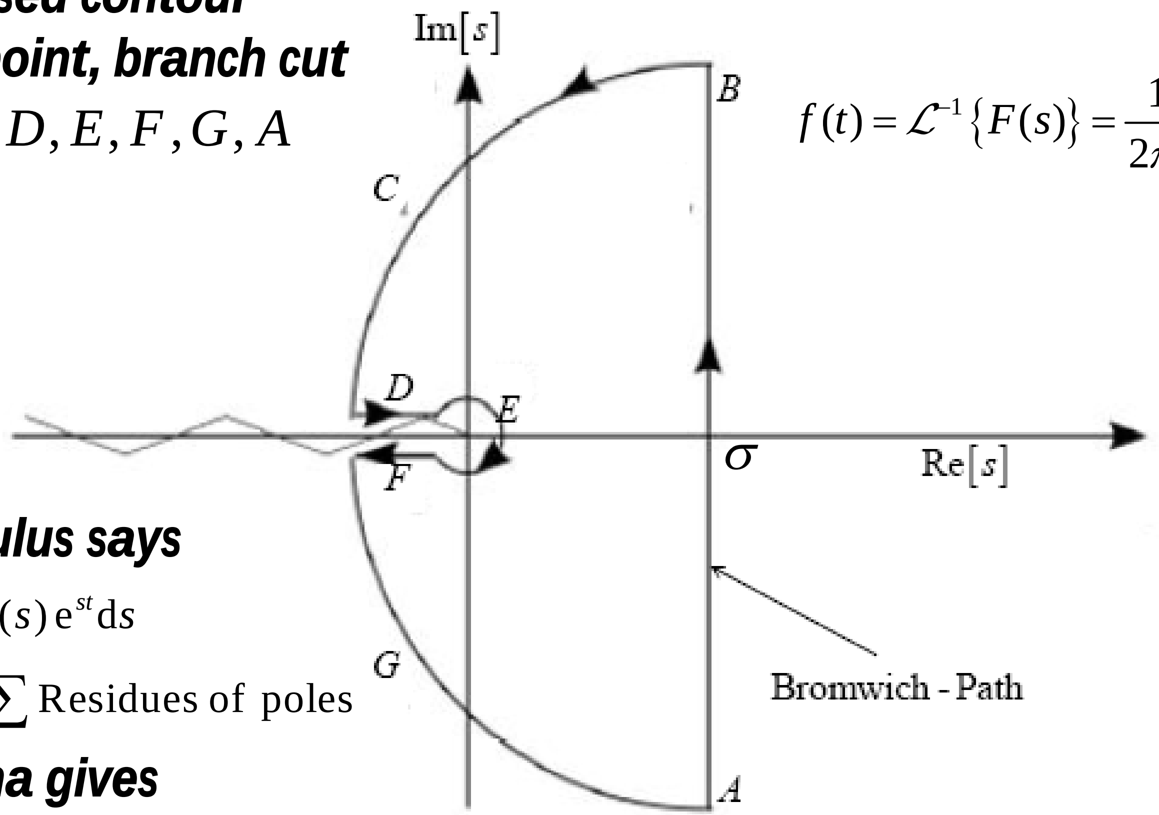
Introduce Branch-Cut which forms barriers through which z ***cannot go and thus*** $f(z)$ ***remains single valued***



For multi-valued function say $\sqrt{z-a}\,;\quad \ln z\,;\quad \frac{1}{z^{\alpha}+a},\ldots$ ***we cut the complex plane and take only primary Riemann-Sheet into consideration***

Contour integration on Branch-cut in Complex Plane
-for Inverse Laplace transform for multi-valued functions

We have closed contour via branch-point, branch cut as A, B, C, D, E, F, G, A



$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds$$

Residue Calculus says

$$\int_{A,B,C,D,E,F,G,A} F(s)\,e^{st}ds = 2\pi i \sum \text{Residues of poles}$$

Jordan Lemma gives

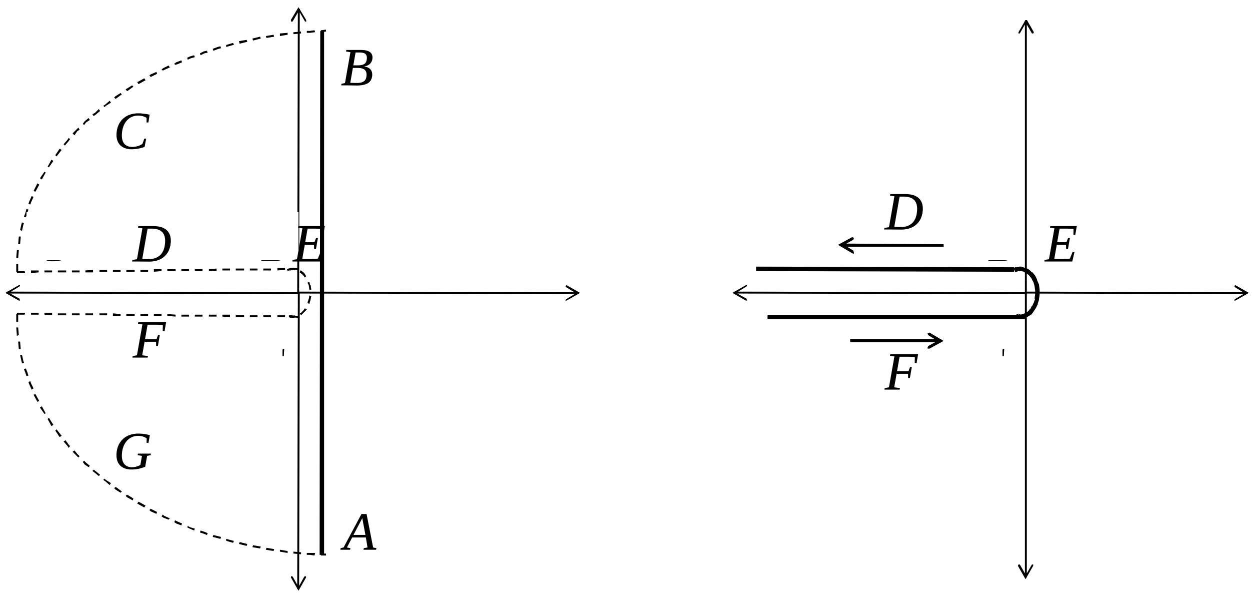
$$t > 0 \quad : \lim_{R\uparrow\infty} \int_{C \text{ and } G} e^{st}F(s)ds = 0$$

$$\int_{A,B} F(s)\,e^{st}ds = -\int_C F(s)\,e^{st}ds - \int_D F(s)\,e^{st}ds - \int_E F(s)\,e^{st}ds - \int_F F(s)\,e^{st}ds - \int_G F(s)\,e^{st}ds + 2\pi i \sum \text{Residues}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \lim_{R\uparrow\infty}\frac{1}{2\pi i}\int_{\text{Bromwich}} F(s)e^{st}ds = -\int_{D+E+F} F(s)e^{st}ds + \sum \text{Residue} = \int_{\text{Hankel}} F(s)e^{st}ds + \sum \text{Residue}$$

Bromwich path deformed to Hankel path for Integration

Bromwich $A \to B$ **Hankel** $F \to E \to D$



Deform the Bromwich path to Hankel path covering Branch Point and Branch cut line-for inverse Laplace Transform

$$\lim_{R\uparrow\infty} \frac{1}{2\pi i}\int_{\text{Bromwich}} F(s)e^{st}ds = \int_{\text{Hankel}} F(s)e^{st}ds + \sum \text{Residue}$$

Consider only poles in primary Riemann sheet

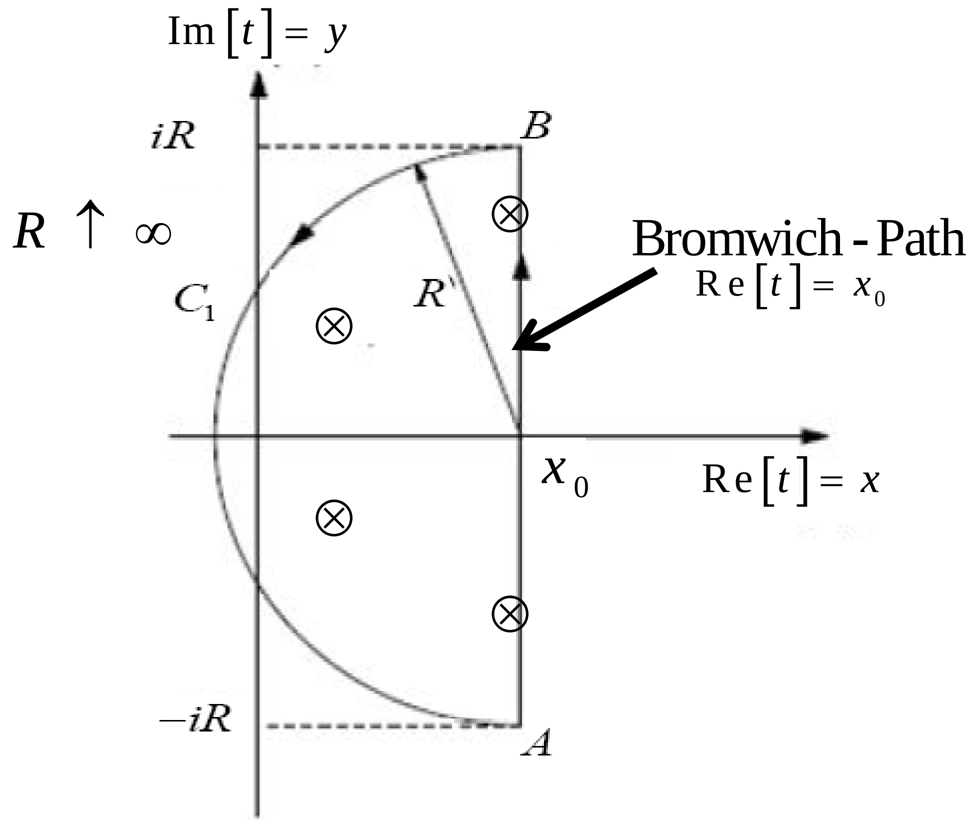
This deformation of Bromwich to Hankel is only for multi-valued functions

Obtaining histogram function of relaxation rates from inverse Laplace transform of time function

In the expression $H(\lambda) = \frac{1}{2\pi i}\int_{x_0 - i\infty}^{x_0 + i\infty} \left(f(t) \right) e^{t\lambda} dt = \mathcal{L}^{-1}\{ f(t) \}$

x_0 ***being such that*** $f(t)$ ***has some form of singularity on the real line*** $\mathrm{Re}[t] = x_0$ ***but is analytic in the complex plane to the right of that line,***

i.e. for $\mathrm{Re}[t] > x_0$



Same procedure but here the contour integration is done on complex time plane

Some standard result of histograms of relaxation rates obtained from standard Laplace transform tables

$$H(\lambda) = \mathcal{L}^{-1}\{f(t)\}$$

S.No.	$f(t),\quad t \geq 0$	$H(\lambda);\quad \lambda \geq 0$
1	e^{-t}	$\delta(\lambda - 1)$ **Delta**
2	$\frac{1}{t+1}$	$e^{-\lambda}$ **Exponential**
3	$\frac{1}{t}$	1 **Uniform**
4	$\frac{1}{t^2}$	λ
5	$\frac{1}{t^n},\quad n = 1, 2, 3 \ldots$	$\frac{1}{(n-1)!}\lambda^{n-1}$
6	$\frac{1}{t^\alpha};\quad 0 < \alpha < 1$	$\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$ **Zipf's**
7	$\frac{1}{t^2+1}$	$\sin\lambda$
8	$\frac{t}{t^2+1}$	$\cos\lambda$

Some relaxation decay functions of non-oscillatory type and their rate relaxation histograms

Relaxation function as Mittag-Leffler function

We have following Laplace identity for one parameter Mittag-Leffler function i.e.

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha + k}; \quad k > 0$$

This is verified as follows by series definition of Mittag-Leffler function

$$E_\alpha(-kt^\alpha) = 1 - \frac{kt^\alpha}{\Gamma(1+\alpha)} + \frac{k^2 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{k^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots$$

and then taking term by term Laplace transform, with known Laplace identity

$$\mathcal{L}\left\{t^n\right\} = \frac{\Gamma(1+n)}{s^{1+n}}$$

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \mathcal{L}\{1\} - \mathcal{L}\left\{\frac{kt^\alpha}{\Gamma(1+\alpha)}\right\} + \mathcal{L}\left\{\frac{k^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right\} - \mathcal{L}\left\{\frac{k^3 t^{3\alpha}}{\Gamma(1+3\alpha)}\right\} + \ldots$$

we get the result as

$$\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{1}{s} - \frac{k}{s^{1+\alpha}} + \frac{k^2}{s^{1+2\alpha}} - \frac{k^3}{s^{1+3\alpha}} + \ldots$$

$$= \frac{1}{s}\left(1 - \frac{k}{s^\alpha} + \left(\frac{k}{s^\alpha}\right)^2 - \left(\frac{k}{s^\alpha}\right)^3 + \ldots\right)$$

$$= \frac{1}{s}\left(\frac{1}{1+(k/s^\alpha)}\right) \quad \text{using} \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 .. \quad |x| < 1$$

$$= \frac{s^{\alpha-1}}{s^\alpha + k}; \quad \left|\frac{k}{s^\alpha}\right| < 1$$

Thus we get $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \dfrac{s^{\alpha-1}}{s^\alpha + k}$ ***with*** $\mathrm{Re}[s] > k^{1/\alpha}$

Expressing Mittag-Leffler function in integral on Hankel path

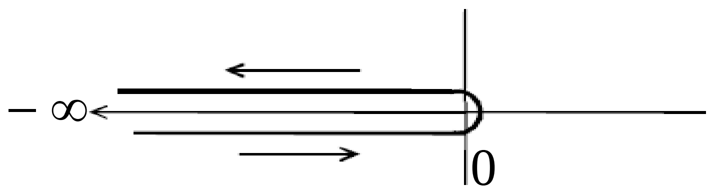
In terms of inverse Laplace integral, we write the following

$$E_\alpha(-kt^\alpha) = \mathcal{L}^{-1}\left\{\tfrac{s^\alpha}{s^\alpha + k}\right\}; \quad k > 0$$

$$E_\alpha(-kt^\alpha) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s + \sum \mathrm{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\right]$$

We bend the Bromwich path of integration into Hankel path , a loop which starts from $-\infty$ along the lower side of negative real axis encircles the circular disc $|s| = \in$ in positive (anticlockwise sense) and ends at $-\infty$ along the upper side of negative real axis. So we write

$$f(t) = E_\alpha(-kt^\alpha) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s$$



$$= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s + \sum \mathrm{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\right]$$

$$= f_1(t) + f_2(t)$$

For the function $\frac{s^{\alpha-1}}{s^\alpha + k}$ ***the poles are at*** $s = |k|^{1/\alpha}\exp\left(i\left(\tfrac{(2m+1)}{\alpha}\right)\pi\right); \quad m = 0, \pm1, \pm2, \ldots$

For $0 < \alpha < 1$ ***we have*** $\left|(2m+1)\tfrac{\pi}{\alpha}\right| > \pi$ ***meaning that no poles are in*** $-\pi < \arg[s] < \pi$

Therefore there are no poles in the 'primary Riemann-sheet' . Thus for $0 < \alpha < 1$

$f_2(t) = \sum \mathrm{Residues}\left[\mathrm{e}^{st}\left(\tfrac{s^{\alpha-1}}{s^\alpha+k}\right)\right] = 0$ ***, and we have only the contribution from Hankel i.e.***

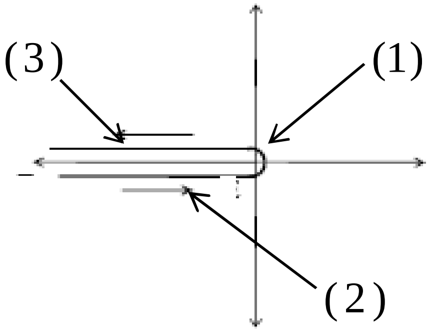
$$E_\alpha(-kt^\alpha) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)\mathrm{d}s = f_1(t); \quad 0 < \alpha < 1$$

Doing integration on Hankel's path

There are three contributions on the Hankel path. The one (1) ***is on the circle*** $s=\in e^{i\theta}$

as $\in\downarrow 0$ ***that is***

$$\frac{1}{2\pi i}\int_{s=\in e^{i\theta}} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s=\frac{1}{2\pi i}\lim_{\in\downarrow 0}\int_{s=\in e^{i\theta}} e^{\in t\exp(i\theta)}\left(\frac{\in^{\alpha-1}\,e^{i(\alpha-1)\theta}}{\in^{\alpha}\,e^{i\alpha\theta}+k}\right)\left(\in i e^{i\theta}\right)\mathrm{d}\theta$$



$$=\frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in e^{i\theta}}(1)\left(\frac{\in^{\alpha}\,e^{i(\alpha-1)\theta}}{\in^{\alpha}\,e^{i\alpha\theta}+k}\right)\left(e^{i\theta}\right)\mathrm{d}\theta;\quad \alpha>0$$

$$=\frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in e^{i\theta}}\frac{\in^{\alpha}\,e^{i\alpha\theta}}{k}\mathrm{d}\theta=0$$

The second (2) ***contribution is from line below negative real axis we call*** $s=re^{-i\pi}$

r ***varying from*** ∞ ***to*** 0 ***that gives***

$$\frac{1}{2\pi i}\int_{s=re^{-i\pi}} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s=\frac{1}{2\pi i}\int_{\infty}^{0} e^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}}(-\mathrm{d}r)$$

$$=\frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}}(\mathrm{d}r)$$

The third (3) ***contribution is from line above negative real axis we call*** $s=re^{i\pi}$

r ***varying from*** 0 ***to*** ∞ ***that gives***

$$\frac{1}{2\pi i}\int_{s=re^{i\pi}} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s=\frac{1}{2\pi i}\int_{0}^{\infty} e^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{i\pi}}(-\mathrm{d}r)$$

Total contribution of Hankel's path

Thus total contributions from Hankel path is sum of the three components

$$E_\alpha(-kt^\alpha) = \frac{1}{2\pi i}\int_{Ha} e^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)ds$$

$$= -\frac{1}{2\pi i}\int_0^\infty e^{-rt}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)_{s=re^{i\pi}} dr + \frac{1}{2\pi i}\int_0^\infty e^{-rt}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)_{s=re^{-i\pi}} dr$$

Since $E_\alpha(-kt^\alpha)$ ***is a real function, we can write the above as***

$$E_\alpha(-kt^\alpha) = -\frac{1}{2\pi}\int_0^\infty e^{-rt}\,\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{i\pi}} dr + \frac{1}{2\pi}\int_0^\infty e^{-rt}\,\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{-i\pi}} dr$$

$$= \int_0^\infty e^{-rt}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{i\pi}}\right)dr = f_1(t); \quad 0<\alpha<1$$

We compare with the formula of Laplace transform i.e. $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$

and we write the following

$$\mathcal{L}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{i\pi}}\right) = E_\alpha(-kt^\alpha); \quad 0<\alpha<1; \quad k>0$$

$$\mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\} = -\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{i\pi}}$$

$$= \frac{1}{\pi}\left(\frac{kr^{\alpha-1}\sin\alpha\pi}{r^{2\alpha}+2kr^\alpha\cos\alpha\pi+k^2}\right), \quad 0<\alpha<1$$

Thus we have obtained $\mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\};\ 0<\alpha<1$ ***as a function of*** r ***varying from*** $0-\infty$

The histogram of relaxation rates for decaying Mittag-Leffler function

We compare with the formula derived i.e. $f(t)=\int_0^\infty (H(\lambda))e^{-t\lambda}d\lambda$ ***we may write the derived expression*** $\mathcal{L}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=re^{i\pi}}\right)=E_\alpha(-kt^\alpha);\quad 0<\alpha<1$ ***by changing variable*** r ***to*** λ ***as following***

$$E_\alpha(-kt^\alpha)=\int_0^\infty e^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=\lambda e^{i\pi}}\right)d\lambda=\int_0^\infty e^{-t\lambda}\left(H_\alpha(\lambda)\right)d\lambda;\quad 0<\alpha<1;\quad k>0$$

That gives the histogram of relaxation rates $H_\alpha(\lambda)$ ***as follows for*** $f(t)=E_\alpha(-kt^\alpha)$

$$f(t)=E_\alpha(-kt^\alpha),\quad 0<\alpha<1;\quad t>0,\quad k=1$$

$$H_\alpha(\lambda)=-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}=\frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right)=\mathcal{L}^{-1}\left\{f_1(t)\right\};\quad 0<\alpha<1$$

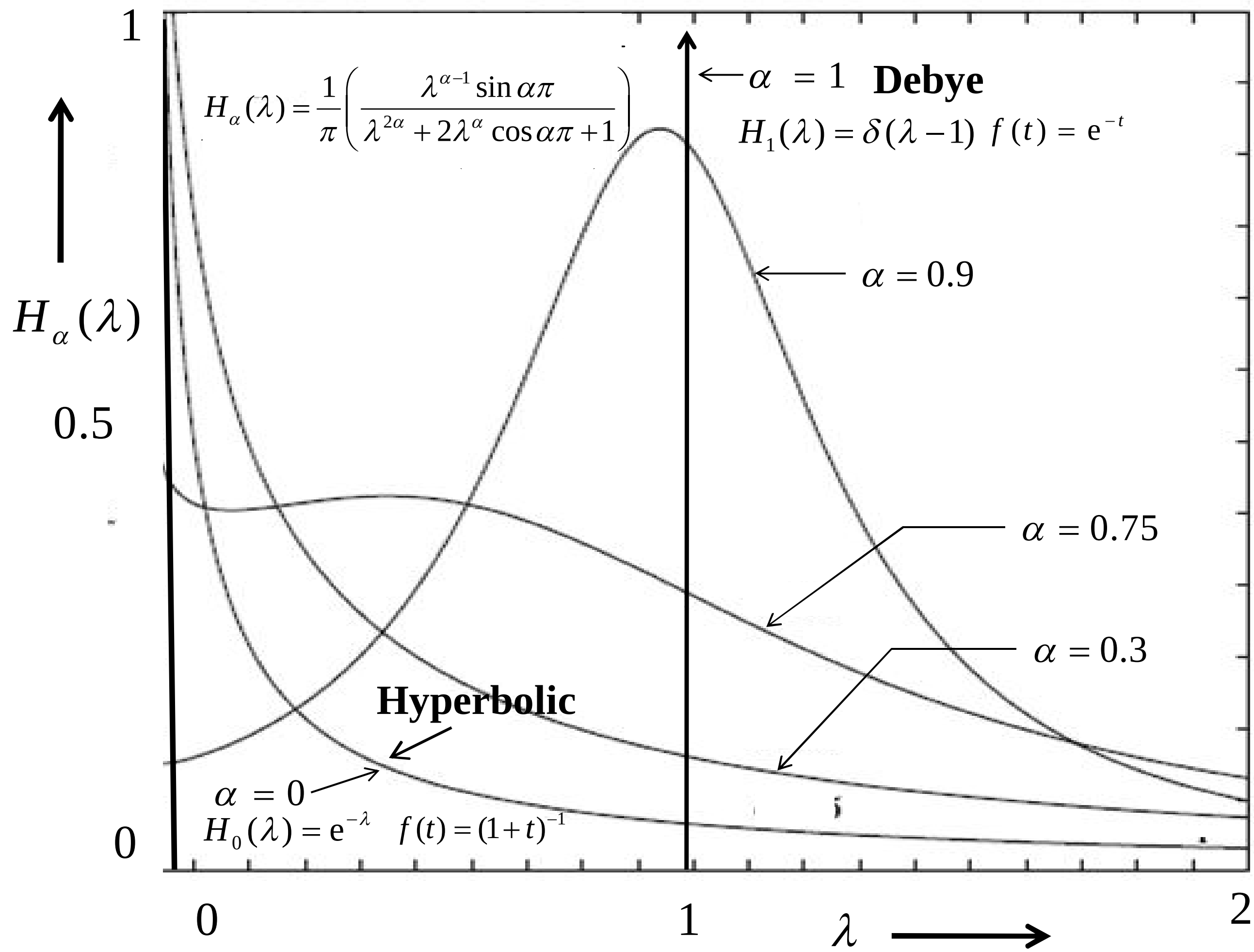
We observe that the histogram function is positive for all λ ***for*** $0<\alpha<1$

and the histogram function is negative for all λ ***for*** $1<\alpha<2$

The denominator of histogram function is $>(\lambda^\alpha-1)^2\geq 0$

Histograms are ordering the relaxation rates in particular functional form for disordered relaxation decay

$f(t) = E_\alpha(-t^\alpha), \quad 0 < \alpha < 1; \quad t > 0$



Relaxation rate distribution histogram plots for non-Debye relaxation given by Mittag-Leffler function for order $0 < \alpha < 1$

Further points regarding this method of contour integration

Thus we write inverse Laplace transform of Mittag-Leffler function of one-parameter as following ways

$$\mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{1}{\pi}\left(\frac{k\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2k\lambda^\alpha\cos\alpha\pi+k^2}\right); \quad 0<\alpha<1: \quad t\to\lambda$$

$$\mathcal{L}^{-1}\left\{E_\alpha(-ks^\alpha)\right\} = \frac{1}{\pi}\left(\frac{kt^{\alpha-1}\sin\alpha\pi}{t^{2\alpha}+2kt^\alpha\cos\alpha\pi+k^2}\right); \quad 0<\alpha<1: \quad s\to t$$

In the case of one-parameter Mittag-Leffler function, one gets a compact representation as above due to the fact that we have Laplace pair i.e. $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha+k}$

If we do not have this type of compact representation, then this described method is difficult to apply. For example $\mathcal{L}\left\{E_{\alpha,\beta}(-kt^\alpha)\right\}$ ***does not have the compact representation.***

We have a pair i.e. $\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\} = \frac{s^{\alpha-\beta}}{s^\alpha+k}$ ***For this type of relaxation, we can use the above described technique of integration on Hankel path and get the following relaxation rate distribution histogram***

$$H_{\alpha,\beta}(\lambda) = \mathcal{L}^{-1}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\} = -\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-\beta}}{s^\alpha+k}\right]_{s=\lambda e^{i\pi}}, \quad \alpha,\beta>0; \quad 0<\alpha<1$$

$$E_{\alpha,\beta}(-z) = \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\alpha n+\beta)}$$

$$= \frac{1}{\pi}\left(\lambda^{\alpha-\beta}\frac{k\sin(\beta-\alpha)\pi+\lambda^\alpha\sin\beta\pi}{\lambda^{2\alpha}+2k\lambda^\alpha\cos\alpha\pi+k^2}\right)$$

Using the M-Wright function as histogram function

Here we mention that for $f(t)=E_\alpha(-t^\alpha);\quad 0<\alpha<1$ ***we get the similar result by using*** $\mathcal{L}\{M_\alpha(t)\}=E_\alpha(-s)$ ***; M-Wright function***

$$M_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\qquad 0<\alpha<1$$

$$M_0(z)=e^{-z}$$

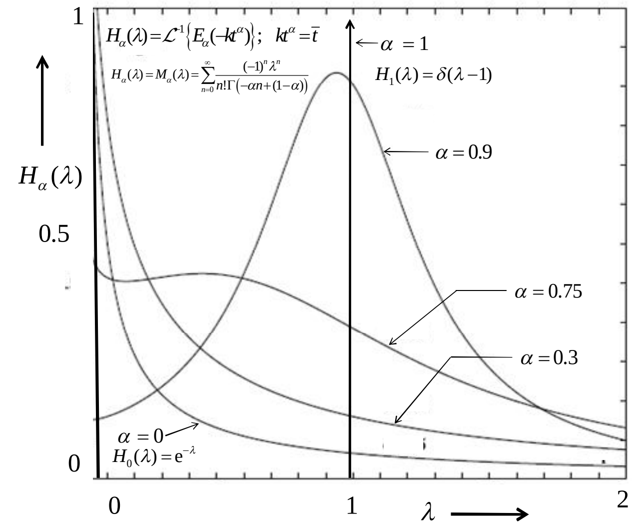
$$M_{\frac{1}{2}}(z)=\frac{1}{\sqrt{\pi}}e^{\left(-z^2/4\right)}$$

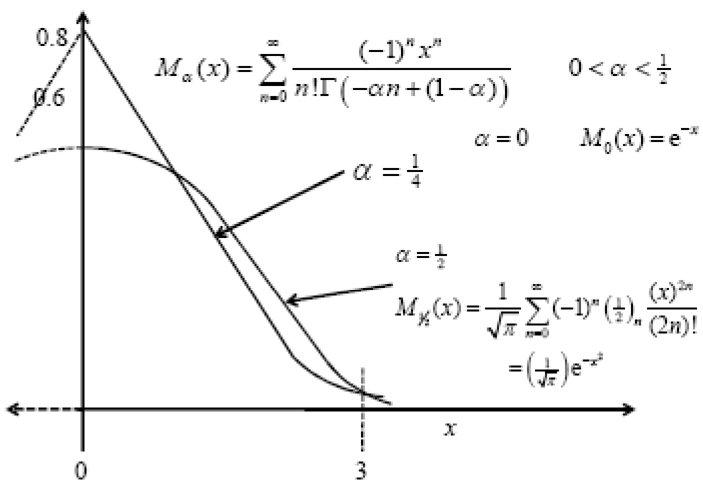
$$M_1(z)=\delta(t-1)$$

By using

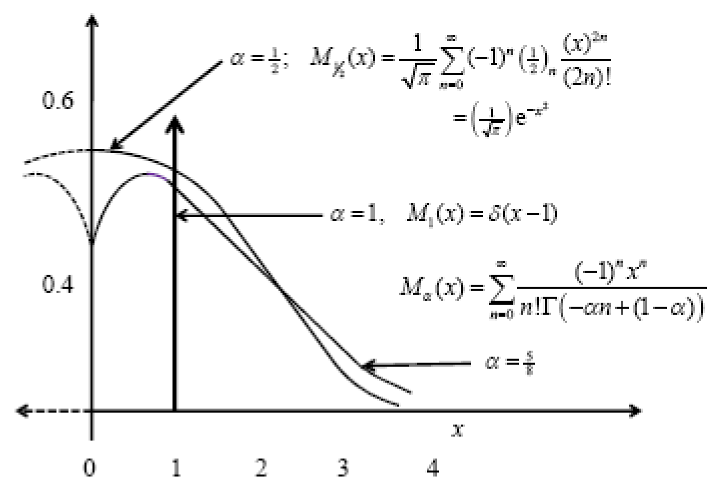
$$f(t)=E_\alpha(-kt^\alpha);\quad kt^\alpha=\bar{t}$$

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\{E_\alpha(-\bar{t})\}=M_\alpha(\lambda)$$





|: Plot of symmetric M-Wright function for α = zero to 0.5



Plot of symmetric M-Wright function for α = 0.5 to 1.0

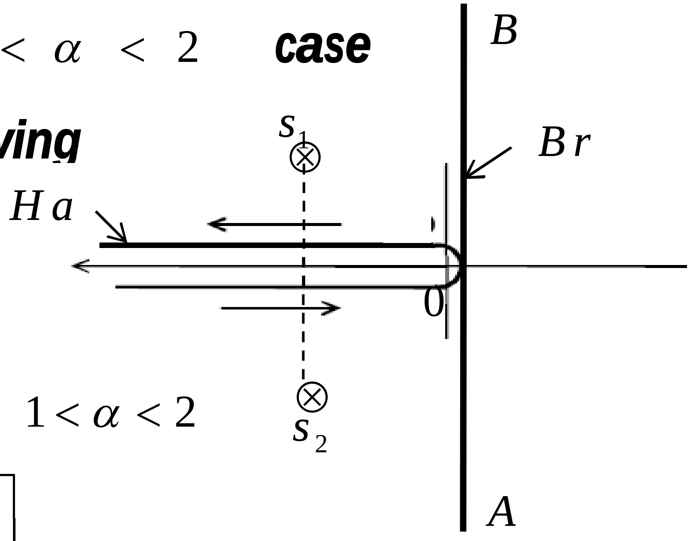
Oscillatory relaxation with Mittag-Leffler function

For, the case $1 < \alpha < 2$ ***the function*** $\frac{s^{\alpha-1}}{s^{\alpha}+k}$ ***the poles are at*** $s = |k|^{1/\alpha} \exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right)$

For the case $1 < \alpha < 2$ ***for*** $m = 0$ ***we have pole at*** $s_1 = |k|^{1/\alpha} \exp\left(i\frac{\pi}{\alpha}\right)$ ***and for*** $m = -1$ ***the pole is at*** $s_2 = |k|^{1/\alpha} \exp\left(-i\frac{\pi}{\alpha}\right)$ ***these two are complex conjugate and remain in the primary Riemann sheet- responsible for response. We represent them as*** $s_{1,2} = \sigma_0 \pm i\omega_0$ ***where*** $\sigma_0 = |k|^{1/\alpha} \cos\left(\frac{\pi}{\alpha}\right)$ ***and*** $\omega_0 = |k|^{1/\alpha} \sin\left(\frac{\pi}{\alpha}\right)$ ***Thus for*** $1 < \alpha < 2$ ***case***

$$f_2(t) = \sum \text{Residues}\left[e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\right] \neq 0$$

and we thus have following



$$E_\alpha(-kt^\alpha) = f_1(t) + f_2(t) = \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha}+k}\right\} = \frac{1}{2\pi i}\int_{Br} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)ds \qquad 1 < \alpha < 2$$

$$= \frac{1}{2\pi i}\int_{Ha} e^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)ds + \sum \text{Residues}\left[e^{st}\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s_1,s_2}$$

$$= \int_0^\infty e^{-\lambda t}\left(-\frac{1}{\pi}\text{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=\lambda e^{i\pi}}\right)d\lambda + \sum \text{Residues}\left[e^{st}\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s_1,s_2}$$

$$s_1 = \sigma_0 + i\omega_0 = |k|^{1/\alpha} e^{i\pi/\alpha}, \quad s_2 = \sigma_0 - i\omega_0 = |k|^{1/\alpha} e^{-i\pi/\alpha} \qquad \sigma_0 = |k|^{1/\alpha}\cos\frac{\pi}{\alpha}, \quad \omega_0 = |k|^{1/\alpha}\sin\frac{\pi}{\alpha}$$

Residue calculations for poles in primary Riemann-sheet

The residue calculation is following for $1 < \alpha < 2$

$$\sum \text{Residues}\left[e^{st} \frac{s^{\alpha-1}}{s^{\alpha}+k} \right]_{s_1,s_2} = \sum \text{Residues}\left[e^{st} \frac{s^{\alpha-1}}{(s-s_1)(s-s_2)} \right]_{s_1,s_2}$$

$$= \lim_{s\to s_1}(s-s_1)e^{st}\frac{s^{\alpha-1}}{(s-s_1)(s-s_2)} + \lim_{s\to s_2}(s-s_2)e^{st}\frac{s^{\alpha-1}}{(s-s_1)(s-s_2)}$$

$$= e^{s_1 t}\frac{s_1^{\alpha-1}}{s_1-s_2} + e^{s_2 t}\frac{s_2^{\alpha-1}}{s_2-s_1}$$

$$= \frac{e^{\sigma_0 t}e^{i\omega_0 t}\left(|k|^{1/\alpha}e^{i\pi/\alpha}\right)^{\alpha-1}}{2i\omega_0} - \frac{e^{\sigma_0 t}e^{-i\omega_0 t}\left(|k|^{1/\alpha}e^{-i\pi/\alpha}\right)^{\alpha-1}}{2i\omega_0}$$

$$= |k|^{\frac{\alpha-1}{\alpha}}\frac{e^{\sigma_0 t}}{2i|k|^{1/\alpha}\sin\frac{\pi}{\alpha}}\left(e^{i\omega_0 t}e^{i\pi\left(\frac{\alpha-1}{\alpha}\right)} - e^{-i\omega_0 t}e^{-i\pi\left(\frac{\alpha-1}{\alpha}\right)}\right)$$

$$= |k|^{\left(1-\frac{2}{\alpha}\right)}\frac{e^{\sigma_0 t}}{\sin\frac{\pi}{\alpha}}\left(\frac{-e^{i\left(\omega_0 t-\frac{\pi}{\alpha}\right)} + e^{-i\left(\omega_0 t-\frac{\pi}{\alpha}\right)}}{2i}\right)$$

$$= \frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}}e^{\sigma_0 t}\sin\left(\frac{\pi}{\alpha}-\omega_0 t\right)$$

$$= \left(\frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}}\right)e^{\left(|k|^{1/\alpha}\cos\frac{\pi}{\alpha}\right)t}\sin\left(\frac{\pi}{\alpha}-\left(|k|^{1/\alpha}\sin\frac{\pi}{\alpha}\right)t\right)$$

Discussing the oscillatory decay part of Mittag-Leffler function

For $k = 1$ ***we have for*** $1 < \alpha < 2$ ***the decay function is*** $f(t) = E_\alpha(-t^\alpha) = f_1(t) + f_2(t)$

$$f_2(t) = \sum \mathrm{Residues}\left[e^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+1}\right)\right]_{s_1,s_2} = \left(\frac{1}{\sin\frac{\pi}{\alpha}}\right) e^{\left(\cos\frac{\pi}{\alpha}\right)t} \sin\left(\frac{\pi}{\alpha} - \left(\sin\frac{\pi}{\alpha}\right)t\right)$$

The part i.e. $f_2(t)$ ***is oscillatory decaying part, given as*** $f_2(t) = -\mathrm{A}e^{\sigma_0 t}\sin\left(\omega_0 t - \frac{\pi}{\alpha}\right)$

where $\sigma_0 = \cos\frac{\pi}{\alpha} < 0$ ***for*** $1 < \alpha < 2$ ***This factor gives exponential decay of amplitude at*** $t = 0$ ***i.e.*** $f_2(0) = -\mathrm{A}\sin\left(-\frac{\pi}{\alpha}\right) = \left(\frac{1}{\sin\frac{\pi}{\alpha}}\right)\left(\sin\left(-\frac{\pi}{\alpha}\right)\right) = 1$

The oscillatory part is $\sin\left(\omega_0 t - \frac{\pi}{\alpha}\right)$ ***with*** $\omega_0 = \sin\frac{\pi}{\alpha}$

We observe that for $\alpha = 2$ ***we have*** $E_2(-t^2) = \cos t$ ***as*** $\sin\left(\frac{\pi}{2} - \sin\frac{\pi}{2} t\right) = \cos t$ $\mathrm{A} = 1$

and $\sigma_0 = 0$ ***Also we note that for*** $\alpha = 2$ ***the term*** $\mathcal{L}^{-1}\left\{f_1(t)\right\} = \frac{1}{\pi}\left(\frac{r^{\alpha-1}\sin\alpha\pi}{r^{2\alpha}+2r^\alpha\cos\alpha\pi+1}\right)$

i.e. contribution from Hankel path is zero. We note that $\alpha = 2$ ***gives*** $\frac{s}{s^2+1}$

with poles at $s_{1,2} = \pm i$ ***that gives inverse Laplace transform as*** $\cos t = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

Histogram function of decay rates for oscillatory decay of Mittag Leffler function

The histogram function for $f(t)=E_\alpha(-t^\alpha)=f_1(t)+f_2(t)$ ***is following where*** $\omega_0=\sin\frac{\pi}{\alpha}$

$\sigma_0=\cos\frac{\pi}{\alpha}$ $\quad\sigma_0<0\quad$ $\mathrm{A}=\frac{1}{\sin\frac{\pi}{\alpha}}$

$$f(t)=E_\alpha(-t^\alpha);\quad 1<\alpha<2$$

$$E_\alpha(-t^\alpha)=f_1(t)+f_2(t)$$

$$=\int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda+\left(\frac{1}{\sin\frac{\pi}{\alpha}}\right)\mathrm{e}^{\left(\cos\frac{\pi}{\alpha}\right)t}\sin\left(\frac{\pi}{\alpha}-\left(\sin\frac{\pi}{\alpha}\right)t\right)$$

$$=\int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t-\frac{\pi}{\alpha}\right)$$

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\{f_1(t)\}+\mathcal{L}^{-1}\{f_2(t)\}$$

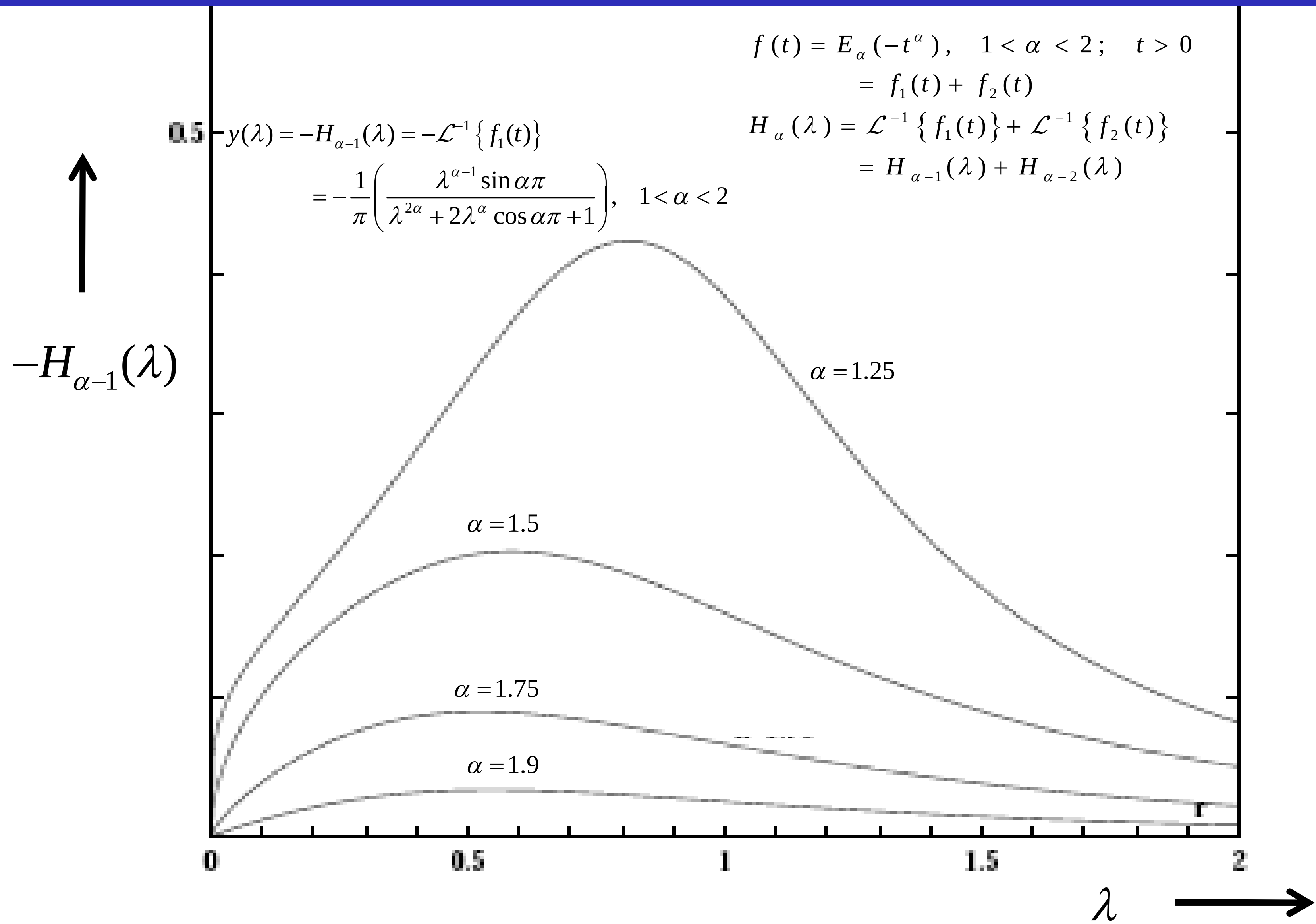
$$=\frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right)+\mathcal{L}^{-1}\left\{-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t-\frac{\pi}{\alpha}\right)\right\}$$

The first term of $H_\alpha(\lambda)$ ***for*** $1<\alpha<2$ ***call it*** $H_{\alpha-1}(\lambda)$ ***comes from contribution from Hankel's path*** ***We see*** $H_{\alpha-1}(\lambda)$ ***is negative for all*** $0<\lambda<\infty$

The second term is $\mathcal{L}^{-1}\left\{-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\frac{\pi}{\alpha}t-\frac{\pi}{\alpha}\right)\right\}$ ***call it*** $H_{\alpha-2}(\lambda)$ ***comes from Residues***

call it $H_{\alpha-2}(\lambda)=\mathcal{L}^{-1}\{f_2(t)\}$ ***we will derive later***

Histogram function for oscillatory decaying Mittag-Leffler function from Hankel's path contribution



We note that as order tends to 2, the $-H_{\alpha-1}(\lambda)$ tends to zero; i.e. for pure cosine decay

Inverse Laplace transform without contour integration

Here there is no requirement of contour integration, but comes directly from Bromwich integral i.e. integration on the line AB

$$f(t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathrm{e}^{st}F(s)\mathrm{d}s\,,\quad s=\sigma+i\omega\,,\quad \mathrm{d}s=i\mathrm{d}\omega$$

$$=\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t+i\sin\omega t\right)\left(F(\sigma+i\omega)\right)\mathrm{d}\omega\,;\quad F=\mathrm{Re}\left[F\right]+i\,\mathrm{Im}\left[F\right]$$

$$f(t)=\mathrm{Re}\left[\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t+i\sin\omega t\right)\left(\mathrm{Re}\left[F\right]+i\,\mathrm{Im}\left[F\right]\right)\mathrm{d}\omega\right]$$

$$=\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[F\right]\cos\omega t-\mathrm{Im}\left[F\right]\sin\omega t\right)\mathrm{d}\omega$$

Since $f(t)$ ***is real function we have only extracted the real part above and say the following***

$$\mathrm{Im}\left[\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\cos\omega t+i\sin\omega t\right)\left(\mathrm{Re}\left[F\right]+i\,\mathrm{Im}\left[F\right]\right)\mathrm{d}\omega\right]=0$$

$$\frac{\mathrm{e}^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}\left(\mathrm{Re}\left[F\right]\sin\omega t+\mathrm{Im}\left[F\right]\cos\omega t\right)\mathrm{d}\omega=0$$

Contd...

Contd…

But we have from definition of Laplace transform i.e. $F(s) = \int_0^{\infty} (f(t))e^{-st}dt$

and by putting $s = \sigma + i\omega$ ***we get following***

$$F(\sigma + i\omega) = \int_0^{\infty} (f(t))e^{-t(\sigma+i\omega)}dt$$

$$= \int_0^{\infty} e^{-\sigma t}(f(t))\cos(\omega t)dt - i\int_0^{\infty} e^{-\sigma t}(f(t))\sin(\omega t)dt$$

This gives following

$$\mathrm{Re}[F] = \int_0^{\infty} e^{-\sigma t}(f(t))\cos(\omega t)dt \qquad \mathrm{Im}[F] = -\int_0^{\infty} e^{-\sigma t}(f(t))\sin(\omega t)dt$$

We find that function $\mathrm{Re}[F]$ ***is even function, call it*** $e(\omega)$ ***in variable*** ω ***and the function*** $\mathrm{Im}[F]$ ***is odd function in variable*** ω ***call it*** $o(\omega)$ ***We get***

$$(\mathrm{Re}[F]\cos\omega t - \mathrm{Im}[F]\sin\omega t) = (e(\omega))\cos\omega t - (o(\omega))\sin\omega t$$

as 'even function' Therefore for overall even function integrand we have the integral as

$$f(t) = \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty}(\mathrm{Re}[F]\cos\omega t - \mathrm{Im}[F]\sin\omega t)d\omega$$

$$= \frac{e^{\sigma t}}{\pi}\int_{0}^{\infty}(\mathrm{Re}[F]\cos\omega t - \mathrm{Im}[F]\sin\omega t)d\omega$$

This is Berberan-Santo formula

Berberan-Santo method for getting histogram function for oscillatory decay part of Mittag-Leffler function

In the derivation of Berberan Santo formula we have derived the formula of inverse Laplace transform from complex frequency domain to time domain. We apply the same to invert time domain function to get histogram of rate relaxation

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\{f_2(t)\} = \mathcal{L}^{-1}\left\{-A\,e^{\cos\frac{\pi}{\alpha}t}\sin\left(\sin\frac{\pi}{\alpha}t - \frac{\pi}{\alpha}\right)\right\} = \mathcal{L}^{-1}\left\{B\,e^{-k_0 t}\sin\omega_0 t\right\}$$

$$B = -A = \frac{1}{\sin\frac{\pi}{\alpha}};\quad k_0 = -\cos\frac{\pi}{\alpha},\quad \omega_0 = \sin\frac{\pi}{\alpha}t - \frac{\pi}{\alpha};\quad 1<\alpha<2$$

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\{f_2(t)\} = \frac{e^{c_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[f_2(c_0+iy)\right]\cos\lambda y - \mathrm{Im}\left[f_2(c_0+iy)\right]\sin\lambda y\right)\mathrm{d}y$$

$$f_2(t) = Be^{-k_0 t}\sin\omega_0 t,\quad t = x+iy;\quad \text{choose}\quad x = c_0 = 0$$

$$H_{\alpha-2}(\lambda) = \mathcal{L}^{-1}\left\{Be^{-k_0 t}\sin\omega_0 t\right\} = \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left[f_2\right]\cos\lambda y - \mathrm{Im}\left[f_2\right]\sin\lambda y\right)\mathrm{d}y$$

$$f_2(iy) = Be^{-k_0(iy)}\sin(i\omega_0 y) = B\left(\cos k_0 y - i\sin k_0 y\right)\left(i\sinh\omega_0 y\right)$$

$$\mathrm{Re}\left[f_2\right] = B\sin k_0 y\sinh\omega_0 y \qquad \mathrm{Im}\left[f_2\right] = B\cos k_0 y\sinh\omega_0 y$$

We get

$$H_{\alpha-2}(\lambda) = \frac{B}{\pi}\int_0^\infty \left(\sinh\omega_0 y\right)\left(\sin\left((k_0-\lambda)y\right)\right)\mathrm{d}y$$

Interesting results in usual inverse Laplace transform are following

$$\mathcal{L}^{-1}\left\{e^{as}\sin bs\right\} = -\frac{1}{\pi}\int_0^\infty \left(\sinh b\omega\right)\left(\sin\left((a+t)\omega\right)\right)\mathrm{d}\omega$$

$$\mathcal{L}^{-1}\left\{\sin bs\right\} = -\frac{1}{\pi}\int_0^\infty \left(\sinh b\omega\right)\left(\sin\omega t\right)\mathrm{d}\omega$$

Berberan-Santo method for getting histogram function for Mittag-Leffler decay

$$H_\alpha(\lambda) = \mathcal{L}^{-1}\{f(t)\} = \frac{e^{c_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\left[f(c_0+iy)\right]\cos\lambda y - \mathrm{Im}\left[f(c_0+iy)\right]\sin\lambda y\right)\mathrm{d}y$$

$$f(t) = E_\alpha(-kt^\alpha), \quad kt^\alpha \equiv \bar{t} = x+iy; \quad \text{choose} \quad x = c_0 = 0$$

$$H_\alpha(\lambda) = \mathcal{L}^{-1}\{E_\alpha(-\bar{t})\} = \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left[E_\alpha(iy)\right]\cos\lambda y - \mathrm{Im}\left[E_\alpha(iy)\right]\sin\lambda y\right)\mathrm{d}y$$

$$E_\alpha(-\bar{t}) = 1 - \frac{\bar{t}}{\Gamma(1+\alpha)} + \frac{\bar{t}^2}{\Gamma(1+2\alpha)} - \frac{\bar{t}^3}{\Gamma(1+3\alpha)} + \ldots$$

$$E_\alpha(iy) = 1 - \frac{iy}{\Gamma(1+\alpha)} + \frac{i^2y^2}{\Gamma(1+2\alpha)} - \frac{i^3y^3}{\Gamma(1+3\alpha)} + \ldots$$

We get the following

$$\mathrm{Re}\left[E_\alpha(-\bar{t})\right] = 1 - \frac{y^2}{\Gamma(1+2\alpha)} + \frac{y^4}{\Gamma(1+4\alpha)} - \frac{y^6}{\Gamma(1+6\alpha)} + \ldots \qquad \mathrm{Im}\left[E_\alpha(-\bar{t})\right] = -\left(\frac{y}{\Gamma(1+\alpha)} - \frac{y^3}{\Gamma(1+3\alpha)} + \frac{y^5}{\Gamma(1+5\alpha)} - \ldots\right)$$

Thus we can calculate the following

$$H_\alpha(\lambda) = \mathcal{L}^{-1}\{E_\alpha(-\bar{t})\} = \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left[E_\alpha(iy)\right]\cos\lambda y - \mathrm{Im}\left[E_\alpha(iy)\right]\sin\lambda y\right)\mathrm{d}y$$

$$= \frac{1}{\pi}\int_0^\infty \left(1 - \frac{y^2}{\Gamma(1+2\alpha)} + \frac{y^4}{\Gamma(1+4\alpha)} - \frac{y^6}{\Gamma(1+6\alpha)} + \ldots\right)\cos(\lambda y)\mathrm{d}y + \frac{1}{\pi}\int_0^\infty \left(\frac{y}{\Gamma(1+\alpha)} - \frac{y^3}{\Gamma(1+3\alpha)} + \frac{y^5}{\Gamma(1+5\alpha)} - \ldots\right)\sin(\lambda y)\mathrm{d}y$$

Berberan-Santo method for getting histogram function for other non-Debye decay function

Kohlraush's Stretched exponential decay

$$f(t) = \mathrm{e}^{-(t/\tau_0)^{\alpha}}$$

$$H(\lambda) = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}y \left(\mathrm{e}^{-(y/\tau_0)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\right)\cos\left(\lambda y - \left(\frac{y}{\tau_0}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)$$

Becquerel's Compressed hyperbolic decay

$$f(t) = \frac{1}{\left(1+\frac{(1-\alpha)t}{\tau_0}\right)^{1/(1-\alpha)}}; \quad 0<\alpha<1$$

$$H(\lambda) = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}y \left(1+\left(\frac{(1-\alpha)y}{\tau_0}\right)^2\right)^{-\frac{1}{2(1-\alpha)}} \cos\left(\lambda y - \frac{\tan^{-1}\left(\frac{(1-\alpha)y}{\tau_0}\right)}{1-\alpha}\right)$$

Asymptotic power law decay

$$f(t) = \frac{1}{1+\left(\frac{t}{\tau_0}\right)^{\alpha}}, \quad 0<\alpha<1$$

$$H(\lambda) = \tau_0^{\alpha}\lambda^{\alpha-1}E_{\alpha,\alpha}\left(-(\tau_0\lambda)^{\alpha}\right)$$

$$\alpha = 1$$

$$f(t) = \frac{1}{1+\left(\frac{t}{\tau_0}\right)}; \quad H(\lambda) = \tau_0\mathrm{e}^{-\tau_0\lambda} = \tau_0 E_{1,1}\left(-(\tau_0\lambda)\right)$$

Curie von Schweidler power law decay

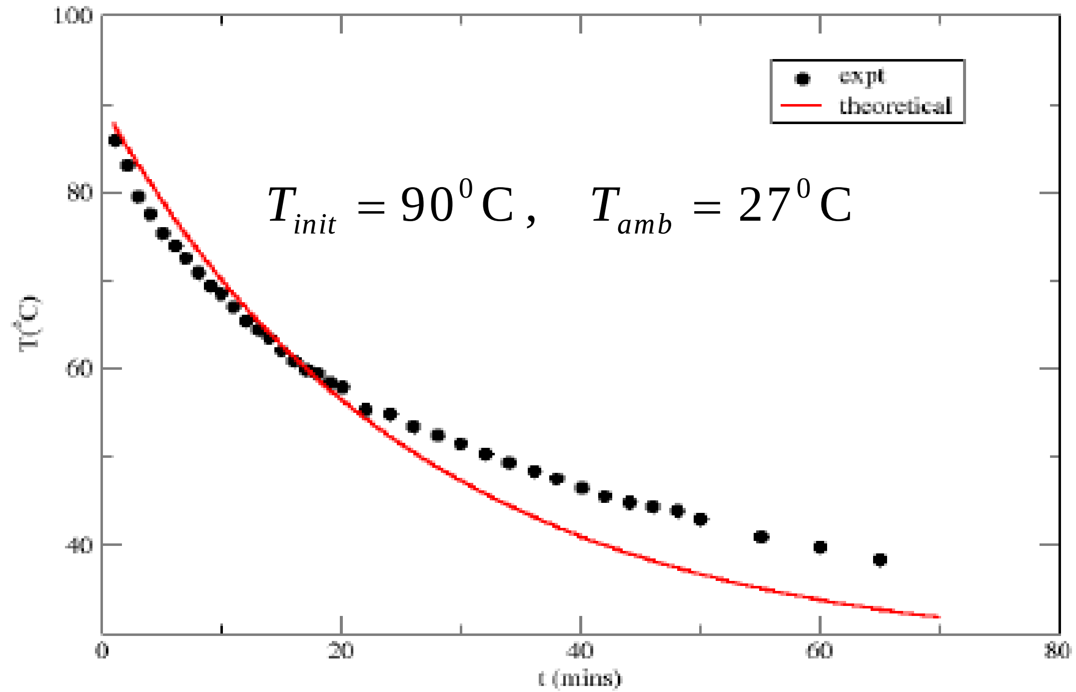
$$f(t) = t^{-\alpha}; \quad 0<\alpha<1$$

$$H(\lambda) = \frac{1}{\pi}\int_0^{\infty} y^{-\alpha}\cos\left(\lambda y - \frac{\alpha\pi}{2}\right)\mathrm{d}y = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$$

Inverse Laplace transform results of Berberan-Santo

S. No.	$G(s)$ The transfer function as a function of complex frequency $s=\sigma_0+\mathrm{i}\omega$	$g(t)=\mathcal{L}^{-1}\{G(s)\}$ In integral representation of function in time domain by inverse Laplace transform by Berberan-Santos method
1	$\frac{1}{s+a}$	$\frac{1}{\pi}\int_0^\infty \frac{a\cos(\omega t)+\omega\sin(\omega t)}{\omega^2+a^2}\mathrm{d}\omega$
2	$\frac{s}{s^2+1}$	$\frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega$
3	e^{-sT_d}	$\frac{1}{\pi}\int_0^\infty \cos\left(\omega(t-T_d)\right)\mathrm{d}\omega$
4	1	$\frac{1}{\pi}\int_0^\infty \cos(\omega t)\mathrm{d}\omega$
5	$\frac{1}{s}$	$\frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega$
6	$s^{-\alpha}$	$\frac{1}{\pi}\int_0^\infty \frac{\cos\left(\omega t-\frac{\alpha\pi}{2}\right)}{\omega^\alpha}\mathrm{d}\omega$
7	$\left(1+(1-\beta)\left(\frac{s}{a}\right)\right)^{-\frac{1}{(1-\beta)}}$	$\frac{a}{\pi(1-\beta)}\int_0^\infty \left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{aut-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u;\quad u=\frac{(1-\beta)\omega}{a}$
8	$e^{-(s/a)^\beta}$	$\frac{a}{\pi}\int_0^\infty \left(e^{-u^\beta\cos(\beta\pi/2)}\right)\cos\left(atu-u^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u;\quad u=\frac{\omega}{a}$
9	$\frac{k}{k+s^\alpha}$	$\frac{\sqrt[\alpha]{k}}{\pi}\int_0^\infty \left(\frac{\left(u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u;\quad u=\frac{\omega}{\sqrt[\alpha]{k}}$
10	$\frac{s^\alpha}{s(1+s^\alpha)}$	$\frac{1}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega$
11	$\frac{k}{s(s^\alpha+k)}$	$\frac{k}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega$

Newton's law of cooling-is it via Newtonian Calculus?



Comparison of experimental cooling curve for water with theoretical solution using classical calculus

$$\frac{\mathrm{d}\,T(t)}{\mathrm{d}t} = -a\left(T(t) - T_{amb}\right) \qquad a > 0; \qquad T(t) = T_{amb} + \left(T_{init} - T_{amb}\right)\mathrm{e}^{-at}$$

So is it $\dfrac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}} \propto \left(T(t) - T_a\right)$ **if so then is it RL or Caputo derivative?**

Courtesy :Dept. Physics St Xavier's Univ. Kolkata BSc Students- Adreja, Rivu, & Prof Tapati Dutta (Journal of Applied Physics 123, 2018)

Classical Newton's law of cooling

The classical Newton's law of cooling gives the relaxation function as

$$T(t) = T_{amb} + \left(T_{init} - T_{amb}\right)\mathrm{e}^{-at}$$

The differential equation describing the above classical cooling is

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -a\left(T(t) - T_{amb}\right), \quad a > 0, \quad T(0) = T_{init}$$

This is valid for only solid

The rate distribution histogram is by taking inverse Laplace transform, gives

$$H_1(\lambda) = \mathcal{L}^{-1}\left\{T_{amb}\right\} + \left(T_{init} - T_{amb}\right)\mathcal{L}^{-1}\left\{\mathrm{e}^{-at}\right\}$$

$$H_1(\lambda) = T_{amb}\delta(t) + \left(T_{init} - T_{amb}\right)\left(\delta(\lambda - a)\right)$$

Fractional Newton's law of cooling-via Caputo Fractional derivative

If the cooling law is composed with Caputo fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}\right|^{C} = -b\left(T(t)-T_{amb}\right), \quad 0<\alpha<1; \quad b>0; \quad T(0)=T_{init}$$

The solution is

$$T(t)\big|_{C} = T_{amb} + \left(T_{init} - T_{amb}\right)E_{\alpha}(-bt^{\alpha})$$

The rate distribution function histogram is

$$H_2(\lambda) = \mathcal{L}^{-1}\left\{T_{amb}\right\} + \left(T_{init} - T_{amb}\right)\mathcal{L}^{-1}\left\{E_{\alpha}(-bt^{\alpha})\right\}$$

$$H_2(\lambda)\big|_{\text{Contour-Integration}} = T_{amb}\delta(t) + \frac{\left(T_{init} - T_{amb}\right)}{\pi}\left(\frac{b\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha} + 2b\lambda^{\alpha}\cos\alpha\pi + b^2}\right), \quad 0<\alpha<1$$

$$H_2(\lambda)\big|_{\text{Berberan Santo}} = T_{amb}\delta(t) + \frac{(T_{init} - T_{amb})}{\pi}\int_0^{\infty}\left(1 - \frac{y^2}{\Gamma(1+2\alpha)} + \frac{y^4}{\Gamma(1+4\alpha)} - \frac{y^6}{\Gamma(1+6\alpha)} + \ldots\right)\cos\lambda y\,dy$$

$$+ \frac{(T_{init} - T_{amb})}{\pi}\int_0^{\infty}\left(\frac{y}{\Gamma(1+\alpha)} - \frac{y^3}{\Gamma(1+3\alpha)} + \frac{y^5}{\Gamma(1+5\alpha)} - \ldots\right)\sin\lambda y\,dy$$

Fractional Newton's law of cooling-via Riemann-Liouville Fractional derivative

If the cooling law is composed with Riemann-Liouville fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^{\alpha}T(t)}{\mathrm{d}t^{\alpha}}\right|^{RL} = -c\left(T(t)-T_{amb}\right), \quad 0<\alpha<1; \quad c>0; \quad T(0)=T_{init}$$

The solution is $\left.T(t)\right|_{RL} = T_{amb}\left(\Gamma(\alpha)\right)E_{\alpha,\alpha}(-ct^{\alpha}) + T_{init}\left(1-E_{\alpha}(-ct^{\alpha})\right)$

The histogram of rate distribution is thus

$$H_3(\lambda) = T_{amb}\left(\Gamma(\alpha)\right)\mathcal{L}^{-1}\left\{E_{\alpha,\alpha}(-ct^{\alpha})\right\} + \mathcal{L}^{-1}\left\{T_{init}\right\} - \mathcal{L}^{-1}\left\{E_{\alpha}\left\{-ct^{\alpha}\right\}\right\}$$

$$\begin{aligned} H_3(\lambda) = {} & \frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{1}{\Gamma(\alpha)} - \frac{y^2}{\Gamma(3\alpha)} + \frac{y^4}{\Gamma(5\alpha)} - \frac{y^6}{\Gamma(7\alpha)} + \ldots\right)\cos\lambda y\,\mathrm{d}y \\ & + \frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{y}{\Gamma(2\alpha)} - \frac{y^3}{\Gamma(4\alpha)} + \frac{y^5}{\Gamma(6\alpha)} - \ldots\right)\sin\lambda y\,\mathrm{d}y \\ & + T_{init}\delta(t) - \frac{T_{init}}{\pi}\left(\frac{c\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha} + 2c\lambda^{\alpha}\cos\alpha\pi + c^2}\right) \end{aligned}$$

Summary of fractional Newton's law of cooling with various liquids

Liquid	Volume (ml)	$T_0(°C)$	$T_a(°C)$	λ	κ	Area (cm^2)	α
Water	40	92.5	26.1	-0.114	-0.075	11.345	0.79
Water	80	100	23.5	-0.104	-0.070	14.527	0.79
Water	300	100	23.5	-0.070	-0.048	60.845	0.79
Mustard Oil	X	105.0	23.4	-0.115	-0.075	14.527	0.88
Mercury	X	105.0	23.2	-0.32	-0.19	14.527	0.92

Constants and parameters for experiments with various liquids

$$\left.\frac{\mathrm{d}^{\alpha} T(t)}{\mathrm{d}t^{\alpha}}\right|^{C} = \lambda\left(T(t) - T_a\right) \qquad \lambda < 0 \qquad T(t)\big|^{C} = T_a + \left(T_0 - T_a\right) E_{\alpha,1}\left(\lambda t^{\alpha}\right)$$

$$\left.\frac{\mathrm{d}^{\alpha} T(t)}{\mathrm{d}t^{\alpha}}\right|^{RL} = \kappa\left(T(t) - T_a\right) \qquad \kappa < 0 \qquad T(t)\big|^{RL} = T_a\left(\Gamma(\alpha)\right) E_{\alpha,\alpha}\left(\kappa t^{\alpha}\right) + T_0\left(1 - E_{\alpha,1}\left(\kappa t^{\alpha}\right)\right)$$

Courtesy :Dept. Physics St Xavier's Univ. Kolkata BSc Students- Adreja, Rivu, & Prof Tapati Dutta (Journal of Applied Physics 123, 2018)

Why non-Newtonian calculus in Newton's law of cooling?

The reason that we have to have fractional rate is due to mixed modes of heat transfer at different time scales happens in liquids-i.e. heterogeneous dynamics, whereas in solids the heat transfer is via homogeneous time scales with one mode. Thus the elemental infinitesimal unit in case of liquids is $(\Delta t)^{\alpha} > \Delta t, \quad 0 < \alpha < 1$ ***gives comfortable time slice to view the heterogeneous process-and this gives fractional rate of change with limit*** $\Delta t \downarrow 0$

The use of fractional differentials in case of non-homogeneous dynamics gives fractional derivatives and fractional integrals and thus non-Newtonian calculus

$$\frac{\Delta T}{(\Delta t)^{\alpha}} \propto \left(T(t) - T_{amb}\right)$$

$$\frac{d^{\alpha} T}{dt^{\alpha}} = K \left(T(t) - T_{amb}\right)$$

Conclusions

These mathematical techniques discussed to extract ordered relaxation rate distribution function histogram functions-for disordered relaxation observed in many systems, are very important for applied mathematics physics and engineering.

Though various reaction curves (responses) are obtained in various system studies where we have given reasons for the disordered anomalous responses observed – to Fractional Differential Equations; yet these discussed mathematical techniques based on theories of complex analysis are yet to be applied to get the spectra or ordered histograms of these experimental records.

Especially the crisp physical interpretation is missing for the negative histogram values for oscillatory decaying Mittag-Leffler function-and other non-Debye decays

The modern method of inverse Laplace transformation by Berberan-Santos formula is yet to be applied in various cases

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First email from Prof. Caputo after publication of Edition-I of "Functional Fractional Calculus" in 2007

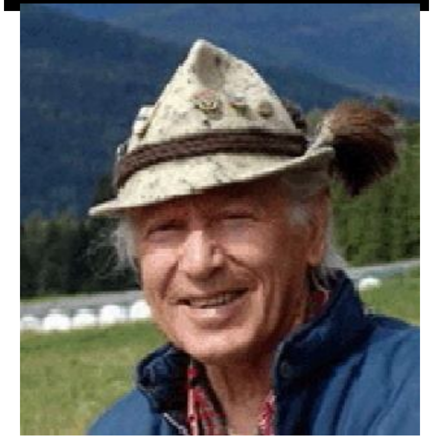
Subject: Book on fractional calculus
From: "Prof. Michele Caputo" <mic.caput@tiscali.it>
Date: Tue, April 13, 2010 12:47 am
To: shantanu@barc.gov.in

Dear Doctor Das
I read with great interest your book Functional Fractional Calculus for System Identification and Controls. I liked it, especially the modern presentation of the matter and the applications you designed. I would like to call your attention to the fact that the distributed order fractional derivative (you call it continuous order distribution page 207) was already in my 1967 original paper which, probably, since the paper is so old and you were oriented to applications, you did not have time to read. You made it good extension on this continuous order system however a new innovative thought.

If you are interested I may send you email with some of the papers.

I spent about 15 years in the US universities where I also had the pleasure to have some excellent students from India. If you are interested I will be glad to correspond with you on any matter where I have some competence. With best regards. Michele Caputo

Michelle Caputo 1927-



I did all my advance work in Mathematics, Physics and Engineering on Fractional Calculus due to blessings from Prof. Caputo founder of Caputo Fractional Derivative 1967

Thank you ISNA & Gokhale Memorial College

for

giving a nonmathematical & not so qualified person the opportunity to

present this topic in Mathematical Science Seminar