Presenting how disordered decaying relaxation via Mittag-Leffler function manifests ordered relaxation rate distribution histograms

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Abstract

We study the disordered relaxation, which is non-Debye relaxation. We form the function describing the distribution of relaxation rates that are distributed from rate zero to infinity. Primarily we see that Debye relaxation that is a pure exponential relaxation or decay curve, has a unique relaxation rate that gets spread when the relaxation is via Mittag-Leffler function. We study the decay curves which are monotonically decaying given by Mittag-Leffler function with its order varying from zero to one, and oscillatory decay function with Mittag-Leffler function with order varying from one to two. Thus this Mittag-Leffler function maps from order zero i.e. a hyperbolic decay to order one i.e. pure exponential decay (Debye-type) and then to order equals two a pure sustained oscillatory relaxation. In order to get the histogram function for distribution of relaxation rates for these non-Debye relaxations, one has to get inverse Laplace transform of decay functions. We will describe various ways to get inverse Laplace transform of Mittag-Leffler function, thus describing the histogram of relaxation rates. These types of non-Debye relaxations are observed in systems with fractional order dynamics. At the end as example we will use these results to obtain relaxation rate histograms for a disordered relaxation of temperature of a cooling body, where the cooling law is governed by Caputo or Riemann-Liouville fractional derivative, instead of classical integer order derivative.

Key-words

Mittag-Leffler function, non-Debye relaxation, disordered relaxation, M-Wright Function, Hankel path, Bromwich path, Residues, histogram function, Laplace integral, Berberan-Santo method

Introduction

The Curie-von Schweidler law relates to relaxation current in dielectric when a step DC voltage is applied and is given by $f(t) \sim t^{-\alpha}$, where $t > 0$ and the power (exponent) i.e. α is called relaxation constant or decay constant, where $0 < \alpha < 1$. This relaxation law is taken as universal law, at least for dielectric relaxations. Whereas we are used to Debye type of relaxation i.e. exponential decay law given by $f(t) \sim \mathrm{e}^{-k_0 t}$ where k_0 denotes the relaxation rate of the process. The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments .We will discuss the non-Debye type relaxation i.e. $f(t) \sim E_\alpha(-k_0 t^\alpha)$, i.e. via Mittag-Leffler function, for cases where $0 < \alpha < 1$, i.e. the decay function is monotonically decaying; also for cases $1 < \alpha < 2$ with oscillatory decay. The non-Debye relaxation has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of relaxation rates λ varying from near zero to infinity. The observations of non-Debye relaxations are also made in the experiments and studies with super-capacitors, cooling of bodies and visco-elastic experiments. In this presentation, we are giving the derivation of the distribution of relaxation rates (λ) particularly for Mittag-Leffler function. We observe the histogram or distribution of relaxation rates is of various types ranging from pure exponential distribution, then taking Gaussian distribution in between with mean at zero, and then going towards delta distribution where the many body relaxation relaxes simultaneously with a single rate with $\lambda = k_0$. This is case when Mittag-Leffler function is non-oscillatory decay curve. Similarly we will extend this method to describe histogram of relaxation rates with Mittag-Leffler function as oscillatory decay curve, and see that the components of rate distribution function has negative values in histogram function. Though by experiments one cannot make histogram directly for the rates of relaxation for any non-Debye processes, yet this mathematical procedure that we develop helps in extracting this information from the observations relaxation decay functions. This is new treatment, and much more research is required, across various dynamic processes. This is what we call ordering the disordered non-Debye relaxation processes-via histogram of relaxation rate distribution. We will discuss various methods to obtain relaxation rate distribution histogram, i.e. several ways we will derive inverse Laplace transformation of Mittag-Leffler function. We will use these results to obtain relaxation rate histograms for a disordered relaxation of temperature of a cooling body, where the cooling law is governed by Caputo or Riemann-Liouville fractional derivative, instead of classical integer order derivative.

Composing complex non-Debye relaxation with summing of several Debye relaxations with various relaxation rates

We call the relaxation as complex process, of non-Debye type, that is $f(t) \neq \mathrm{e}^{-\lambda_0 t}$. In this section, we formulate the method to extract the histogram of the relaxation rates call it $H(\lambda)$, for a complex non-Debye relaxation process $f(t)$, which we assume to be composed of several Debye type relaxations $\mathrm{e}^{-\lambda t}$, with λ varying from zero to infinity. This means the system is having multi-body relaxation, where by many bodies are relaxing with Debye exponential law but with different rates of relaxation-thus giving disordered relaxation. Some rates have more numbers of relaxing bodies, and some rates have less number of relaxing bodies, thus we get a rate relaxation distribution function or a histogram function.

The complex decay is expressed as following with several Debye rate constants $\lambda_1, \lambda_2, \lambda_3, \ldots$ with weights $a_1, a_2, a_3, \ldots$, where λ is having units in $\sec^{-1}$ i.e. 'per second'. We write following composite relaxation expression as sum of several 'discrete' relaxations of Debye type i.e.

$$
\begin{aligned}
f(t) &= a_1\mathrm{e}^{-\lambda_1 t} + a_2\mathrm{e}^{-\lambda_2 t} + a_3\mathrm{e}^{-\lambda_3 t} + \ldots \\
&= \sum a_j \mathrm{e}^{-\lambda_j t} \\
f(0) &= a_1 + a_2 + a_3 + \ldots
\end{aligned}
$$

In continuum limit we may write the above as following

$$
f(t) = \int_0^\infty \left(H(\lambda)\right) \mathrm{e}^{-\lambda t} \mathrm{d}\lambda
$$

The function i.e. $H(\lambda)$ is the distribution-function of the rate of the relaxation (λ) of the process, or we may call it as histogram of relaxation rates. We note here the weights a_j can be positive or negative, and so $H(\lambda)$ can be too have positive or negative values. This negative value we will discuss subsequently.

While for the case with discrete set of relaxation rates i.e. $\lambda_j = k_1, k_2, k_3, ...$ the rate distribution function would be having discrete delta functions ($\delta(\lambda - \lambda_j)$, $j = 1, 2, 3, ...$) at points $k_1, k_2, k_3, ...$; which we write like following expression

$$H(\lambda) = a_1\delta(\lambda - k_1) + a_2\delta(\lambda - k_2) + a_3\delta(\lambda - k_3) + ...$$
$$= \sum a_j \delta(\lambda - \lambda_j); \qquad \lambda_j\Big|_{j=1,2,3,...} = k_1, k_2, k_3, ...$$

From above formulation if we have only one single Debye relaxation i.e. having only one rate constant say $\lambda = k_0$ i.e. $f(t) = a_0 \mathrm{e}^{-k_0 t}$ then $H(\lambda) = a_0\delta(\lambda - k_0)$. This is verified in the following expression

$$f(t) = \int_0^\infty \left((H_\lambda \lambda)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$$
$$= \int_0^\infty \left(a_0\delta(\lambda - k_0)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda = a_0\mathrm{e}^{-k_0 t}$$

In above expression we used the property of delta function i.e. $\int \left(\delta(x - x_0)\right) g(x)\mathrm{d}x = g(x_0)$.

Figure-1 demonstrates this idea presented in this section

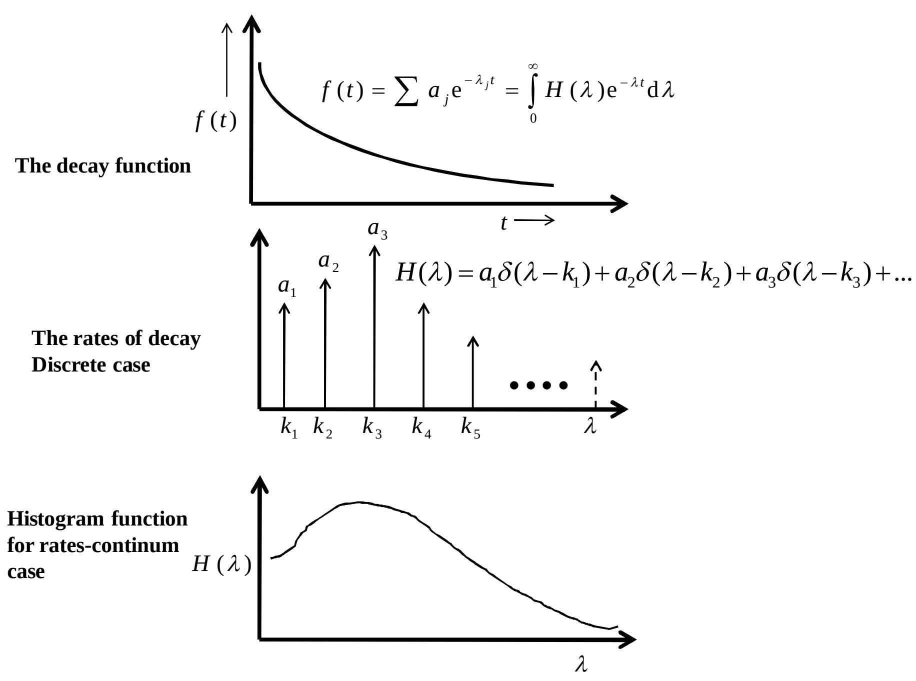


Figure-1: Composition of non-Debye relaxation function with several relaxation functions of Debye type in discrete and continuum realization

The above explanation pertains to monotonically decaying function either via single exponential function or via summation of several exponential functions.

If the decaying function is a constant in time, i.e. there is no decay say $f(t) = A$, then by above explanation we have $f(t) = A\mathrm{e}^{-k_0 t}; \quad k_0 = 0$, and the histogram function is $H(\lambda) = A\delta(\lambda)$. This means all the excited multi-bodies have zero rates of decay, i.e. given by delta function at $\lambda = 0$.

If the relaxation is of oscillatory type say a decaying oscillation i.e. $f(t) = \mathrm{e}^{-k_0 t}\cos\omega_0 t$, then with the above explanation, we see that we have a time changing histogram function given as $H(\lambda) = A(t)\delta(\lambda - k_0)$ with $A(t) = \cos\omega_0 t$. Means the unit-delta function at $\lambda = k_0$ will rise and fall with time!! Similarly for a pure non decaying oscillatory relaxation function, we have $f(t) = \cos\omega_0 t$ the histogram of rate relaxation is $H(\lambda) = A(t)\delta(t)$, with time varying unit-delta function at origin whose height changes with time as $A(t) = \cos\omega_0 t$. These simple explanations we will use subsequently.

Formulation of Laplace integral and extraction of relaxation rate distribution function for non-Debye relaxation process

In this section we formulate Laplace integral and getting inverse Laplace transform of time domain response i.e. $\mathcal{L}^{-1}\{f(t)\}$ to get relaxation rate distribution function i.e. $H(\lambda)$. That is $\mathcal{L}^{-1}\{f(t)\} = H(\lambda)$

The Laplace transform $G(s)$ of a function in time domain $g(t)$ is defined as following integral transform relation i.e. called Laplace integral

$$G(s) \overset{\text{def}}{=} \int_0^\infty \left(g(t)\right)\mathrm{e}^{-st}\mathrm{d}t$$

$$G(s) = \mathcal{L}\{g(t)\} \qquad \mathcal{L}^{-1}\{G(s)\} = g(t)$$

This is standard integral transform of a function $g(t)$ from a time domain (t) to a complex frequency domain i.e. $s = \mathrm{Re}[s] + i\omega; \quad i = \sqrt{-1}$; where real part is significant in the transient response and the imaginary part of the frequency corresponds to 'steady-state' response. Here $g(t)$ is 'inverse Laplace transform' of $G(s)$, and we write $\mathcal{L}^{-1}\{G(s)\} = g(t)$ and $\mathcal{L}\{g(t)\} = G(s)$.

In earlier section we formulated the complex non-Debye relaxation of $f(t)$ as in continuum case as an integral i.e. $f(t) = \int_0^\infty \left(H(\lambda)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$; defining $H(\lambda)$ as rate distribution function or histogram of relaxation rates. Compare this expression with defined Laplace integral expression as follows

$$f(t) = \int_0^\infty \left(H(\lambda)\right)\mathrm{e}^{-t\lambda}\mathrm{d}\lambda \qquad\qquad G(s) = \int_0^\infty \left(g(t)\right)\mathrm{e}^{-st}\mathrm{d}t$$

Both above are Laplace transform expressions, (or Laplace integrals). The first expression is transforming the function $H(\lambda)$ from λ domain to 'complex' t time domain; while the second one is transforming $g(t)$ from t domain to 'complex' s frequency domain. Thus, both are Laplace integral expressions with change of variable and symbol. Therefore we say $H(\lambda)$ is inverse Laplace Transform of $f(t)$ in the first expression, i.e. $H(\lambda)=\mathcal{L}^{-1}\left\{f(t)\right\}$; as $g(t)$ is inverse Laplace of $G(s)$ in the second expression i.e. $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$.

Therefore in order to get the rate distribution-function $H(\lambda)$ from the decay curve (or relaxation-function $f(t)$), we need to perform inverse Laplace Transform of the time function $f(t)$. The definition of inverse Laplace Transform is described as following integral expressions on Bromwich path.

$$g(t)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\left(G(s)\right)\mathrm{e}^{st}\mathrm{d}s \qquad H(\lambda)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\left(f(t)\right)\mathrm{e}^{t\lambda}\mathrm{d}t$$

In the expression $H(\lambda)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\left(f(t)\right)\mathrm{e}^{t\lambda}\mathrm{d}t$; x_0 being such that $f(t)$ has some form of singularity on the real line $\mathrm{Re}\left[t\right]=x_0$ but is analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\left[t\right]>x_0$. Thus in the formulation we treat time variable as complex quantity say $t=x_0+iy$ in the expression of inverse Laplace Transform i.e. $H(\lambda)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}\left(f(t)\right)\mathrm{e}^{t\lambda}\mathrm{d}t$.

We have observed in the previous section that a Debye relaxation of $f(t)\sim\mathrm{e}^{-k_0t}$ has rate distribution as $H(\lambda)=\delta(\lambda-k_0)$ i.e. it is given by a delta function at point $\lambda=k_0$. This we verify with known Laplace relation i.e. $\mathcal{L}\left\{g(t-t_0)\right\}=\mathrm{e}^{-st_0}G(s)$, where $\mathcal{L}\left\{g(t)\right\}=G(s)$. In addition, we have $\mathcal{L}\left\{\delta(t)\right\}=1$; thus, we can write $\mathcal{L}\left\{\delta(t-t_0)\right\}=\mathrm{e}^{-st_0}$. From here, we can write with change of variable for $f(t)=\mathrm{e}^{-k_0t}$ the inverse Laplace of this time domain function in λ domain we get as $H(\lambda)=\delta(\lambda-k_0)$, i.e. the rate distribution function. The Table-1 describes various decay functions $f(t)$ where inverse Laplace transformation is giving the relaxation rate distribution function i.e. $H(\lambda)$.

S.No.	$f(t),\quad t\ge 0$	$H(\lambda);\quad \lambda\ge 0$
1	e^{-t}	$\delta(\lambda-1)$
2	$\frac{1}{t+1}$	$\mathrm{e}^{-\lambda}$
3	$\frac{1}{t}$	1
4	$\frac{1}{t^2}$	λ
5	$\frac{1}{t^n},\quad n=1,2,3...$	$\frac{1}{(n-1)!}\lambda^{n-1}$
6	$\frac{1}{t^\alpha};\quad 0<\alpha<1$	$\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$
7	$\frac{1}{t^2+1}$	$\sin\lambda$
8	$\frac{t}{t^2+1}$	$\cos\lambda$

Table-1: Some relaxation decay functions of non-oscillatory type and their rate relaxation histograms

Description about histogram of rate relaxation functions for ordered and disordered systems for a non-oscillatory decay function

As mentioned earlier, relaxation process is a complex process involving a several relaxing bodies simultaneously relaxing to excitation stimuli. Earlier we described a Debye relaxation depicted in S.No.1 of Table-1, where the all the relaxing bodies that are relaxing simultaneously relaxes with a single rate of relaxation. This Debye relaxation is ordered relaxation process.

The other entries in Table-1 are disordered relaxation processes. The S.No.2 $f(t) = (1+t)^{-1}$ is a hyperbolic decay process, where it has infinite relaxing bodies with different relaxation rates λ from zero to infinity disordered relaxation where relaxation rate histogram is depicted as pure exponential distribution function. Interestingly S.No.3 is also a disordered relaxation, but the relaxation rates are distributed as uniform distribution from relaxation rate zero to infinity. The S.No.6 gives Curie-von Schweidler relaxation observed universally in dielectric relaxations (since late 19th century); having relaxation rate histogram as Zipf's power law distribution i.e. $H(\lambda) \sim \lambda^{\alpha-1}; \quad 0 < \alpha < 1$. S.No.1 to S.No.7 all are non-oscillating decaying functions. We notice that for S.No.1 to S.No.6 the histogram function $H(\lambda)$ is positive, whereas the S.No.6 has $H(\lambda) = \sin\lambda$ which has both positive and negative values. Therefore we may say, for monotonically decaying non-oscillatory relaxation function the histogram function need not be always be positive. Whereas the S.No.8 is a decay function $f(t)$, which starts at $t = 0$ increases to maximum and then falls off at late times and tends to zero at very-very large time. This too has a histogram of relaxation rates that is cyclic $H(\lambda) = \cos\lambda$ and has negative values also.

Can histogram of relaxation rate distribution have negative values and possible reasons?

The complex non-Debye process of relaxation may be considered to be having simultaneously infinite bodies relaxing at several rates of relaxations and can be grouped by number of relaxing bodies as a function of relaxation rates: that we called as $H(\lambda)$. In the previous section and Table-1 we have seen that $H(\lambda)$ need not be always positive, it can have negative values, as appears in for S.No.7 and S.No.8.

Now what are the negative and positive values indicate in histogram of $H(\lambda)$ for a non-Debye relaxing function? For positive values say infinite number of relaxing bodies are releasing energy from an excited state to ground state, with several rates. If all the bodies at every possible rate of relaxation are releasing energy then we have a histogram that is positive always at all relaxing rates; from zero to infinity. Now a complex multi-body system may have bodies that are releasing energy from higher state to ground level, but also may have bodies that are gaining energy at particular region of rates λ. The number of bodies gaining energy is represented as negatives. For example in the case of $H(\lambda) = \sin\lambda$ the relaxing bodies are releasing energy from high energy state to ground state with relaxation rates $\lambda = 0$ to $\lambda = \pi$; with maximum number of bodies clustered at rate $\lambda = \frac{\pi}{2}$ and while in the higher rate region $\lambda = \pi$ to $\lambda = 2\pi$ the bodies are gaining energy from lower energy level state to ground state.

Another picture that comes to our mind is that multi-body relaxations are having a positive decay currents and negative decay currents. At say range of rate $0 - \pi$ the positive decay currents appear and in the range of rates $\pi - 2\pi$ negative decay currents appear. This seems the picture of histogram of migration, positive migration is into the country and negative migration out of the country, and together they give cyclic nature!!

Therefore we say the positive histogram is associated with decay of bodies with relaxing function as $a_m e^{-\lambda_m t}$, where $a_m > 0$ say for $H(\lambda) \sim \sin\lambda$ with $0 < \lambda_m < \pi$ and for $\pi < \lambda_n < 2\pi$ we have decaying bodies having decay as $-a_n e^{-\lambda_n t}$ where $a_n > 0$, giving negative histogram. This $\pm a_k e^{-\lambda_k t}$ gives the picture of bodies loosing/gaining energy, or positive decaying current or negative decaying current source body. Bodies having positive or negative current are analogous to migration of charges-showing positive migration giving positive histogram and negative migration giving negative histogram.

We will mainly concentrate in this article about the relaxation in between S.No.1 and S.No.2 i.e. between hyperbolic and pure exponential decay given by Mittag-Leffler function of one parameter $E_\alpha(-kt^\alpha)$, with $0 < \alpha < 1$ and then discuss oscillatory decay between exponential decay $f(t) = e^{-kt}$ to pure oscillatory response $f(t) = \cos kt$, with Mittag-Leffler function $E_\alpha(-kt^\alpha)$ with $1 < \alpha < 2$.

Non-Debye Relaxation via Mittag-Leffler Function

The Mittag-Leffler function of one parameter is generalization of exponential function. We can describe a non-Debye decay function by α order Mittag-Leffler function-i.e. $f(t) = E_\alpha(-kt^\alpha)$, $0 < \alpha < 1$ and $k > 0$. Let us call $kt^\alpha = z$. The Mittag-Leffler function $E_\alpha(-z)$ is described as

$$E_\alpha(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + 1)}$$

Where z is a complex variable $z = x + iy$ and $\alpha \geq 0$. Though we have cases with negative arguments α for Mittag-Leffler function; that we are not discussing in relaxation phenomena. For $\alpha = 0$ we have

$$E_0(-z) = \frac{1}{1+z}$$

For $\alpha = 1$ we have

$$E_1(-z) = e^{-z}$$

For $0 < \alpha < 1$ the Mittag-Leffler function interpolates between a pure exponential non-Debye relaxation function i.e. $E_1(-z) = e^{-z}$ and a hyperbolic relaxation function $E_0 = (1+z)^{-1}$; $z \geq 0$. The generalized Mittag-Leffler function having two parameters α and β is described as following

$$E_{\alpha,\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha n + \beta)}$$

We have $E_{\alpha,1}(-z) = E_{\alpha}(-z)$. Figure-2 gives decay curves $f(t) = E_{\alpha}(-t^{\alpha})$ for various arguments of Mittag-Leffler function, for $0 < \alpha < 1$. The case $f(t) = E_{\alpha}(-t^{\alpha})$ for $1 < \alpha < 2$, that describes oscillatory decay is discussed later (Figure-7).

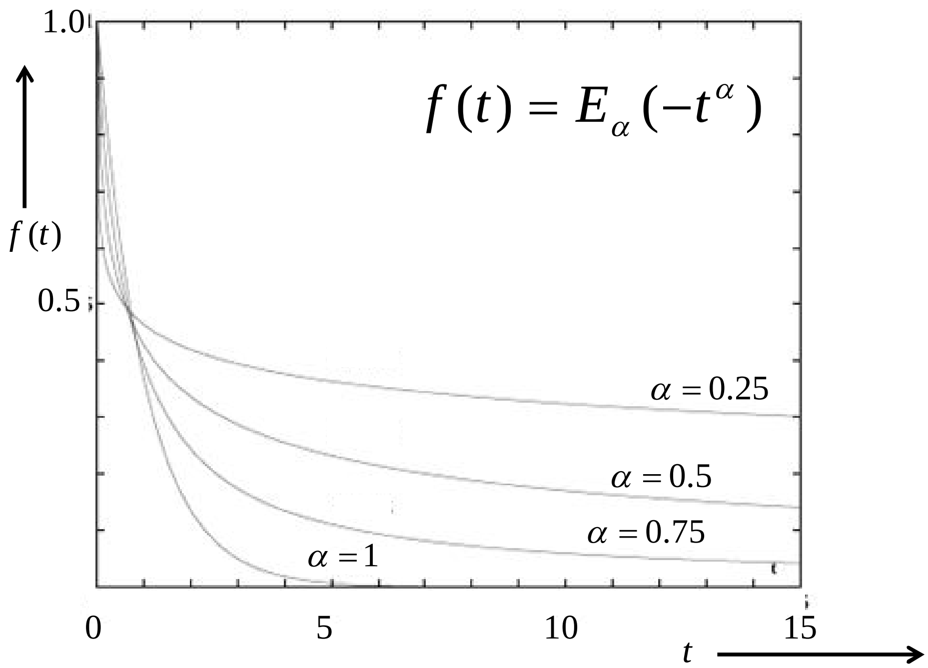


Figure-2: Mittag-Leffler function $E_{\alpha}(-t^{\alpha})$ **as decay function for** $0 < \alpha < 1$

From Figure-2 we observe as the value of α decrease from one towards zero, the relaxation function $f(t) = E_{\alpha}(-t^{\alpha})$ from pure exponential decay $f(t) = \mathrm{e}^{-t}$; $\alpha = 1$ tends towards a hyperbolic decay $f(t) = (1 + t^{\alpha})^{-1}$ for α close to zero.

What we also observe that as α is smaller than one, the initial fall of the curve is steep with faster rate and as the time increases, the fall becomes at slower rate. At $\alpha = 1$ we are having $f(t) = \mathrm{e}^{-1}$ that means it is a pure Debye decay with constant rate $\lambda = k_0 = 1$ always at all times; with all the relaxing bodies in a multi-body relaxation has one rate of relaxation i.e. k_0 .This also imply that the Histogram of rate distribution is $H(\lambda) = \delta(\lambda - 1)$; and also there is no time dependency on the decay rate i.e. $\lambda(t) = k_0 = 1$.

Whereas for the cases $0 < \alpha < 1$ we have initially at $t \sim 0$ a higher rate of decay and as time increases the decay rate λ becomes smaller thus there is a time dependent decay rate $\lambda(t)$. This implies that there are multiple decaying rates governing a complex-non Debye relaxation process; implies that we have a distribution or histogram for several decay rates, i.e. $H(\lambda)$ depending on α : we will call that as $H_{\alpha}(\lambda)$

We will study how $H_\alpha(\lambda)$ the distribution for decay rates or the histogram takes shape from delta distribution i.e. $H_1(\lambda)=\delta(\lambda-1)$ for pure exponential relaxation with $\alpha=1$, $f(t)=E_1(-t)=\mathrm{e}^{-t}$ as α changes from one towards zero. For that we need to find out inverse Laplace transform of Mittag-Leffler function i.e. $H_\alpha(\lambda)=\mathcal{L}^{-1}\left\{E_\alpha(-t^\alpha)\right\}; \quad t\geq 0, \quad 0<\alpha<1$

There are several techniques to find out the inverse Laplace transform; first we will discuss the use of M-Wright function in finding the inverse Laplace transform of the Mittag-Leffler function. For that we need to describe M-Wright function.

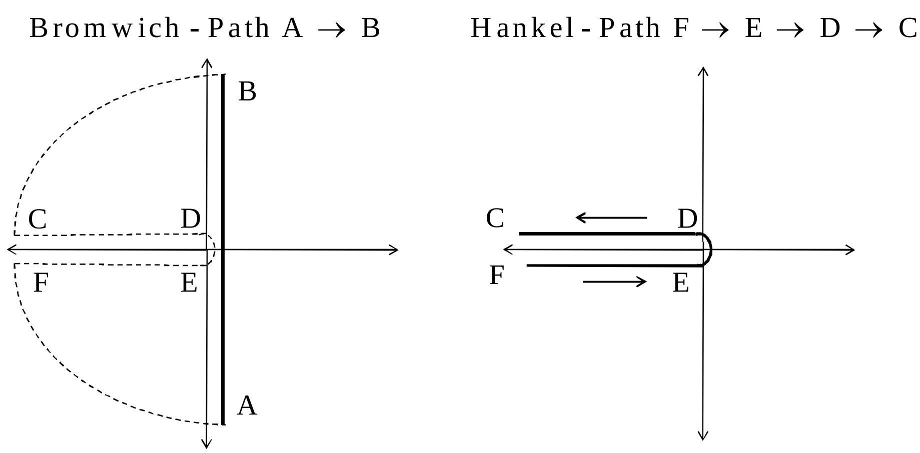
Bromwich path and Hankel Path in Complex Analysis

The inverse Laplace transform integral is

$$f(t)=\frac{1}{2\pi i}\int_{Br}\mathrm{e}^{\sigma t}\bar{F}(\sigma)\mathrm{d}\sigma \qquad f(t)=\mathcal{L}^{-1}\left\{\bar{F}(\sigma)\right\}$$

The Br denotes Bromwich part, i.e. A to B in Figure-3.

The Hankel contour that begins at $\sigma=-\infty-ia$, $a>0$, encircles the branch cut that lies along the negative real axis, and ends up at $\sigma=-\infty+ib$, $b>0$ with $a\downarrow 0$ and $b\downarrow 0$ which is equivalent to original path of Bromwich. This is shown in Figure-3.



$$\int_{\text{Bromwich}} F(\sigma)\,\mathrm{d}\sigma = \int_{\text{Hankel}} F(\sigma)\,\mathrm{d}\sigma$$

Figure-3: Showing Bromwich Path and Hankel Path

The inverse Laplace transform is carried out as integral on the Bromwich path which is A to B as indicated in Figure-3. Notionally we represent as following ($F(\sigma)$ is inclusive of exponential kernel in inverse Laplace transform formula) i.e. $f(t)=\frac{1}{2\pi i}\int_{A\to B}F(\sigma)\mathrm{d}\sigma;\quad F(\sigma)=\mathrm{e}^{\sigma t}\bar{F}(\sigma)$. We make a closed contour across the branch cut line (i.e. the negative real axis) and call the closed contour Ω as $\mathrm{A,B,C,D,E,F,A}$ in counterclockwise direction. Assuming the contour thus made does not include any poles, and then we have

$$\int_{\Omega}F(\sigma)\mathrm{d}\sigma=\int_{\mathrm{A\ B\ C\ D\ E\ F\ A}}F(\sigma)\mathrm{d}\sigma=0$$

$$\int\limits_{A\to B}F(\sigma)\mathrm{d}\sigma+\int\limits_{B\to C}F(\sigma)\mathrm{d}\sigma+\int\limits_{C\to D}F(\sigma)\mathrm{d}\sigma+\int\limits_{D\to E}F(\sigma)\mathrm{d}\sigma+\int\limits_{E\to F}F(\sigma)\mathrm{d}\sigma+\int\limits_{F\to A}F(\sigma)\mathrm{d}\sigma=0$$

As the radius of the arcs BC and FA grows to infinity, the function $F(\sigma)\downarrow 0$, (for a well behaved function to have inverse Laplace transform, this condition is satisfied), therefore the integrals on these arcs vanishes. So we are left with the following paths on which we do the integration

$$\int\limits_{A\to B}F(\sigma)\mathrm{d}\sigma+\int\limits_{C\to D}F(\sigma)\mathrm{d}\sigma+\int\limits_{D\to E}F(\sigma)\mathrm{d}\sigma+\int\limits_{E\to F}F(\sigma)\mathrm{d}\sigma=0$$

$$\int\limits_{\text{Bromwich}}F(\sigma)\mathrm{d}\sigma=\int\limits_{A\to B}F(\sigma)\mathrm{d}\sigma=-\left(\int\limits_{C\to D}F(\sigma)\mathrm{d}\sigma+\int\limits_{D\to E}F(\sigma)\mathrm{d}\sigma+\int\limits_{E\to F}F(\sigma)\mathrm{d}\sigma\right)$$

$$=\int\limits_{D\to C}F(\sigma)\mathrm{d}\sigma+\int\limits_{E\to D}F(\sigma)\mathrm{d}\sigma+\int\limits_{F\to E}F(\sigma)\mathrm{d}\sigma$$

$$=\int\limits_{F\to E}F(\sigma)\mathrm{d}\sigma+\int\limits_{E\to D}F(\sigma)\mathrm{d}\sigma+\int\limits_{D\to C}F(\sigma)\mathrm{d}\sigma=\int\limits_{\text{Hankel}}F(\sigma)\mathrm{d}\sigma$$

In addition if the contour $\mathrm{ABCDEFA}$ encloses poles we write

$$\int\limits_{\text{Bromwich}}F(\sigma)\mathrm{d}\sigma=\int\limits_{\text{Hankel}}F(\sigma)\mathrm{d}\sigma+2\pi i\sum\text{Residues of poles}$$

$$\frac{1}{2\pi i}\int\limits_{\text{Bromwich}}F(\sigma)\mathrm{d}\sigma=\frac{1}{2\pi i}\int\limits_{\text{Hankel}}F(\sigma)\mathrm{d}\sigma+\sum\text{Residues of poles}$$

$$f(t)=\mathcal{L}^{-1}\left\{\bar{F}(\sigma)\right\}$$

$$=\frac{1}{2\pi i}\int\limits_{\text{Bromwich}}\mathrm{e}^{\sigma t}\bar{F}(\sigma)\mathrm{d}\sigma=\frac{1}{2\pi i}\int\limits_{\text{Hankel}}\mathrm{e}^{\sigma t}\bar{F}(\sigma)\mathrm{d}\sigma+\sum\text{Residues of poles}$$

This is the technique to get inverse Laplace transform by contour integration and residue theorem of complex variables. The poles that we will consider, should belong to only the primary Riemann-sheet, in case we have a branch cut. The poles in other Riemann-sheet will not contribute to the response.

The M-Wright function a Generalization of Gaussian Function

Like Mittag-Leffler function is generalization of exponential function; the M-Wright function of order α is described as follows

$$M_\alpha(z) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\sigma - z\sigma^\alpha}\left(\frac{1}{\sigma^{1-\alpha}}\right)\mathrm{d}\sigma$$

for $z > 0$, and $0 < \alpha < 1$. The representation Br is Bromwich path on which Laplace inversion integration is done is a line on complex plane from $\sigma = \gamma - i\infty$ to $\sigma = \gamma + i\infty$. This above integral representation of $M_\alpha(z)$ is Bromwich representation can be analytically continued for all z in $\mathbb{C}$, adopting suitable integral and series representations valid in all of $\mathbb{C}$. For this purpose let us deform the Bromwich path Br into the Hankel path Ha. With this description of above section we can represent the formula i.e. $M_\alpha(z) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{\sigma - z\sigma^\alpha}\left(\frac{1}{\sigma^{1-\alpha}}\right)\mathrm{d}\sigma, \quad z > 0, \quad 0 < \alpha < 1$ with Hankel path integral as

$$M_\alpha(z) = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma - z\sigma^\alpha}\left(\frac{1}{\sigma^{1-\alpha}}\right)\mathrm{d}\sigma; \qquad 0 < \alpha < 1$$

Getting series representation of $M_\alpha(z)$ from its integral representation

Therefore this equivalence of Bromwich path and Hankel path redefines M-Wright function on Hankel path as $M_\alpha(z) = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma - z\sigma^\alpha}\left(\frac{1}{\sigma^{1-\alpha}}\right)\mathrm{d}\sigma$, $0 < \alpha < 1$. This representation is Integral representation of the M-Wright function. M-Wright's function, as it is very similar to Wright's function here (M-for Mainardi). We study this $M_\alpha(z)$, and get its series representation as described in the following steps

$$\begin{aligned}
M_\alpha(z) &= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma - z\sigma^\alpha}\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}} \\
&= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\left(e^{-z\sigma^\alpha}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}} \qquad \text{use} \qquad \mathrm{e}^{-z\sigma^\alpha} = \sum_{n=0}^{\infty}\frac{(-1)^n (z\sigma^\alpha)^n}{n!} \\
&= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\sigma^{\alpha n}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\sigma^{\alpha n+\alpha-1}\mathrm{d}\sigma\right] \qquad \text{use} \qquad \frac{1}{\Gamma(z)} := \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{x}x^{-z}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\,\Gamma\left(-\alpha n + (1-\alpha)\right)}
\end{aligned}$$

We have used, in above derivation the Hankel representation of reciprocal Gamma function i.e. $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{x}x^{-z}\mathrm{d}x$ .Therefore we have a series representation of this function $M_\alpha(z)$ as mentioned below

$$M_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\qquad 0<\alpha<1$$

Proof of property of i.e. $\int_0^\infty M_\alpha(z)\mathrm{dz}=1$

Here we show that we get; $\int_0^\infty M_\upsilon(z)\mathrm{dz}=1$; via following derivation.

$$\int_0^\infty M_\alpha(z)\mathrm{dz}=\int_0^\infty \frac{1}{2\pi i}\int_{Ha}\left(\mathrm{e}^{\sigma-\sigma^\alpha z}\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}}\right)\mathrm{dz}=\frac{1}{2\pi i}\int_{Ha}\mathrm{e}^{\sigma}\left(\int_0^\infty \mathrm{e}^{-\sigma^\alpha z}\mathrm{dz}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}}$$

$$=\frac{1}{2\pi i}\int_{Ha}\mathrm{e}^{\sigma}\left(\frac{\mathrm{e}^{-\sigma^\alpha z}}{-\sigma^\alpha}\right)_{z=0}^{z=\infty}\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}}$$

$$=\frac{1}{2\pi i}\int_{Ha}\frac{\mathrm{e}^{\sigma}}{\sigma}\mathrm{d}\sigma$$

Now we need evaluate integral on Hankel path for the function $F(\sigma)=\frac{\mathrm{e}^\sigma}{\sigma}$. First leg is section FE (Figure-3). Here we write $\sigma=r\mathrm{e}^{-i\pi}=-r,\quad \mathrm{d}\sigma=-\mathrm{d}r$. The r varies from ∞ to 0^+; thus the integration on FE is $\int_{\mathrm{F}\to\mathrm{E}}\mathrm{e}^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma=\int_\infty^{0^+}\mathrm{e}^{-r}\left(\frac{1}{r}\right)\mathrm{d}r$. Similarly second leg of Hankel path DC with $\sigma=r\mathrm{e}^{i\pi}=-r;\quad \mathrm{d}\sigma=-\mathrm{d}r$ and with r varying from 0^+ to ∞, yields the integration on path DC (Figure-3) as $\int_{\mathrm{D}\to\mathrm{C}}\mathrm{e}^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma=\int_{0^+}^{\infty}\mathrm{e}^{-r}\left(\frac{1}{r}\right)\mathrm{d}r$. Summing these two above, the integrations on the legs of Hankel path FE and DC gives zero. Thus we are left with small circle on the Hankel path as ED, where we write $\sigma=\in\mathrm{e}^{i\theta}$, with $\in$ a small constant such that $\in\downarrow 0$; and θ varying from $-\pi$ to $+\pi$. With this we have $\mathrm{d}\sigma=\in i\mathrm{e}^{i\theta}\mathrm{d}\theta$, and integration on this small circle is

$$\int_{\mathrm{ED}}\frac{\mathrm{e}^\sigma}{\sigma}\mathrm{d}\sigma=\lim_{\in\downarrow 0}\int_{\theta=-\pi}^{\theta=\pi}\frac{\mathrm{e}^{\in\exp(i\theta)}}{\in\mathrm{e}^{i\theta}}\in i\mathrm{e}^{i\theta}\mathrm{d}\theta=\lim_{\in\downarrow 0}\int_{-\pi}^{\pi}\mathrm{e}^{\in\exp(i\theta)}i\mathrm{d}\theta=i\int_{-\pi}^{\pi}\mathrm{d}\theta=2\pi i$$

Therefore the total integration on Hankel path gives

$$\int_0^\infty M_\alpha(z)\mathrm{dz}=\frac{1}{2\pi i}\int_{Ha}\mathrm{e}^\sigma\left(\frac{1}{\sigma}\right)\mathrm{d}\sigma=1$$

Moments of M-Wright function $M_\alpha(z)$

Very important deduction is as follows that is the absolute moments of this M-Wright's function derived for β order moment with z in $\mathbb{R}^+$, for $M_\alpha(z)$ with $\beta>-1$ and $0\le\alpha<1$.

$$\int_0^\infty z^\beta M_\alpha(z)\mathrm{dz} = \int_0^\infty z^\beta \left(\frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma - z\sigma^\alpha} \frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}} \right)\mathrm{dz}$$

$$= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\sigma}\left(\int_0^\infty \mathrm{e}^{-z\sigma^\alpha} z^\beta \mathrm{dz}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\alpha}} \qquad \text{use} \qquad \int_0^\infty \mathrm{e}^{-z\sigma^\alpha} z^\beta \mathrm{dz} = \frac{\Gamma(\beta+1)}{(\sigma^\alpha)^{\beta+1}}$$

$$= \frac{\Gamma(\beta+1)}{2\pi i}\int_{Ha} \frac{\mathrm{e}^{\sigma}}{\sigma^{\alpha\beta+1}}\mathrm{d}\sigma \qquad \text{use} \qquad \frac{1}{\Gamma(\alpha\beta+1)} = \frac{1}{2\pi i}\int_{Ha}\frac{\mathrm{e}^{\sigma}}{\sigma^{\alpha\beta+1}}\mathrm{d}\sigma$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \qquad \beta > -1; \quad 0 \le \alpha < 1$$

In the above steps we used Gamma function as $\Gamma(z) = \int_0^\infty \mathrm{e}^{-y} y^{z-1}\mathrm{d}y$, by taking $z\sigma^\alpha = y$ and $\mathrm{dz} = \mathrm{d}y/\sigma^\alpha$ we get $\int_0^\infty \mathrm{e}^{-z\sigma^\alpha} z^\beta \mathrm{dz} = \left(\Gamma(\beta+1)\right)/(\sigma^\alpha)^{\beta+1}$; that we have used. So we have generalized moment expression for auxiliary function as following

$$\int_0^\infty x^\beta M_\alpha(x)\mathrm{d}x = \frac{\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)} \qquad x \in \mathbb{R}^+; \quad \beta > -1, \quad 0 \le \alpha < 1$$

Function $M_\alpha(z)$ for α at $\frac{1}{2}$, $\frac{1}{3}$ and 0

$$M_{\frac{1}{2}}(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!\Gamma\left(\frac{1-n}{2}\right)}$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{z}{\Gamma(0)} + \frac{z^2}{2!\Gamma\left(-\frac{1}{2}\right)} - \frac{z^3}{3!\Gamma(-1)} + \frac{z^4}{4!\Gamma\left(-\frac{3}{2}\right)} + \ldots$$

$$= \frac{1}{\sqrt{\pi}} - 0 + \frac{z^2}{2!(-2)\sqrt{\pi}} - 0 + \frac{z^4}{4!\left(\frac{4}{3}\right)\sqrt{\pi}} + \ldots$$

$$= \frac{1}{\sqrt{\pi}}\left(1 + \left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)} + \left(\frac{3}{4}\right)\frac{(+1)z^4}{4!} + \ldots\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(1 + \left(\frac{1}{2}\right)\frac{(-1)^1(z)^{2\times1}}{(2\times1)!} + \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\frac{(-1)^2(z)^{2\times2}}{(2\times2)!} + \ldots\right)$$

$$= \frac{1}{\sqrt{\pi}}\sum_{n=0}^\infty (-1)^n \left(\tfrac{1}{2}\right)_n \frac{(z)^{2n}}{(2n)!}$$

We have rising truncated factorial as $\left(m\right)_n = m(m+1)(m+2)..(m+n-1)$; $\left(m\right)_0 = 1$. The notation $(m)_n$ is also called Pochhammer number which is also $(m)_n = \frac{\Gamma(m+n)}{\Gamma(m)}$.

$$
\begin{aligned}
M_{1/2}(z) &= \frac{1}{\sqrt{\pi}}\left(1+\left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)}+\left(\frac{3}{4}\right)\frac{(+1)z^4}{4!}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(1+\frac{(-z^2)}{2\times 2\times 1}+\frac{3}{4}\frac{(-z^2)^2}{(4\times 3\times 2\times 1)}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(1+\frac{\left(-\frac{z^2}{4}\right)}{1!}+\frac{\left(-\frac{z^2}{4}\right)^2}{2!}+\ldots\right)\\
&= \frac{1}{\sqrt{\pi}}\left(\sum_{n=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^n}{n!}\right)\\
&= \frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-z^2/4\right)}
\end{aligned}
$$

Therefore we write

$$
M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)_n\left(\frac{z^{2n}}{(2n)!}\right) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-z^2/4\right)}
$$

For $\alpha = \frac{1}{3}$, we have the following formula

$$
\begin{aligned}
M_{1/3}(z) &= \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma\left(\frac{2-n}{3}\right)}\\
&= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{1}{3}\right)_n\frac{z^{3n}}{(3n)!}\\
&= \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)_n\frac{z^{3n+1}}{(3n+1)!}
\end{aligned}
$$

In above derivations the formula $M_\alpha(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma(-\alpha n+(1-\alpha))}$ is used. From this series representation, we get $M_0(z) = \mathrm{e}^{-z}$ for $\alpha = 0$

$$
\begin{aligned}
M_\alpha(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\\
M_0(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\\
&= \mathrm{e}^{-z}
\end{aligned}
$$

We know that Mittag-Leffler function is (fractional) generalization of exponential function; similarly we may call this $M_\alpha(z)$ as fractional generalization of the Gaussian function.

Laplace transform of $M_\alpha(t)$ as Mittag-Leffler function, i.e. $E_\alpha(-s)$

We now write a Laplace transform pair for the M-Wright function, that is $\mathcal{L}\{M_\alpha(t)\}=E_\alpha(-s)$ for $0<\alpha<1$, which we derive in following steps where we expand in series the exponential kernel e^{-st} and use the absolute moments of the $M_\alpha(z)$ obtained earlier that is $\int_0^\infty x^\beta M_\alpha(x)\mathrm{d}x=\frac{\Gamma(\beta+1)}{\Gamma(\alpha\beta+1)}$,and then use the series definition of the one-parameter Mittag-Leffler function as follows

$$\begin{aligned}
\mathcal{L}\{M_\alpha(t)\} &= \int_0^\infty \mathrm{e}^{-st} M_\alpha(t)\mathrm{d}t \\
&= \int_0^\infty \left(\sum_{n=0}^\infty \frac{(-st)^n}{n!}\right) M_\alpha(t)\mathrm{d}t \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!}\int_0^\infty t^n M_\alpha(t)\mathrm{d}t \qquad \text{use} \qquad \int_0^\infty t^n M_\alpha(t)\mathrm{d}t=\frac{\Gamma(n+1)}{\Gamma(\alpha n+1)} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(\alpha n+1)} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{n!}\frac{n!}{\Gamma(\alpha n+1)} \qquad \text{use} \qquad \Gamma(n+1)=n! \qquad \text{for} \quad n\in\mathbb{N} \\
&= \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma(\alpha n+1)}=E_\alpha(-s) \qquad \text{by using} \qquad E_\alpha(z)=\sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n+1)}
\end{aligned}$$

Therefore we have the relation, i.e. Laplace inverse transform of Mittag-Leffler function $E_\alpha(-s)$ is M-Wright function

$$M_\alpha(t)=\mathcal{L}^{-1}\{E_\alpha(-s)\}$$

$$\alpha=0 \qquad M_0(t)=\mathrm{e}^{-t} \qquad \mathcal{L}^{-1}\{E_0(-s)\}=\mathrm{e}^{-t}$$

$$\alpha=\tfrac{1}{3} \qquad M_{1/3}(t)=\frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^\infty\left(\tfrac{2}{3}\right)_n\frac{t^{3n+1}}{(3n+1)!} \qquad \mathcal{L}^{-1}\{E_{1/3}(-s)\}=\frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^\infty\left(\tfrac{2}{3}\right)_n\frac{t^{3n+1}}{(3n+1)!}$$

$$\alpha=\tfrac{1}{2} \qquad M_{1/2}(t)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-t^2/4\right)} \qquad \mathcal{L}^{-1}\{E_{1/2}(-s)\}=\frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-t^2/4\right)}$$

$$\alpha=1 \qquad E_1(-s)=\mathrm{e}^{-s} \qquad \mathcal{L}^{-1}\{E_1(-s)\}=\delta(t-1)$$

For extreme case $\alpha=1$ we have $E_1(-s)=\mathrm{e}^{-s}$ and from standard Laplace transformation table we write $\mathcal{L}^{-1}\{\mathrm{e}^{-as}\}=\delta(t-a)$ and thus $\mathcal{L}^{-1}\{E_1(-s)\}=\delta(t-1)$. With this observation we can write a limit formula for $\lim_{\alpha\to1}M_\alpha(z)$ as

$$\lim_{\alpha\to 1}\left(M_\alpha(z)\right)=\lim_{\alpha\to 1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\right)=\delta(z-1)$$

$$\lim_{\alpha\to 1}\left(M_\alpha(x+1)\right)=\lim_{\alpha\to 1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n (x+1)^n}{n!\Gamma\left(-\alpha n+(1-\alpha)\right)}\right)=\delta(x)$$

is another expression for delta function, as summation of series as expressed above. However we cannot put $\alpha=1$ since $\Gamma(-n)$ blows up at $n=0,1,2,3...$; therefore above are defined in limit.

Though for $\alpha=0$ we are getting the value from $M_0(t)=\mathrm{e}^{-t}$, we can see that via Mittag-Leffler function at $\alpha=0$ which is $E_0(-s)=\frac{1}{1+s}$. From standard Laplace transform tables we have $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=\mathrm{e}^{at}$, thus we have $\mathcal{L}^{-1}\left\{E_0(-s)\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}=\mathrm{e}^{-t}$.

Graphical representation of M-Wright function

We describe the graphs $M_\alpha(x)$ by schematic diagrams. First we take the case for $0<\alpha<\frac{1}{2}$, depicted schematically in Figure-4. For $\alpha=\frac{1}{2}$ we have seen $M_{1/2}(x)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}$. This is standard Gaussian bell shaped curve centered at $x=0$ (when extended to negative axis by reflecting about $x=0$ line, shown by dotted extension in Figure-4). The other extreme is at $\alpha=0$ where $M_0(x)=\mathrm{e}^{-x}$. Shown in the Figure-4 is curve for $\alpha=\frac{1}{4}$. The observation points as, while α is much less than $\frac{1}{2}$ say near zero, the $M_\alpha(x)$ starts from $x=0$ and has a steep fall and as x increases, the curve goes to zero. While α is nearing $\frac{1}{2}$, the curve becomes Gaussian in nature; with slow initial fall followed by sudden fall and eventually goes to zero. For the case α varying from 0 to $\frac{1}{2}$ we observe the maximum value of $M_\alpha(x)$ is at $x=0$ and accordingly to value of α the curve of $M_\alpha(x)$ decays and goes to zero at say at $x=5$ i.e. for large x.

Now we take the case of $M_\alpha(x)$ for α varying from $\frac{1}{2}$ to 1, in which is depicted in schematic of Figure-5. At the value of $\alpha=\frac{1}{2}$, the curve $M_{1/2}(x)$ is Gaussian curve. As α increases from value $\frac{1}{2}$ the peak value of the curve shifts from $x=0$ to $x>0$, and after attaining the peak value the curve falls towards zero. This is as shown for the case $\alpha=\frac{5}{8}$, in Figure-8.3. Still increasing the value of α towards 1, makes the peak of the curve shift towards $x=1$, making it steeper and sharper. At the value $\alpha=1$, it is expected to take the form of delta function $M_1(x)=\delta(x-1)$. For the negative x the function $M_\alpha(x)$ gets plotted by reflection of theses curves about the line $x=0$ as $M_\alpha(x)$ is symmetric function.

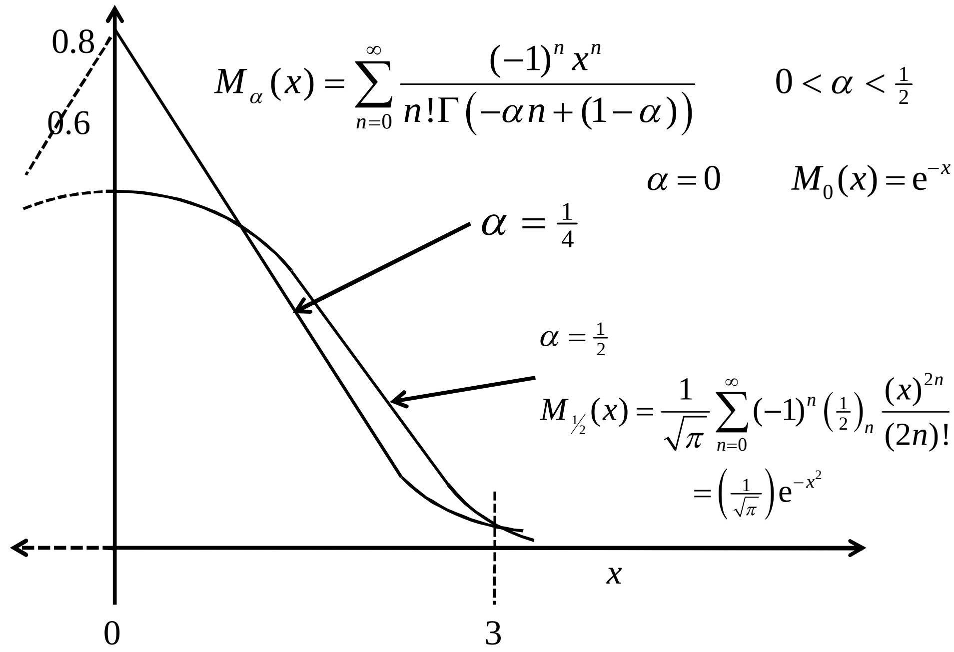


Figure-4: Plot of symmetric M-Wright function for α = zero to 0.5

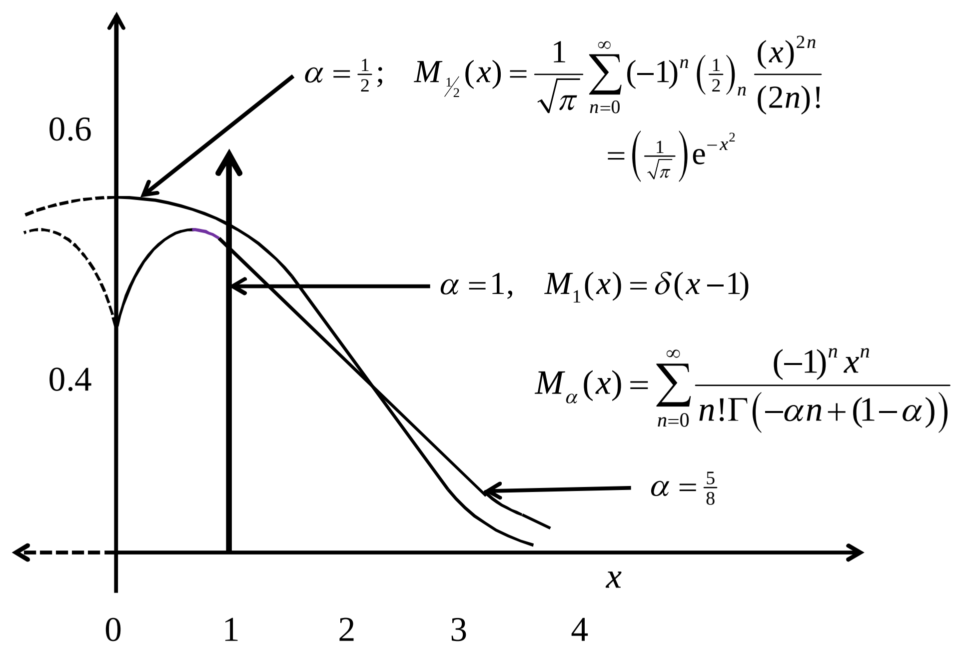


Figure-5: Plot of symmetric M-Wright function for α = 0.5 to 1.0

Obtaining relaxation rate distribution histogram for Mittag-Leffler decay in non oscillatory non-Debye system

For a decay of $f(t) = \mathrm{e}^{-kt}$, where k is a constant and $k > 0$ we have shown the histogram of the decay rates is concentrated as Delta distribution function i.e. $H(\lambda) = \delta(\lambda - k)$. This is a pure Debye relaxation system. Now we analyze $f(t) = E_\alpha(-kt^\alpha)$ and get $H_\alpha(\lambda)$ for different α , with $0 < \alpha < 1$. As per our discussion the histogram of rate relaxation is

$$H_\alpha(\lambda) = \mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\}$$

With change of variables say $kt^\alpha = \overline{t}$ we can write

$$H_\alpha(\lambda) = \mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\}$$

With the obtained inverse Laplace transform i.e. $M_\alpha(t) = \mathcal{L}^{-1}\left\{E_\alpha(-s)\right\}$ we write variables re-named as $s \equiv \overline{t}$ and $t \equiv \lambda$, to get rate relaxation histogram function as

$$\begin{aligned} &f(t) = E_\alpha(-kt^\alpha); \quad kt^\alpha = \overline{t} \\ &H_\alpha(\lambda) = \mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\} = M_\alpha(\lambda); \quad 0 < \alpha < 1 \end{aligned}$$

Following are some interesting observations

$$\begin{aligned} &H_\alpha(\lambda) = \mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\} \\ &\alpha = 0 \qquad H_0(\lambda) = \mathrm{e}^{-\lambda} \qquad \mathcal{L}^{-1}\left\{E_0(-\overline{t})\right\} = \mathrm{e}^{-\lambda} \\ &\alpha = \tfrac{1}{3} \qquad H_{1/3}(\lambda) = \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)_n \frac{\lambda^{3n+1}}{(3n+1)!} \qquad \mathcal{L}^{-1}\left\{E_{1/3}(-\overline{t})\right\} = \frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\left(\tfrac{2}{3}\right)_n \frac{\lambda^{3n+1}}{(3n+1)!} \\ &\alpha = \tfrac{1}{2} \qquad H_{1/2}(\lambda) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-\lambda^2/4\right)} \qquad \mathcal{L}^{-1}\left\{E_{1/2}(-\overline{t})\right\} = \frac{1}{\sqrt{\pi}}\mathrm{e}^{\left(-\lambda^2/4\right)} \\ &\alpha = 1 \qquad E_1(-\overline{t}) = \mathrm{e}^{-\overline{t}} \qquad H_1(\lambda) = \mathcal{L}^{-1}\left\{E_1(-\overline{t})\right\} = \delta(\lambda - 1) \end{aligned}$$

With the discussions about the plot of M-Wright function we draw the histogram function $H_\alpha(\lambda)$ for non-Debye relaxation with Mittag-Leffler function $f(t) = E_\alpha(-kt^\alpha)$ in the Figure-6.

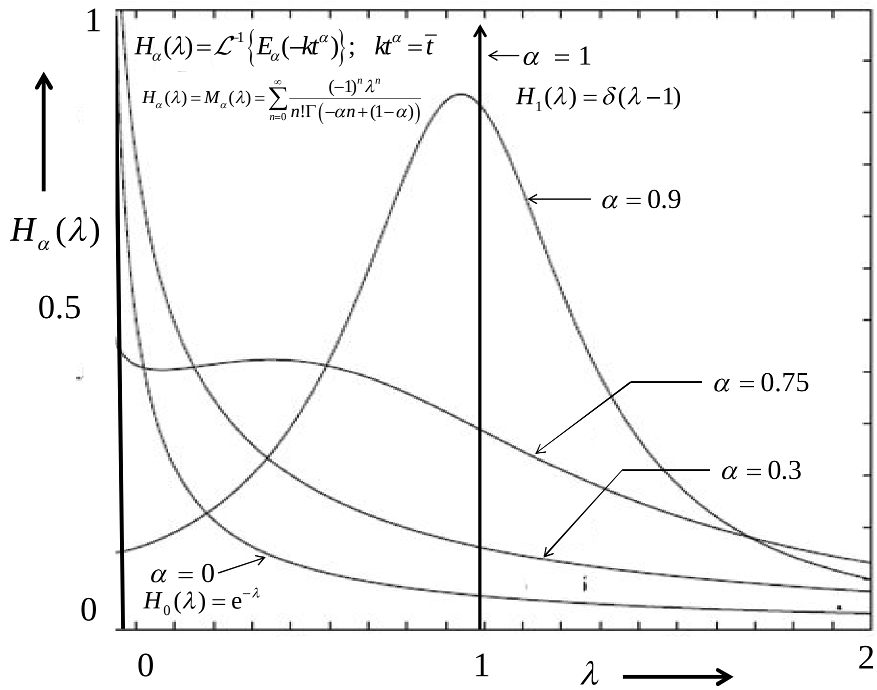


Figure-6: Relaxation rate distribution histogram plots for non-Debye relaxation given by Mittag-Leffler function for order $0<\alpha<1$

Observed disordering of multi-body relaxation for non-oscillatory non-Debye decay

Figure-6 gives interesting interpretation about non-Debye relaxation function. The Mittag-Leffler function maps from pure exponential decay for $\alpha=1$, to pure hyperbolic decay for $\alpha=0$. The rate distribution is delta distribution function for Debye relaxation $\alpha=1$, and a pure exponential distribution for pure hyperbolic decay for $\alpha=0$ and in between $0<\alpha<1$, we get $H_\alpha(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!\Gamma(-\alpha n+(1-\alpha))}$

We observe the concentrated delta distribution peaked at $\lambda=1$ in Figure-6 is actually at the rate $\lambda=k$ pertaining to exponential relaxation e^{-kt} with single rate k -gets spread towards zero rate and towards infinite rate. That is a multi-body system in a Debye relaxation with ordered rate of unique rates for all the bodies gets disordered with the defined histogram function, where the number of multi-body relaxation rates are spread according to the rate from zero to infinity. So single rate of relaxation; k i.e. for Debye process gets spread from $\lambda=0$ to $\lambda=\infty$. So a non-Debye relaxation is a multi-body discharge with several relaxation rates distributed as per α. As the α goes lower towards zero we get more number of bodies in a multi body system discharging at lower rate of relaxation λ and less number of bodies relaxing at high and very high rate.

We observe that peak of distribution function $H_\alpha(\lambda)$ moves away from the point $\lambda=1$ (or we call this point as k) towards zero and the width gets wider, as α becomes lower than one, attains a Guassian distribution (Figure 4 and 5) with peak at $\lambda = 0$, at $\alpha = \frac{1}{2}$-and then tends towards exponential distribution as α approaches zero.

The other ways to describe inverse Laplace transform of Mittag-Leffler functions-for obtaining rate distribution histogram

We have following Laplace identity for one parameter Mittag-Leffler function

$$\mathcal{L}\left\{E_{\alpha}(-kt^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}+k}$$

This is verified as follows by series definition of Mittag-Leffler function

$$E_{\alpha}(-kt^{\alpha}) = 1 - \frac{kt^{\alpha}}{\Gamma(1+\alpha)} + \frac{k^2t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{k^3t^{3\alpha}}{\Gamma(1+3\alpha)} + \ldots$$

and then taking term by term Laplace transform, with known Laplace identity $\mathcal{L}\left\{t^n\right\} = \frac{\Gamma(1+n)}{s^{1+n}}$, gives following

$$\mathcal{L}\left\{E_{\alpha}(-kt^{\alpha})\right\} = \mathcal{L}\left\{1\right\} - \mathcal{L}\left\{\frac{kt^{\alpha}}{\Gamma(1+\alpha)}\right\} + \mathcal{L}\left\{\frac{k^2t^{2\alpha}}{\Gamma(1+2\alpha)}\right\} - \mathcal{L}\left\{\frac{k^3t^{3\alpha}}{\Gamma(1+3\alpha)}\right\} + \ldots$$

$$\mathcal{L}\left\{E_{\alpha}(-kt^{\alpha})\right\} = \frac{1}{s} - k\frac{1}{s^{1+\alpha}} + k^2\frac{1}{s^{1+2\alpha}} - k^3\frac{1}{s^{1+3\alpha}} + \ldots$$

$$= \frac{1}{s}\left(1 - \frac{k}{s^{\alpha}} + \left(\frac{k}{s^{\alpha}}\right)^2 - \left(\frac{k}{s^{\alpha}}\right)^3 + \ldots\right)$$

$$= \frac{1}{s}\left(\frac{1}{1+\frac{k}{s^{\alpha}}}\right) \quad \text{using} \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 .. \quad |x| < 1$$

$$= \frac{s^{\alpha-1}}{s^{\alpha}+k}; \quad \left|\frac{k}{s^{\alpha}}\right| < 1$$

Thus we get $\mathcal{L}\left\{E_{\alpha}(-kt^{\alpha})\right\} = \frac{s^{\alpha-1}}{s^{\alpha}+k}$ with $\mathrm{Re}[s] > k^{1/\alpha}$. In terms of inverse Laplace integral, we write $E_{\alpha}(-kt^{\alpha})$ as

$$E_{\alpha}(-kt^{\alpha}) = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s + \sum \mathrm{Residues}\left[\mathrm{e}^{st}\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]$$

Where Br denotes a Bromwich path i.e. a line with $\mathrm{Re}[s] = \sigma > k^{1/\alpha}$ with $\mathrm{Im}[s]$ running from $-\infty$ to $+\infty$. We bend the Bromwich path of integration Br into Hankel path Ha , a loop which starts from $-\infty$

along the lower side of negative real axis encircles the circular disc $|s| = \in$ in positive (anticlockwise sense and ends at $-\infty$ along the upper side of negative real axis. So we write

$$\begin{aligned} E_\alpha(-kt^\alpha) &= \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\mathrm{d}s + \sum \text{Residues}\left[\mathrm{e}^{st}\,\frac{s^{\alpha-1}}{s^\alpha + k}\right] \\ &= f_1(t) + f_2(t) \end{aligned}$$

For the function $\frac{s^{\alpha-1}}{s^\alpha + k}$ the poles are at $s = |k|^{1/\alpha}\exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right), m = 0, \pm 1, \pm 2, \ldots$.. For $0 < \alpha < 1$ we have $\left|(2m+1)\frac{\pi}{\alpha}\right| > \pi$, meaning that no poles are in $-\pi < \arg[s] < \pi$. Therefore there are no poles in the 'primary Riemann-sheet'. Thus for $0 < \alpha < 1$ case $f_2(t) = \sum \text{Residues}\left[\mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\right] = 0$ is zero, and we thus have following

$$\begin{aligned} E_\alpha(-kt^\alpha) &= \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\mathrm{d}s \\ &= \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\mathrm{d}s = f_1(t); \quad 0 < \alpha < 1 \end{aligned}$$

The case $0 < \alpha < 1$ we have $E_\alpha(-kt^\alpha) = f_1(t) = \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + 1}\right\}$ .We will analyze subsequently for $1 < \alpha < 2$ that $E_\alpha(-kt^\alpha) = f_1(t) + f_2(t)$, with $f_2(t) \neq 0$; for oscillatory decay function.

There are three contributions on the Hankel path. The one is on the circle $s = \in \mathrm{e}^{i\theta}$ as $\in \downarrow 0$, that is

$$\begin{aligned} \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^\alpha + k}\right)\mathrm{d}s &= \frac{1}{2\pi i}\lim_{\in\downarrow 0}\int_{s=\in\mathrm{e}^{i\theta}} \mathrm{e}^{\in t\exp(i\theta)}\left(\frac{\in^{\alpha-1}\,\mathrm{e}^{i(\alpha-1)\theta}}{\in^\alpha\,\mathrm{e}^{i\alpha\theta} + k}\right)\left(\in i\mathrm{e}^{i\theta}\right)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in\mathrm{e}^{i\theta}} (1)\left(\frac{\in^\alpha\,\mathrm{e}^{i(\alpha-1)\theta}}{\in^\alpha\,\mathrm{e}^{i\alpha\theta} + k}\right)\left(\mathrm{e}^{i\theta}\right)\mathrm{d}\theta; \quad \alpha > 0 \\ &= \frac{1}{2\pi}\lim_{\in\downarrow 0}\int_{s=\in\mathrm{e}^{i\theta}} \frac{\in^\alpha\,\mathrm{e}^{i\alpha\theta}}{k}\mathrm{d}\theta = 0 \end{aligned}$$

The second contribution is from line below negative real axis we call $s = r\mathrm{e}^{-i\pi}$ i.e. $s = -r$, and r varying from ∞ to $\in\downarrow 0$ that gives

$$\frac{1}{2\pi i}\int_{s=re^{-i\pi}} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{\infty}^{0} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}} (-\mathrm{d}r)$$
$$= \frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}} (\mathrm{d}r)$$

The third contribution is from line above negative real axis we call $s = r\mathrm{e}^{i\pi}$ i.e. $s = -r$, and r varying from $\in \downarrow 0$ to ∞ that gives

$$\frac{1}{2\pi i}\int_{s=re^{i\pi}} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s = \frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{i\pi}} (-\mathrm{d}r)$$

Thus total contributions from Hankel path is sum of the three components

$$E_{\alpha}(-kt^{\alpha}) = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{st}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)\mathrm{d}s$$
$$= -\frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{i\pi}} \mathrm{d}r + \frac{1}{2\pi i}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(\frac{s^{\alpha-1}}{s^{\alpha}+k}\right)_{s=re^{-i\pi}} \mathrm{d}r$$

Since $E_{\alpha}(-kt^{\alpha})$ is a real function, we can write the above as

$$E_{\alpha}(-kt^{\alpha}) = -\frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-rt}\,\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}} \mathrm{d}r + \frac{1}{2\pi}\int_{0}^{\infty} \mathrm{e}^{-rt}\,\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{-i\pi}} \mathrm{d}r$$
$$= \frac{1}{\pi}\int_{0}^{\infty} \mathrm{e}^{-rt}\left(-\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=re^{i\pi}}\right)\mathrm{d}r = f_1(t); \quad 0<\alpha<1$$

Since for $0<\alpha<1$, $f_2(t)=0$ we compare with the formula derived i.e. $f(t) = \int_0^{\infty}\left(H(\lambda)\right)\mathrm{e}^{-t\lambda}\mathrm{d}\lambda$ we may write the above derived expression by changing variable r to λ as

$$E_{\alpha}(-kt^{\alpha}) = \int_{0}^{\infty} \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda = \int_{0}^{\infty} \mathrm{e}^{-t\lambda}\left(H_{\alpha}(\lambda)\right)\mathrm{d}\lambda; \quad 0<\alpha<1$$

That gives the histogram of relaxation rates $H_{\alpha}(\lambda)$ as follows

$$H_{\alpha}(\lambda) = -\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^{\alpha}+k}\right]_{s=\lambda e^{i\pi}} = \frac{1}{\pi}\left(\frac{k\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2k\lambda^{\alpha}\cos\alpha\pi+\lambda^{2}}\right) = \mathcal{L}^{-1}\left\{f_1(t)\right\}; \quad 0<\alpha<1$$

Thus we write inverse Laplace transform of Mittag-Leffler function of one-parameter as following (i) from t domain to λ domain that is our rate distribution histogram function and (ii) from s domain to t domain.

$$\mathcal{L}^{-1}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{1}{\pi}\left(\frac{k\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2k\lambda^\alpha\cos\alpha\pi+\lambda^2}\right);\quad 0<\alpha<1$$

$$\mathcal{L}^{-1}\left\{E_\alpha(-ks^\alpha)\right\}=\frac{1}{\pi}\left(\frac{kt^{\alpha-1}\sin\alpha\pi}{t^{2\alpha}+2kt^\alpha\cos\alpha\pi+t^2}\right);\quad 0<\alpha<1$$

In the case of one-parameter Mittag-Leffler function, one gets a compact representation as above due to the fact that we have Laplace pair i.e. $\mathcal{L}\left\{E_\alpha(-kt^\alpha)\right\}=\frac{s^{\alpha-1}}{s^\alpha+k}$. If we do not have this type of compact representation, then this described method is difficult to apply.

Looking at the obtained expression, taking $k=1$

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\left\{E_\alpha(-t^\alpha)\right\}=\frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right)=\mathcal{L}^{-1}\left\{f_1(t)\right\};\quad 0<\alpha<1$$

For $0<\alpha<1$ we have $\mathcal{L}^{-1}\left\{f_1(\lambda)\right\}=H_\alpha(\lambda)>0$ i.e. as positive and for $1<\alpha<2$ we get $\mathcal{L}^{-1}\left\{f_1(t)\right\}<0$ i.e. negative.

We have a pair i.e. $\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\}=\frac{s^{\alpha-\beta}}{s^\alpha+k}$. For this type of relaxation, we can use the above described technique of integration on Hankel path and get the following relaxation rate distribution histogram

$$H_{\alpha,\beta}(\lambda)=\mathcal{L}^{-1}\left\{t^{\beta-1}E_{\alpha,\beta}(-kt^\alpha)\right\}=-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-\beta}}{s^\alpha+k}\right]_{s=\lambda e^{i\pi}},\quad \alpha,\beta>0;\quad 0<\alpha<1$$

$$=\frac{1}{\pi}\left(\lambda^{\alpha-\beta}\frac{k\sin(\beta-\alpha)\pi+\lambda^\alpha\sin\beta\pi}{\lambda^{2\alpha}+2k\lambda^\alpha\cos\alpha\pi+\lambda^2}\right)$$

The oscillatory decay of Mittag-Leffler Function and its analysis

The $E_\alpha(-t^\alpha)$ for $0<\alpha<1$, relaxation function is shown in Figure-2, that is a non-Debye (non-oscillatory nature of relaxation. For $1<\alpha<2$. Figure-7 gives oscillatory relaxation curves, for $E_\alpha(-t^\alpha)$.

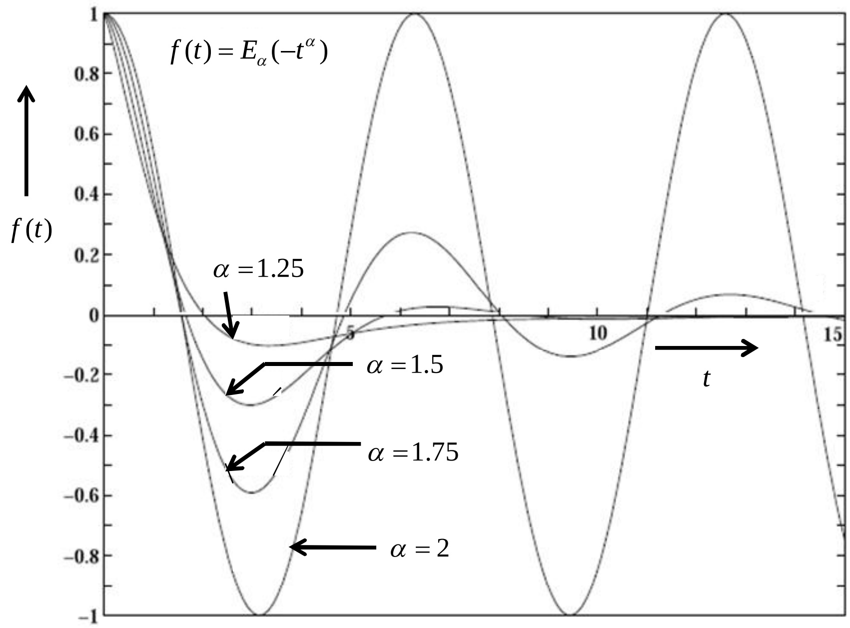


Figure-7: Relaxation decay with oscillation with Mittag-Leffler function $f(t)=E_\alpha(-t^\alpha)$ for $1<\alpha<2$

For, the case $1<\alpha<2$ the function $\frac{s^{\alpha-1}}{s^\alpha+k}$ the poles are at $s=|k|^{1/\alpha}\exp\left(i\left(\frac{(2m+1)}{\alpha}\right)\pi\right)$. For the case $1<\alpha<2$, at $m=0$ we have pole at $s_1=|k|^{1/\alpha}\exp\left(i\frac{\pi}{\alpha}\right)$ and for $m=-1$, the pole is at $s_2=|k|^{1/\alpha}\exp\left(-i\frac{\pi}{\alpha}\right)$; these two are complex conjugate and remain in the primary Riemann sheet-responsible for response. We represent them as $s_{1,2}=\sigma\pm i\omega$, where $\sigma=|k|^{1/\alpha}\cos\left(\frac{\pi}{\alpha}\right)$ and $\omega=|k|^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right)$. Thus for $1<\alpha<2$ case $f_2(t)=\sum\text{Residues}\left[e^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)\right]\neq 0$ is zero, and we thus have following

$$\begin{aligned}
E_\alpha(-kt^\alpha) &= f_1(t)+f_2(t) \qquad\qquad 1<\alpha<2\\
&=\frac{1}{2\pi i}\int_{Ha} e^{st}\left(\frac{s^{\alpha-1}}{s^\alpha+k}\right)ds+\sum\text{Residues}\left[e^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s_1,s_2}\\
&=\int_0^\infty e^{-\lambda t}\left(-\frac{1}{\pi}\text{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s=\lambda e^{i\pi}}\right)d\lambda+\sum\text{Residues}\left[e^{st}\,\frac{s^{\alpha-1}}{s^\alpha+k}\right]_{s_1,s_2}
\end{aligned}$$

$$s_1=\sigma+i\omega=|k|^{1/\alpha}e^{i\pi/\alpha},\quad s_2=\sigma-i\omega=|k|^{1/\alpha}e^{-i\pi/\alpha}\qquad \sigma=|k|^{1/\alpha}\cos\tfrac{\pi}{\alpha},\quad \omega=|k|^{1/\alpha}\sin\tfrac{\pi}{\alpha}$$

The residue calculation is following for $1<\alpha<2$

$$\sum \text{Residues}\left[e^{st} \frac{s^{\alpha-1}}{s^\alpha + k} \right]_{s_1,s_2} = \sum \text{Residues}\left[e^{st} \frac{s^{\alpha-1}}{(s-s_1)(s-s_2)} \right]_{s_1,s_2}$$

$$= \lim_{s\to s_1} (s-s_1) e^{st} \frac{s^{\alpha-1}}{(s-s_1)(s-s_2)} + \lim_{s\to s_2} (s-s_2) e^{st} \frac{s^{\alpha-1}}{(s-s_1)(s-s_2)}$$

$$= e^{s_1 t} \frac{s_1^{\alpha-1}}{s_1 - s_2} + e^{s_2 t} \frac{s_2^{\alpha-1}}{s_2 - s_1}$$

$$= \frac{e^{\sigma t} e^{i\omega t} \left(|k|^{1/\alpha} e^{i\pi/\alpha} \right)^{\alpha-1}}{2i\omega} - \frac{e^{\sigma t} e^{-i\omega t} \left(|k|^{1/\alpha} e^{-i\pi/\alpha} \right)^{\alpha-1}}{2i\omega}$$

$$= |k|^{\frac{\alpha-1}{\alpha}} \frac{e^{\sigma t}}{2i|k|^{1/\alpha} \sin\frac{\pi}{\alpha}} \left(e^{i\omega t} e^{i\pi\left(\frac{\alpha-1}{\alpha}\right)} - e^{-i\omega t} e^{-i\pi\left(\frac{\alpha-1}{\alpha}\right)} \right)$$

$$= |k|^{\left(1-\frac{2}{\alpha}\right)} \frac{e^{\sigma t}}{\sin\frac{\pi}{\alpha}} \left(\frac{-e^{i\left(\omega t - \frac{\pi}{\alpha}\right)} + e^{-i\left(\omega t - \frac{\pi}{\alpha}\right)}}{2i} \right)$$

$$= \frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}} e^{\sigma t} \sin\left(\frac{\pi}{\alpha} - \omega t \right)$$

$$= \left(\frac{|k|^{\left(1-\frac{2}{\alpha}\right)}}{\sin\frac{\pi}{\alpha}} \right) e^{\left(|k|^{1/\alpha} \cos\frac{\pi}{\alpha} \right) t} \sin\left(\frac{\pi}{\alpha} - \left(|k|^{1/\alpha} \sin\frac{\pi}{\alpha} \right) t \right)$$

For $k = 1$ we have for $1 < \alpha < 2$; $\sum \text{Residues}\left[e^{st} \left(\frac{s^{\alpha-1}}{s^\alpha + 1} \right) \right]_{s_1,s_2} = \left(\frac{1}{\sin\frac{\pi}{\alpha}} \right) e^{\left(\cos\frac{\pi}{\alpha}\right)t} \sin\left(\frac{\pi}{\alpha} - \left(\sin\frac{\pi}{\alpha} \right) t \right)$. From the above we write rate relaxation histogram for $E_\alpha(-t^\alpha) = f_1(t) + f_2(t)$ as $H_\alpha(\lambda) = \mathcal{L}^{-1}\left\{ E_\alpha(-t^\alpha) \right\} = \mathcal{L}^{-1}\left\{ f_1(t) \right\} + \mathcal{L}^{-1}\left\{ f_2(t) \right\}$ for oscillatory decay function where $f(t) = E_\alpha(-kt^\alpha);\quad 1 < \alpha < 2$.

Rate relaxation histogram function for oscillatory decay function

The function $f(t) = E_\alpha(-t^\alpha);\quad 1 < \alpha < 2$ has two parts $f_1(t)$ and $f_2(t)$. The part i.e. $f_2(t)$ is oscillatory decaying part, given as $f_2(t) = \left(\frac{1}{\sin\frac{\pi}{\alpha}} \right) e^{\left(\cos\frac{\pi}{\alpha}\right)t} \sin\left(\frac{\pi}{\alpha} - \left(\sin\frac{\pi}{\alpha} \right) t \right) = -\text{A} e^{\sigma_0 t} \sin\left(\omega_0 t - \frac{\pi}{\alpha} \right)$. Where $\sigma_0 = \cos\frac{\pi}{\alpha} < 0$ for $1 < \alpha < 2$. This factor gives exponential decay of amplitude, $\text{A} = \frac{1}{\sin\frac{\pi}{\alpha}}$ at $t = 0$. The oscillatory part is $\sin\left(\omega_0 t - \frac{\pi}{\alpha} \right)$ with $\omega_0 = \sin\frac{\pi}{\alpha}$. We observe that for $\alpha = 2$ we have $E_2(-t^2) = \cos t$ as $\sin\left(\frac{\pi}{2} - \sin\frac{\pi}{2} t \right) = \cos t$, $\text{A} = 1$ and $\sigma_0 = 0$. Also we note that for $\alpha = 2$, the term $\mathcal{L}\left\{ f_1(t) \right\} = \frac{1}{\pi} \left(\frac{r^{\alpha-1} \sin\alpha\pi}{r^{2\alpha} + 2r^\alpha \cos\alpha\pi + 1} \right)$ i.e. contribution from Hankel path is zero. We note that $\alpha = 2$ gives $\frac{s}{s^2+1}$ with poles at $s_{1,2} = \pm i$, that gives inverse Laplace transform as $\cos t$. i.e. $\mathcal{L}^{-1}\left\{ \frac{s}{s^2+1} \right\} = \cos t$.

The histogram function for $f(t)=E_\alpha(-t^\alpha)=f_1(t)+f_2(t)$ is following where $\omega_0=\sin\frac{\pi}{\alpha}$, $\sigma_0=\cos\frac{\pi}{\alpha}$, $\sigma_0<0$ and $\mathrm{A}=\frac{1}{\sin\frac{\pi}{\alpha}}$ for $1<\alpha<2$.

$$
\begin{aligned}
&f(t)=E_\alpha(-t^\alpha);\quad 1<\alpha<2\\
&E_\alpha(-t^\alpha)=f_1(t)+f_2(t)\\
&\qquad\qquad=\int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda+\left(\tfrac{1}{\sin\frac{\pi}{\alpha}}\right)\mathrm{e}^{\left(\cos\frac{\pi}{\alpha}\right)t}\sin\left(\tfrac{\pi}{\alpha}-\left(\sin\tfrac{\pi}{\alpha}\right)t\right)\\
&\qquad\qquad=\int_0^\infty \mathrm{e}^{-\lambda t}\left(-\frac{1}{\pi}\mathrm{Im}\left[\frac{s^{\alpha-1}}{s^\alpha+1}\right]_{s=\lambda e^{i\pi}}\right)\mathrm{d}\lambda-\mathrm{A}\mathrm{e}^{\sigma_0 t}\sin\left(\sin\tfrac{\pi}{\alpha}t-\tfrac{\pi}{\alpha}\right)\\
&H_\alpha(\lambda)=\mathcal{L}^{-1}\{f_1(t)\}+\mathcal{L}^{-1}\{f_2(t)\}\\
&\qquad\qquad=\frac{1}{\pi}\left(\frac{\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha}+2\lambda^\alpha\cos\alpha\pi+1}\right)+(-\mathrm{A})\sin\left(\omega_0 t-\tfrac{\pi}{\alpha}\right)\delta(\lambda+\sigma_0)
\end{aligned}
$$

The first term of $H_\alpha(\lambda)$ for $0<\alpha<1$ is negative for all $0<\lambda<\infty$, second term is an oscillating delta function of height $-\mathrm{A}$ located at $\lambda_0=\cos\frac{\pi}{\alpha}$ ringing with time with frequency $\omega_0=\sin\frac{\pi}{\alpha}$ phase shifted by $-\frac{\pi}{\alpha}$. The reasons of this we have discussed in previous section.

Berberan-Santo method to get inverse Laplace transform of Mittag-Leffler function

We demonstrate a simple method of obtaining inverse Laplace transform of $E_\alpha(-z)$ can be obtained via Berberan-Santo method for getting inverse Laplace transform of $f(t)$, with following formula

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\{f(t)\}=\frac{\mathrm{e}^{c_0\lambda}}{\pi}\int_0^\infty\left(\mathrm{Re}\left[f(c_0+iy)\right]\cos\lambda y-\mathrm{Im}\left[f(c_0+iy)\right]\sin\lambda y\right)\mathrm{d}y$$

Where in the function $f(t)$ we write $t=x+iy$ with $x=c_0$, to right of which $f(t)$ does not have any singularity. In case of $f(t)=E_\alpha(-kt^\alpha)$, take $kt^\alpha\equiv\overline{t}=x+iy$ we choose $x=c_0=0$, and write the following

$$H_\alpha(\lambda)=\mathcal{L}^{-1}\{E_\alpha(-\overline{t})\}=\frac{1}{\pi}\int_0^\infty\left(\mathrm{Re}\left[E_\alpha(iy)\right]\cos\lambda y-\mathrm{Im}\left[E_\alpha(iy)\right]\sin\lambda y\right)\mathrm{d}y$$

With following series definition of $E_\alpha(-\overline{t})$ we write

$$
\begin{aligned}
E_\alpha(-\overline{t})&=1-\frac{\overline{t}}{\Gamma(1+\alpha)}+\frac{\overline{t}^2}{\Gamma(1+2\alpha)}-\frac{\overline{t}^3}{\Gamma(1+3\alpha)}+\ldots\\
E_\alpha(iy)&=1-\frac{iy}{\Gamma(1+\alpha)}+\frac{i^2y^2}{\Gamma(1+2\alpha)}-\frac{i^3y^3}{\Gamma(1+3\alpha)}+\ldots
\end{aligned}
$$

We get the following

$$\mathrm{Re}\left[E_\alpha(-\overline{t})\right]=1-\frac{y^2}{\Gamma(1+2\alpha)}+\frac{y^4}{\Gamma(1+4\alpha)}-\frac{y^6}{\Gamma(1+6\alpha)}+\dots$$

$$\mathrm{Im}\left[E_\alpha(-\overline{t})\right]=-\left(\frac{y}{\Gamma(1+\alpha)}-\frac{y^3}{\Gamma(1+3\alpha)}+\frac{y^5}{\Gamma(1+5\alpha)}-\dots\right)$$

Thus we can calculate the following

$$\begin{aligned}H_\alpha(\lambda)=\mathcal{L}^{-1}\left\{E_\alpha(-\overline{t})\right\}&=\frac{1}{\pi}\int_0^\infty\left(\mathrm{Re}\left[E_\alpha(iy)\right]\cos\lambda y-\mathrm{Im}\left[E_\alpha(iy)\right]\sin\lambda y\right)\mathrm{d}y\\&=\frac{1}{\pi}\int_0^\infty\left(1-\frac{y^2}{\Gamma(1+2\alpha)}+\frac{y^4}{\Gamma(1+4\alpha)}-\frac{y^6}{\Gamma(1+6\alpha)}+\dots\right)\cos\lambda y\mathrm{d}y\\&+\frac{1}{\pi}\int_0^\infty\left(\frac{y}{\Gamma(1+\alpha)}-\frac{y^3}{\Gamma(1+3\alpha)}+\frac{y^5}{\Gamma(1+5\alpha)}-\dots\right)\sin\lambda y\mathrm{d}y\end{aligned}$$

The two-parameter Mittag-Leffler function is defined with $\alpha,\beta>1$ as

$$E_{\alpha,\beta}(-\overline{t})=\frac{1}{\Gamma(\beta)}-\frac{\overline{t}}{\Gamma(\alpha+\beta)}+\frac{\overline{t}^2}{\Gamma(2\alpha+\beta)}-\frac{\overline{t}^3}{\Gamma(3\alpha+\beta)}+\dots$$

With the application of Berberan-Santo method, we write

$$\begin{aligned}H_{\alpha,\beta}(\lambda)=\mathcal{L}^{-1}\left\{E_{\alpha,\beta}(-\overline{t})\right\}&=\frac{1}{\pi}\int_0^\infty\left(\mathrm{Re}\left[E_{\alpha,\beta}(iy)\right]\cos\lambda y-\mathrm{Im}\left[E_{\alpha,\beta}(iy)\right]\sin\lambda y\right)\mathrm{d}y\\&=\frac{1}{\pi}\int_0^\infty\left(\frac{1}{\Gamma(\beta)}-\frac{y^2}{\Gamma(\beta+2\alpha)}+\frac{y^4}{\Gamma(\beta+4\alpha)}-\frac{y^6}{\Gamma(\beta+6\alpha)}+\dots\right)\cos\lambda y\mathrm{d}y\\&+\frac{1}{\pi}\int_0^\infty\left(\frac{y}{\Gamma(\beta+\alpha)}-\frac{y^3}{\Gamma(\beta+3\alpha)}+\frac{y^5}{\Gamma(\beta+5\alpha)}-\dots\right)\sin\lambda y\mathrm{d}y\end{aligned}$$

Since we do not get any compact form Laplace transform pair for $E_{\alpha,\beta}(-\overline{t})$ we do inverse Laplace transform via the above technique of Berberan-Santo.

Application of the described technique to get rate distribution histogram function of relaxation in Newton's law of cooling

The classical Newton's law of cooling gives the relaxation function as

$$T(t)=T_{amb}+\left(T_{init}-T_{amb}\right)\mathrm{e}^{-at}$$

This above law is only true for metals where cooling is from heated temperature T_{init} to T_{amb} ambient (room temperature) where $T_{init} > T_{amb}$ .The differential equation describing the above classical cooling is

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -a\left(T(t) - T_{amb}\right), \quad a > 0, \quad T(0) = T_{init}$$

The rate distribution histogram is by taking inverse Laplace transform, gives

$$H_1(\lambda) = \mathcal{L}^{-1}\left\{T_{amb}\right\} + \left(T_{init} - T_{amb}\right)\mathcal{L}^{-1}\left\{\mathrm{e}^{-at}\right\}$$
$$H_1(\lambda) = T_{amb}\delta(t) + \left(T_{init} - T_{amb}\right)\left(\delta(\lambda - a)\right)$$

If the cooling law is composed with Caputo fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^\alpha T(t)}{\mathrm{d}t^\alpha}\right|^C = -b\left(T(t) - T_{amb}\right), \quad 0 < \alpha < 1; \quad b > 0; \quad T(0) = T_{init}$$

The solution is

$$T(t)\big|_C = T_{amb} + \left(T_{init} - T_{amb}\right)E_\alpha(-bt^\alpha)$$

The rate distribution function histogram is

$$H_2(\lambda) = \mathcal{L}^{-1}\left\{T_{amb}\right\} + \left(T_{init} - T_{amb}\right)\mathcal{L}^{-1}\left\{E_\alpha(-bt^\alpha)\right\}$$
$$H_2(\lambda) = T_{amb}\delta(t) + \frac{\left(T_{init} - T_{amb}\right)}{\pi}\left(\frac{b\lambda^{\alpha-1}\sin\alpha\pi}{\lambda^{2\alpha} + 2b\lambda^\alpha\cos\alpha\pi + \lambda^2}\right), \quad 0 < \alpha < 1$$
$$= T_{amb}\delta(t) + \left(T_{init} - T_{amb}\right)M_\alpha(\lambda); \quad M_\alpha(\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!\Gamma\left(-\alpha n + (1-\alpha)\right)}$$
$$= T_{amb}\delta(t) + \frac{(T_{init} - T_{amb})}{\pi}\int_0^\infty\left(1 - \frac{y^2}{\Gamma(1+2\alpha)} + \frac{y^4}{\Gamma(1+4\alpha)} - \frac{y^6}{\Gamma(1+6\alpha)} + \ldots\right)\cos\lambda y\mathrm{d}y$$
$$+ \frac{(T_{init} - T_{amb})}{\pi}\int_0^\infty\left(\frac{y}{\Gamma(1+\alpha)} - \frac{y^3}{\Gamma(1+3\alpha)} + \frac{y^5}{\Gamma(1+5\alpha)} - \ldots\right)\sin\lambda y\mathrm{d}y$$

We have given various alternative expressions as obtained in previous sections

If the cooling law is composed with Riemann-Liouville fractional derivative with following fractional differential equation

$$\left.\frac{\mathrm{d}^\alpha T(t)}{\mathrm{d}t^\alpha}\right|^{RL} = -c\left(T(t) - T_{amb}\right), \quad 0 < \alpha < 1; \quad c > 0; \quad T(0) = T_{init}$$

The solution is

$$T(t)\big|_{RL} = T_{amb}\left(\Gamma(\alpha)\right)E_{\alpha,\alpha}(-ct^{\alpha}) + T_{init}\left(1 - E_{\alpha}(-ct^{\alpha})\right)$$

The histogram of rate distribution is thus

$$\begin{aligned}
H_3(\lambda) &= T_{amb}\left(\Gamma(\alpha)\right)\mathcal{L}^{-1}\left\{E_{\alpha,\alpha}(-ct^{\alpha})\right\} + \mathcal{L}^{-1}\left\{T_{init}\right\} - \mathcal{L}^{-1}\left\{E_{\alpha}\left\{-ct^{\alpha}\right\}\right\} \\
H_3(\lambda) &= \frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{1}{\Gamma(\alpha)} - \frac{y^2}{\Gamma(3\alpha)} + \frac{y^4}{\Gamma(5\alpha)} - \frac{y^6}{\Gamma(7\alpha)} + \ldots\right)\cos\lambda y \mathrm{d}y \\
&+ \frac{T_{amb}\left(\Gamma(\alpha)\right)}{\pi}\int_0^{\infty}\left(\frac{y}{\Gamma(2\alpha)} - \frac{y^3}{\Gamma(4\alpha)} + \frac{y^5}{\Gamma(6\alpha)} - \ldots\right)\sin\lambda y \mathrm{d}y \\
&+ T_{init}\delta(t) - T_{init}M_{\alpha}(\lambda); \quad ; \quad M_{\alpha}(\lambda) = \sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!\Gamma\left(-\alpha n + (1-\alpha)\right)}
\end{aligned}$$

Some other type of non-Debye relaxation functions

Stretched exponential decay

$$\begin{aligned}
f(t) &= \mathrm{e}^{-(t/\tau_0)^{\alpha}} \\
H(\lambda) &= \frac{1}{\pi}\int_0^{\infty}\mathrm{d}y\left(\mathrm{e}^{-(y/\tau_0)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\right)\cos\left(\lambda y - \left(\tfrac{y}{\tau_0}\right)^{\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\right)
\end{aligned}$$

Above obtained by Berberan-Santo formula

Compressed hyperbolic decay of Becquerel

$$\begin{aligned}
f(t) &= \frac{1}{\left(1 + \frac{(1-\alpha)t}{\tau_0}\right)^{1/(1-\alpha)}}; \quad 0 < \alpha < 1 \\
H(\lambda) &= \frac{1}{\pi}\int_0^{\infty}\mathrm{d}y\left(1 + \left(\tfrac{(1-\alpha)y}{\tau_0}\right)^2\right)^{-\frac{1}{2(1-\alpha)}}\cos\left(\lambda y - \frac{\tan^{-1}\left(\frac{(1-\alpha)y}{\tau_0}\right)}{1-\alpha}\right)
\end{aligned}$$

Above is obtained by Berberan-Santo formula

Asymptotic power law decay

$$\begin{aligned}
f(t) &= \frac{1}{1 + \left(\frac{t}{\tau_0}\right)^{\alpha}}, \quad 0 < \alpha < 1 \\
H(\lambda) &= \tau_0^{\alpha}\lambda^{\alpha-1}E_{\alpha,\alpha}\left(-(\tau_0\lambda)^{\alpha}\right)
\end{aligned}$$

Above we obtain as by power series expansion and then doing term by term inverse Laplace transform. In above $\alpha = 1$, we get

$$f(t)=\frac{1}{1+\left(\frac{t}{\tau_0}\right)}$$

$$H(\lambda)=\tau_0 E_{1,1}\left(-(\tau_0\lambda)\right)=\tau_0 \mathrm{e}^{-\tau_0\lambda}$$

This is exponential distribution that we have discussed for hyperbolic decay function.

Conclusion

This note gives insight into disordered decaying system i.e. non-Debye relaxation, formed by Mittag-Leffler function. The rate relaxation histogram gives the disorder in a non-Debye type relaxation, in which we find how the multi-body relaxation with simultaneously relaxing bodies gets distributed and gives a histogram as a function of rate of relaxation. The ordering of histogram for several types of disordered decay functions we have discussed. We have discussed the histogram of rate relaxation rates for a perfectly decaying relaxation and oscillatory decay relaxation. Mainly here Mittag-Leffler function is discussed, and we have discussed various methods to find analytically the inverse Laplace transformation of relaxation function.

References

Analytical inversion of the Laplace transform without contour integration: application to luminescence decay laws and other relaxation functions, Mario. N. Berberan-Santos, Journal of Mathematical Chemistry, Vol. 38, No. 2 August 2005

Properties of the Mittag-Leffler function, Mario. N. Berberan-Santos, Journal of Mathematical Chemistry, Vol. 38, No. 4, 2005.

On the distinguished role of the Mittag-Leffler and Wright functions in fractional calculus, Francesco Mainardi, Special Functions in 21st Century: Theory & Applications, Washington DC USA, 6-8 April 2011.

FRACTIONAL CALCULUS: Integral and Differential Equations of Fractional Order, Rudolf Gorenflo, Franscesco Mainardi, CISM LECTURE NOTES International Centre for Mechanical Sciences, Palazzo del Torso, Piazza Garibaldi, Udine, Italy, URL: www.fracalmo.org FRACALMO PRE-PRINT pp. 223-276, 2008.

An insight into Newton's cooling law using fractional calculus, Adreja Mondol, Rivu Gupta, Shantanu Das, Tapati Dutta, Journal of Applied Physics 123, 2018.

Kindergarten of fractional calculus, (Lecture notes of fractional calculus-in use as limited prints in Jadavpur university and Calcutta university)-under publication process.

Revisiting the Curie-von Schweidler law for dielectric relaxation and derivation of distribution function for relaxation rates as per Zipf's power law and manifestation of fractional differential equations for capacitors, Shantanu Das, Journal of Modern Physics 8, 1988-2012, 2017.