Theory of capacitors with a new formulation of charge storage concept and observed relaxation current as per Curie-von Schweidler law & determining its rate histogram function as Zipf's distribution

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Here in this discussion we will introduce in very short
FRACTIONAL CALCULUS
and apply this while deliberating the topic that is

Theory of capacitors with a new formulation of charge storage concept and observed relaxation current as per Curie-von Schweidler law & determining its rate histogram function as Zipf's distribution

PART-A**: **New formulation of Capacitor with Convolution Integral as Charge Storage instead of usual Capacitance-Voltage multiplication, and its advantages for a Classical Capacitor and Fractional Capacitor (that follows Curie-von Schweidler relaxation law)-and formation of Fractional Derivative Getting loss-tangent, New way to look at capacitor break down, Memory in capacitors etc.

PART-B**: **Insight into Curie-von Schweidler relaxation and reasons why Zipf's power law is histogram of relaxation rates, and appearance of Fractional derivative with scale dependence relaxation rate for charging of a capacitor etc.

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture48.pdf

PART-A

From fractional Laplace operator we find fractional derivative & fractional integration operation

Classically we have $\mathcal{L}\left\{f^{(1)}(t)\right\} = sF(s) - f(0)$ ***where*** $\mathcal{L}\left\{f(t)\right\} = F(s)$

Say for case of RHS if we have $s^n F(s) - f(0)$ ***for*** $0 < n < 1$

Then we relate to $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - f(0)$ ***as*** $n-$ ***th derivative of*** $f(t)$

For n ***- negative we have*** $n-$ ***order integration i.e. with*** $n = -m$

$$\mathcal{L}\left\{f^{(-m)}(t)\right\} = s^{-m} F(s)$$

So there is possibility that we have in between operations for integration & differentiation, like half, one third etc

$$D^{1/2} f(t) \qquad D^{1/3} f(t) \qquad D^{1\frac{1}{2}} f(t) \qquad D^{-1/2} f(t)$$

We are having thus fractional Laplace variables like $s^{1/2}$ $s^{-1/2}$ ***etc***

Functional Fractional Calculus-2007 https://doi.org/10.1007/978-3-642-20545-3

Fractional integration and Fractional Derivative formulas

Fractional integration

Riemann-formula

$$ {}_0I_t^m f(t) = {}_0D_t^{-m} f(t) = \frac{1}{\Gamma(m)} \int_0^t \frac{f(x)}{(t-x)^{1-m}} \mathrm{d}x , \quad m > 0 ; \quad m \in \mathbb{R}^+ $$

Comes as Generalization of repeated integration formula by Cauchy i.e.

$$ {}_aI_x^n \left[f(x) \right] = \int_a^x \int_a^{x_1} \ldots\ldots \int_a^{x_n-1} f(x_n) \mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_2 \mathrm{d}x_1 = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y) \mathrm{d}y $$

Fractional derivative

RL-Formula

$$ {}_0D_t^n \left[f(t) \right] = \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x , \quad 0 < n < 1 $$

$$ = \frac{1}{\Gamma(1-n)} \frac{f(0)}{t^n} + \frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} \mathrm{d}x , \quad 0 < n < 1 $$

Caputo Formula

$$ {}_0^CD_t^n \left[f(t) \right] = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} \mathrm{d}x , \quad 0 < n < 1 $$

Euler's Formula

$$ {}_0D_x^n x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} x^{p-n} ; \quad p > -1 $$

Caputo Method

$$f^{(1/2)}(x) = D^{-(1/2)}\left(D^{(1)} f(x)\right)$$

$f(x) = x$ $\quad$ $f^{(1/2)}(x) = \frac{2}{\sqrt{\pi}}\sqrt{x}$ $\quad$ $f^{(1)}(x) = 1$

0 $\quad$ $\frac{1}{2}$ $\quad$ 1

1. Do one whole derivative of $f(x) = x$ →

← 2. Do half-integration of $f^{(1)}(x) = 1$

Riemann-Liouville Method

$$f^{(1/2)}(x) = D^{(1)}\left(D^{-1/2} f(x)\right)$$

$f^{(-1/2)}(x) = \frac{4}{3\sqrt{\pi}}\sqrt{x^3}$ $\quad$ $f(x) = x$ $\quad$ $f^{(1/2)}(x) = \frac{2}{\sqrt{\pi}}\sqrt{x}$

$-1/2$ $\quad$ 0 $\quad$ $1/2$

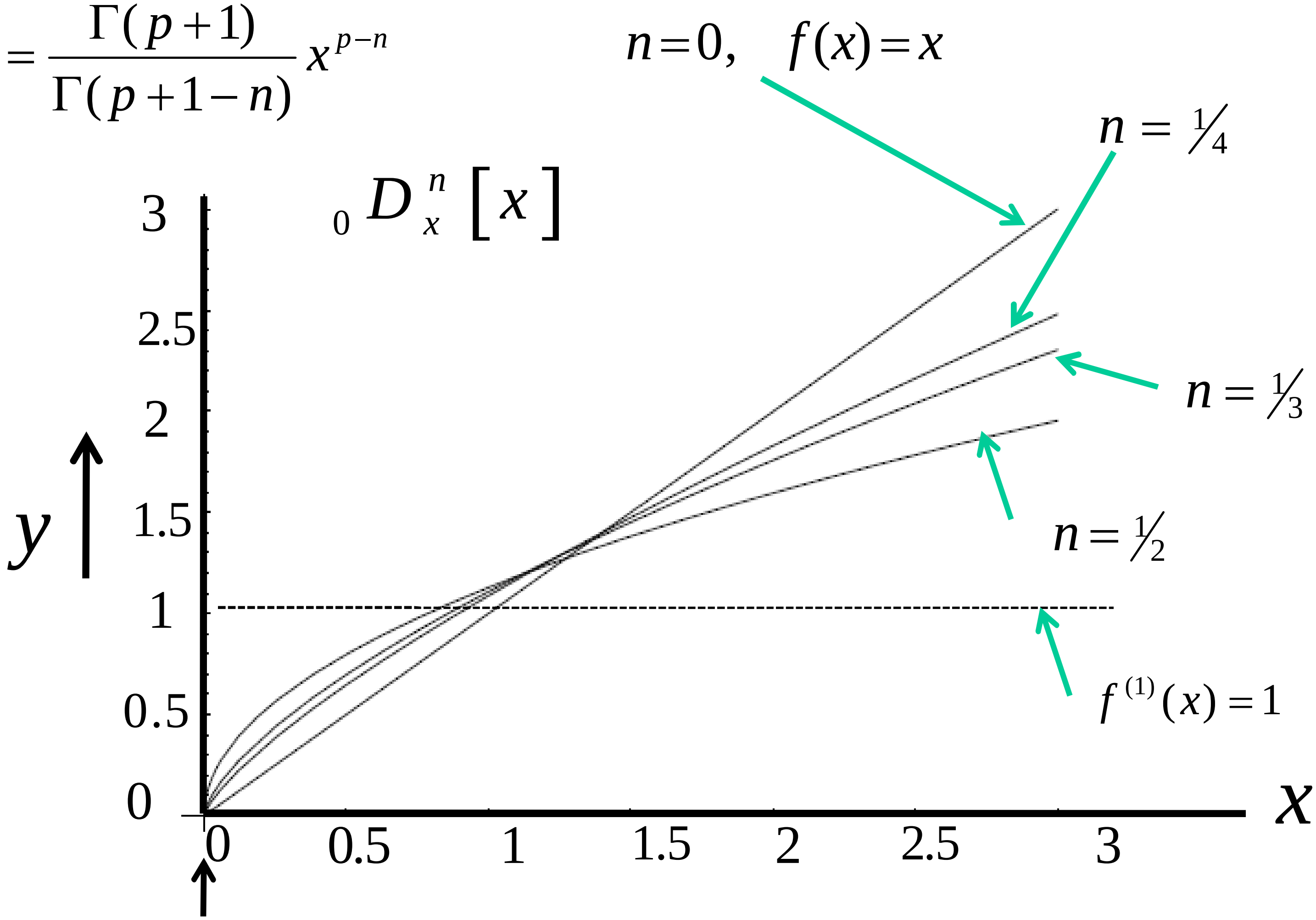
← 1. Do half integration for $f(x) = x$

2. Do one whole derivation of $f^{(-1/2)}(x) = \left(4/3\sqrt{\pi}\right)\sqrt{x^3}$ →

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture44.pdf

Fractional derivatives of $f(x) = x$ for orders 0, ¼, 1/3, ½, 1

$$ {}_0D_x^n x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} x^{p-n} $$

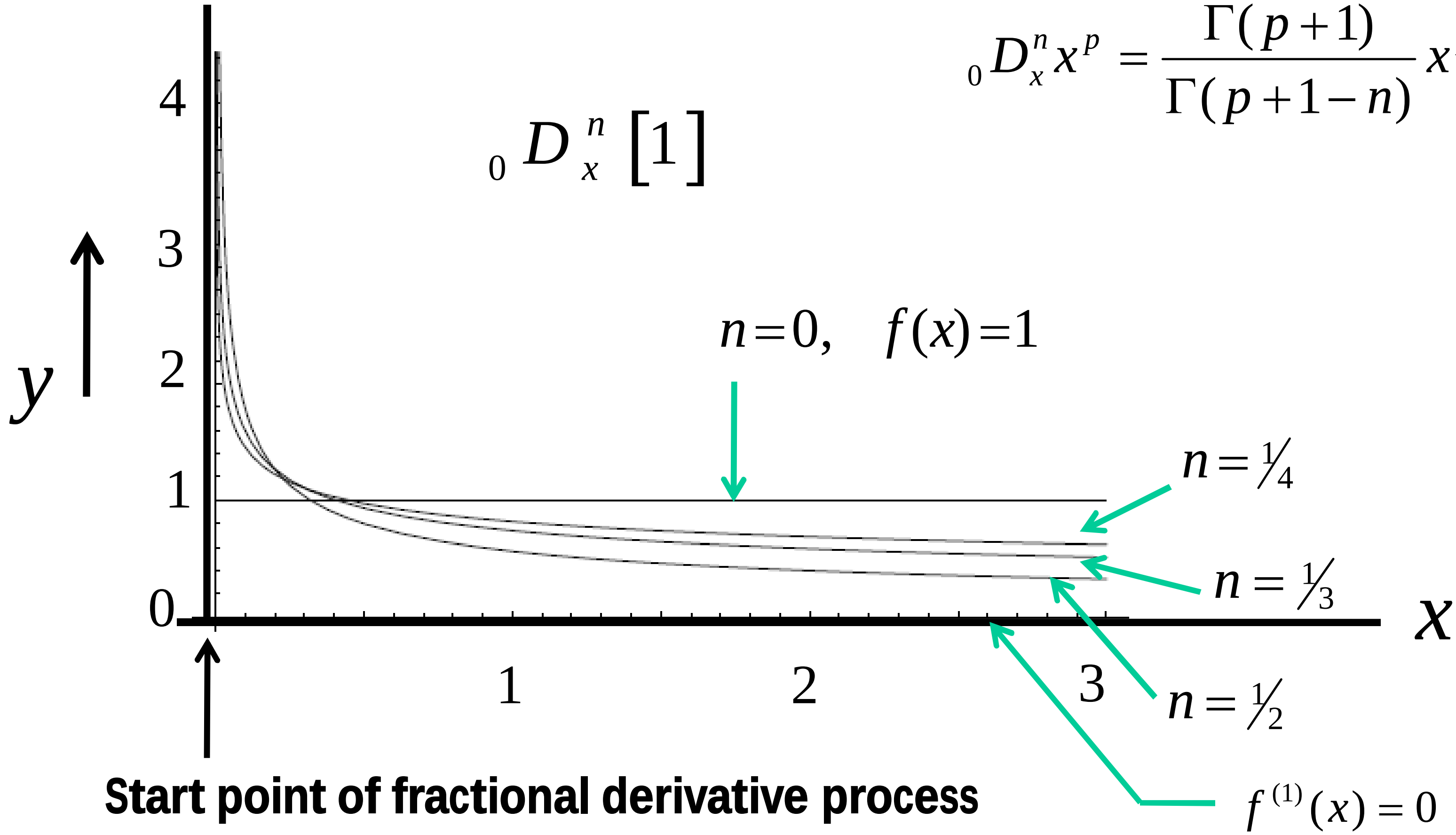


Start point of fractional derivative process

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture43.pdf

Fractional derivatives of $f(x) = 1$ for orders 0, ¼, 1/3, ½, 1

$$ {}_0D_x^n x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-n)} x^{p-n} $$



Fractional derivative of constant is not-zero!!

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture43.pdf

For a constant charging current the voltage across capacitor is not changing proportional to time !

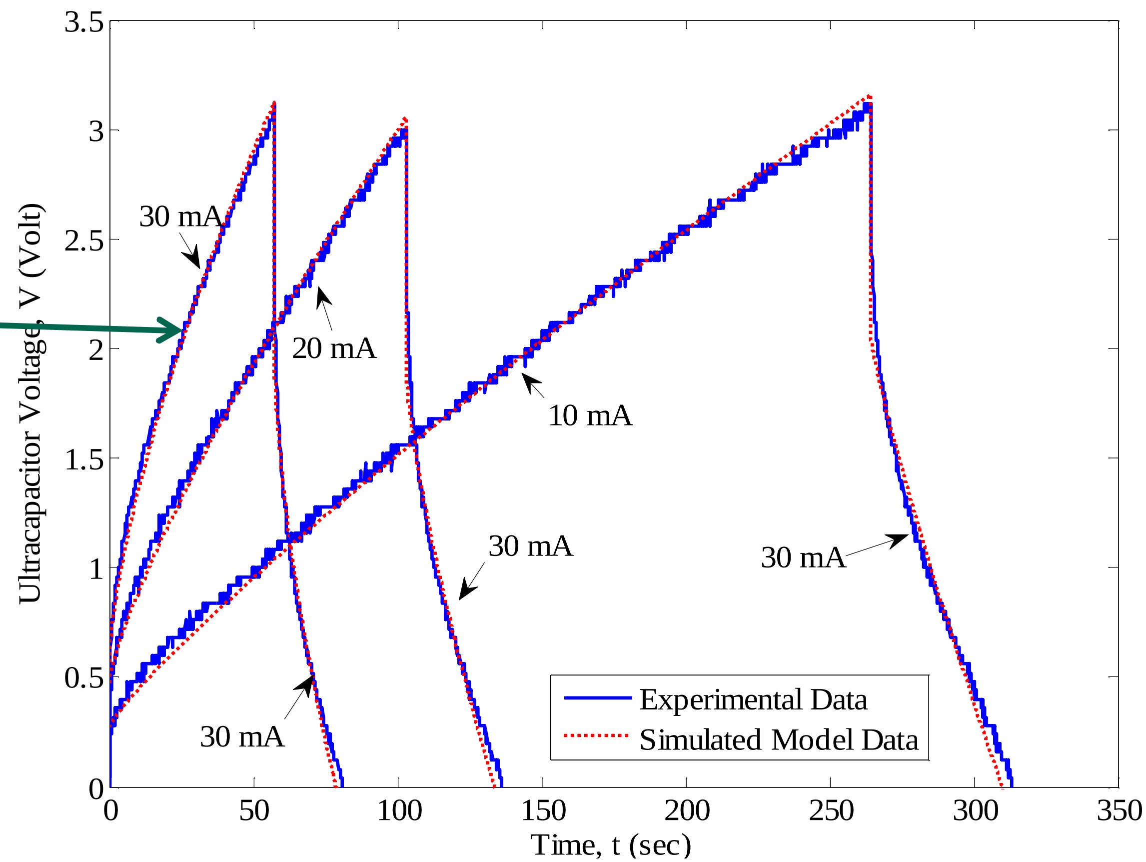
$$v(t) \propto t^n; \quad 0 < n < 1; \quad i(t) = I \text{ mA} \qquad v(t) \neq \frac{1}{C}\int_{t_1}^{t_2} i(\tau)\,d\tau$$

So $i(t) \neq C\frac{dv(t)}{dt}$ ***does fractional derivative relates the voltage-current?***

Is $i(t) = C_n \frac{d^n v(t)}{dt^n}, \quad n \in \mathbb{R}^+ ?$

$v(t) \propto t^n$

Not usual linear rise ?



This observation $v(t) \propto t^n$ ***is more pronounced when charging rate is quick !!***

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Universal law, for dielectric relaxations

The Curie-von Schweidler law relates to the relaxation current in dielectric when a step DC voltage is applied and is given by

$$i(t) \sim t^{-n} \quad \textit{where} \quad t > 0$$

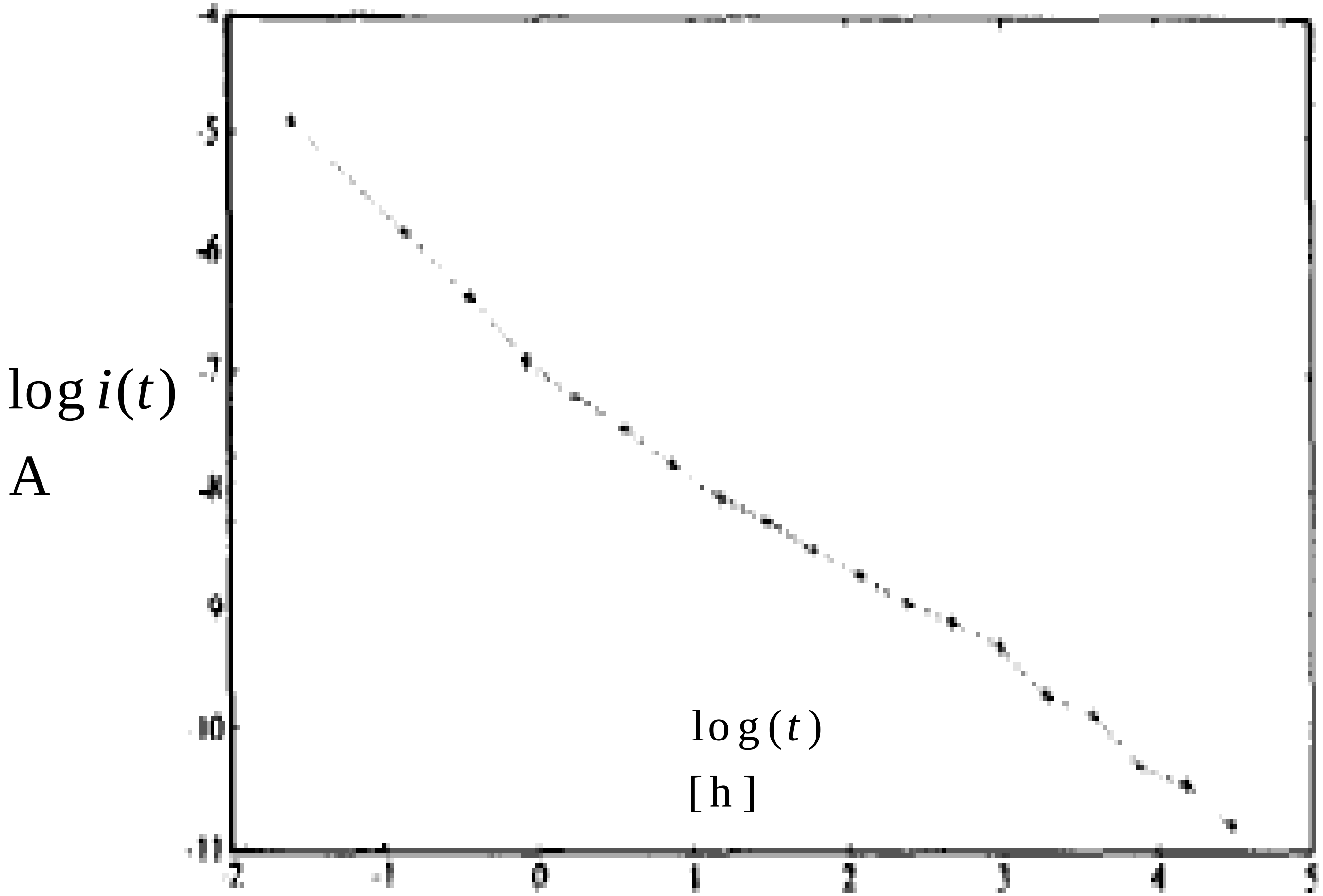
and the power (exponent) n ***is called relaxation constant or decay constant with*** $0 < n < 1$

We note that n ***is non-integer***

The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments

Jaques, C. (1889) Annales de Chimie et de Physique , 17, 384-434.
Schweidler, E.R. (1907) Annalen der Physik, 329, 711-770.
Jonscher, A.K. (1983) Dielectric Relaxation in Solids. Chelsea Dielectrics Press Limited.

Experimental evidence of Universal law, for dielectric relaxations



Current vs Time. At time zero a voltage of 100V is connected to a 0.47uF metalized paper dielectric capacitor; in log-log scales
Average slope is -0.86

IEEE Trans on Dielectrics and Insulation, 1,826. https://doi.org/10.1109/94.326654

A brief about ideal loss-less capacitor

What we know about geometric capacitor or a constant capacitor of say value C_1 is a constant value of Farad at all the frequencies from DC value of zero Hertz to infinite Hertz

This is ideal capacitor as though the dielectric used ε_r is lossless and is constant at all frequencies; and the capacity is given as $C_1 = \varepsilon_r A/d$

Therefore in Laplace domain we have $C(s) = C_1 \qquad s = \mathrm{i}\omega; \qquad C(\omega) = C_1$

$$C(\omega) = \mathrm{Re}\left[C(\omega)\right] - \mathrm{i}\,\mathrm{Im}\left[C(\omega)\right] = C_1 - \mathrm{i}(0)$$

Loss tangent as $\tan\phi = \mathrm{Im}\left[C(\omega)\right] / \mathrm{Re}\left[C(\omega)\right] = 0$

We say the 'time varying capacity function' call it $c(t)$ of geometric capacitor (ideal-capacitor) is $c(t) = \mathcal{L}^{-1}\left\{C(s)\right\}; \quad c(t) = C_1\delta(t)$

Therefore, we say that a constant ideal capacitor has a 'capacity function' as Dirac delta function—at the time of application of voltage stress

A ‘constant’ capacity function in time is lossy capacitor

The capacity function is constant for $t \geq 0$

$$c(t) = C_0$$

only if the frequency function is

$$C(s) = C_0 s^{-1}$$

in frequency domain in complex notation is

$$C(\omega) = 0 - \mathrm{i}\omega^{-1} C_0$$

with loss tangent as infinity.

Therefore, we say that a constant ideal loss less capacitor has a ‘capacity function’

$$c(t) = C_1 \delta(t)$$

and not $c(t) = C_0 ; \quad t \geq 0$

Practical capacitor

In reality the capacity of a capacitor, say of $1\mu\mathrm{F}$ ***means this value***

is at particular frequency of measurement standard is at $1\mathrm{kHz}$

(also depends on application)

Practically due to losses in ε_r ***the value of capacity of capacitor***

is varying in frequency; therefore in reality we have time varying capacity function $c(t)$

IEEE Trans on Dielectrics and Insulation, 1,826. https://doi.org/10.1109/94.326654

Reviewing concept of charge storage in constant capacitor-the classical theory

Impedance expression we write as in Laplace domain is

$$Z(s)=\frac{V(s)}{I(s)}=\frac{1}{C_1 s};\quad V(s)=\mathcal{L}\{v(t)\};\quad I(s)=\mathcal{L}\{i(t)\}$$

From this we have $C_1=\left(s^{-1}I(s)\right)/V(s);\quad C_1=\mathcal{L}\{c(t)\};\quad c(t)=C_1\,\delta(t)$

Call $C(s)=\mathcal{L}\{c(t)\}$ ***to write above as***

$$C(s)=\frac{s^{-1}I(s)}{V(s)},\quad C(s)=\frac{Q(s)}{V(s)};\quad Q(s)=s^{-1}I(s)=\mathcal{L}\{q(t)\}$$

We have $\mathcal{L}^{-1}\left\{s^{-1}I(s)\right\}=\int_0^t i(x)\,\mathrm{d}x = q(t)$ ***a charge function***

Using convolution theorem $\mathcal{L}^{-1}\left\{\left(F_1(s)\right)\left(F_2(s)\right)\right\}=\left(\mathcal{L}^{-1}\{F_1(s)\}\right)*\left(\mathcal{L}^{-1}\{F_2(s)\}\right)=f_1(t)*f_2(t)$

we get following $q(t)=\mathcal{L}^{-1}\{C(s)V(s)\}$

$$=c(t)*v(t)=\int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x$$

We are used to relation $q(t)=\left(c(t)\right)\left(v(t)\right)$

Charge storage in constant capacitor-the classical theory

Let an uncharged capacitor of constant capacity at $t = 0$ ***of value*** C_1

is charged with a constant step V_{BB} ***i.e.*** $v(t) = V_{BB}\left(u(t)\right); \quad t \geq 0$

Where $u(t)$ ***unit step function***

Thus the charge in time domain is a step function i.e. $q(t) = C_1 V_{BB}\left(u(t)\right)$

Laplace transform of this step charge is $Q(s) = \mathcal{L}\left\{C_1 V_{BB} u(t)\right\} = \dfrac{C_1 V_{BB}}{s}$

The first derivative of charge i.e.

$$i(t) = q^{(1)}(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} C_1 V_{BB} u(t) = C_1 V_{BB}\left(\delta(t)\right)$$

This is classical result that we all know is as per classical capacitor-theory

Charge storage in constant capacitor-by new formulation of Convolution for constant step voltage application

Capacity function $c(t) = C_1\left(\delta(t)\right)$

Voltage function $v(t) = V_{BB}\ ; \quad t \geq 0$

Charge function is $q(t) = c(t) * v(t) = \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x$

$$q(t) = c(t) * v(t) = \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x) = V_{BB}\,, \quad x \geq 0$$

$$= \int_{0}^{t} C_1\left(\delta(t-x)\right)\left(V_{BB}\right)\mathrm{d}x = C_1 V_{BB}\int_{0}^{t}\left(\delta(t-x)\right)\mathrm{d}x; \qquad t \geq 0$$

$$= C_1 V_{BB}\ ; \qquad t \geq 0$$

We have used identity $\int\left(\delta(x_0 - x)\right)\mathrm{d}x = 1$

Thus from above we get charge as step function at $q(t) = C_1 V_{BB}\left(u(t)\right)$

Charge storage in constant capacitor-by new formulation of Convolution for constant sinusoidal voltage application

Capacity function $c(t) = C_1\left(\delta(t)\right)$

Voltage function $v(t) = v(t) = \cos at; \quad t \ge 0$

Charge function is $q(t) = c(t) * v(t) = \int_{-\infty}^{t} \left(c(t-x)\right)\left(v(x)\right) \mathrm{d}x$

$$q(t) = c(t) * v(t) = \int_{-\infty}^{t} \left(c(t-x)\right)\left(v(x)\right) \mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x) = \cos ax; \quad x \ge 0$$

$$= \int_{0}^{t} C_1\left(\delta(t-x)\right)\left(\cos ax\right) \mathrm{d}x; \qquad t \ge 0$$

$$= C_1 \cos at; \qquad t \ge 0$$

We have used identity $\int \left(\delta(x_0 - x)\right) \mathrm{d}x = 1$

Thus from above we get charge as sinusoidal function as

$$q(t) = C_1 \cos at; \qquad t \ge 0$$

with same phase as voltage function

Interpretation of loss-less ideal geometrical capacitor in terms of charge

Thus we observe for a constant capacitor, there is no phase difference between voltage function and charge function for ideal loss-less capacitor given by capacity function in time as Dirac delta function

Generalizing the charge store expression

Thus, we have general expression for any time varying voltage applied at uncharged capacitor with geometrical capacity given by capacity function as $c(t) = C_1\left(\delta(t)\right)$

$$q(t) = c(t) * v(t) = \int_{-\infty}^{t} \left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x); \quad x \geq 0$$

$$= \int_{0}^{t} C_1\left(\delta(t-x)\right)\left(v(x)\right)\mathrm{d}x; \qquad t \geq 0$$

$$= C_1\left(v(t)\right); \qquad t \geq 0$$

Now we differentiate the expression above and write following

$$i(t) = q^{(1)}(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1\left(v(t)\right)\right), \qquad t \geq 0$$

$$= v(t)\frac{\mathrm{d}C_1}{\mathrm{d}t} + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

$$= \left(v(t)\right)\left(C_1\left(\delta(t)\right)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C_1\left(v(0)\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

$$= i(0) + i(t), \qquad t \geq 0$$

Interpretation of current expression

$$i(t) = q^{(1)}(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

$$= \left(v(t)\right)\left(C_1\left(\delta(t)\right)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C_1\left(v(0)\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} = i(0) + i(t), \qquad t \geq 0$$

The first term at RHS , indicate the value of current at $t = 0$

This unit delta functions when multiplied by $v(t)$ ***gives*** $v(0)\delta(t)$

This is from property $\int\left(\delta(x_0 - x)\right)\left(f(x)\right)\mathrm{d}x = f(x_0)$ ***differentiation gives***

$$\left(\delta(x_0 - x)\, f(x)\right) = \tfrac{\mathrm{d}}{\mathrm{d}x}\, f(x_0) = f(x_0)\delta(x)$$

The second term is $i(t)$ ***for*** $t \neq 0$ ***i.e.*** $i(t) = C_1\left(v^{(1)}(t)\right)$

For a constant step excitation at $t = 0$ ***we have*** $i(t) = C_1\left(v^{(1)}(t)\right) = 0$

thus we have $i(t) = C_1 v(0)\delta(t) = C_1 V_{BB}\left(\delta(t)\right), \quad t \geq 0; \quad v(0) = V_{BB}$

The obtained expression via the formulation $q(t) = c(t) * v(t)$

Capacity function as convolution relation

$$C(s) = \frac{Q(s)}{V(s)}$$

$$\mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\left(Q(s)\left(V(s)\right)^{-1}\right)\right\}$$

$$c(t) = \mathcal{L}^{-1}\{Q(s)\} * \mathcal{L}^{-1}\{(V(s)^{-1}\}$$

$$= \left(q(t)\right) * \left(v(t)\right)^{-1} = \int_{-\infty}^{t} \frac{q(t-x)}{v(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{q(x)}{v(t-x)} \mathrm{d}x$$

we say that capacity i.e. $c(t) \neq q(t)/v(t)$ ***not the usual ratio of charge to voltage in time domain, but it is given as convolution expression .***

For $q(t) = \left(C_1 V_{BB}\right)\left(u(t)\right)$ ***and*** $v(t) = V_{BB} u(t)$

$$c(t) = \left(q(t)\right) * \left(v(t)\right)^{-1}$$

$$= \int_{-\infty}^{t} \frac{C_1 V_{BB} u(t-x)}{V_{BB} u(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{C_1 V_{BB} u(x)}{V_{BB} u(t-x)} \mathrm{d}x$$

$$= C_1 \left(u(t) * u(t)^{-1}\right)$$

$$= C_1 \left(\delta(t)\right)$$ ***we got the ideal capacity***

We have used inverse identity $f * f^{-1} = \delta$

The Curie-von Schweidler Law origin of -Fractional Capacitor

Practically on applying a step input voltage $v(t) = V_{BB}u(t)$ ***to a capacitor which is initially uncharged; we get a power-law decay of current given by empirical Curie-von Schweidler as*** $i(t) \sim t^{-n}\,; \quad 0 < n < 1$ ***.Following we get***

$$i(t) = K_n \frac{V_{BB}}{t^n} \qquad t > 0 \qquad I(s) = \mathcal{L}\left\{i(t)\right\} = \mathcal{L}\left\{K_n V_{BB} t^{-n}\right\} \quad \textbf{\textit{use}}\ \mathcal{L}\left\{t^m\right\} = m!\,s^{-(m+1)}$$

$$= K_n V_{BB}\left(\frac{(-n)!}{s^{-n+1}}\right) \qquad n \neq 1; \quad 0 < n < 1$$

Laplace transform of Voltage input is $V(s) = \mathcal{L}\left\{v(t)\right\} = \mathcal{L}\left\{V_{BB}\left(u(t)\right)\right\} = V_{BB}s^{-1}$

Use $(\alpha - 1)! = \Gamma(\alpha)$ ***to write*** $I(s) = K_n \dfrac{\Gamma(1-n)V_{BB}}{s^{1-n}}$

$$= K_n \frac{\Gamma(1-n)}{s^{-n}}\left(\frac{V_{BB}}{s}\right)$$

Transfer function of capacitor as

$$G(s) = \frac{I(s)}{V(s)} = \frac{K_n \dfrac{\Gamma(1-n)}{s^{-n}}\left(\dfrac{V_{BB}}{s}\right)}{\left(\dfrac{V_{BB}}{s}\right)}$$

$$= K_n\left(\Gamma(1-n)\right)s^n = C_n s^n \qquad C_n = K_n\left(\Gamma(1-n)\right)$$

Fractional Derivative directly from Curie-von Schweidler Law -Fractional Capacitor

$$G(s) = \frac{I(s)}{V(s)} = C_n s^n \qquad C_n = K_n\left(\Gamma(1-n)\right)$$

$$s = \mathrm{i}\omega \qquad I(\omega) = \left(\cos\tfrac{n\pi}{2} + \mathrm{i}\sin\tfrac{n\pi}{2}\right)\omega^n C_n V(\omega)$$

$$\mathrm{i}^n = \left(\mathrm{e}^{\mathrm{i}\pi/2}\right)^n = \mathrm{e}^{\mathrm{i}n\pi/2}$$

This means current leads voltage in fractional capacitor by angle $n\left(90^0\right)$

Impedance expression is $Z(s) = \dfrac{1}{C_n s^n}, \quad 0 < n < 1$

With use of Generalized Laplace identity $\mathcal{L}^{-1}\left\{s^n F(s)\right\} = {}_0D_t^n\left[f(t)\right]$

We get fractional derivative as in expression $i(t) = C_n\left({}_0D_t^n\left[v(t)\right]\right), \quad 0 < n < 1$

The 'fractional capacity' C_n ***is in unit of*** $\mathrm{Farad}/\mathrm{sec}^{1-n}$

Ideal & Fractional capacitor expressions are following

$$i(t) = C_1\left({}_0D_t^{(1)}v(t)\right) = C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \qquad i(t) = C_n\left({}_0D_t^n v(t)\right) = C_n\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n}$$

$$v(t) = \frac{1}{C_1}\,{}_0D_t^{(-1)}i(t) = \frac{1}{C_1}\int_0^t i(x)\,\mathrm{d}x \qquad v(t) = \frac{1}{C_n}\,{}_0D_t^{-n}i(t) = \frac{1}{C_n}\int_0^t i(x)\left(\mathrm{d}x\right)^n$$

Charge stored in a fractional capacitor and capacity function from Curie-von Schweidler law

We have $i(t)=K_nV_{BB}t^{-n}\,,\quad 0<n<1$ ***for*** $v(t)=V_{BB}\left(u(t)\right)$

We write the above as incremental relation of Δq ***and*** Δt

$$\Delta q=\frac{K_nV_{BB}\Delta t}{t^n}\qquad\qquad \mathrm{d}q=\frac{K_nV_{BB}\mathrm{d}t}{t^n}$$

Integrating this we get $q(t)=\int_0^t \mathrm{d}q=\int_0^t\frac{K_nV_{BB}}{x^n}\mathrm{d}x$

$$=\frac{K_nV_{BB}}{(1-n)}t^{1-n}\,,\qquad 0<n<1\qquad t>0$$

From the expression in frequency domain i.e. $C(s)=\left(s^{-1}I(s)\right)/\left(V(s)\right)=\left(Q(s)\right)/\left(V(s)\right)$

we have for $i(t)=K_nV_{BB}t^{-n}\,;\quad I(s)=K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}\,;\quad V(s)=V_{BB}/s$

the following expression for $C(s)$

$$C(s)=\frac{\left(s^{-1}I(s)\right)}{V(s)}=\frac{s^{-1}\left(K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}\right)}{V_{BB}s^{-1}}$$

$$=\frac{K_n\left(\Gamma(1-n)\right)}{s^{1-n}};\qquad m!=\Gamma(1+m)$$

$$=K_n\frac{(-n)!}{s^{1+(-n)}}$$

Doing inverse Laplace transform by using $\mathcal{L}^{-1}\left\{(m!)/s^{(1+m)}\right\}=t^m$

We get time dependent capacity function

$$c(t)=K_nt^{-n}\,;\quad 0<n<1,\quad t>0$$

Charge stored in a fractional capacitor by use of convolution integral

Capacity function of a fractional capacitor $c(t) = K_n t^{-n};\quad 0 < n < 1,\quad t > 0$

Stressed by voltage $v(t) = V_{BB} u(t)$

We perform convolution as follows

$$q(t) = \big(c(t)\big) * \big(v(t)\big) = \int_{-\infty}^{t} \big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x, \qquad c(x) = K_n x^{-n}, \quad v(x) = V_{BB}; \quad t > 0$$

$$= \int_0^t K_n \left((t-x)^{-n}\right)\left(V_{BB}\right)\mathrm{d}x, \qquad 0 < n < 1$$

$$= -V_{BB} K_n \frac{(t-x)^{1-n}}{1-n}\Bigg|_{x=0}^{x=t} = \frac{V_{BB} K_n}{1-n} t^{1-n}$$

The same we got from Curie-von Schweidler law in previous slide

Note that $\lim_{t \uparrow \infty} q(t) = \infty$ ***for*** $c(t) = K_n t^{-n}$ ***for fractional capacitor***

$\lim_{t \uparrow \infty} q(t) = C_1 V_{BB}$ ***for*** $c(t) = C_1 \delta(t)$ ***for classical capacitor***

Observations on breakdown mechanism of a fractional capacitor

$$\lim_{t \uparrow \infty} q(t) = \infty$$

This is the new idea of breakdown of capacitors due to accumulation of enough charge (electrostatic breakdown) at a constant voltage even though voltage is less than the breakdown limit of dielectric proposed by S. Westerlund; we got same via convolution formula.

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Loss tangent with earlier theory of Fractional Capacitor

In S. Westerlund theory the charge formula used is $c(t) = q(t)/v(t)$

With Curie-von Schweidler law taken as $i(t) = V_{BB}/h_1 t^n\,; \quad t > 0, \quad 0 < n < 1$

for $v(t) = V_{BB}u(t)$ ***the capacity function is got as***

$$c(t) = \frac{t^{1-n}}{h_1(1-n)}, \qquad t > 0; \qquad 0 < n < 1$$

The frequency domain representation for above is

$$C(s) = \frac{(1-n)!}{h_1(1-n)} s^n, \qquad 0 < n < 1, \quad s = \mathrm{i}\omega$$

$$C(\omega) = \left(\frac{\omega^n (1-n)!}{h_1(1-n)} \right) \left(\cos \tfrac{n\pi}{2} + \mathrm{i} \sin \tfrac{n\pi}{2} \right)$$

We get $\tan\phi = \frac{\mathrm{Im}[C(\omega)]}{\mathrm{Re}[C(\omega)]} = \tan\left(\frac{n\pi}{2}\right)$ ***which is not correct, as the loss tangent is***

$\tan\phi = \tan(1-n)\frac{\pi}{2}$

Therefore, by S. Westerlund the loss tangent is not calculated by the capacity function instead, phase difference ψ ***is calculated between current*** $I(\omega)$ ***and voltage*** $V(\omega)$ ***by using admittance expression*** $G(s) = C_n s^n$ ***and then doing steady state (sinusoidal) analysis, and then writing loss tangent as*** $\tan\left(\frac{\pi}{2} - \psi\right)$ ***, which is*** $\tan\left(\frac{(1-n)\pi}{2}\right)$

Physica Scripta. Vol. 43, 174-179, 1991

Loss tangent with new theory of Fractional Capacitor

Whereas we have from our new derivation we get the following for a fractional capacitor

$$c(t) = K_n t^{-n}; \qquad t > 0, \quad 0 < n < 1$$

$$C(s) = K_n \left(\Gamma(1-n)\right) s^{-(1-n)}; \qquad s = \mathrm{i}\omega$$

$$C(\omega) = K_n \left(\Gamma(1-n)\right) \omega^{-(1-n)} \left(\cos\tfrac{(1-n)\pi}{2} - \mathrm{i}\sin\tfrac{(1-n)\pi}{2}\right)$$

where the capacity function tends towards zero for large time and at large frequency-gives a stable property

From above we get loss tangent as

$$\tan\phi = \frac{\mathrm{Im}\left[C(\omega)\right]}{\mathrm{Re}\left[C(\omega)\right]} = \tan\left(\tfrac{(1-n)\pi}{2}\right)$$

This is true loss tangent also reported by Westerlund

Regarding capacity function of fractional capacitor in conjugation to classical capacitor

Impedance of fractional capacitor $Z(s) = s^{-n}\frac{1}{C_n}; \quad 0 < n < 1 \qquad C_n = K_n\Gamma(1-n)$

with $C_n(s) = \mathcal{L}\left\{c_n(t)\right\} = C_n = K_n\left(\Gamma(1-n)\right)$ ***as obtained earlier a constant***

in units of $\text{Farad}/\text{sec}^{1-n}$

Thus, we expect that in time domain the fractional capacity call it $c_n(t)$

be given by delta function $c_n(t) = \left(K_n\left(\Gamma(1-n)\right)\right)\left(\delta(t)\right) = C_n\delta(t)$

The meaning of capacity function in time domain is an impulse of height C_n ***in units*** Farad/sec^{1-n} ***at the time of application of voltage excitation***

Whereas, in the frequency domain, the definition of fractional capacity is $C_n(s) = C_n$ ***i.e.*** $C_n(s) = \mathcal{L}\left\{C_n\delta(t)\right\} = C_n$ ***that is a constant in unit of*** Farad/sec^{1-n} ***value at all frequencies***

Further derivations regarding fractional capacitor in conjugation to classical capacitor

$$Z(s) = \frac{V(s)}{I(s)} = s^{-n}\frac{1}{C_n(s)}; \qquad 0 < n < 1$$

$$c_n(t) = \mathcal{L}^{-1}\left\{C_n(s)\right\}$$

Use $\mathcal{L}^{-1}\left\{s^{-n}F(s)\right\} = {}_0\mathcal{I}_t^{\,n} f(t)$

where ${}_0\mathcal{I}_t^n$ ***is defining fractional integration operation***

Get the following

$$C_n(s) = \frac{s^{-n}I(s)}{V(s)} = \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{\,n}\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad 0 < n < 1$$

$$= \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{\,n-1}\,{}_0\mathcal{I}_t^{\,1}\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}} = \frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{\,n-1}\int_0^t i(x)\mathrm{d}x\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad q(t) = \int_0^t i(x)\mathrm{d}x$$

$$= \frac{\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}; \qquad {}_0\mathcal{I}_t^{\,n-1} f(t) = {}_0D_t^{1-n} f(t)$$

$$\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\} = \left(\mathcal{L}\left\{v(t)\right\}\right)\left(\mathcal{L}\left\{c_n(t)\right\}\right)$$

$$\mathcal{L}\left\{c_n(t)\right\} = \mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}\left(\mathcal{L}\left\{v(t)\right\}\right)^{-1}$$

$$c_n(t) = \left({}_0D_t^{1-n}\left[q(t)\right]\right) * \left(v(t)\right)^{-1}$$

Capacity function of fractional capacitor in conjugation to classical capacitor

Therefore, we write following formulas for fractional capacitor with conjugation to classical capacitor theory

$$c_n(t) = \left({}_0D_t^{1-n}\left[q(t)\right]\right) * \left(v(t)\right)^{-1}; \qquad 0 < n < 1$$

$${}_0D_t^{1-n}\left[q(t)\right] = \left(c_n(t)\right) * \left(v(t)\right)$$

$$q(t) = {}_0D_t^{n-1}\left[\left(c_n(t)\right) * \left(v(t)\right)\right]$$

We are not decomposing above instead we write the following expected relation in expansion

$$c(t) = {}_0D_t^{n-1}\left[\left(c_n(t)\right)\right] = {}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)\right]$$

$$q(t) = \left(c(t)\right) * \left(v(t)\right)$$

Using obtained expression i.e. $q(t) = \frac{K_n V_{BB}}{1-n} t^{1-n}$ ***in above we get***

$${}_0D_t^{1-n}\left[q(t)\right] = \frac{K_n V_{BB}}{1-n}\left(\frac{\Gamma(1-n+1)}{\Gamma(1-n+1-1+n)} t^{1-n-1+n}\right) = K_n V_{BB}\left(\Gamma(1-n)\right)$$

that is a constant function for $t > 0$

This we have got by formula of fractional derivative i.e. ${}_0D_x^m x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-m)} x^{p-m}$

Thus, we write ${}_0D_t^{1-n}\left[q(t)\right] = K_n V_{BB}\left(\Gamma(1-n)\right) u(t)$

Conversion Fractional Capacity units to Farad units

$$c_n(t) = \left({}_0D_t^{1-n}\left[q(t)\right]\right) * \left(v(t)\right)^{-1} = \left(K_nV_{BB}\left(\Gamma(1-n)\right)\left(u(t)\right)\right) * \left(V_{BB}\left(u(t)\right)\right)^{-1}$$

$$= K_n\left(\Gamma(1-n)\right)\delta(t)$$

We used identity i.e. $f * f^{-1} = \delta$

We consider the following relation for time varying capacity function

$$c(t) = {}_0D_t^{(n-1)}\left[c_n(t)\right]; \quad t > 0, \quad 0 < n < 1; \qquad {}_0D_t^{(n-1)} \equiv {}_0\mathcal{I}_t^{(1-n)}$$

i.e. time varying capacity function defined as fractional integral of the order $1-n$ ***for the fractional capacity function i.e.*** $c_n(t)$ ***i.e. in units of*** $\text{Farad}/\sec^{1-n}$ ***which is constant in frequency domain as*** $C_n(\omega) = K_n\left(\Gamma(1-n)\right)$ ***. Using*** $c_n(t) = K_n\left(\Gamma(1-n)\right)\delta(t)$ ***as obtained above we get following***

$$c(t) = {}_0D^{n-1}\left[c_n(t)\right]; \quad c_n(t) \equiv \text{Farad}/\sec^{1-n}$$

$$= {}_0\mathcal{I}_t^{1-n}\left[\left(K_n\left(\Gamma(1-n)\right)\left(\delta(t)\right)\right)\right]$$

$$= K_n\left(\Gamma(1-n)\right)\left({}_0\mathcal{I}_t^{1-n}\left[\delta(t)\right]\right); \qquad {}_0\mathcal{I}_x^m\left[\delta(x)\right] = \frac{1}{\Gamma(m)}x^{m-1}$$

$$= K_n\left(\Gamma(1-n)\right)\left(\frac{t^{1-n-1}}{\Gamma(1-n)}\right) = K_nt^{-n}; \quad \text{Farad}$$

we had obtained earlier too

General Charge expression for arbitrary voltage to a fractional capacitor

We obtain a general expression of charge for Curie-von Schweidler relaxing current in a capacitor that is having capacity function as $c(t) = K_n t^{-n}$ ***when stressed with a time varying voltage*** $v(t)$ ***applied at*** $t = 0$ ***is by convolution process as*** $q(t) = \left(K_n t^{-n}\right) * \left(v(t)\right)$

$$q(t) = \left(c(t)\right) * \left(v(t)\right) = \int_{-\infty}^{t} \left(c(t-x)\right)\left(v(x)\right) \mathrm{d}x \qquad c(x) = K_n x^{-n} \qquad x > 0$$

$$q(t) = \int_0^t K_n \frac{v(x)}{(t-x)^n} \mathrm{d}x$$

As we did for geometrical capacity in previously we differentiate this

$$i(t) = q^{(1)}(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t}$$

$$= K_n \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{v(x)}{(t-x)^n} \mathrm{d}x$$

General current expression for arbitrary voltage to a fractional capacitor

$$i(t) = K_n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)}{(t-x)^n}\,\mathrm{d}x$$

We apply formula of integration by parts i.e.

$$\int_0^t \big(f_1(x)\big)\big(f_2(x)\big)\mathrm{d}x = \Big[f_1(x)\int f_2(x)\mathrm{d}x\Big]_{x=0}^{x=t} - \int_0^t \bigg(\Big(f_1^{(1)}(x)\Big)\int_0^t \big(f_2(x)\big)\mathrm{d}x\bigg)\mathrm{d}x$$

to evaluate $\int_0^\infty \frac{v(x)}{(t-x)^n}\mathrm{d}x$

i.e.

$$\int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} = \left[v(x)\int \frac{\mathrm{d}x}{(t-x)^n}\right]_{x=0}^{x=t} - \int_0^t \left(v^{(1)}(x)\int \frac{\mathrm{d}x}{(t-x)^n}\right)\mathrm{d}x$$

$$= v(x)\left(-\frac{(t-x)^{1-n}}{1-n}\right)\Bigg|_{x=0}^{x=t} - \int_0^t v^{(1)}(x)\left(\frac{(-1)(t-x)^{1-n}}{1-n}\right)\mathrm{d}x$$

$$= \frac{v(0)}{1-n}t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\,\mathrm{d}x$$

Contd…

…Contd

Now we differentiate and write the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(0)}{1-n}t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\mathrm{d}x\right)$$

$$= v(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{1-n}}{1-n}\right) - \int_0^t \frac{v^{(1)}(x)}{1-n}\frac{\mathrm{d}\left((-1)(t-x)^{1-n}\right)}{\mathrm{d}t}\mathrm{d}x$$

$$= \frac{v(0)}{t^n} - \int_0^t \frac{v^{(1)}(x)}{1-n}\left((-1)(1-n)(t-x)^{1-n-1}\right)\mathrm{d}x$$

$$= \frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)}{(t-x)^n}\mathrm{d}x$$

This gives

$$i(t) = K_n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)}{(t-x)^n}\mathrm{d}x = K_n\frac{v(0)}{t^n} + K_n\int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}; \quad K_n = \frac{C_n}{\Gamma(1-n)}; \qquad 0<n<1$$

This expression is consistent with S. Westerlund's expression but here this is obtained via convolution formula

For geometrical loss less capacitor we have

$$i(t) = C_1\left(v(0)\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}$$

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Current expression for step constant voltage to a fractional capacitor –and recovery of Curie-von Schweidler law

For $v(t) = V_{BB}\left(u(t)\right)$ ***i.e. a constant step voltage applied at time*** $t = 0$

to a time varying capacity $c(t) = K_n t^{-n}$ ***we have for*** $t > 0$ $\quad v^{(1)}(t) = 0$

with

$v(0) = V_{BB}$

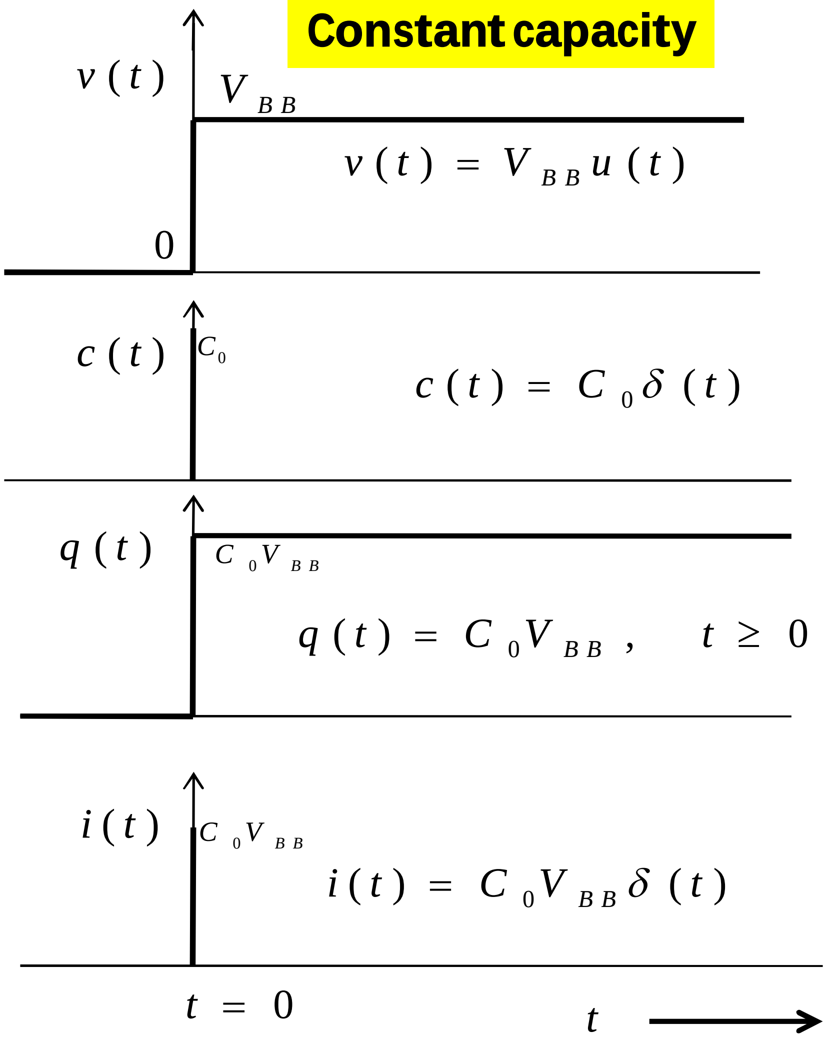
$$i(t) = K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad v(0) = V_{BB}; \quad v^{(1)}(x) = 0, \quad x > 0$$

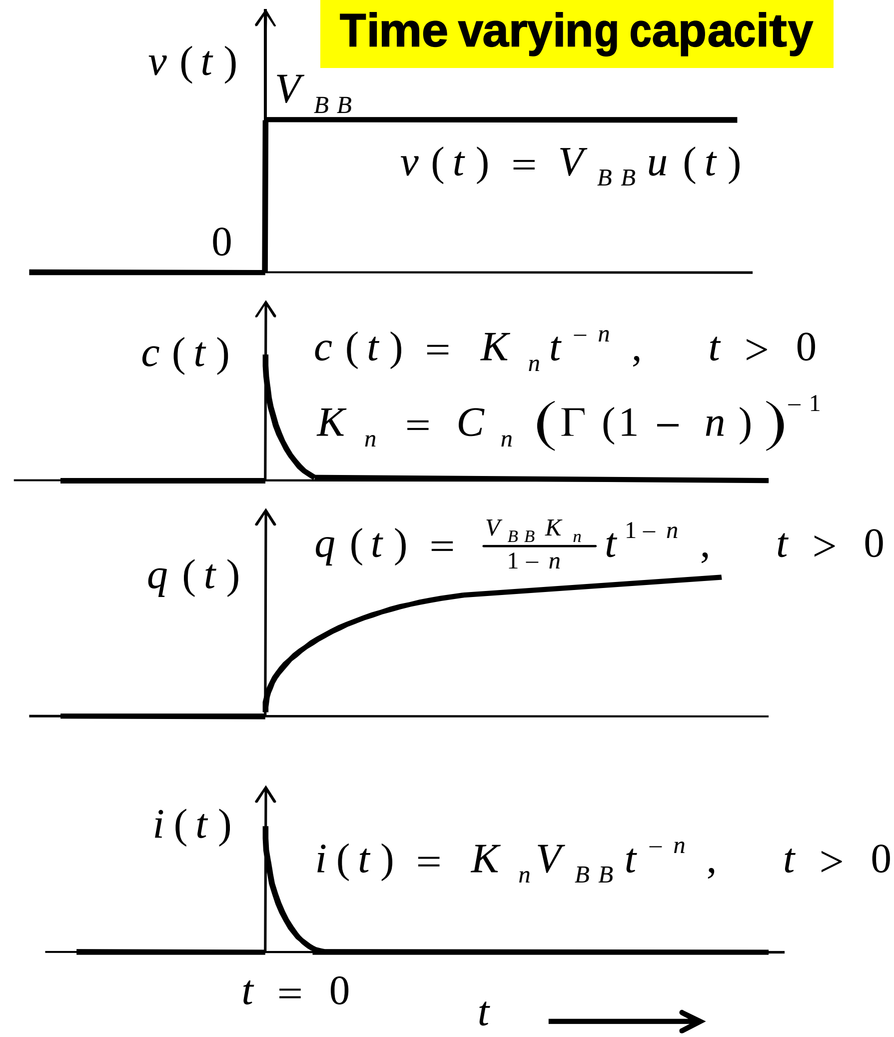
$$= K_n \frac{V_{BB}}{t^n} + K_n \int_0^t \frac{\left(0\right)\mathrm{d}x}{(t-x)^n} = K_n \frac{V_{BB}}{t^n}$$

We recover the Curie-von Schweidler law

Summary of discussion about constant capacity vis-à-vis time varying capacity

Capacity, charge, current for constant capacitor vis-a-vis time varying capacitor to a step voltage excitation





Appearance of fractional derivative in Fractional Capacitor

Current voltage related by fractional derivative

$$i(t) = K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad 0<n<1$$

$$= K_n \left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\left(\frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}\right)\right), \qquad K_n\left(\Gamma(1-n)\right) = C_n$$

$$= C_n\left({}_0D_t^n\left[v(t)\right]\right), \qquad 0<n<1$$

Charge voltage related by fractional integration

$$q(t) = \left(c(t)\right) * \left(v(t)\right) = \left(K_n t^{-n}\right) * \left(v(t)\right)$$

$$= \int_0^t K_n \frac{v(x)}{(t-x)^n}\mathrm{d}x$$

$$= K_n\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\int_0^t \frac{v(x)}{(t-x)^n}\mathrm{d}x\right); \qquad K_n\left(\Gamma(1-n)\right) = C_n$$

$$= C_n\left({}_0\mathcal{I}_t^{(1-n)}\left[v(t)\right]\right)$$

$$= C_n\left({}_0D_t^{n-1}\left[v(t)\right]\right) \qquad t>0 \qquad 0<n<1$$

Charge always growing with time to a constant voltage for a Fractional Capacitor

We apply a constant step voltage $v(t) = V_{BB}$ ***at*** $t = 0$ ***to an uncharged fractional capacitor with capacity function*** $c(t) = \frac{C_n}{\Gamma(1-n)} t^{-n}$

$$q(t) = C_n \left({}_0D_t^{n-1} \left[v(t) \right] \right) \qquad t > 0 \qquad 0 < n < 1$$

$$= C_n \left({}_0D_t^{n-1} \left[V_{BB} \right] \right) \qquad {}_0D_t^{\alpha} \left[\mathrm{C} \right] = \mathrm{C} \frac{\Gamma(1)}{\Gamma(1-\alpha)} t^{-\alpha}$$

$$= C_n V_{BB} \frac{\Gamma(1)}{\Gamma\left(1+(1-n)\right)} t^{1-n} = \frac{C_n}{(1-n)\left(\Gamma(1-n)\right)} V_{BB} t^{1-n}, \qquad K_n \left(\Gamma(1-n) \right) = C_n$$

$$= \frac{K_n V_{BB}}{(1-n)} t^{1-n}$$

The same expression we showed earlier too-a new break down law of capacitor due to electrostatic forces

Capacity function of an ideal capacitor and explanation vis-à-vis a pitcher with non-porous walls holding water

We take example of a pitcher, which holds water, of volume V. Let the pitcher be made of metal walls so that there are no pores. It is fully filled with water from empty state, hence once full it has no capacity left. This is like ideal capacitor, where the volume of water V remains fixed as constant after filling, with no left over capacity

Thus, an ideal capacitor described by capacity function $c(t) = C_1\delta(t)$ ***after it is charged at*** $t = 0$ ***with a constant voltage holds the constant charge*** $q(t) = C_1 V_{BB}$ ***at times*** $t > 0$ ***and at time,*** $t > 0$ ***this capacitor has zero capacity function, i.e.*** $c(t) = 0$ ***that is like no more capacity left to fill, like pitcher*** $q_{\max} = \lim_{t\uparrow\infty} q(t) = C_1 V_{BB}$

Therefore we can say the capacity function indicates the left over capacity to fill from maximum charge $q_{\max} = \lim_{t\uparrow\infty} q(t)$

Capacity function of a fractional capacitor and explanation vis-à-vis a pitcher with porous walls holding water

Now let the walls of the pitcher be made of clay with an infinitely porous material. As the pitcher gets the water volume V the pitcher walls too starts seepage of water into its pores . This water filling process in the porous walls we call fractional capacity. Now due to infinite nature of these pores, we have a situation, that infinite amount of water keeps seeping into the walls. This is analogous to charging porous walls with water as charging a fractional capacitor where we derived $q_{\max} = \lim_{t\uparrow\infty} q(t) = \infty$ ***Yet as we go on with charging process the remaining capacity of holding the charge from maximum value (in this case infinity) keeps on decreasing but will never be going to zero -capacity function for a fractional capacitor,***

$$c(t) = K_n t^{-n}$$

$$\lim_{t\uparrow\infty} c(t) = 0$$

$$q(t) = \frac{K_n V_{BB}}{(1-n)} t^{1-n}, \quad 0 < n < 1$$

Integrated Capacity defined from Capacity function

We define integral capacity as following from the capacity function

$$c_{\text{int}}(t) = \int_0^t c(x)\mathrm{d}x\,; \quad t > 0$$

Thus for a classical capacitor $c_{\text{int}}(t) = \int_0^t \left(C_1\delta(x)\right)\mathrm{d}x = C_1\,, \qquad t > 0$

$$\lim_{t\uparrow\infty} c_{\text{int}}(t) = C_1$$

This integrated capacity is what is discussed in classical theory that we derived from capacity function

For the case of fractional capacitor $c_{\text{int}}(t) = \int_0^t K_n x^{-n}\mathrm{d}x = \dfrac{K_n}{(1-n)} t^{1-n}\,; \qquad t > 0$

$$\lim_{t\uparrow\infty} c_{\text{int}}(t) = \infty$$

This is same as by S. Westerlund

Thus, the term 'integrated capacity' of capacitor is analogous to 'total' water holding capacity of pitcher- constant at all times for ideal capacitor and infinity for fractional capacitor

Use of Capacity Function in obtaining Loss-tangent

We mention here the expressions for

$$C_{\text{int}}(\omega) = \mathcal{L}\left\{c_{\text{int}}(t)\right\}\Big|_{s=\mathrm{i}\omega}$$

cannot be used to determine the loss tangent, while from capacity function

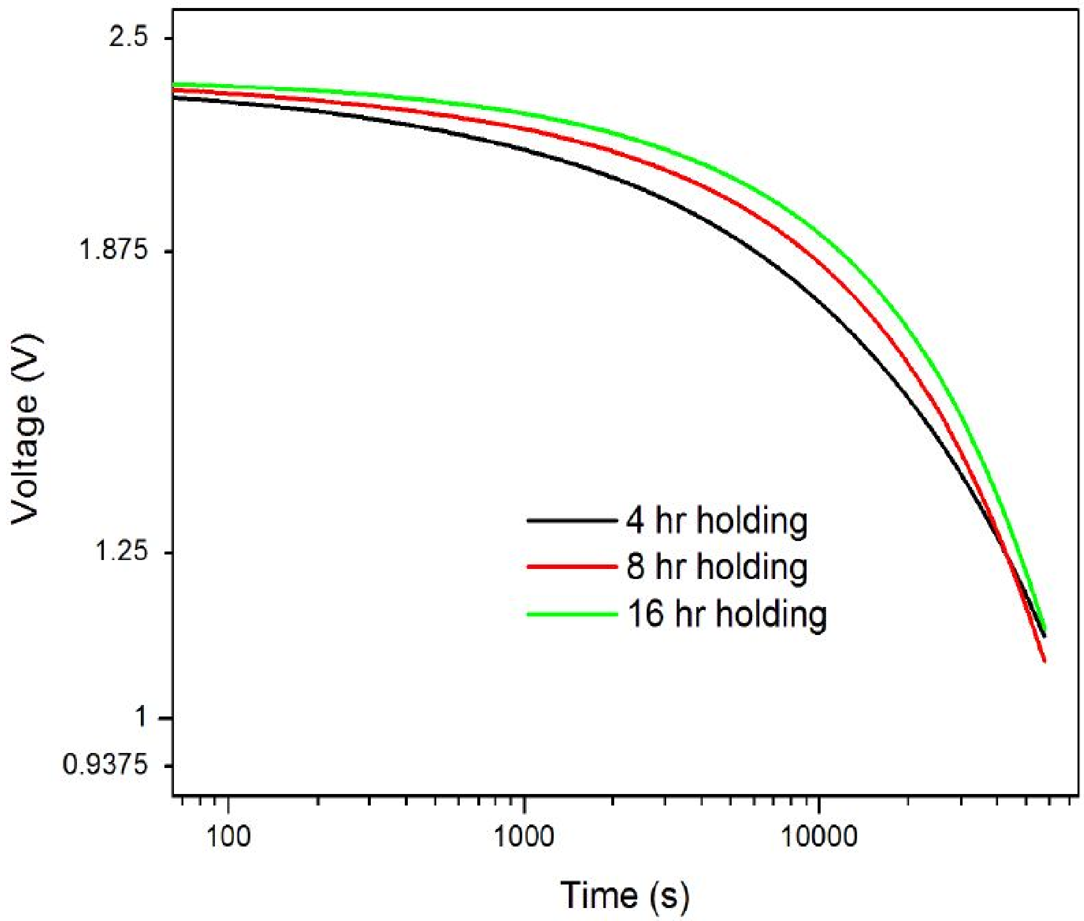
with

$$C(\omega) = \mathcal{L}\left\{c(t)\right\}\Big|_{s=\mathrm{i}\omega}$$

is used to determine loss tangent value.

The charging time is memorized in fractional capacitor

We charge a ultra-capacitor to maximum limit voltage and keep for time T on float charge, then it is kept under open-circuit. The self-discharge plot is depending on T , i.e. capacitor memorizes the time history



The open circuit self discharge voltage is

$$v(t)=\frac{V_{BB}}{\Gamma(1-n)\Gamma(n)}\int_0^T \frac{\mathrm{d}x}{x^n(T+t-x)^{1-n}}$$

Only possible if

$$i(t)=C_n\left({}_0D_t^n\left[v(t)\right]\right), \qquad 0<n<1$$

due to fractional capacitor

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture26.pdf

PART-B

Complex relaxation with several exponential decay functions

We call the Curie-von Shweidler relaxation law $i(t) \sim t^{-n}; \quad 0 < n < 1$ ***as complex process, of non-Debye type. We formulate the method to extract the histogram of the relaxation rates call it*** $H_\lambda(\lambda)$ ***for a complex non-Debye relaxation process*** $i(t)$ ***which we assume to be composed of several Debye type relaxations*** $\mathrm{e}^{-\lambda t}$ ***with*** λ ***varying from zero to infinity***

$$i(t) = a_1\mathrm{e}^{-\lambda_1 t} + a_2\mathrm{e}^{-\lambda_2 t} + a_3\mathrm{e}^{-\lambda_3 t} + \ldots = \sum a_k \mathrm{e}^{-\lambda_k t}$$

$$i(0) = a_1 + a_2 + a_3 + \ldots$$

In continuum limit we may write the above as following

$$i(t) = \int_0^\infty \left(H_\lambda(\lambda)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$$

From above formulation if we have only one single Debye relaxation i.e. having only one rate constant λ_0 ***i.e.*** $i(t) = \mathrm{e}^{-\lambda_0 t}$ ***then*** $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$

i.e.

$$i(t) = \int_0^\infty \left((H_\lambda \lambda)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda; \quad \int \left(\delta(x - x_0)\right) f(x)\mathrm{d}x = f(x_0)$$

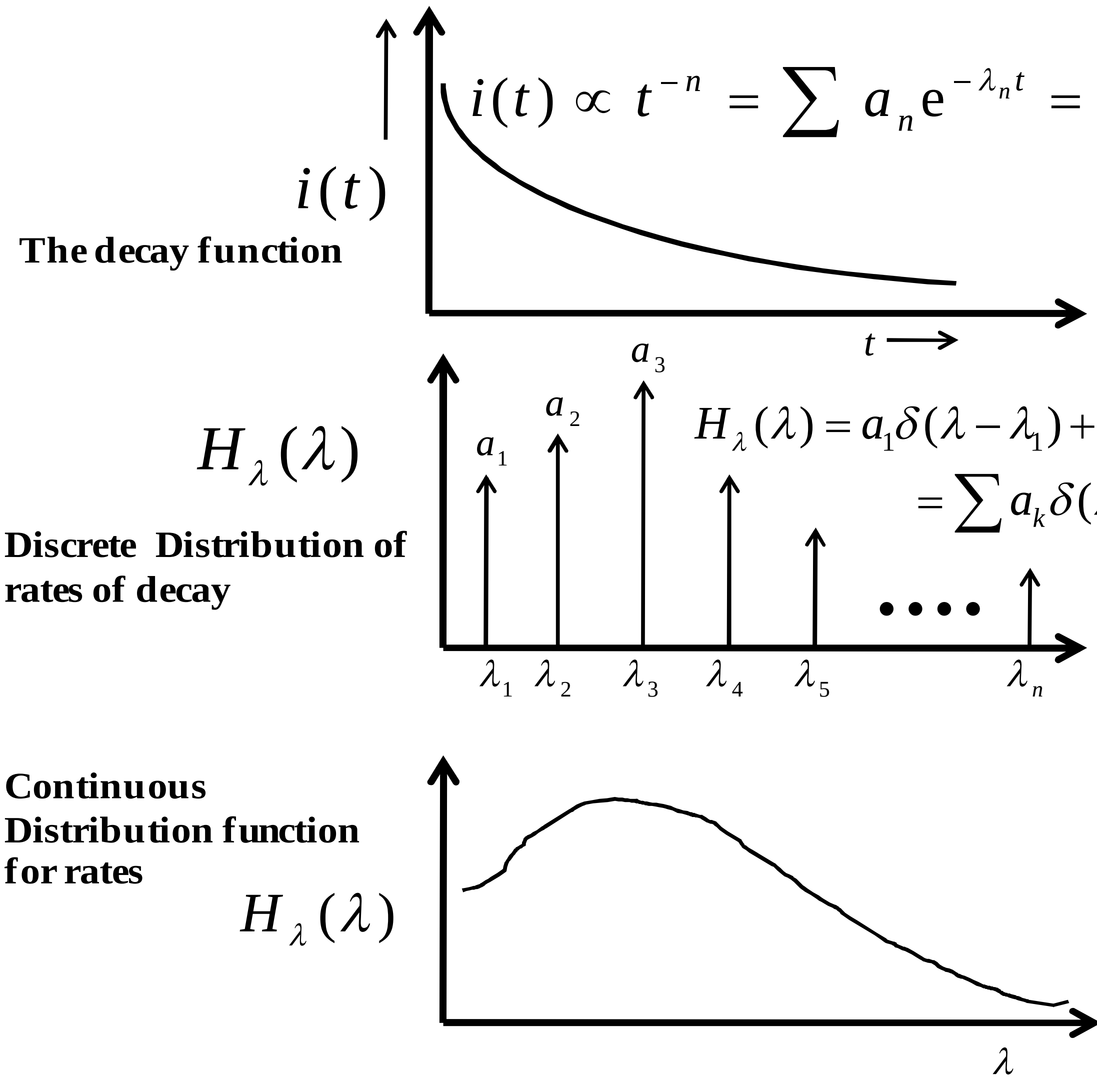
$$= \int_0^\infty \left(\delta(\lambda - \lambda_0)\right)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda = \mathrm{e}^{-\lambda_0 t}$$

https://doi.org/10.4236/jmp.2017.812120

For complex relaxation function discrete and continuous distribution of rate distributions

$$i(t) \propto t^{-n} = \sum a_n \mathrm{e}^{-\lambda_n t} = \int_0^\infty H_\lambda(\lambda)\mathrm{e}^{-\lambda t}\mathrm{d}\lambda$$

$$H_\lambda(\lambda) = a_1\delta(\lambda-\lambda_1) + a_2\delta(\lambda-\lambda_2) + a_3\delta(\lambda-\lambda_3) + \dots = \sum a_k \delta(\lambda-\lambda_k)$$



Extraction of rate distribution function by formulating Laplace integral

The Laplace transform $F(s)$ ***of a function in time domain*** $f(t)$ ***is defined as following integral transform relation i.e. called Laplace integral***

$$F(s) = \mathcal{L}\left\{f(t)\right\} \qquad F(s) \overset{\text{def}}{=} \int_0^{\infty} \left(f(t)\right) e^{-st} dt \qquad F(s) = 0 \quad \text{for} \quad s < 0$$

We have derived $i(t) = \int_0^{\infty} \left(H_\lambda(\lambda)\right) e^{-\lambda t} d\lambda$ ***or*** $i(t) = \mathcal{L}\left\{H_\lambda(\lambda)\right\}$

Therefore we can say $H_\lambda(\lambda)$ ***is inverse Laplace Transform of*** $i(t)$

$$H_\lambda(\lambda) = \mathcal{L}^{-1}\left\{i(t)\right\}$$

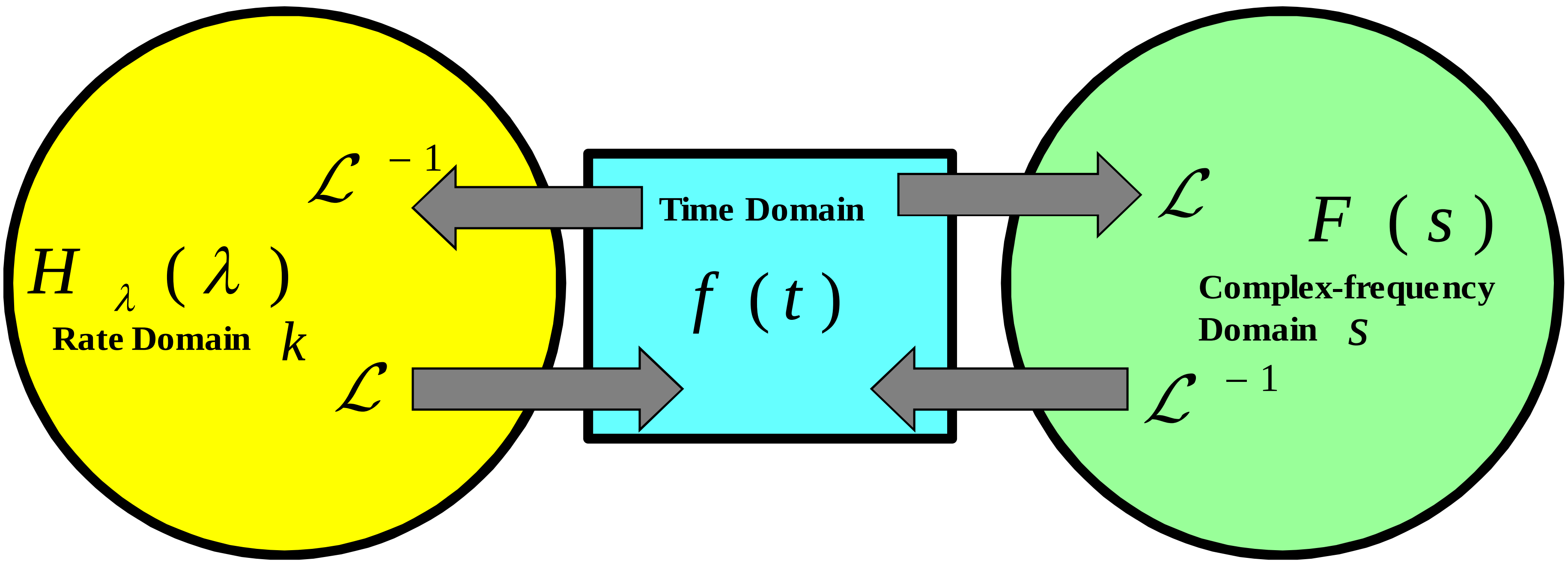
like $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$

Integral transform of Laplace inverse is

$$f(t) = \frac{1}{2\pi i} \int_{x - i\infty}^{x + i\infty} \left(F(s)\right) e^{st} ds \qquad H_\lambda(\lambda) = \frac{1}{2\pi i} \int_{x - i\infty}^{x + i\infty} \left(i(t)\right) e^{t\lambda} dt$$

https://doi.org/10.4236/jmp.2017.812120

Inverse Laplace Transformation of time function to get rate distribution function



$$H_{\lambda}(\lambda) = \mathcal{L}^{-1}\{f(t)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(t)e^{t\lambda}dt$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$$

$$f(t) = \mathcal{L}\{H_{\lambda}(\lambda)\} = \int_0^{\infty} H_{\lambda}(\lambda)e^{-t\lambda}d\lambda$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt$$

Derivation of rate distribution function for Curie-von Schweidler relaxation

For the Curie-von Schweidler relaxation of type i.e. $i(t) \sim t^{-n}$ ***then rate distribution function is*** $H_\lambda(\lambda) = \mathcal{L}^{-1}\left\{t^{-n}\right\}$ ***. With the known Laplace pair i.e.*** $\mathcal{L}^{-1}\left\{s^{-(\alpha+1)}\right\} = \frac{1}{\alpha!}t^{\alpha}$ ***we can write the following***

$$H_\lambda(\lambda) = \mathcal{L}^{-1}\left\{t^{-n}\right\}$$

$$= \frac{1}{(n-1)!}\lambda^{(n-1)} = \frac{1}{m!}\lambda^{m}; \quad m = n-1; \quad \Gamma(\alpha) = (\alpha-1)!, \quad \alpha \in \mathbb{R}$$

$$= \frac{1}{\Gamma(n)}\lambda^{n-1} = \frac{1}{\Gamma(m+1)}\lambda^{m}; \quad \lambda > 0$$

Therefore above discussion suggests that for a power law type relaxation, i.e. Curie-von Schweidler law $i(t) \propto t^{-n}; \quad 0 < n < 1$ ***the relaxation rates are also having a power law distribution of type i.e.***

$$H_\lambda(\lambda) \sim \lambda^{m}, \quad m = n-1, \quad -1 < m < 0, \quad \lambda > 0$$

This is Zipf's power law with $m < 0$

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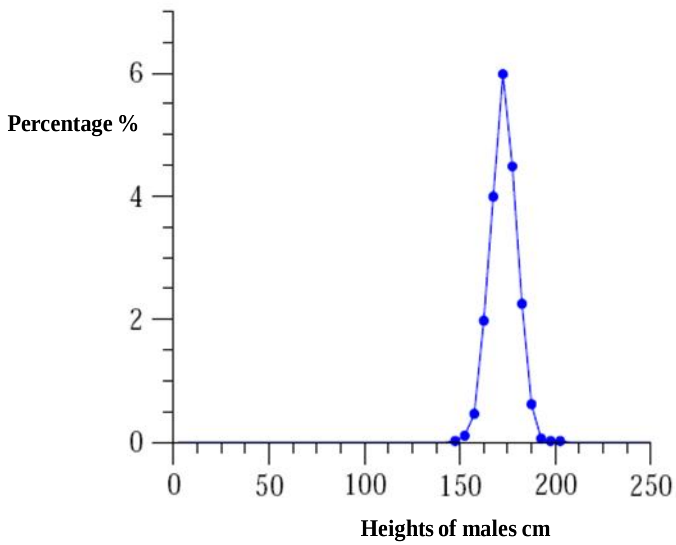
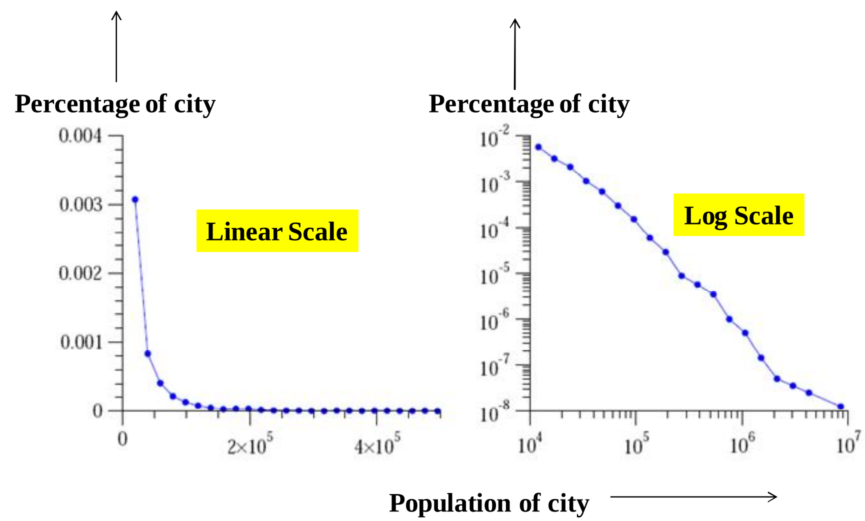
Normal Distribution and Power Law Distribution



Histogram of height of males



Histogram of population of all cities with population of 10,000 or more in linear and log scale

Qualitative analysis Normal Distribution and Power Law Distribution

The histogram of cities & population is highly 'right-skewed', meaning that while the bulk of distribution occurs for fairly small sizes-i.e. most cities have small population-there is small number of cities with population much higher than a said typical value, producing the long tail to the right of histogram. This right skewed form is qualitatively quite different from histogram of person's height. That is because we know that there is large dynamic range from smallest to largest city sizes, we can immediately infer that there can only be a small number of very large cities. For example the ratio of population of largest town to population of smallest town is about 2, 50,000. For example as per Guinness book of records tallest person was having height 272cm and shortest person was having the height of 57cm, making this ratio 4.8. This ratio is relatively low value. We see the most adults are about 180cm tall-there is some variations around this figure notably depending on sex, but we never measure persons having height of 10cm or 500cm.

Zipf's power law distribution for relaxation rates of a relaxing current of fractional capacitors – a highly right skewed with long tail

For dielectric relaxation as observed that $0 < n < 1$ in Curie-von Schweidler relaxation $i(t) \sim t^{-n}$ the rate relaxation distribution function $H_\lambda(\lambda) \sim \lambda^m$ has exponent in power $-1 < m < 0$

Considering graph of $H_\lambda(\lambda) = \lambda^m ; \quad m < 0$ as histogram, we infer that for Curie-von Schweidler relaxation function i.e. $i(t) \sim t^{-n}$ there are very large number of simultaneous relaxations with small λ i.e. large number of slower decay takes place, compared to fewer faster decay rates-and the histogram is highly right skewed with long tail.

Same as with power law distribution is large dynamic range from smallest rate of relaxation to largest rate of relaxation

Zipfian power law distribution due to connected exponential processes-a hypothesis

In this complex relaxation mechanism i.e. $i(t) \propto t^{-n}$ ***that we are discussing simultaneously, in different time scales*** T ***. We consider that a complex relaxation mechanism and a quantity*** T ***say survival time of a relaxing body, has exponential distribution of probability*** $p(\mathrm{T}) \sim \mathrm{e}^{-a\mathrm{T}}$

This means that a probability for a body having very large survival time (age) is very low; and vice-versa. Then $p(\mathrm{T})\,\mathrm{d}\mathrm{T}$ ***indicates the fraction of survival numbers of bodies between survival time*** T ***and*** $\mathrm{T}+\mathrm{dT}$

Now suppose that the real quantity we are interested is not T ***but other quantity*** λ ***say the relaxation rate of discharge which is exponentially related to*** T ***; thus*** $\lambda \sim \mathrm{e}^{-b\mathrm{T}}$ ***. That implies the surviving bodies with very large time of survival (age) have a very low rate of relaxation. This also states that*** $\mathrm{d}\lambda = -\mathrm{dT}$ ***. Then if probability distribution of*** λ ***is*** $p(\lambda)$ ***then we have*** $p(\lambda)\mathrm{d}\lambda = -p(\mathrm{T})\,\mathrm{d}\mathrm{T}$ ***(statement about conservation of probability)***

The negative sign indicates opposite movement, as T ***is increased from*** T ***to*** $\mathrm{T}+\mathrm{dT}$ ***then*** λ ***is decreased from*** λ ***to*** $\lambda+(-\mathrm{d}\lambda)$

Applying conservation of probability to connected exponential processes giving Zipfian distribution -the derivation

This means that number of simultaneously discharging units having relaxation rates between λ ***and*** $\lambda + \mathrm{d}\lambda$ ***is equal to number of surviving bodies having survival time between*** T ***and*** $\mathrm{T} + \mathrm{d}\mathrm{T}$

Thus, we write following derivation

$$p(\lambda) = -p(\mathrm{T})\frac{\mathrm{dT}}{\mathrm{d}\lambda} = -\frac{p(\mathrm{T})}{\left(\frac{\mathrm{d}\lambda}{\mathrm{dT}}\right)} \sim \frac{\mathrm{e}^{-a\mathrm{T}}}{b\mathrm{e}^{-b\mathrm{T}}} = \frac{\mathrm{e}^{\frac{-a}{-b}(-b\mathrm{T})}\mathrm{e}^{b\mathrm{T}}}{b}$$

$$= \frac{\lambda^{\frac{a}{b}}\lambda^{-1}}{b} \sim \lambda^{-\left(1-\frac{a}{b}\right)}$$

$$\sim \lambda^{-m}; \qquad m = 1 - \frac{a}{b}$$

This could be one explanation in physical sense, in line with exponential distribution in the Boltzmann distribution of energies in statistical mechanics

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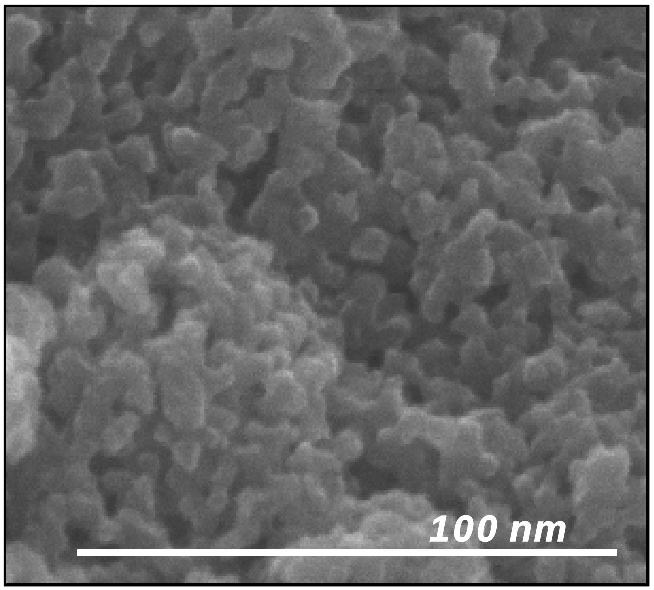
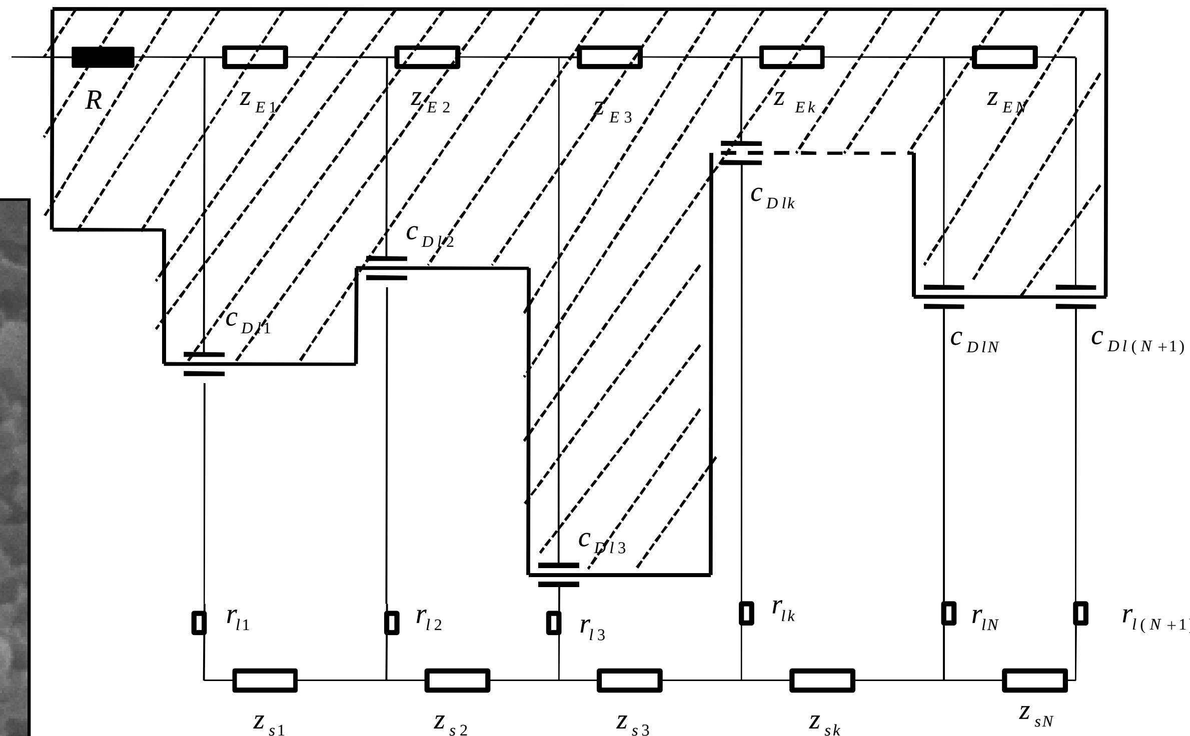
Cause of several simultaneous relaxations with several relaxation rates giving Zipfian histogram

There are several relaxation rates simultaneously from h^{-1} ***to*** ps^{-1}





The SEM image of super-capacitor electrode showing roughness & porous nature (Courtesy CMET Govt. of India Thrissur, Kerala and possible equivalent circuit of half cell

http://shantanudaslecture.com/wp-content/uploads/2017/09/Lecture47.pdf

A simple case of uncharged capacitor getting charged

Circuit equation is $\left(\frac{1}{C}\int_0^t i(x)\,\mathrm{d}x\right) + R\,i(t) = v(t)$

The above integral equation may be differentiated and is put as

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \lambda_0 i(t) = \left(\frac{1}{R}\right)\frac{\mathrm{d}v(t)}{\mathrm{d}t} \qquad \lambda_0 = (RC)^{-1} \qquad \tau_0 = RC$$

$$i^{(1)}(t) + \lambda_0 i(t) = f(t); \qquad f(t) = \tfrac{1}{R}v^{(1)}(t)$$

$$v(t) = V_{BB}u(t)$$

$$v^{(1)}(t) = V_{BB}\frac{\mathrm{d}\big(u(t)\big)}{\mathrm{d}t} = V_{BB}\big(\delta(t)\big)$$

We write in following way

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \lambda_0 i(t) = \frac{V_{BB}}{R}\delta(t) \qquad \lambda_0 = (RC)^{-1} \qquad \tau_0 = RC$$

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \lambda_0 i(t) = I_0\delta(t) \qquad I_0 = \frac{V_{BB}}{R}; \qquad i(t) = I_0 \mathrm{e}^{-\lambda_0 t}$$

The solution to the above equation gives Debye relaxation function i.e.

$$i(t) = \mathrm{e}^{-\lambda_0 t}; \quad I_0 = 1$$

The rate histogram is $H_\lambda(\lambda) = \delta(\lambda - \lambda_0); \quad I_0 = 1$

Charging of capacitor with scale dependent relaxation rate

There are several relaxation rates simultaneously from $\lambda \equiv \mathrm{h}^{-1}$ ***to*** $(\mathrm{ps})^{-1}$

We modify the capacitor discharge current equation, with $\lambda_{eq} = \lambda = \lambda_0$

i.e. with one relaxation rate at any scale of relaxation λ

$$\frac{\mathrm{d}\,i(t)}{\mathrm{d}\,t} + \lambda_0 i(t) = \delta(t)$$

to following i.e. variable $\lambda_{eq} = \lambda^{1/\alpha}$ ***at any scale of relaxation rate*** λ

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \left(\lambda_{eq}\right) i(t) = \delta(t) \qquad \frac{\mathrm{d}i(t)}{\mathrm{d}t} + (\lambda)^{1/\alpha} i(t) = \delta(t)$$

The above equation is having a free 'scale' parameter λ ***varying from***

zero to infinity. The solution of the above $i(t) = \mathrm{e}^{-\lambda_{eq} t}$ ***. We call this***

$i(t) = \exp(-\lambda_{eq} t) = \exp\left(-\lambda^{1/\alpha} t\right)$ ***as 'impulse response function' at a particular***

scale λ ***i.e.*** $i(t) = h(\lambda, t) = \mathrm{e}^{-\left(\lambda^{(1/\alpha)} t\right)}; \qquad 0 < \alpha < 1$

Solution to the scale dependent charging equation , the green's function is

$$g(t) = \int_0^\infty h(\lambda, t)\,\mathrm{d}\lambda = \int_0^\infty \mathrm{e}^{-\left(\lambda^{(1/\alpha)} t\right)}\mathrm{d}\lambda = \frac{\Gamma(1+\alpha)}{t^\alpha}$$

The proof of green function as inverse power law in time

$$g(t) = \int_0^\infty \mathrm{e}^{-\left(\lambda^{(1/\alpha)}t\right)}\mathrm{d}\lambda$$

substitute $\lambda^{1/\alpha}t = x$

$$\lambda = \left(\tfrac{x}{t}\right)^\alpha; \qquad \left(\tfrac{x}{t}\right) = \lambda^{1/\alpha}$$

$$\mathrm{d}\lambda = \alpha x^{\alpha-1}\left(\tfrac{1}{t}\right)^\alpha \mathrm{d}x$$

$$= \left(\tfrac{\alpha}{t}\right)\left(\tfrac{x}{t}\right)^{\alpha-1}\mathrm{d}x = \left(\tfrac{\alpha}{t}\right)\left(\lambda^{1/\alpha}\right)^{\alpha-1}\mathrm{d}x$$

Then by using definition of Gamma function i.e. $\Gamma(\upsilon) = \int_0^\infty e^{-y}y^{\upsilon-1}\mathrm{d}y$ ***and its property***

$\upsilon\left(\Gamma(\upsilon)\right) = \Gamma(1+\upsilon)$ ***we have***

$$g(t) = \int_0^\infty \mathrm{e}^{-\left(\lambda^{(1/\alpha)}t\right)}\mathrm{d}\lambda = \int_0^\infty \mathrm{e}^{-x}\lambda^{1-(1/\alpha)}\left(\tfrac{\alpha}{t}\right)\mathrm{d}x$$

$$= \int_0^\infty \mathrm{e}^{-x}\left(\tfrac{\alpha}{t}\right)\lambda\,\lambda^{-(1/\alpha)}\mathrm{d}x, \qquad \lambda = \left(\tfrac{x}{t}\right)^\alpha$$

$$= \int_0^\infty \mathrm{e}^{-x}\left(\tfrac{\alpha}{t}\right)\left(\tfrac{\alpha}{t}\right)^\alpha\left(\tfrac{x}{t}\right)^{-1}\mathrm{d}x$$

$$= \left(\tfrac{\alpha}{t}\right)\int_0^\infty \mathrm{e}^{-x}\left(\tfrac{x}{t}\right)^\alpha\left(\tfrac{x}{t}\right)^{-1}\mathrm{d}x$$

$$= \left(\tfrac{\alpha}{t}\right)\int_0^\infty \mathrm{e}^{-x}\left(\frac{x^{\alpha-1}}{t^{\alpha-1}}\right)\mathrm{d}x = \left(\frac{\alpha}{t^\alpha}\right)\int_0^\infty \mathrm{e}^{-x}x^{\alpha-1}\mathrm{d}x$$

$$= \frac{\alpha\left(\Gamma(\alpha)\right)}{t^\alpha} = \frac{\Gamma(1+\alpha)}{t^\alpha}$$

Universal dielectric relaxation is scale dependent relaxation

From our solution of green's function with scale dependence rate we write

$$i(t) = t^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} \mathrm{e}^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda \qquad \text{and} \qquad i(t) = \mathrm{e}^{-\lambda_0 t} \qquad \text{for} \quad n = 1$$

For $n = 1$ ***case we have scale invariance*** λ ***thus*** $i(t) = h(\lambda_0, t) = \exp\left(-\lambda_0 t\right)$

where $\lambda_{eq} = \lambda_0$ ***at all scales for*** $\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \left(\lambda_{eq}\right)i(t) = \delta(t)$

We find that $i(t) \sim t^{-n}; \quad 0 < n < 1$ ***for a system where the equivalent relaxation rate is*** $\lambda_{eq} = \lambda^{1/n}$ ***similar to a distribution function that we obtained as*** $H_{\lambda}(\lambda) \sim \lambda^{n-1}$ ***(Zipf's distribution) gives current as*** $i(t) \sim t^{-n}$

We write the two currents expressions for $i(t) \sim t^{-n}; \quad 0 < n < 1$

$$t^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty} \lambda^{n-1}\mathrm{e}^{-t\lambda}\mathrm{d}\lambda \qquad\qquad t^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} \mathrm{e}^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda$$

Therefore we infer that the Curie-von Schweidler relaxation current for dielectric excited by a step voltage that follows the relation has distribution function $H_{\lambda}(\lambda) \sim \lambda^{n-1}$ ***with scale variable relaxation rate*** $\lambda_{eq} = \lambda^{1/n}$

…another interpretation of Curie-von Schweidler law

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Appearance of fractional derivative-in the system having Zipfian power law distribution in relaxation rates, where the equivalent relaxation rate is scale dependent

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + (\lambda)^{1/n} i(t) = \delta(t); \quad 0 < n < 1; \qquad i(t) = g(t) = \frac{\Gamma(1+n)}{t^n}$$

Now let the system described above be excited by a signal proportional to $f(t) \sim v^{(1)}(t)$ ***so we write this as following***

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + (\lambda)^{1/n} i(t) = v^{(1)}(t)$$

Note that if $v(t) = u(t)$ ***that is unit-step-function then*** $v^{(1)}(t) = \delta(t)$ ***we recover the above homogeneous differential equation.***

Then the response to this new excitation function $v^{(1)}(t)$

$$i(t) = \big(g(t)\big) * \big(f(t)\big) = \big(g(t)\big) * \big(v^{(1)}(t)\big)$$

$$= \int_0^t \big(g(t-x)\big)\big(v^{(1)}(x)\big)\mathrm{d}x; \qquad g(t) = \frac{\Gamma(1+n)}{t^n}$$

$$= \Gamma(1+n)\int_0^t \frac{v^{(1)}(x)}{(t-x)^n}\mathrm{d}x \qquad 0 < n < 1$$

Contd…

Contd…

$$i(t) = \Gamma(1+n)\int_0^t (t-x)^{-n} v^{(1)}(x)\mathrm{d}x$$

$$= \left(\Gamma(1+n)\right)\left(\Gamma(1-n)\right)\int_0^t \frac{(t-x)^{-n}}{\Gamma(1-n)} v^{(1)}(x)\mathrm{d}x$$

$${}_0\mathcal{I}_t^{\upsilon}\left(f(t)\right) = \int_0^t \frac{1}{\Gamma(\upsilon)}(t-x)^{\upsilon-1} f(x)\mathrm{d}x \qquad \upsilon > 0$$

$$i(t) = \Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[v^{(1)}(t)\right]\right); \qquad (1-n) > 0$$

$$= \Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{-(1-n)}\left[v^{(1)}(t)\right]\right)$$

$$= \Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\, {}_0D_t^{-1}\left[v^{(1)}(t)\right]\right)$$

$$= \Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\left[v(t)\right]\right), \qquad n < 1$$

This derivation implies the appearance of fractional derivative for cases where several relaxation rates (ideally infinite of them) define a relaxation process; which are having a scale dependence behavior, with histogram distributed as Zipf's power law , and the relaxation is by Curie-von Schweidler law.

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Inverse Laplace Transform by Analytical Approach

$$H_\lambda(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} \left(i(t)\right) \mathrm{e}^{t\lambda} \mathrm{d}t \qquad H_\lambda(\lambda) = \frac{1}{\Gamma(n)} \lambda^{n-1} \quad \text{for} \quad i(t) = t^{-n}$$

$t = x_0 + \mathrm{i}y$ ***as the decay function is not expected to have singularity at time*** $t > 0$ ***.Choosing*** $x_0 = 0$ ***we have for*** $i(t) = t^{-n}$

$$i(\mathrm{i}y) = \frac{1}{(\mathrm{i}y)^n} = \frac{y^{-n}}{\left(\cos\left(\frac{n\pi}{2}\right) + \mathrm{i}\sin\left(\frac{n\pi}{2}\right)\right)}; \quad \mathrm{i}^n = \mathrm{e}^{(\mathrm{i}n\pi/2)} = \cos\left(\frac{n\pi}{2}\right) + \mathrm{i}\sin\left(\frac{n\pi}{2}\right)$$

$$= y^{-n}\cos\left(\frac{n\pi}{2}\right) - \mathrm{i}y^{-n}\sin\left(\frac{n\pi}{2}\right)$$

$$\mathrm{Re}\left\{i(\mathrm{i}y)\right\} = y^{-n}\cos\left(\frac{n\pi}{2}\right); \quad \mathrm{Im}\left\{i(\mathrm{i}y)\right\} = -y^{-n}\sin\left(\frac{n\pi}{2}\right)$$

Now using the Berberan–Snatos formula we get following steps

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{\pi}\left(\int_0^\infty \left(\mathrm{Re}\left\{i(x_0+\mathrm{i}y)\right\}\right)\cos(\lambda y) - \mathrm{Im}\left\{i(x_0+\mathrm{i}y)\right\}\sin(\lambda y)\right)\mathrm{d}y; \qquad x_0 = 0$$

$$= \frac{1}{\pi}\int_0^\infty \left(\left(y^{-n}\cos\left(\frac{n\pi}{2}\right)\right)\cos(\lambda y) + \left(y^{-n}\sin\left(\frac{n\pi}{2}\right)\right)\sin(\lambda y)\right)\mathrm{d}y; \quad u = y, \quad \mathrm{d}u = \mathrm{d}y$$

$$= \frac{1}{\pi}\int_0^\infty u^{-n}\cos\left(\lambda u - \frac{n\pi}{2}\right)(\mathrm{d}u)$$

We can write another integral expression for t^{-n}

$$t^{-n} = \frac{\Gamma(1-n)}{\pi}\int_0^\infty \lambda^{(n-1)}\cos\left(\lambda t - \left(\frac{(1-n)\pi}{2}\right)\right)(\mathrm{d}\lambda)$$

Berberan-Santos, M.N. (2005) Journal of Mathematical Chemistry , 38, No. 2.

Three integral representations for Curie-von Schweidler law

Lastly

$$t^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \lambda^{n-1} \mathrm{e}^{-t\lambda} \mathrm{d}\lambda$$

From Laplace Table

$$t^{-n} = \frac{1}{n\left(\Gamma(n)\right)} \int_0^\infty \mathrm{e}^{-\left(\lambda^{(1/n)} t\right)} \mathrm{d}\lambda$$

From scale variance relaxation

$$t^{-n} = \frac{\Gamma(1-n)}{\pi} \int_0^\infty \lambda^{(n-1)} \cos\left(\lambda t - \left(\tfrac{(1-n)\pi}{2}\right)\right)\left(\mathrm{d}\lambda\right)$$

From Berberan –Santo method

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Thank you

for

giving opportunity

to

present this new thinking