Inverse Laplace Transformation by Analytical method without contour integration

Shantanu Das

Tutorial CMPRC Department of Physics Jadavpur University (2013)

Reactor Control Division B.A.R.C
Mumbai-400085
Email: shantanu@barc.gov.in

The Laplace transform technique is a very important aspect in solution of differential equations. What is tough is the taking inverse Laplace transform. It is fine if tables give the result, if not we resort to classical technique of contour integration. Sometimes this contour integration is tough to do so. Here we describe the method of Berberan-Santos where the analytical method is obtained to find out inverse Laplace transformation, without going for contour integration method. The result is placed in integral representation, the closed form of which is difficult to obtain, however can be plotted via numerical integration.

The Laplace integral

The Laplace transform $G(s)$ of a function in time domain $g(t)$ is defined as following integral transform relation i.e. called Laplace integral

$$\begin{aligned} G(s) &\overset{\text{def}}{=} \int_0^\infty \big(g(t)\big)e^{-st}\mathrm{d}t \\ G(s) &= \mathcal{L}\{g(t)\} \end{aligned} \tag{1}$$

This is standard integral transform of a function $g(t)$ from a time domain (t) to a complex frequency domain i.e. $s = \sigma + \mathrm{i}\omega$; $\mathrm{i} = \sqrt{-1}$; where real part is significant in the transient response and the imaginary part of the frequency corresponds to 'steady-state' response; in classical 'Control Science'. Here $g(t)$ is 'inverse Laplace transform' of $G(s)$, and we write $\mathcal{L}^{-1}\{G(s)\} = g(t)$ and $\mathcal{L}\{g(t)\} = G(s)$ We need to find $g(t) = \mathcal{L}^{-1}\{G(s)\}$, that is following

$$g(t) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \big(G(s)\big)e^{st}\mathrm{d}s \tag{2}$$

Usually the solution of (2) is obtained via contour integration. Here we do analytical technique to find (2) with out using the usual contour integration method.

Analytical inversion of Laplace Transform- the Berberan-Santos method

We describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. Here the real part i.e. σ is constant as a vertical line calls it $\sigma = \sigma_0$ a constant. The σ_0 is a real number being such that $G(s)$ has some form of singularity on the line $\mathrm{Re}\{s\} = \sigma_0$ but analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\{s\} > \sigma_0$. The equation (2) is usually evaluated via contour integration. Performing the variable change on (2) to $s = \sigma_0 + \mathrm{i}\omega$ we get following steps

$$\begin{aligned} g(t) &= \frac{1}{2\pi\mathrm{i}} \int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty} \big(G(s)\big)e^{st}\mathrm{d}s; \qquad s \equiv \sigma_0 + \mathrm{i}\omega \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty} \big(G(\sigma_0+\mathrm{i}\omega)\big)\big(e^{(\sigma_0+\mathrm{i}\omega)t}\big)\big(\mathrm{d}(\sigma_0+\mathrm{i}\omega)\big); \qquad \mathrm{d}\sigma_0 = 0 \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\sigma_0-\mathrm{i}\infty}^{\sigma_0+\mathrm{i}\infty} \big(G(\sigma_0+\mathrm{i}\omega)\big)\big(e^{t\sigma_0}e^{\mathrm{i}t\omega}\big)\big(\mathrm{i}\mathrm{d}\omega\big) \\ &= \frac{e^{\sigma_0 t}}{2\pi} \int_{-\infty}^{+\infty} \big(G(\sigma_0+\mathrm{i}\omega)\big)e^{\mathrm{i}\omega t}\mathrm{d}\omega \end{aligned} \tag{3}$$

Writing $e^{\mathrm{i}t\omega} = \cos\omega t + \mathrm{i}\sin\omega t$ we get the following form

$$g(t)=\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(G(\sigma_0+\mathrm{i}\omega)\right)\cos(\omega t)\mathrm{d}\omega+\mathrm{i}\int_{-\infty}^{+\infty}\left(G(\sigma_0+\mathrm{i}\omega)\right)\sin(\omega t)\mathrm{d}\omega\right) \tag{4}$$

Write

$$G(\sigma_0+\mathrm{i}\omega)=\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}+\mathrm{i}\,\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\} \tag{5}$$

and place in above expression (4) to get the following expression

$$\begin{aligned} g(t)=&\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\cos(\omega t)\right)\right)-\left(\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right)\\ &+\mathrm{i}\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\cos(\omega t)\right)\right)+\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right) \end{aligned} \tag{6}$$

Given that $g(t)$ is a real function, we get the following (i.e. equating the imaginary part to zero)

$$\left(\int_{-\infty}^{+\infty}\left(\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\cos(\omega t)\right)+\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right)=0 \tag{7}$$

Thus the above expression for $g(t)$ reduces to following (i.e. considering only real part) we get

$$g(t)=\frac{e^{\sigma_0 t}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega\right) \tag{8}$$

But we have from (1) $G(s)=\int_0^{\infty}\left(g(t)\right)e^{-st}\mathrm{d}t$ and by putting $s=\sigma_0+\mathrm{i}\omega$ we get following

$$\begin{aligned} G(\sigma_0+\mathrm{i}\omega)&=\int_0^{\infty}\left(g(t)\right)e^{-t(\sigma_0+\mathrm{i}\omega)}\mathrm{d}t\\ &=\int_0^{\infty}e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t-\mathrm{i}\int_0^{\infty}e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t \end{aligned} \tag{9}$$

This gives following

$$\begin{aligned} \mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}&=\int_0^{\infty}e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t\\ \mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}&=-\int_0^{\infty}e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t \end{aligned} \tag{10}$$

Using this (10) in obtained expression for $g(t)$ in (8) we observe that integrand of (8) i.e.

$$f_1(t) = \left(\mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\cos(\omega t) - \mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\sin(\omega t)\right)$$

$$f_1(t) = \left(\int_0^{\infty} e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t\right)\cos(\omega t) + \left(\int_0^{\infty} e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t\right)\sin(\omega t)$$

$$f_1(-t) = \left(\int_0^{-\infty} e^{-\sigma_0(-t)}\left(g(-t)\right)\cos(\omega(-t))\mathrm{d}(-t)\right)\cos(\omega(-t))$$

$$+\left(\int_0^{-\infty} e^{-\sigma_0(-t)}\left(g(-t)\right)\sin(\omega(-t))\mathrm{d}(-t)\right)\sin(\omega(-t)) \qquad (11)$$

Form (11) we see if we do $t \equiv -t$, the integrand $f_1(t) = f_1(-t)$ indicates that integrand of (8) is even parity. Therefore we re-write the formula (8) as following that is taking only one sided limit of integration that is from 0 to ∞.

$$g(t) = \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\cos(\omega t) - \mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \qquad (12)$$

We have integrand of (7) as following

$$f_2(t) = \left(\mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\cos(\omega t)\right) + \mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right) \qquad (13)$$

Using (10) we see the following steps

$$f_2(t) = \left(\mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\cos(\omega t)\right) + \mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right)$$

$$f_2(t) = \left(-\int_0^{\infty} e^{-\sigma_0 t}\left(g(t)\right)\sin(\omega t)\mathrm{d}t\right)\cos(\omega t) + \left(\int_0^{\infty} e^{-\sigma_0 t}\left(g(t)\right)\cos(\omega t)\mathrm{d}t\right)\sin(\omega t)$$

$$f_2(-t) = \left(-\int_0^{-\infty} e^{-\sigma_0(-t)}\left(g(-t)\right)\sin(\omega(-t))\mathrm{d}(-t)\right)\cos(\omega(-t))$$

$$+\left(\int_0^{-\infty} e^{-\sigma_0(-t)}\left(g(-t)\right)\cos(\omega(-t))\mathrm{d}(-t)\right)\sin(\omega(-t)) \qquad (14)$$

We observe the integrand of (7) is also even parity. Thus we get for (7) the following i.e. one sided limit of integration

$$\left(\int_0^{\infty}\left(\mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\cos(\omega t)\right) + \mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\left(\sin(\omega t)\right)\right)\mathrm{d}\omega\right) = 0 \qquad (15)$$

We will use formula (12) in following discussion and use to solve several cases. Write (12) in polar form as described below

$$G(\sigma_0 + \mathrm{i}\omega) = \rho(\omega)e^{\mathrm{i}\theta(\omega)} = \rho(\omega)\Big(\cos\big(\theta(\omega)\big) + \mathrm{i}\sin\big(\theta(\omega)\big)\Big)$$
$$\rho(\omega) = \left|G(\sigma_0 + \mathrm{i}\omega)\right| \qquad \theta(\omega) = \angle G(\sigma_0 + \mathrm{i}\omega) \tag{16}$$

to get following formulas

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \Big(\mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\cos(\omega t) - \mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \Big(\rho(\omega)\big(\cos\theta(\omega)\big)\cos(\omega t) - \rho(\omega)\big(\sin\theta(\omega)\big)\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \big(\rho(\omega)\big)\Big(\cos\big(\omega t + \theta(\omega)\big)\Big)\mathrm{d}\omega \end{aligned} \tag{17}$$

<u>Examples of Laplace inversion for known Laplace pairs</u>

Example-1

Take

$$G(s) = \frac{1}{s-a} \tag{18}$$

Put $s = \sigma_0 + \mathrm{i}\omega$ to, with $\sigma_0 > a$ write

$$G(\sigma_0 + \mathrm{i}\omega) = \frac{1}{(\sigma_0 - a) + \mathrm{i}\omega} = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2} - \mathrm{i}\frac{\omega}{(\sigma_0 - a)^2 + \omega^2}$$
$$\mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\} = \frac{\sigma_0 - a}{(\sigma_0 - a)^2 + \omega^2}; \qquad \mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\} = -\frac{\omega}{(\sigma_0 - a)^2 + \omega^2} \tag{19}$$

Applying (12) we get following

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \Big(\mathrm{Re}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\cos(\omega t) - \mathrm{Im}\left\{G(\sigma_0 + \mathrm{i}\omega)\right\}\sin(\omega t)\Big)\mathrm{d}\omega \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega\,\mathrm{t})\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}(\sigma_0 - a)}{\pi}\int_0^\infty \frac{\cos(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} + \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{\omega\sin(\omega t)\mathrm{d}\omega}{(\sigma_0 - a)^2 + \omega^2} \\ &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \frac{(\sigma_0 - a)\cos(\omega t) + \omega\sin(\omega t)}{(\sigma_0 - a)^2 + \omega^2}\mathrm{d}\omega \end{aligned} \tag{20}$$

From Laplace tables we know that

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at} \tag{21}$$

So we have integral representation of e^{at} as following

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\frac{(\sigma_0-a)\cos(\omega t)+\omega\sin(\omega t)}{(\sigma_0-a)^2+\omega^2}\mathrm{d}\omega=e^{at} \tag{22}$$

For $a=-b,\quad b>0$, we choose $\sigma_0=0>-b$, to write following

$$\begin{aligned}&\mathcal{L}^{-1}\left\{\frac{1}{s+b}\right\}=e^{-bt}\\&\frac{1}{\pi}\int_0^{\infty}\frac{b\cos(\omega t)+\omega\sin(\omega t)}{b^2+\omega^2}\mathrm{d}\omega=e^{-bt}\end{aligned} \tag{23}$$

In polar form we have

$$\begin{aligned}&G(\sigma_0+\mathrm{i}\omega)=\frac{\sigma_0-a}{(\sigma_0-a)^2+\omega^2}-\mathrm{i}\frac{\omega}{(\sigma_0-a)^2+\omega^2}\\&\theta(\omega)=\angle G(\sigma_0+\mathrm{i}\omega)=\tan^{-1}\left(-\tfrac{\omega}{(\sigma_0-a)}\right);\qquad \rho(\omega)=\left|G(\sigma_0+\mathrm{i}\omega)\right|=\tfrac{1}{\sqrt{(\sigma_0-a)^2+\omega^2}}\end{aligned} \tag{24}$$

We apply (17) to get following

$$\begin{aligned}g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\rho(\omega)\right)\left(\cos\left(\omega t+\theta(\omega)\right)\right)\mathrm{d}\omega\\&=\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\tfrac{1}{\sqrt{(\sigma_0-a)^2+\omega^2}}\right)\cos\left(\omega t+\tan^{-1}\left(-\tfrac{\omega}{(\sigma_0-a)}\right)\right)\mathrm{d}\omega\end{aligned} \tag{25}$$

Thus we have another formulation for e^{at} as

$$\frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\tfrac{1}{\sqrt{(\sigma_0-a)^2+\omega^2}}\right)\cos\left(\omega t+\tan^{-1}\left(-\tfrac{\omega}{(\sigma_0-a)}\right)\right)\mathrm{d}\omega=e^{at} \tag{26}$$

For a case

$$G(s)=\frac{1}{s+1} \tag{27}$$

The above expression has singularity at $s=-1$ .We choose $\sigma_0=0>-1$ we write the following

$$\frac{1}{\pi}\int_0^\infty \frac{\cos\omega t+\omega\sin\omega t}{1+\omega^2}\mathrm{d}\omega = e^{-t}$$

$$\frac{1}{\pi}\int_0^\infty \left(\tfrac{1}{\sqrt{(1+\omega^2}}\right)\cos\left(\omega t+\tan^{-1}(-\omega)\right)\mathrm{d}\omega = e^{at} \tag{28}$$

Example-2

$$G(s)=\frac{s}{s^2+1}, \qquad G(1+\mathrm{i}\omega)=\frac{1+\mathrm{i}\omega}{(1+\mathrm{i}\omega)^2+1} \tag{29}$$

The above expression has singularity at $s=\pm\mathrm{i}$ at line $\mathrm{Re}\{s\}=0$. We choose $\sigma_0=1$, in this case, we get

$$\mathrm{Re}\{G(1+\mathrm{i}\omega)\}=\frac{2+\omega^2}{4+\omega^4} \qquad \mathrm{Im}\{G(1+\mathrm{i}\omega)\}=-\frac{\omega^2}{4+\omega^4} \tag{30}$$

Using (12) we write

$$\begin{aligned}
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\{G(\sigma_0+\mathrm{i}\omega)\}\cos(\omega t)-\mathrm{Im}\{G(\sigma_0+\mathrm{i}\omega)\}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{e^{t}}{\pi}\int_0^\infty \left(\mathrm{Re}\{G(1+\mathrm{i}\omega)\}\cos(\omega t)-\mathrm{Im}\{G(1+\mathrm{i}\omega)\}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{e^{t}}{\pi}\int_0^\infty \left(\frac{2+\omega^2}{4+\omega^4}\cos(\omega t)+\frac{\omega^2}{4+\omega^4}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{e^{t}}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos(\omega t)\mathrm{d}\omega}{4+\omega^4}+\frac{e^{t}}{\pi}\int_0^\infty \frac{\omega^2\sin(\omega t)\mathrm{d}\omega}{4+\omega^4} \\
&= \frac{e^{t}}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega
\end{aligned} \tag{31}$$

The known Laplace transform is $\mathcal{L}^{-1}\left\{s/(s^2+1)\right\}=\cos t$, so we have integration representation of $\cos t$ as

$$\frac{e^{t}}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega=\cos t \tag{32}$$

In polar form we have

$$\rho(\omega)=\tfrac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}; \quad \theta(\omega)=\tan^{-1}\left(-\tfrac{\omega^2}{\omega^2+2}\right) \tag{33}$$

Using (17) we write following

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty(\rho(\omega))\left(\cos\left(\omega t+\theta(\omega)\right)\right)\mathrm{d}\omega$$
$$=\frac{e^{t}}{\pi}\int_0^\infty\left(\frac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}\right)\left(\cos\left(\omega t+\tan^{-1}\left(-\frac{\omega^2}{\omega^2+2}\right)\right)\right)\mathrm{d}\omega \tag{34}$$

We have another representation of $\cos t$ as follows

$$\frac{e^{t}}{\pi}\int_0^\infty\left(\frac{\sqrt{2(\omega^4+\omega^2+2)}}{\omega^4+4}\right)\left(\cos\left(\omega t+\tan^{-1}\left(-\frac{\omega^2}{\omega^2+2}\right)\right)\right)\mathrm{d}\omega=\cos t \tag{35}$$

Example-3

$G(s)=e^{-\lambda_0 s}$ has no singularity at $s>0$ so choose $\sigma_0=0$. Thus we write in complex variable the following

$$G(\sigma_0+\mathrm{i}\omega)=G(\mathrm{i}\omega)=e^{-\mathrm{i}\omega\lambda_0}=\cos(\omega\lambda_0)-\mathrm{i}\sin(\omega\lambda_0) \tag{36}$$

Thus we have

$$\mathrm{Re}\{G(\sigma_0+i\omega)\}=\cos(\omega\lambda_0);\quad \mathrm{Im}\{G(\sigma_0+i\omega)\}=-\sin(\omega\lambda_0) \tag{37}$$

Applying the formula (12) we get

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\{G(\sigma_0+\mathrm{i}\omega)\}\cos(\omega t)-\mathrm{Im}\{G(\sigma_0+\mathrm{i}\omega)\}\sin(\omega t)\right)\mathrm{d}\omega$$
$$=\frac{1}{\pi}\int_0^\infty\left(\cos(\omega\lambda_0)\cos(\omega t)+\sin(\omega\lambda_0)\sin(\omega t)\right)\mathrm{d}\omega$$
$$=\frac{1}{\pi}\int_0^\infty\cos\left(\omega(t-\lambda_0)\right)\mathrm{d}\omega \tag{38}$$

From standard Laplace tables we have $\mathcal{L}^{-1}\left\{e^{-s\lambda_0}\right\}=\delta(t-\lambda_0)$, thus we get integral representation as

$$\frac{1}{\pi}\int_0^\infty\cos\left(\omega(t-\lambda_0)\right)\mathrm{d}\omega=\delta(t-\lambda_0) \tag{39}$$

In polar form we have $\rho(\omega)=1,\quad \theta(\omega)=-\omega\lambda_0$, using (17) we write

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty(\rho(\omega))\left(\cos\left(\omega t+\theta(\omega)\right)\right)\mathrm{d}\omega$$
$$=\frac{1}{\pi}\int_0^\infty\left(\cos\left(\omega t-\omega\lambda_0\right)\right)\mathrm{d}\omega \tag{40}$$

This is same that we obtained via (12).

Example-4

$G(s)=s^{-\alpha}$ this function does not have singularity at $s>0$. Choosing $\sigma_0=0$ we obtain

$$G(\sigma_0+\mathrm{i}\omega)=(\mathrm{i}\,\omega)^{-\alpha}=\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)-\mathrm{i}\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right) \tag{41}$$

Applying the formula (12) we get the following

$$\begin{aligned} g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\ &=\frac{1}{\pi}\int_0^\infty\left(\omega^{-\alpha}\cos\left(\tfrac{\alpha\pi}{2}\right)\cos(\omega t)+\omega^{-\alpha}\sin\left(\tfrac{\alpha\pi}{2}\right)\sin(\omega t)\right)\mathrm{d}\omega \\ &=\frac{1}{\pi}\int_0^\infty\omega^{-\alpha}\cos\left(\omega t-\tfrac{\alpha\pi}{2}\right)\mathrm{d}\omega \end{aligned} \tag{42}$$

The known Laplace transform is $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\}=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$ we write following

$$\frac{1}{\pi}\int_0^\infty\omega^{-\alpha}\cos\left(\omega t-\tfrac{\alpha\pi}{2}\right)\mathrm{d}\omega=\frac{1}{\Gamma(\alpha)}t^{\alpha-1} \tag{43}$$

Putting $(\alpha-1)=-n$ we write integral representation of t^{-n} as following

$$\frac{\Gamma(1-n)}{\pi}\int_0^\infty\omega^{n-1}\cos\left(\omega t-\tfrac{(1-n)\pi}{2}\right)\mathrm{d}\omega=t^{-n} \tag{44}$$

With use of polar form (17) we will get same result.

Example-5

Take $G(s)=1$ choose $\sigma_0=0$, we have for $s=\sigma_0+\mathrm{i}\omega$, $G(\sigma_0+\mathrm{i}\omega)=1+\mathrm{i}(0)$. Using (12) we write

$$\begin{aligned} g(t)&=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\ &=\frac{1}{\pi}\int_0^\infty\left((1)\cos(\omega t)-(0)\sin(\omega t)\right)\mathrm{d}\omega \\ &=\frac{1}{\pi}\int_0^\infty\cos(\omega t)\mathrm{d}\omega \end{aligned} \tag{45}$$

Knowing $\mathcal{L}^{-1}\left\{1\right\}=\delta(t)$ we write integral representation of $\delta(t)$ as

$$\frac{1}{\pi}\int_0^\infty \cos(\omega t)\mathrm{d}\omega = \delta(t) \tag{46}$$

Example-6

We take $G(s)=s^{-1}$ and with $\sigma_0=0$ we write $G(\sigma_0+\mathrm{i}\omega)=(\mathrm{i}\omega)^{-1}=\omega^{-1}\left(0-\mathrm{i}(1)\right)$. We use formula (12) to write following

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^\infty \left(\omega^{-1}(0)\cos(\omega t)+(1)\omega^{-1}\sin(\omega t)\right)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega \end{aligned} \tag{47}$$

We know that $\mathcal{L}^{-1}\left\{s^{-1}\right\}=u(t)$ i.e. Heaviside unit step function at $t=0$, so we write integral representation of $u(t)$ as

$$\begin{aligned} \frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega &= u(t)=1, \quad t\ge 0 \\ \frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega &= u(t)=0, \quad t<0 \end{aligned} \tag{48}$$

Laplace inversion of some functions that are not listed in standard tables

Example-7

For $G(s)=e^{-(s/s_0)^\beta}$, in complex variable we get with $s=\sigma_0+\mathrm{i}\omega$ with $\sigma_0=0$

$$\begin{aligned} G(\mathrm{i}\omega) &= e^{-(\mathrm{i}\omega/s_0)^\beta} = e^{-(\omega/s_0)^\beta(\mathrm{i})^\beta} \\ &= e^{-(\omega/s_0)^\beta\left[\cos\left(\frac{\beta\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\beta\pi}{2}\right)\right]} \\ &= e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta\cos\left(\frac{\beta\pi}{2}\right)\right]}e^{\left[-\mathrm{i}\left(\frac{\omega}{s_0}\right)^\beta\sin\left(\frac{\beta\pi}{2}\right)\right]} \\ &= \left(\rho(\omega)\right)\left(e^{\mathrm{i}(\theta(\omega))}\right) \end{aligned} \tag{49}$$

Where in this case we have in polar form representation for use of (17)

$$\left|G(\sigma_0+\mathrm{i}\omega)\right| = \rho(\omega) = e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta\cos\left(\frac{\beta\pi}{2}\right)\right]} \qquad \angle G(\sigma_0+\mathrm{i}\omega)=\theta(\omega)=-\left(\frac{\omega}{s_0}\right)^\beta\sin\left(\frac{\beta\pi}{2}\right) \tag{50}$$

Therefore we write inverse Laplace transform $\mathcal{L}^{-1}\left\{e^{-(s/s_0)^\beta}\right\}$ as $g(t)$ in following expression

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\cos\left(\omega t+\left(\theta(\omega)\right)\right)\mathrm{d}\omega$$

$$=\frac{1}{\pi}\int_0^\infty e^{\left[-\left(\frac{\omega}{s_0}\right)^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(\omega t-\left(\frac{\omega}{s_0}\right)^\beta \sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}\omega \qquad (51)$$

Doing change of variable $u=\omega/s_0$ we obtain

$$g(t)=\frac{s_0}{\pi}\int_0^\infty e^{\left[-u^\beta \cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left(ts_0 u-u^\beta \sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u \qquad (53)$$

Example-8

We take

$$G(s)=\frac{1}{\left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{1/(1-\beta)}} \qquad (54)$$

We have following steps to find inverse Laplace transform of above transfer function by putting $s=\sigma_0+\mathrm{i}\omega$ with $\sigma_0=0$

$$G(\mathrm{i}\omega)=\frac{1}{\left(1+(1-\beta)\left(\frac{\mathrm{i}\omega}{s_0}\right)\right)^{1/1-\beta}}$$

$$\left|G(\mathrm{i}\omega)\right|=\rho(\omega)=\left(1+\left(\frac{(1-\beta)\omega}{s_0}\right)^2\right)^{-1/2(1-\beta)};\quad \angle G(\mathrm{i}\omega)=\theta(\omega)=-\frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta} \qquad (55)$$

So we write the inverse Laplace transform i.e. $\mathcal{L}^{-1}\left\{\left(1+(1-\beta)\left(\frac{s}{s_0}\right)\right)^{-1/(1-\beta)}\right\}=g(t)$ as following

$$g(t)=\frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\rho(\omega)\right)\cos\left(\omega t+\left(\theta(\omega)\right)\right)\mathrm{d}\omega;\qquad \sigma_0=0$$

$$=\frac{1}{\pi}\int_0^\infty \left[1+\left(\frac{(1-\beta)\omega}{s_0}\right)^2\right]^{-\frac{1}{2(1-\beta)}}\cos\left(\omega t-\frac{\tan^{-1}\left(\frac{(1-\beta)\omega}{s_0}\right)}{1-\beta}\right)\mathrm{d}\omega \qquad (56)$$

With change of variable $u=\frac{(1-\beta)\omega}{s_0}$, we get following expression

$$g(t)=\frac{s_0}{\pi(1-\beta)}\int_0^\infty \left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{t\tau_0 u-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u;\qquad u=\frac{(1-\beta)\omega}{s_0} \qquad (57)$$

Example-9

We do inverse Laplace transform of simple power law in s domain i.e. $G(s)=\frac{1}{1+(s/a)^{\alpha}}$; $\alpha<1$ to get function $g(t)=\mathcal{L}^{-1}\left\{\left(1+\left(\frac{s}{a}\right)^{\alpha}\right)^{-1}\right\}$, will be expressed via same rule as above .Put $s=0+\mathrm{i}\omega$

$$\begin{aligned} G(\mathrm{i}\omega) &= \frac{1}{1+\left(\frac{\mathrm{i}\omega}{a}\right)^{\alpha}} = \frac{1}{1+\left(\frac{\omega}{a}\right)^{\alpha}(\mathrm{i})^{\alpha}} \\ &= \frac{1}{1+\left(\left(\frac{\omega}{a}\right)^{\alpha} e^{\mathrm{i}\frac{\alpha\pi}{2}}\right)} = \frac{1}{1+\left(\left(\frac{\omega}{a}\right)^{\alpha}\left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)} \\ &= \frac{1}{\left(1+\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+\mathrm{i}\left(\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)} \end{aligned} \tag{58}$$

From above we have following

$$\mathrm{Re}\left[G(\mathrm{i}\omega)\right]=\frac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1} \qquad \mathrm{Im}\left[G(\mathrm{i}\omega)\right]=-\frac{\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1} \tag{59}$$

We do inverse Laplace transform i.e. $g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$ with $G(s)=\frac{1}{1+(s/a)^{\alpha}}$ as follows using the formula (12), as follows

$$\begin{aligned} g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\left(\begin{aligned}&\left(\frac{\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\cos(\omega t) \\ &\quad+\left(\frac{\left(\frac{\omega}{a}\right)^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\sin(\omega t)\end{aligned}\right)\mathrm{d}\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\left(\frac{\left(\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\omega t+\left(\frac{\omega}{a}\right)^{\alpha}\left(\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\omega t}{\left(\frac{\omega}{a}\right)^{2\alpha}+2\left(\frac{\omega}{a}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}\omega; \quad u=\frac{\omega}{a} \\ &= \frac{a}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos aut+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u \end{aligned} \tag{60}$$

Integral representations of Mittag-Leffler function-via inverse Laplace transform

With the listed inverse Laplace transform of two-parameter Mittag-Leffler function defined as $E_{\alpha,\beta}(z)=\sum_{m=0}^{\infty}\frac{(z)^m}{\Gamma(\alpha m+\beta)}$ we have the following

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{1+(s/a)^{\alpha}}\right\} = a\left(at\right)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)\right) \tag{61}$$

Therefore we write using (60)

$$a(at)^{\alpha-1}\left(E_{\alpha,\alpha}\left(-(at)^{\alpha}\right)\right) = \frac{a}{\pi}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(atu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u \qquad u=\frac{\omega}{a}$$

$$E_{\alpha,\alpha}\left(-(at)^{\alpha}\right) = \frac{1}{\pi}(at)^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(atu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin aut}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u; \qquad at=z$$

$$E_{\alpha,\alpha}\left(-z^{\alpha}\right) = \frac{1}{\pi}z^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(zu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin zu}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u \tag{62}$$

We got integral representation of Mittag-Leffler function via the Berberan-Santos method. With $\alpha=1$ and $a=1$ we get $g(t)=\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=E_{1,1}\left(-t\right)=e^{-t}$ is correct one and the integral representation is following

$$E_{\alpha,\alpha}\left(-z^{\alpha}\right) = \frac{1}{\pi}z^{1-\alpha}\int_0^{\infty}\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\left(\cos(zu)\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin zu}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\mathrm{d}u; \quad \alpha=1$$

$$E_{1,1}\left(-z\right) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos(zu)+u\sin zu}{u^2+1}\mathrm{d}u = e^{-t} \tag{63}$$

Earlier in Example-1, we have obtained $\mathcal{L}^{-1}\left\{\frac{1}{1+s}\right\}=\frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega t+\omega\sin\omega t}{1+\omega^2}\mathrm{d}\omega=e^{-t}$.The same we got from Mittag-Leffler function as above (63)

Also from the derivation we can write the following

$$\mathcal{L}^{-1}\left\{\frac{k}{k+s^{\alpha}}\right\} = \frac{\sqrt[\alpha]{k}}{\pi}\int_0^{\infty}\left(\frac{\left(u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u; \quad u=\frac{\omega}{\sqrt[\alpha]{k}} \tag{64}$$

Example-10

We will apply this to known Laplace pair of Mittag-Leffler function $\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\}=s^{\alpha-1}(s^{\alpha}+1)^{-1}$.

$$\mathcal{L}\left\{E_{\alpha}(-t^{\alpha})\right\} = \int_0^{\infty}E_{\alpha}(-t^{\alpha})e^{-st}\mathrm{d}t$$

$$= \frac{s^{\alpha-1}}{1+s^{\alpha}} = \frac{s^{\alpha}}{s(s^{\alpha}+1)} \tag{65}$$

Here now we apply the Berberan-Santo technique on $G(s)=s^{\alpha-1}(s^{\alpha}+1)^{-1}$, to get in integral representation of inverse Laplace transformed result.

In this technique we write $s=\sigma_0+\mathrm{i}\omega$ with $\sigma_0=0$. That is because we do not expect singularity in the right half plane of complex frequency s i.e. $\mathrm{Re}\left[s\right]>0$ for function $G(s)$ for its well meaning behavior. With this substitution we get the following steps

$$
\begin{aligned}
G(s) &= \frac{s^{\alpha-1}}{1+s^{\alpha}} \qquad s=0+\mathrm{i}\omega \\
G(\mathrm{i}\omega) &= \frac{(\mathrm{i}\omega)^{\alpha-1}}{1+(\mathrm{i}\omega)^{\alpha}} = \frac{\omega^{\alpha-1}(\mathrm{i})^{\alpha-1}}{1+(\mathrm{i}\omega)^{\alpha}} \\
&= \frac{\omega^{\alpha-1}(\mathrm{i})^{-1}\left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+\mathrm{i}\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)} \\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)-\mathrm{i}\cos\left(\frac{\alpha\pi}{2}\right)}{\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)+\mathrm{i}\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)} \\
&= \omega^{\alpha-1}\frac{\left(\sin\left(\frac{\alpha\pi}{2}\right)-\mathrm{i}\cos\left(\frac{\alpha\pi}{2}\right)\right)\left(\left(1+\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\right)-\mathrm{i}\left(\omega^{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)\right)\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \\
&= \omega^{\alpha-1}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} - \mathrm{i}\omega^{\alpha-1}\frac{\omega^{\alpha}+\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}
\end{aligned} \tag{66}
$$

From above we write the following

$$
\mathrm{Re}\left[G(\mathrm{i}\omega)\right]=\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}; \quad \mathrm{Im}\left[G(\mathrm{i}\omega)\right]=-\frac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}} \tag{67}
$$

The inverse Laplace transform by applying (12) is following

$$
\begin{aligned}
E_{\alpha}(-t^{\alpha}) &= g(t) = \mathcal{L}^{-1}\left\{G(\mathrm{i}\omega)\right\} \\
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\left(\left(\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\cos(\omega t)+\left(\frac{\omega^{2\alpha-1}+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\right)\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}-\frac{\omega^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)\cos(\omega t)+\omega^{2\alpha-1}\sin(\omega t)+\omega^{\alpha-1}\cos\left(\frac{\alpha\pi}{2}\right)\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega
\end{aligned} \tag{68}
$$

From here we write integral representation of $E_\alpha(-x)$ as following (i.e. placing $t^\alpha = z$)

$$E_\alpha(-z) = \frac{1}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z} + \frac{\alpha\pi}{2}\right) + \omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1 + 2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right) + \omega^{2\alpha}}\mathrm{d}\omega \tag{69}$$

Therefore we can write from above derivation (69) the following relation

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left\{\frac{s^\alpha}{s(1+s^\alpha)}\right\} \\ &= \frac{1}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\sin\left(\omega\sqrt[\alpha]{z} + \frac{\alpha\pi}{2}\right) + \omega^{2\alpha-1}\sin(\omega\sqrt[\alpha]{z})}{1 + 2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right) + \omega^{2\alpha}}\mathrm{d}\omega, \qquad t^\alpha = z \end{aligned} \tag{70}$$

Example-11

We take

$$G(s) = \frac{k}{s(s^\alpha + k)} \tag{71}$$

Put $s = \mathrm{i}\omega$ in $G(s) = \frac{k}{s(s^\alpha + k)}$ Thus we have following steps

$$\begin{aligned} G(\mathrm{i}\omega) &= \frac{k}{(\mathrm{i}\omega)\left((\mathrm{i}\omega)^\alpha + k\right)} = \frac{k}{\omega\left(\omega^\alpha \mathrm{i}^{\alpha+1} + \mathrm{i}k\right)} \\ &= \frac{k}{\omega\left(\omega^\alpha\left(\cos\left(\frac{(\alpha+1)\pi}{2}\right) + \mathrm{i}\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + \mathrm{i}k\right)\right)} \\ &= \frac{k}{\omega\left(\omega^\alpha\cos\left(\frac{(\alpha+1)\pi}{2}\right) + \mathrm{i}\left(\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)} \\ &= \frac{k\left(\omega^\alpha\cos\left(\frac{(\alpha+1)\pi}{2}\right) - \mathrm{i}\left(\omega^\alpha\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)}{\omega\left(\omega^{2\alpha}\cos^2\left(\frac{(\alpha+1)\pi}{2}\right) + \omega^{2\alpha}\sin^2\left(\frac{(\alpha+1)\pi}{2}\right) + 2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2\right)} \\ &= \frac{k\left(\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right) - \mathrm{i}\left(\omega^{\alpha-1}\left(\sin\left(\frac{(\alpha+1)\pi}{2}\right)\right) + k\right)\right)}{\omega^{2\alpha} + 2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2} \end{aligned} \tag{72}$$

We get

$$\mathrm{Re}\left[G(\mathrm{i}\omega)\right] = \frac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha} + 2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}, \quad \mathrm{Im}\left[G(\mathrm{i}\omega)\right] = -\frac{k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2}{\omega^{2\alpha} + 2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2} \tag{73}$$

Now using the formula of Laplace inversion by Berberan-Santos method we get the integral representation of $g(t)=\mathcal{L}^{-1}\left\{\frac{k}{s(s^{\alpha}+k)}\right\}$

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{G(\mathrm{i}\omega)\right\} \\
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^{\infty}\left(\mathrm{Re}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\cos(\omega t)-\mathrm{Im}\left\{G(\sigma_0+\mathrm{i}\omega)\right\}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\left(\begin{array}{l}\left(\dfrac{k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\right)\cos(\omega t) \\ \qquad +\left(\dfrac{\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\right)\sin(\omega t)\end{array}\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^{\infty}\frac{\left(k\omega^{\alpha-1}\cos\left(\frac{(\alpha+1)\pi}{2}\right)\right)\cos(\omega t)+\left(k\omega^{\alpha-1}\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2\right)\sin(\omega t)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega \\
&= \frac{k}{\pi}\int_0^{\infty}\frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^{\alpha}k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega
\end{aligned}
\tag{74}
$$

Table-1 gives integral representation obtained via Berberan Santos Method

S. No.	$G(s)$ **The transfer function as a function of complex frequency** $s=\sigma_0+\mathrm{i}\omega$	$g(t)=\mathcal{L}^{-1}\{G(s)\}$ **In integral representation of function in time domain by inverse Laplace transform by Berberan-Santos method**
1	$\frac{1}{s+a}$	$\frac{1}{\pi}\int_0^\infty \frac{a\cos(\omega t)+\omega\sin(\omega t)}{\omega^2+a^2}\mathrm{d}\omega$
2	$\frac{s}{s^2+1}$	$\frac{e^t}{\pi}\int_0^\infty \frac{(2+\omega^2)\cos\omega t+\omega^2\sin\omega t}{4+\omega^4}\mathrm{d}\omega$
3	e^{-sT_d}	$\frac{1}{\pi}\int_0^\infty \cos\left(\omega(t-T_d)\right)\mathrm{d}\omega$
4	1	$\frac{1}{\pi}\int_0^\infty \cos(\omega t)\mathrm{d}\omega$
5	$\frac{1}{s}$	$\frac{1}{\pi}\int_0^\infty \frac{\sin\omega t}{\omega}\mathrm{d}\omega$
6	$s^{-\alpha}$	$\frac{1}{\pi}\int_0^\infty \frac{\cos\left(\omega t-\frac{\alpha\pi}{2}\right)}{\omega^\alpha}\mathrm{d}\omega$
7	$\left(1+(1-\beta)\left(\frac{s}{a}\right)\right)^{-\frac{1}{(1-\beta)}}$	$\frac{a}{\pi(1-\beta)}\int_0^\infty \left(1+u^2\right)^{-\frac{1}{2(1-\beta)}}\cos\left(\frac{aut-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u;\quad u=\frac{(1-\beta)\omega}{a}$
8	$e^{-(s/a)^\beta}$	$\frac{a}{\pi}\int_0^\infty \left(e^{-u^\beta\cos(\beta\pi/2)}\right)\cos\left(atu-u^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u;\quad u=\frac{\omega}{a}$
9	$\frac{k}{k+s^\alpha}$	$\frac{\sqrt[\alpha]{k}}{\pi}\int_0^\infty \left(\frac{\left(u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1\right)\cos\left(ut\sqrt[\alpha]{k}\right)+\left(u^\alpha\sin\left(\frac{\alpha\pi}{2}\right)\right)\sin\left(ut\sqrt[\alpha]{k}\right)}{u^{2\alpha}+2u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\right)\mathrm{d}u;\quad u=\frac{\omega}{\sqrt[\alpha]{k}}$
10	$\frac{s^\alpha}{s(1+s^\alpha)}$	$\frac{1}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\sin\left(\omega t+\frac{\alpha\pi}{2}\right)+\omega^{2\alpha-1}\sin(\omega t)}{1+2\omega^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+\omega^{2\alpha}}\mathrm{d}\omega$
11	$\frac{k}{s(s^\alpha+k)}$	$\frac{k}{\pi}\int_0^\infty \frac{\omega^{\alpha-1}\cos\left(\omega t-\left(\frac{(\alpha+1)\pi}{2}\right)\right)+k\sin(\omega t)}{\omega^{2\alpha}+2\omega^\alpha k\sin\left(\frac{(\alpha+1)\pi}{2}\right)+k^2}\mathrm{d}\omega$

Table-1: List of inverse Laplace transforms in integral representation

Laplace inversion of function having log term

Example-12

We take

$$G(s)=\frac{s^b-s^a}{\ln s} \tag{75}$$

Put $s=\sigma_0+\mathrm{i}\omega$ with $\sigma_0=0$, this makes the following steps

$$
\begin{aligned}
G(\mathrm{i}\omega) &= \frac{(\mathrm{i}\omega)^b - (\mathrm{i}\omega)^a}{\ln(\mathrm{i}\omega)} \\
&= \frac{\omega^b\left(\cos\left(\frac{b\pi}{2}\right) + \mathrm{i}\sin\left(\frac{b\pi}{2}\right)\right) - \omega^a\left(\cos\left(\frac{a\pi}{2}\right) + \mathrm{i}\sin\left(\frac{a\pi}{2}\right)\right)}{\ln\omega + \mathrm{i}\left(\frac{\pi}{2}\right)} \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right) + \mathrm{i}\left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)}{\ln\omega + \mathrm{i}\left(\frac{\pi}{2}\right)} \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right) + \mathrm{i}\left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\left(\ln\omega - \mathrm{i}\left(\frac{\pi}{2}\right)\right) \\
&= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)(\ln\omega) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}} \\
&\quad - \mathrm{i}\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}
\end{aligned} \tag{76}
$$

This gives real and imaginary parts as

$$
\begin{aligned}
\mathrm{Re}\left[G(\mathrm{i}\omega)\right] &= \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)(\ln\omega) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}} \\
\mathrm{Im}\left[G(\mathrm{i}\omega)\right] &= -\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}
\end{aligned} \tag{77}
$$

We apply (12) to get the following

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1}\left\{\frac{s^b - s^a}{\ln s}\right\} \\
g(t) &= \frac{e^{\sigma_0 t}}{\pi}\int_0^\infty \left(\mathrm{Re}\{G(\sigma_0 + \mathrm{i}\omega)\}\cos(\omega t) - \mathrm{Im}\{G(\sigma_0 + \mathrm{i}\omega)\}\sin(\omega t)\right)\mathrm{d}\omega \\
&= \frac{1}{\pi}\int_0^\infty \left(\begin{gathered}\frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)(\ln\omega) + \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right)}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\cos(\omega t) \\ + \frac{\left(\omega^b\cos\left(\frac{b\pi}{2}\right) - \omega^a\cos\left(\frac{a\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) - \left(\omega^b\sin\left(\frac{b\pi}{2}\right) - \omega^a\sin\left(\frac{a\pi}{2}\right)\right)\ln\omega}{\sqrt{\left((\ln\omega)^2 + \frac{\pi^2}{4}\right)}}\end{gathered}\right)\mathrm{d}\omega
\end{aligned} \tag{78}
$$

Summary

The technique discussed give a possible way to represent inverse Laplace transform of function via integral representation, not by using a tough method of contour integration. The functions that were taken as example may arise in solution of differential equations, fractional differential equations and also continuous order differential equations. This new proposed method by Berberan-Santos.

References

Mario N Berberan-Santos, "**Properties of the Mittag-Leffler Relaxation Function**", Journal of Mathematical Chemistry, Vol. 38, No. 4, 2005.

Berberan-Santos Mario. N, "**Analytical inversion of Laplace transforms without contour integration: application to luminescence decay laws and other relaxation functions**", Journal of mathematical Chemistry, Vol. 38, No.2 August 2005.