The Curie-von Schweidler Law in dielectric relaxation and Zipf's Power Law Distribution for Relaxation Rates-its analysis

Shantanu Das

Reactor Control System Design Section E & I Group B.A.R.C
Mumbai-400085
Email: shantanu@barc.gov.in
Senior Research Professor CMPRC Dept. of Physics Jadavpur University, Kolkata
Adjunct Professor D.I.A.T Pune
UGC-Visiting Fellow University of Calcutta

Lecture notes on insight to complex dielectric relaxations-Condensed Matter Physics Research Centre Dept. of Physics Jadavpur University Kolkata

ABSTRACT

The classical power law relaxation, i.e. relaxation of current with inverse of power of time for a step-voltage excitation to dielectric-as popularly known as Curie-von Schweidler law is empirically derived and is observed via several relaxation experiments on various di-electrics studies since late 19th Century. This decay law is also regarded as 'Universal-Law' for dielectric relaxations. This Curie-von Schewidler power law is then used to derive Fractional Differential equations describing constituent expression for Capacitors. In this note we give simple mathematical treatment to derive the distribution of relaxation rates of this Curie-von Schweidler law, and show that the relaxation rate distribution follows Zipf's power law. We also show the method developed here give Zipfian power law distribution for relaxing time constants also. Then we will show however mathematically correct this may be, but physical interpretation from the obtained time constants distribution are contradictory to the Zipfian rate relaxation distribution. In this note we develop possible explanation that as to why Zipfian distribution of relaxation rates appears for Curie-von Schweidler Law, and relate this law to time variant rate of relaxation. We give appearance of fractional derivative while use of Zifian power law distribution, that gives notion of scale dependent rate relaxation function for Curie-von Schweidler relaxation phenomena. The treatment here in this note is new and requires further investigation.

Keywords: Power law, Relaxation Rate Distribution, fractional derivative, fractional integration, Curie-von Schweidler law, time-constants, Laplace Integral, Zipf's Law, Integral representation, time dependent Relaxation rate, scale dependent relaxation rate, Berberan-Santos Method for analytical Laplace inversion.

Introduction

The Curie-von Schweidler law relates to relaxation current in dielectric when a step DC voltage is applied and is given by $i(t) \propto t^{-n}$, where $t > 0$ and the power (exponent) i.e. n is called relaxation constant or decay constant, where $0 < n < 1$ [1]-[4]. This relaxation law is taken as universal law, at least for dielectric relaxations. Whereas we are used to Debye type of relaxation i.e. exponential decay law given by $i(t) \propto e^{-t/\tau_0}$ or $i(t) \propto e^{-\lambda_0 t}$ where τ_0 is the relaxation time constant, while λ_0 denotes the relaxation rate of the process, and is equal to lumped resistance R in Ohms and lumped capacitance C in Farads i.e. $\tau_0 = RC$; with $\lambda_0 = \tau_0^{-1}$. The radioactive decay is example of ideal Debye law where the exponential decay is governed by 'one-lumped' decay constant i.e. λ_0 . The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments [3] [4], [12], [22]-[26]. This power law relaxation of the type $i(t) \propto t^{-n}$ has been interpreted as a many-body problem but can also be formulated as an infinite number of resistor-capacitor circuits, meaning infinite number of time constants τ or relaxation rates λ varying from near zero to infinity [4], [5], and [6]. The observations of power law relaxation are also made in the experiments and studies with super-capacitors [7], [8], [9], [10], [11]. These studies also indicate the fractional calculus is used as constituent expression to describe super-capacitors. The use of empirical power law i.e. Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12]. Apart from relaxation of current decay in dielectrics and super-capacitors, the power law type or Non-Debye relaxation is observed in visco-elastic experiments strain relaxation in [13], [14], [15]. In this discussion we are giving the derivation of the distribution of relaxation rates particularly for Curie-von Schweidler law, and we observe the distribution nature as Zipf's distribution. We try to reason out as to why this distribution of relaxation rates takes Zipfian nature. We also show that Curie-von Schweidler law has time varying rate of relaxation.

This write-up will not deal with the mathematics of Zipfian Distribution like probability density, cumulative probability density function, and the conditions of finding finite mean, variance or standard deviation for power law distribution. This write-up describes finding the distribution function of relaxation rates (histogram) by formulating Laplace integral, and show that the distribution thus obtained is a Zipf's power law. We try to extend this mathematical approach to get the distribution function for time constants too and we get that time constants are also distributed as Zipf's power law; but the observation points a contrary physical interpretation derived from this obtained power law distribution for the time constants. Thus we can conclude that this method developed by Laplace Integral approach is restricted to get only distribution of relaxation rates and not to get the distribution of time constants. Though we are discussing Curie-von Schweidler law, we will tabulate relaxation rate distributions for some other relaxation functions which are obtained via this Laplace Integral method.

We shall demonstrate the formation of Fractional Derivative in expression of current and voltage considering the relaxation rates as Zifian distribution; and thus forming a scale dependent power law for relaxation rates as the scale varies from zero to infinity. Though via by experiments one cannot make histogram directly about the rates of relaxation in non-Debye processes, yet this mathematical procedure that we develop helps in extracting this information from the observations relaxation function. This is new treatment, and much more research is required, across various dynamic processes.

The Zipf's Law and probable mechanism

Zipf's Law is an empirical law formulated using mathematical statistics that refers to the fact that many types of data studied in the physical and social sciences can be approximated with a Zipfian distribution. This distribution is one of a family of related discrete power law probability distributions [18], [19], [20], [30], [31].

This power law distribution help to describe phenomena where large events are rare, but small ones are quite common. For example, there are few large earthquakes but many small ones. There are a few mega-cities, but many small towns. There are few words, such as 'and' and 'the' that occur very frequently, but many which occur rarely.

The emergence of a complex language is one of the fundamental events of human evolution, and several remarkable features suggest the presence of fundamental principles of organization. These principles seem to be common to all languages. The best known is the so-called Zipf's law, which states that the frequency of a word decays as a (universal) power law of its rank. The possible origins of this law have been controversial, and its meaningfulness is still an open question. One of the early hypotheses of Zipf of a principle of least effort for explaining the law is shown to be sound [30], [31]. But still the exact mechanism how the Zipf's distribution manifests is debated.

Though this law is widely referred in linguistic studies, economics studies, population studies we use this for a dielectric relaxation law (i.e. Curie-von Schweidler law), which is observed as $i(t) \propto t^{-n}, \quad 0 < n < 1$, since late 19^{th} century; and form a histogram of relaxation rates and show that it follows Zipf's power law. We try to give possible reasons as to why Zipfian distribution is observed for the distribution of relaxation rates.

Many of the things that we measure have a typical size or 'scale'. We ask ourselves why the relaxation rates λ cannot be arranged as simple Normal Distribution. Like while we plot the height of person in X-axis and percentage of occurrence of that particular height in Y-axis, we get a Normal Distribution peaked around mean Height with a spread both ways. We find that ratio of maximum height and minimum height of a person is finite (or relatively low value). For example as per Guinness Book of Records tallest person was having height 272cm and shortest person was having the height of 57cm, making this ratio 4.8. This ratio is relatively low value. We see the most adults are about 180cm tall-there is some variations around this figure notably depending on sex, but we never measure persons having height of 10cm or 500cm. Figure-1 gives this histogram of Normal Distribution.

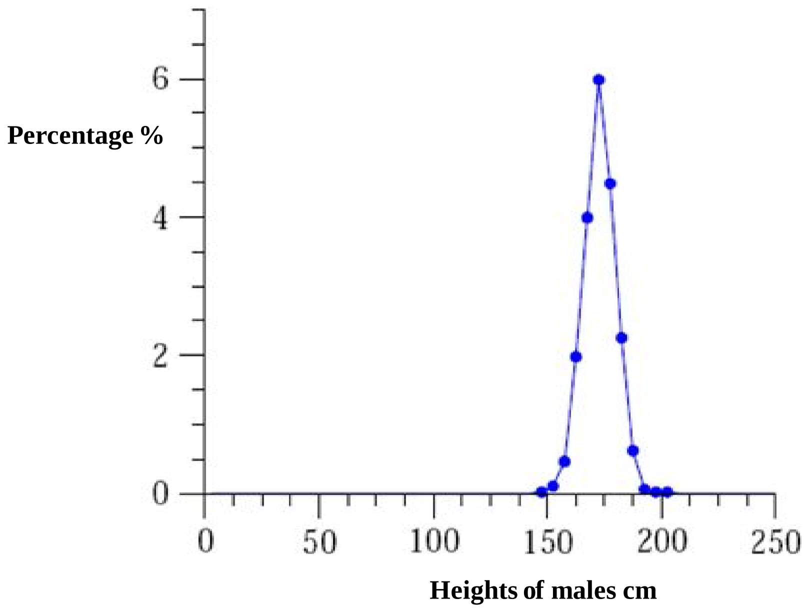

Figure-1: Normal Distribution of Height of Adult Male

But not all things we measure are peaked around a typical value. Some may over a very large dynamic range, sometimes many orders of magnitude. For example the ratio of population of largest town to population of smallest town is about 1, 50,000. The histogram if plotted for X-Axis with population of cities and Y-Axis with percentage of cities having that population; the distribution will not show the 'normal-distribution'. The histogram of cities & population is highly 'right-skewed', meaning that while the bulk of distribution occurs for fairly small sizes-i.e. most cities have small population-there is small number of cities with population much higher than a said typical value, producing the long tail to the right of histogram. This right (Figure-2) skewed form is qualitatively quite different from histogram of person's height. That is because we know that there is large dynamic range from smallest to largest city sizes, we can immediately infer that there can only be a small number of very large cities. The histogram of this sort is like a function i.e. $H_x(x) \sim x^{-\alpha}$. The distribution of this nature is called Zip's power law distribution.

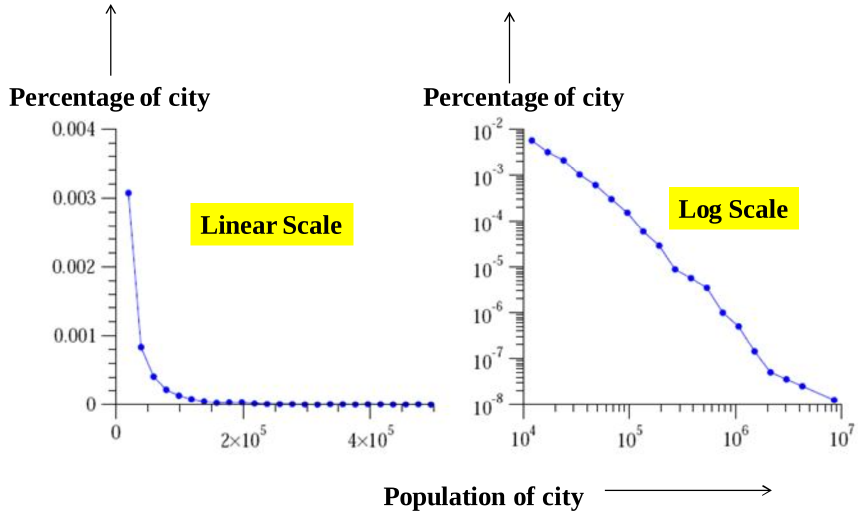

Figure-2: Histogram of Power Law Distribution (Long Tailed Right Skewed)

The same we observe when relaxation rates call them λ having a large (ideally infinite) spreads follow Zipfian distribution, we call that $H_\lambda(\lambda)$ and will show that the histogram follows the function i.e. $H_\lambda(\lambda) \sim \lambda^{-m}$. Thus one reason that this non-Debye relaxation of Curie-von Schweidler Law ($i(t) \propto t^{-n}$) in dielectric is having infinite spread of relaxation rates of λ 's-thus forming a Zipfian power law.

A much more common distribution than power law is the exponential distribution. In this relaxation mechanism that we are discussing, we consider infinite bodies relaxing simultaneously, in different time scales (T). We consider that a complex relaxation mechanism and a quantity T say survival time of a relaxing body, has exponential distribution of probability $p(\mathrm{T}) \sim e^{a\mathrm{T}}$. This means that a probability for a body with very large survival time (age) is very low (consider a as negative), and vice-versa. Then $p(\mathrm{T})\mathrm{dT}$ indicates the fraction of survival numbers of bodies between T and $\mathrm{T}+\mathrm{dT}$. Now suppose that the real quantity we are interested is not T but other quantity λ, say the relaxation rate of discharge which is exponentially related to T; thus $\lambda \sim e^{b\mathrm{T}}$. That implies the surviving bodies with very large time of survival (age) have a very low rate of relaxation, (considering b as negative). This also states that $\mathrm{d}\lambda = -\mathrm{dT}$, considering b as negative. Then if probability distribution of λ is $p(\lambda)$; then we have $p(\lambda)\mathrm{d}\lambda = p(\mathrm{T})\,\mathrm{dT}$. This means that number of discharging units having relaxation rates between λ and $\lambda + \mathrm{d}\lambda$ is equal to number of surviving bodies having survival time between T and $\mathrm{T}+\mathrm{dT}$. Thus we write following steps

$$p(\lambda) = p(\mathrm{T})\frac{\mathrm{dT}}{\mathrm{d}\lambda} = \frac{p(\mathrm{T})}{\left(\frac{\mathrm{d}\lambda}{\mathrm{dT}}\right)} \sim \frac{e^{a\mathrm{T}}}{be^{b\mathrm{T}}} = \frac{e^{\frac{a}{b}b\mathrm{T}}e^{-b\mathrm{T}}}{b}$$
$$= \frac{\lambda^{\frac{a}{b}}\lambda^{-1}}{b} \sim \lambda^{-\left(1-\frac{a}{b}\right)}$$
$$\sim \lambda^{-\alpha}; \qquad \alpha = 1-\tfrac{a}{b}$$

The above discussion gives a power law distribution where there is combination of exponential processes. Thus we expect that in our complex relaxation process governed by Curie-von Scweidler Law $i(t) \propto t^{-n}$ which is having infinite number of discharging bodies will have a power law distribution for relaxation rates as a histogram $H_\lambda(\lambda) \sim \lambda^{-\alpha}$. We proceed with this explanation and hypothesis. This could be one explanation in physical sense, in line with exponential distribution in the Boltzmann distribution of energies in statistical mechanics.

Relaxation with several exponential decay functions

Here we formulate the method to extract the histogram of the relaxation rates call it $H_\lambda(\lambda)$, for a complex non-Debye relaxation process $i(t)$, which we assume to be composed of several Debye type relaxations $e^{-\lambda t}$, with λ varying from zero to infinity. The complex decay may be expressed as following with several rate constants $\lambda_1, \lambda_2, \lambda_3, ...$ with weights $a_1, a_2, a_3, ...$, where λ is having units in 'per second', and is equal to inverse of time constant i.e. $\lambda_k = (\tau_k)^{-1}; \quad k = 1, 2, 3....$ We write following composite relaxation expression with several 'discrete' relaxation rates as

$$i(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + ...$$
$$= \sum a_k e^{-\lambda_k t}$$

In continuum limit we may write the above as following

$$i(t) = \int_0^\infty \left(H_\lambda(\lambda)\right) e^{-\lambda t} \mathrm{d}\lambda \qquad i(0) = a_1 + a_2 + a_3 + ...$$

Where the function i.e. $H_\lambda(\lambda)$ is the distribution-function of the rate of the relaxation decay process, or we may call it as histogram of relaxation rates. While for the case with discrete set of relaxation rates i.e. $\lambda_1, \lambda_2, \lambda_3, ...$ the rate distribution function would be having discrete delta functions ($\delta(t - \lambda_k)$, $k = 1, 2, 3, ...$) at points $\lambda_1, \lambda_2, \lambda_3, ...$; which we write like following expression

$$H_\lambda(\lambda) = a_1\delta(\lambda - \lambda_1) + a_2\delta(\lambda - \lambda_2) + a_3\delta(\lambda - \lambda_3) + ...$$
$$= \sum a_k \delta(\lambda - \lambda_k)$$

From above formulation if we have Debye relaxation i.e. having only one rate constant say λ_0 i.e. $i(t)=e^{-\lambda_0 t}$ then $H_\lambda(\lambda)=\delta(\lambda-\lambda_0)$. This is verified in the following expression

$$
\begin{aligned}
i(t) &= \int_0^\infty \left((H_\lambda\lambda)\right)e^{-\lambda t}\mathrm{d}\lambda \\
&= \int_0^\infty \left(\delta(\lambda-\lambda_0)\right)e^{-\lambda t}\mathrm{d}\lambda = e^{-\lambda_0 t}
\end{aligned}
$$

In above we used the property of delta function [16], [28], [29] i.e. $\int\left(\delta(x-x_0)\right)f(x)\mathrm{d}x = f(x_0)$.

Inverse Laplace Transform of time domain response function to get relaxation rate distribution function

The Laplace transform $F(s)$ of a function in time domain $f(t)$ is defined as following integral transform relation

$$
F(s)\overset{\text{def}}{=}\int_0^\infty \left(f(t)\right)e^{-st}\mathrm{d}t \qquad F(s)=0 \quad \text{for} \quad s<0
$$

This is standard integral transform of a function $f(t)$ from a time domain (t) to a complex frequency domain i.e. $s=\mathrm{Re}\{s\}+\mathrm{i}\omega;\quad \mathrm{i}=\sqrt{-1}$; where real part is significant in the transient response and the imaginary part of the frequency corresponds to 'steady-state' response; in classical 'Control Science'. Here $f(t)$ is 'inverse Laplace transform' of $F(s)$, and we write $\mathcal{L}^{-1}\{F(s)\}=f(t)$ and $\mathcal{L}\{f(t)\}=F(s)$. Compare the two expressions as follows

$$
i(t)=\int_0^\infty \left(H_\lambda(\lambda)\right)e^{-t\lambda}\mathrm{d}\lambda \qquad\qquad F(s)=\int_0^\infty \left(f(t)\right)e^{-st}\mathrm{d}t
$$

Both above are Laplace transform expressions, (or Laplace integrals). The first expression is transforming the function $H_\lambda(\lambda)$ from λ domain to 'complex' t time domain; while the second one is transforming $f(t)$ from t domain to 'complex' s frequency domain. Thus both are Laplace transformation expressions with change of variable and symbol. Therefore we can say $H_\lambda(\lambda)$ is inverse Laplace Transform of $i(t)$ in the first expression, i.e. $H_\lambda(\lambda)=\mathcal{L}^{-1}\{i(t)\}$; as $f(t)$ is inverse Laplace of $F(s)$ in the second expression, i.e. $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

Therefore in order to get the rate distribution-function $H_\lambda(\lambda)$ from the decay curve (or relaxation-function $i(t)$), we need to have inverse Laplace Transform of the time function $i(t)$. The definition of inverse Laplace Transform is described as following integral expressions

$$f(t)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(F(s)\right)e^{st}\mathrm{d}s \qquad\qquad H_\lambda(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda$$

In the above expression x is real number larger than x_0, where x_0 being such that $i(t)$ has some form of singularity on the real line $\mathrm{Re}\{t\}=x_0$ but is analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\{t\}>x_0$. Thus in this formulation we treat time variable as complex quantity say $t=x+\mathrm{i}y$ in the expression of inverse Laplace Transform i.e. $H_\lambda(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda$.

The Laplace inversion is usually carried out by contour integration. But the very modern technique of Berberan-Santo [17] method is the analytical Laplace inversion without the usual contour integration. This we have kept in Appendix, also the use of Berberan-Santo method is employed to find out relaxation rate distributions functions $H_\lambda(\lambda)$ of many non-Debye type relaxation functions (refer Appendix) and Table-1.

If the relaxation is of type i.e. $i(t)\propto t^{-n}$ then rate distribution function is $H_\lambda(\lambda)=\mathcal{L}^{-1}\left\{t^{-n}\right\}$. With the known Laplace pair i.e. $\mathcal{L}^{-1}\left\{s^{-(q+1)}\right\}=\frac{1}{q!}t^q$, we can write following steps

$$\begin{aligned}
H_\lambda(\lambda)&=\mathcal{L}^{-1}\left\{t^{-n}\right\}\\
&=\frac{1}{(n-1)!}\lambda^{(n-1)}=\frac{1}{m!}\lambda^m; \quad m=n-1; \quad \Gamma(k)=(k-1)!, \quad k\in\mathbb{R}\\
&=\frac{1}{\Gamma(n)}\lambda^{n-1}=\frac{1}{\Gamma(m+1)}\lambda^m; \quad \lambda>0
\end{aligned}$$

Therefore above discussion suggests that for a power law type relaxation, i.e. Curie-von Schweidler law i.e. $i(t)\propto t^{-n}; \quad 0<n<1$, the relaxation rates λ's are also having a power law distribution of type i.e. $H_\lambda(\lambda)\sim\lambda^m, \quad m=n-1, \quad -1<m<0, \quad \lambda>0$. This is Zipf's power law with $m<0$.

For dielectric relaxation as observed that $0<n<1$ in Curie-von Schweidler relaxation $i(t)\propto t^{-n}$, the rate relaxation distribution function $H_\lambda(\lambda)\sim\lambda^m$ has exponent in power $-1<m<0$. Considering graph of $H_\lambda(\lambda)=\lambda^m; \quad m<0$ as histogram, we infer that for Curie-von Schweidler relaxation function i.e. $i(t)\propto t^{-n}; \quad 0<n<1$ there are very large number of relaxations with small λ i.e. large number of slower decay takes place, compared to fewer faster decay rates-and the histogram $H_\lambda(\lambda)=\lambda^m; \quad m<0$ is highly right skewed with long tail.

From the above discussion and using our Laplace integral i.e. $i(t)=\int_0^\infty\left(H_\lambda(\lambda)\right)e^{-t\lambda}\mathrm{d}\lambda$ we write for Curie-von Schweidler relaxation function the following

$$t^{-n} = \frac{1}{\Gamma(n)} \int_0^{\infty} \left(\lambda^{(n-1)}\right) e^{-t\lambda} \mathrm{d}\lambda$$

Above expression is integral representation of the t^{-n} shows weighted averaging of infinite Debye relaxations i.e. $e^{-\lambda t}$ with weight $\lambda^{(n-1)}$ applied for all λ from zero to infinity.

We have observed in the previous section that a Debye relaxation of $i(t) \propto e^{-\lambda_0 t}$ has rate distribution as $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$ i.e. it is given by a delta function at point $\lambda = \lambda_0$. This we verify with known Laplace relation i.e. $\mathcal{L}\left\{ f(t-t_0)\right\} = e^{-st_0}F(s)$, where $\mathcal{L}\left\{ f(t)\right\} = F(s)$. Also we have $\mathcal{L}\left\{\delta(t)\right\} = 1$; thus we can write $\mathcal{L}\left\{\delta(t-t_0)\right\} = e^{-st_0}$. From here we can write with change of variable for $i(t) = e^{-\lambda_0 t}$ the inverse Laplace of this time domain function in λ domain we get as $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$, the rate distribution function. If there is no decay then say $i(t) = 1 = e^{-\lambda_0 t}$; $\lambda_0 = 0$; the rate distribution function is delta function at origin $H_\lambda(\lambda) = \delta(t)$.

If the relaxation function is of say $i(t) = 1 - e^{-\lambda_0 t}$; then we have $H_\lambda(\lambda) = \mathcal{L}^{-1}\left\{1 - e^{-\lambda_0 t}\right\}$; giving $H_\lambda(\lambda) = \delta(\lambda) - \delta(\lambda - \lambda_0)$. From these observations we say that for our earlier expression i.e. $H_\lambda(\lambda) = \sum a_k \delta(\lambda - \lambda_k)$, the coefficients a_k 's can have negative values as well for some type of relaxation function. For example if $i(t) = t(t+1)^{-1}$ is a relaxation function that initially grows to a maximum value and then starts falling as time increases, it has rate distribution function as $H_\lambda(\lambda) = \cos\lambda$, a oscillatory one (Refer Table-1). Thus in this case the distribution function i.e. $H_\lambda(\lambda)$ can take positive as well as negative values. One interesting observation is for a relaxation function $i(t) = t^{-1}$ the relaxation function is $H_\lambda(\lambda) = 1$-a 'uniform distribution', for $\lambda \geq 0$.

Using Berberan-Santo method (Refer Appendix), we get the Laplace inversion of $i(t) = t^{-n}$ as $\mathcal{L}^{-1}\left\{t^{-n}\right\} = \frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^{\infty} u^{-n}\cos(\lambda u)\left(\mathrm{d}u\right)$, this is $H_\lambda(\lambda) = \frac{1}{\Gamma(n)}\lambda^{n-1}$, as per our above discussion. So we write the following with n changed to α

$$\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1} = \frac{2}{\pi}\cos\left(\frac{\alpha\pi}{2}\right)\int_0^{\infty} u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

$$\lambda^{\alpha-1} = \frac{2}{\pi}\Gamma(\alpha)\left(\cos\left(\frac{\alpha\pi}{2}\right)\right)\int_0^{\infty} u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

Now we do another trick of changing λ to t, $\alpha - 1$ to $-n$ to get $t^{-n} = \frac{2\Gamma(1-n)}{\pi}\cos\left(\frac{(1-n)\pi}{2}\right)\int_0^{\infty} u^{(n-1)}\cos(tu)\left(\mathrm{d}u\right)$. Considering now u as λ we write another integral representation of t^{-n} as follows

$$t^{-n} = 2\frac{\Gamma(1-n)}{\pi}\cos\left(\frac{(1-n)\pi}{2}\right)\int_0^{\infty} \lambda^{(n-1)}\cos(\lambda t)\left(\mathrm{d}\lambda\right)$$

This means that if we chose basic relaxation function as $\cos(\lambda t)$, then Curie-von Schweidler relaxation $i(t) = t^{-n}$ is weighted sum of all $\cos(\lambda t)$'s with weights λ^{n-1}, as λ is varied from zero to infinity.

Zipf's distribution for relaxation time constants for Curie-von Schweidler relaxation law-a contradiction

Now converting to $\tau = \lambda^{-1}$, we assume the Distribution of time-constants call it $H_\tau(\tau) \sim \tau^{-m}$, is the Zipf's power law distribution. However direct taking of reciprocal of obtained inversion of $H_\lambda(\lambda)$ the rate distribution function obtained via Laplace inversion of $i(t)$ is not possible. This we demonstrate in this section.

As we have formulated Laplace integral i.e. $i(t) = \int_0^\infty \left(H_\lambda(\lambda)\right) e^{-t\lambda} \mathrm{d}\lambda$, just by replacing $\lambda = \tau^{-1}$, $\mathrm{d}\lambda = -(\tau)^{-2}\mathrm{d}\tau$ we will get $i(t) = \int_0^\infty \left(H_\lambda(\lambda)\right)(\tau)^{-2} e^{-t/\tau} \mathrm{d}\tau$ which is not Laplace integral.

Now we do the following steps, for $i(t) = t^{-n}$ and obtained $H_\lambda(\lambda) = \frac{1}{\Gamma(n)}\lambda^{n-1}$

$$
\begin{aligned}
t^{-n} &= \int_0^\infty \left(\tfrac{1}{\Gamma(n)}\lambda^{n-1}\right)(\tau)^{-2} e^{-t/\tau} \mathrm{d}\tau \\
&= \frac{1}{\Gamma(n)} \int_0^\infty \tau^{1-n}\tau^{-2} e^{-t/\tau} \mathrm{d}\tau \\
&= \frac{1}{\Gamma(n)} \int_0^\infty \tau^{-(n+1)} e^{-t/\tau} \mathrm{d}\tau
\end{aligned}
$$

We write the two representations of t^{-n} as following integrals.

$$
t^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \left(\lambda^{(n-1)}\right) e^{-t\lambda} \mathrm{d}\lambda \qquad t^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \left(\tau^{-(n+1)}\right) e^{-t/\tau} \mathrm{d}\tau
$$

Thus we have $H_\tau(\tau) \propto \tau^{-(n+1)}$, as we have $H_\lambda(\lambda) = \lambda^{n-1}$. Now we verify the above obtained result in following discussion.

By the logic that we had constructed $i(t) = \int_0^\infty \left(H_\lambda(\lambda)\right) e^{-t\lambda} \mathrm{d}\lambda$ which is Laplace integral; we will similarly get the integral $i(t) = \int_0^\infty \left(H_\tau(\tau)\right) e^{-t/\tau} \mathrm{d}\tau$ which is not a direct Laplace Transform formula. Following steps will convert this expression into the Laplace Transform formula, and from there we will extract $H_\tau(\tau)$.

$$
\begin{aligned}
i(t) &= \int_0^{\infty} \left(H_\tau(\tau)\right) e^{-t/\tau} \mathrm{d}\tau; \quad \tau = \lambda^{-1}, \quad \mathrm{d}\tau = -\lambda^{-2}\mathrm{d}\lambda \\
&= \int_{\infty}^{0} \left(-\lambda^{-2}\left(H_\tau(\tau)\right)\right)\left(e^{-t\lambda}\right)\mathrm{d}\lambda; \qquad \mathrm{F} = \lambda^{-2}\left(H_\tau(\tau)\right) \\
&= \int_0^{\infty} (\mathrm{F})\left(e^{-t\lambda}\right)\mathrm{d}\lambda
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{F} &= \mathcal{L}^{-1}\left\{i(t)\right\} = \frac{1}{\Gamma(n)}\lambda^{(n-1)} \\
\lambda^{-2}\left(H_\tau(\tau)\right) &= \frac{1}{\Gamma(n)}\lambda^{(n-1)} \\
H_\tau(\tau) &= \frac{\lambda^{n-1}\lambda^{2}}{\Gamma(n)}; \qquad \lambda = \tau^{-1} \\
&= \frac{\tau^{-(n+1)}}{\Gamma(n)}
\end{aligned}
$$

Now we take different approach to verify the above obtained expression for $H_\tau(\tau)$. Let us have set of relaxation functions with various time constants τ ranging from 0 to infinity that is $\left\{e^{-t/\tau_1}, e^{-t/\tau_2}, e^{-t/\tau_3}, \ldots\right\}$, comprising of infinite number of functions, in continuum in τ. The relaxation function varies from very-very quick decay (when $\tau \approx 0$) to very-very slow decay curve (when $\tau \approx \infty$). We construct a weighted decay function as $\tau^{-m}e^{-t/\tau}$. This shows that we are multiplying by weight τ^{-m}; $m > 0$ the decay function $e^{-t/\tau}$. We are assuming Zipf's type distribution of τ, in form of $H_\tau(\tau) \sim \tau^{-m}$, meaning the lowest time constant i.e. fastest decay occurs more frequent than slow decay i.e. large time constant. The time constant parameter τ let vary from 0 to infinity and construct the following integral I, i.e.

$$
\mathrm{I} = \int_0^{\infty} \tau^{-m} e^{-t/\tau} \mathrm{d}\tau
$$

The integral I gives notion of weighted average of infinite relaxation functions. We do the substitution i.e. $\frac{t}{\tau} = y$, i.e. $\tau = \frac{t}{y}$ and $\mathrm{d}\tau = (-y)^{-2} t(\mathrm{d}y)$ in the above integral to get following steps

$$
\begin{aligned}
\mathrm{I} &= \int_0^{\infty} \tau^{-m} e^{-t/\tau} \mathrm{d}\tau \\
&= \int_{\infty}^{0} \left(\tfrac{t}{y}\right)^{-m} e^{-y} (-y)^{-2} t(\mathrm{d}y) \\
&= t^{-m+1} \int_0^{\infty} e^{-y} y^{m-2} \mathrm{d}y
\end{aligned}
$$

By using the definition of the Gamma function in integral form i.e. $\Gamma(\alpha) = \int_0^{\infty} e^{-y} y^{\alpha-1} \mathrm{d}y$, we write the above integral as

$$\mathrm{I} = t^{-(m-1)}\int_0^{\infty} e^{-y} y^{m-2}\mathrm{d}y$$
$$= t^{-(m-1)}\left(\Gamma(m-1)\right)$$
$$\int_0^{\infty} \tau^{-m} e^{-t/\tau}\mathrm{d}\tau = \frac{\Gamma(m-1)}{t^{m-1}}$$

Putting $m-1=n$ in above we get integral representation of the power law t^{-n} and we represent this by time constant distribution function $H_\tau(\tau)$ in following expressions

$$t^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty} \tau^{-(n+1)} e^{-t/\tau}\mathrm{d}\tau$$
$$t^{-n} = \int_0^{\infty} \left(H_\tau(\tau)\right) e^{-t/\tau}\mathrm{d}\tau$$

Earlier we have obtained $t^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty}\left(\lambda^{(n-1)}\right)e^{-t\lambda}\mathrm{d}\lambda$; where we called rate distribution function as $H_\lambda(\lambda) = \frac{1}{\Gamma(n)}\lambda^{(n-1)}$. Now from above weighted average logic we get $H_\tau(\tau) = \frac{1}{\Gamma(n)}\tau^{-(n+1)}$; these two are not reciprocal of each other.

Therefore we can conclude that Curie-von Schweidler law ($\sim t^{-n}$) relates to weighted averaging of several classical Debye relaxations (of type $e^{-t/\tau}$) over several time constants from zero to infinity, that is having Zipf's power-law with time constant distribution as $H_\tau(\tau) \sim \tau^{-(n+1)}$.

What does it say for $0<n<1$, that $H_\tau(\tau) \sim \tau^{-(n+1)}$ is also a right-skewed distribution, where the lower time constants (faster decay) appear more than larger time constant (slower decay). This is contradiction to what we inferred for $H_\lambda(\lambda) \sim \lambda^{n-1}$.

This contradiction we demonstrate. As for $i(t) = e^{-\lambda_0 t}$ we got $H_\lambda(\lambda) = \delta(\lambda-\lambda_0)$; we expect $H_\tau(\tau) = \delta(\tau-\tau_0)$ for $i(t) = e^{-t/\tau_0}$; let us see what happens

$$i(t) = e^{-\lambda_0 t}; \quad \mathrm{F} = \mathcal{L}^{-1}\left\{i(t)\right\}$$
$$\mathrm{F} = \mathcal{L}^{-1}\left\{e^{-\lambda_0 t}\right\} = \delta(\lambda-\lambda_0); \quad \mathrm{F} = \lambda^{-2}\left(H_\tau(\tau)\right)$$
$$\lambda^{-2}\left(H_\tau(\tau)\right) = \delta(\lambda-\lambda_0)$$
$$H_\tau(\tau) = \lambda^2\left(\delta(\lambda-\lambda_0)\right); \qquad \lambda = \tau^{-1}$$
$$= \frac{\delta(\tau-\tau_0)}{\tau^2} \neq \delta(\tau-\tau_0)$$

Though mathematically we can get integral representation for any $i(t)=\int_0^{\infty}\left(H_{\tau}(\tau)\right)e^{-t/\tau}\mathrm{d}\tau$ but physically it will be contradictory to Laplace integral $i(t)=\int_0^{\infty}\left(H_{\lambda}(\lambda)\right)e^{-\lambda t}\mathrm{d}\lambda$. Hence we will deal with the relaxation rate distribution function that we extracted as $H_{\lambda}(\lambda)$ from $i(t)$ via our devised method of Laplace inversion.

Relaxation Rate variation in time is different from Rate Distribution function

Any decay function $i(t)$ is written as a general formulation in following way

$$i(t)=\exp\left(-\int_0^t\left(\lambda(\xi)\right)\mathrm{d}\xi\right)$$

where $\lambda(\xi)$ is the time (ξ) dependent rate coefficient. When the relaxation is pure exponential, one has $\lambda(\xi)$ as constant say λ_0 described as $\lambda(\xi)=\lambda_0$.

$$\begin{aligned}i(t)&=\exp\left(-\int_0^t\left(\lambda(\xi)\right)\mathrm{d}\xi\right)\\&=\exp\left(-\int_0^t\left(\lambda_0\right)\mathrm{d}\xi\right)=\exp\left(-\lambda_0\,\xi\big|_0^t\right)\\&=e^{-\lambda_0 t}\end{aligned}$$

Thus we get a Debye relaxation for a system having constant rate of relaxation. To extract $\lambda(t)$ that is time dependent rate coefficient we have to follow the following steps

$$-\int_0^t\left(\lambda(\xi)\right)\mathrm{d}\xi=\ln\left(i(t)\right)$$
$$\lambda(t)=-\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(i(t)\right)$$

We use the above rule for Mittag-Leffler relaxation function i.e. $i(t)=E_{\alpha}(-t)$.The generalized Mittag-Leffler (GML) function; denoted by $E_{\alpha,\beta}(\xi)$ which reads as follows

$$E_{\alpha,\beta}(\xi)=\sum\nolimits_{k=0}^{\infty}\frac{(\xi)^k}{\Gamma\left(\alpha k+\beta\right)}\qquad \alpha,\beta>0\qquad E_{\alpha,1}(\xi)=E_{\alpha}(\xi),\quad E_1(\xi)=e^{\xi}$$

For negative ξ, we have the following expressions for GML function

$$E_{\alpha,\beta}(-\xi)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\beta)}\xi^k\qquad E_{\alpha,\beta}(-\xi^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\beta)}(\xi)^{\alpha k}$$

For $i(t)=E_{\alpha}(-t)$ the $\lambda(t)$ is extracted as in following steps

$$
\begin{aligned}
\lambda(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(E_\alpha(-t)\right) \\
&= -\frac{1}{E_\alpha(-t)}\frac{\mathrm{d}E_\alpha(-t)}{\mathrm{d}t} = -\frac{1}{E_\alpha(-t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(\alpha k+1)}\right) \\
&= -\frac{1}{E_\alpha(-t)}\left(\sum_{k=0}^{\infty}\frac{(k+1)(-t)^k}{\Gamma(1+\alpha+\alpha k)}\right)
\end{aligned}
$$

For our Curie-von Schweilder relaxation law $i(t) = t^{-n}$ we have time dependent rate relaxation rate as

$$
\begin{aligned}
\lambda(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(t^{-n}\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\left(-n\ln t\right) = n\left(t^{-1}\right) \qquad \tau(t) = \frac{1}{\lambda(t)}
\end{aligned}
$$

$$
\tau(t) = \frac{t}{n}
$$

Therefore we have two observations that Curie-von Schweidler relaxation $i(t) \propto t^{-n}$ has time constant distributed as Zipfian power law $H_\lambda \sim \lambda^{n-1};\ \ 0 < n < 1$, while the relaxation rate constant is variable in time as a function $\lambda(t) = nt^{-1}$. Thus implying starts with very-very fast relaxation at a very-very high rate and as the time goes the rate constant decreases indicating slow rate of current decay.

S No.	**Relaxation function** $i(t), \quad t>0$	**Rate Distribution function** $H_\lambda(\lambda), \quad \lambda>0$
1	$i(t)=A$:Constant Function	$H_\lambda(\lambda)=A\left(\delta(\lambda)\right)$
2	$i(t)=t^{-1}$	$H_\lambda(\lambda)=1, \quad \lambda>0$
3	$i(t)=t^{-n}$	$H_\lambda(\lambda)=\frac{1}{(n-1)!}\lambda^{n-1}$
4	$i(t)=(t+a)^{-1}$	$H_\lambda(\lambda)=e^{-a\lambda}$
5	$i(t)=(t+a)^{-n}$	$H_\lambda(\lambda)=\frac{1}{(n-1)!}\lambda^{n-1}e^{-a\lambda}$
6	$i(t)=at^{-1}(t+a)^{-1}$	$H_\lambda(\lambda)=1-e^{-a\lambda}$
7	$i(t)=(t+a)^{-1}(t+b)^{-1}$	$H_\lambda(\lambda)=\frac{1}{b-a}\left(e^{-a\lambda}-e^{-b\lambda}\right)$
8	$i(t)=e^{-\lambda_0 t}$	$H_\lambda(\lambda)=\delta(\lambda-\lambda_0)$
9	$i(t)=t(t^2+a)^{-1}$	$H_\lambda(\lambda)=\cos\left(\lambda\sqrt{a}\right)$
10	$\sqrt{a}(t^2+a)^{-1}$	$H_\lambda(\lambda)=\sin\left(\lambda\sqrt{a}\right)$
11	$i(t)=\sqrt{a}\left((t+b)^2+a\right)^{-1}$	$H_\lambda(\lambda)=e^{-b\lambda}\sin\left(\lambda\sqrt{a}\right)$
12	$i(t)=e^{-(t/\tau_0)^\beta}$	$H_\lambda(\lambda)=\frac{\tau_0}{\pi}\int_0^\infty\left(e^{\left(-u^\beta\cos(\beta\pi/2)\right)}\cos\left(\lambda\tau_0 u-u^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\right)\mathrm{d}u$
13	$i(t)=\left(1+(1-\beta)\left(\frac{t}{\tau_0}\right)\right)^{-1/(1-\beta)}$	$H_\lambda(\lambda)=\frac{\tau_0}{\pi(1-\beta)}\int_0^\infty(\mathrm{d}u)(1+u^2)^{-1/(2(1-\beta))}\cos\left(\frac{\lambda\tau_0 u-\tan^{-1}u}{1-\beta}\right)$
14	$i(t)=\left(1+\left(\frac{t}{\tau_0}\right)^\alpha\right); \quad 0<\alpha<1$	$H_\lambda(\lambda)=\frac{2\tau_0}{\pi}\int_0^\infty\mathrm{d}u\,\frac{u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{u^{2\alpha}+2u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda\tau_0 u)$
15	$i(t)=E_\alpha\left(-t/\tau_0\right)$	$H_\lambda(\lambda)=\frac{2}{\pi}\int_0^\infty E_{2\alpha}(-y^2)\cos(\lambda y)\mathrm{d}y$
16	$i(t)=E_{1/2}\left(-t/\tau_0\right)$	$H_\lambda(\lambda)=\frac{1}{\sqrt{\pi}}e^{-\left(\lambda^2/4\right)}$

Table-1: Several relaxation functions and corresponding rate distribution functions

Scale dependence relaxation rates give Capacitors charging current as per Curie-von Schweidler Law

Let a capacitor C be connected to a voltage source $v(t)$ Volts, at time $t=0$; obviously this capacitor will get charged to the battery voltage. Let this capacitor is uncharged at $t<0$, thus there is no charge held by it, therefore the voltage across the capacitor is zero at $t<0$, and the circuit current is $i(t)=0; \quad t<0$. The voltage balance equation assuming R be the total resistance of the circuit (including internal resistance of Capacitor) at $t>0$ is the following

$$\frac{1}{C}\int_0^t i(x)\mathrm{d}x + Ri(t) = v(t)$$

Where $i(t)$ is the charging current flowing into the capacitor. The above integral equation may be differentiated and is put as following, for $t > 0$

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \lambda_0 i(t) = \left(\frac{1}{R}\right)\frac{\mathrm{d}v(t)}{\mathrm{d}t} \qquad \lambda_0 = (RC)^{-1} \qquad \tau_0 = RC$$

We take $v(t) = V_{BB}$ a constant, then considering $u(t) = 1; \quad t \geq 0$ and $u(t) = 0; \quad t < 0$, i.e. 'unit-step function', we have for RHS of the above equation as following

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = v^{(1)}(t)$$
$$v^{(1)}(t) = V_{BB}\frac{\mathrm{d}\left(u(t)\right)}{\mathrm{d}t}$$
$$= V_{BB}\left(\delta(t)\right)$$

Thus substituting the above we get following equation for $v(t) = V_{BB}$ for $t \geq 0$ the following

$$\frac{\mathrm{d}}{\mathrm{d}t}i(t) + \lambda_0 i(t) = \frac{V_{BB}}{R}\delta(t) \qquad \lambda_0 = (RC)^{-1} \qquad \tau_0 = RC$$
$$\frac{\mathrm{d}}{\mathrm{d}t}i(t) + \lambda_0 i(t) = I_0\delta(t) \qquad I_0 = \frac{V_{BB}}{R}$$

The solution to the above equation gives Debye relaxation function i.e.

$$i(t) = I_0 e^{-\lambda_0 t}$$

This solution $i(t) \sim e^{-\lambda_0 t}$ is the 'impulse response' of the circuit equation. The relaxation current of the above system follows Debye's relaxation, with one relaxation rate λ_0 (also termed as Debye law). The rate distribution function is $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$; that we discussed in previous sections. With this $i(t) = e^{-\lambda_0 t}$ as Green's function i.e. $g(t) = e^{-\lambda_0 t}$ i.e. solution of differential equation with 'unit' impulse excitation ($I_0 = 1$) or say Homogeneous solution, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}i(t) + \lambda_0 i(t) = \delta(t) \qquad i(t) = g(t) = e^{-\lambda_0 t}$$

Now we find if the input is say step function, call it $I_0 u(t)$, where $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$; then we get relaxation function for current as convolution integral, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}i(t)+\lambda_0 i(t)=I_0 u(t) \qquad g(t)=e^{-\lambda_0 t}$$
$$i(t)=\left(I_0 u(t)\right)*\left(g(t)\right)$$
$$=\int_{-\infty}^{t}\left(I_0 u(x)\right)\left(e^{-\lambda_0(t-x)}\right)\mathrm{d}x=I_0\int_0^t e^{-\lambda_0(t-x)}\mathrm{d}x$$
$$i(t)=\frac{I_0}{\lambda_0}\left(1-e^{-\lambda_0 t}\right)$$

We saw in earlier sections the relaxation rates (λ) distribution, for a Curie-von Schweidler relaxation law, i.e. $i(t)\sim t^{-n}$ is $H_\lambda(\lambda)\sim\lambda^{n-1}; \quad 0<n<1$; for relaxations in di-electrics. This is histogram of rates; it says that the relaxation of current is with several relaxation rates, which are distributed as discussed–in Zipf's law fashion with right-skewed-histogram. Thus if we represent the equivalent relaxation rate say $\lambda_{eq}\sim\lambda^{1/q}; \quad 0<q<1$ as scale of relaxation λ varies from zero to infinity; we will not be incorrect in assuming this. That is as we slide from a low scale λ to high scale λ the equivalent relaxation rate λ_{eq} will be different at different scales of relaxation (λ). If the index parameter i.e. $q=1$ then we have single rate constant system given by $\lambda_{eq}=\lambda$ always at all scales of relaxation i.e. $\lambda=\lambda_0=\tau_0^{-1}=(RC)^{-1}$, and with solution as $i(t)=e^{-\lambda_0 t}$.

We thus modify the capacitor discharge current equation, with $\lambda_{eq}=\lambda=\lambda_0$ i.e. with one relaxation rate at any scale of relaxation (λ)

$$\frac{\mathrm{d}}{\mathrm{d}t}i(t)+\lambda_0 i(t)=\delta(t)$$

to following i.e. variable $\lambda_{eq}=\lambda^{1/q}$ at any scale of relaxation rate (λ)

$$\frac{\mathrm{d}}{\mathrm{d}t}i(t)+\left(\lambda_{eq}\right)i(t)=\delta(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}i(t)+\left(\lambda\right)^{1/q}i(t)=\delta(t)$$

The initial condition is given as $i(t)=0$ for $t<0$. The above equation is having a free 'scale' parameter λ varying from zero to infinity. The solution of the above is $i(t)=e^{-\lambda_{eq}t}$; we call this $i(t)=e^{-\lambda_{eq}t}$ as 'impulse response function' at a particular scale λ, i.e. call it $h(\lambda,t); \quad \lambda\in\left(0,\infty\right)$

$$i(t)=h(\lambda,t)=\exp\left(-\lambda^{1/q}t\right); \qquad 0<q<1$$

The above expression actually is valid for all scale λ varying from zero to infinity. Thus on integrating this 'impulse response function' on the free variable (λ) from 0 to ∞, we get the function of time and that is called 'impulse response' or the Green's function $g(t)$

$$g(t) = \int_0^\infty h(\lambda, t)\mathrm{d}\lambda$$
$$= \int_0^\infty \exp\left(-\lambda^{1/q} t\right)\mathrm{d}\lambda = \frac{\Gamma(1+q)}{t^q}$$

For $q = 1$ case we have scale invariance λ thus $i(t) = h(\lambda_0, t) = \exp\left(-\lambda_0 t\right)$; $q = 1$, where $\lambda_{eq} = \lambda_0$ at all scales. For this case $q = 1$ 'impulse response' or Green's function is $g(t) = h(\lambda_0, t) = e^{-\lambda_0 t}$ same as 'impulse response function' i.e. $h(\lambda_0, t)$.

To get above expression we substitute in $g(t) = \int_0^\infty e^{-\left(\lambda^{(1/q)} t\right)}\mathrm{d}\lambda$, $\lambda^{1/q} t = x$ that makes following changes

$$\lambda = \left(\frac{x}{t}\right)^q; \quad \left(\frac{x}{t}\right) = \lambda^{1/q}$$
$$\mathrm{d}\lambda = q x^{q-1} \left(\tfrac{1}{t}\right)^q \mathrm{d}x$$
$$= \left(\frac{q}{t}\right)\left(\frac{x}{t}\right)^{q-1} \mathrm{d}x = \left(\frac{q}{t}\right)\left(\lambda^{1/q}\right)^{q-1} \mathrm{d}x,$$
$$\mathrm{d}\lambda = \lambda^{1-(1/q)}\left(\frac{q}{t}\right)\mathrm{d}x$$

Then by using definition of Gamma function i.e. $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\mathrm{d}y$, and its property $\alpha\left(\Gamma(\alpha)\right) = \Gamma(1+\alpha)$ the following steps are followed to get the desired expression i.e. $g(t) = \frac{\Gamma(1+q)}{t^q}$

$$g(t)=\int_0^{\infty} e^{-\left(\lambda^{(1/q)}t\right)}\mathrm{d}\lambda=\int_0^{\infty} e^{-x}\lambda^{1-\left(\frac{1}{q}\right)}\left(\frac{q}{t}\right)\mathrm{d}x$$

$$=\int_0^{\infty} e^{-x}\left(\frac{q}{t}\right)\lambda\lambda^{-\left(\frac{1}{q}\right)}\mathrm{d}x, \qquad \lambda=\left(\frac{x}{t}\right)^{q}$$

$$=\int_0^{\infty} e^{-x}\left(\frac{q}{t}\right)\left(\frac{x}{t}\right)^{q}\left(\frac{x}{t}\right)^{-1}\mathrm{d}x$$

$$=\left(\frac{q}{t}\right)\int_0^{\infty} e^{-x}\left(\frac{x}{t}\right)^{q}\left(\frac{x}{t}\right)^{-1}\mathrm{d}x$$

$$=\left(\frac{q}{t}\right)\int_0^{\infty} e^{-x}\frac{x^{q-1}}{t^{q-1}}\mathrm{d}x=\left(\frac{q}{t^q}\right)\int_0^{\infty} e^{-x}x^{q-1}\mathrm{d}x$$

$$=\frac{q\left(\Gamma(q)\right)}{t^q}=\frac{\Gamma(1+q)}{t^q}, \qquad q=n$$

$$t^{-n}=\frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} e^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda; \qquad 0<n<1$$

We find that $i(t)\propto t^{-n}; \quad 0<n<1$ for a system where the equivalent relaxation rate is $\lambda_{eq}=\lambda^{\frac{1}{n}}$; similar to a distribution function that we obtained as $H_\lambda(\lambda)\sim\lambda^{n-1}$ gives current as $i(t)\propto t^{-n}$. We write the two currents expressions $i(t)\sim t^{-n}$ obtained as following for $0<n<1$

$$t^{-n}=\frac{1}{\Gamma(n)}\int_0^{\infty}\lambda^{n-1}e^{-t\lambda}\mathrm{d}\lambda \qquad\qquad t^{-n}=\frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} e^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda$$

Therefore we infer that the Curie-von Schweidler relaxation current for dielectric excited by a step voltage that follows the relation $i(t)\sim t^{-n}; \quad 0<n<1$ has distribution function $H_\lambda(\lambda)\sim\lambda^{n-1}$ a power law or Zipfian distribution, with scale variable relaxation rate described as $\lambda_{eq}=\lambda^{\frac{1}{n}}$.

Appearance of Fractional derivative-in the system having Zipfian power law distribution in relaxation rates, where the equivalent relaxation rate is varying with scale

The delta-function for excitation as shown in above section gives Homogeneous system i.e.

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda\right)^{\frac{1}{n}}i(t)=\delta(t); \quad 0<n<1; \qquad i(t)=g(t)=\frac{\Gamma(1+n)}{t^n}$$

Now let the system described above be excited by a signal proportional to $v^{(1)}(t)$, a derivative of voltage excitation function $v(t)$; so we write this as following

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda\right)^{\frac{1}{n}}i(t)=v^{(1)}(t)$$

Note that if $v(t)=u(t)$, that is unit-step-function then $v^{(1)}(t)=\delta(t)$, we recover the above homogeneous differential equation. Then the response to this new excitation function $v^{(1)}(t)$ is convolution of Green's function obtained i.e. $g(t)=\frac{\Gamma(1+n)}{t^n}$ above, with the forcing function i.e. now $v^{(1)}(t)$ that is as follows

$$i(t)=\left(g(t)\right)*\left(v^{(1)}(t)\right)=\int_0^t \left(g(t-x)\right)\left(v^{(1)}(x)\right)\mathrm{d}x;\qquad g(t)=\frac{\Gamma(1+n)}{t^n}$$

$$=\Gamma(1+n)\int_0^t \frac{v^{(1)}(x)}{(t-x)^n}\mathrm{d}x \qquad 0<n<1$$

Multiplying and dividing the above expression with $\Gamma(1-n)$ and using the definition of fractional integral [6] that is

$${}_0\mathcal{I}_t^{\alpha}\left(f(t)\right)={}_0D_t^{-\alpha}\left(f(t)\right)=\int_0^t \frac{1}{\Gamma(\alpha)}(t-x)^{\alpha-1}f(x)\mathrm{d}x \qquad \alpha>0$$

we have following

$$i(t)=\Gamma(1+n)\Gamma(1-n)\int_0^t \frac{(t-x)^{(-n)}}{\Gamma(1-n)}v^{(1)}(x)\mathrm{d}x$$

$$=\Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[v^{(1)}(t)\right]\right);\qquad (1-n)>0$$

$$=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{-(1-n)}\left[v^{(1)}(t)\right]\right)$$

$$=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\,{}_0D_t^{-1}\left[v^{(1)}(t)\right]\right)$$

$$=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\left[v(t)\right]\right),\qquad n<1$$

In above we have used ${}_0\mathcal{I}_t^{\alpha}={}_0D_t^{-\alpha}$; $\alpha>0$ and ${}_0D_t^{\alpha-\beta}={}_0D_t^{\alpha}\,{}_0D_t^{-\beta}$; $\alpha,\beta>0$ [6]. Implying the appearance of fractional derivative for cases where several relaxation rates (ideally infinite of them) define a relaxation process; which are having a scale dependence behavior, i.e. $\lambda_{eq}=\lambda^{1/n}$ and histogram distributed as Zipf's power law i.e. $H_\lambda(\lambda)\sim\lambda^{n-1}$, and the relaxation is by Curie-von Schweidler law i.e. $i(t)\sim t^{-n}$, $0<n<1$. Thus we have current through a system (having a complex relaxation process with several rate distributed as power law excited by a voltage $v(t)$ as fractional derivative of it, i.e. $i(t)\propto {}_0D_t^{n}\left[v(t)\right]$.

Let this system i.e. $\frac{\mathrm{d}}{\mathrm{d}t}i(t)+\lambda^{1/n}i(t)=v^{(1)}(t)$; $0<n<1$ be excited by a source which is a delta function say $v^{(1)}(t)=\left(\left(\frac{V_{BB}}{R}\right)\delta(t)\right)$. This means $v(t)=\left(\frac{V_{BB}}{R}\right)u(t)$; where $u(t)$ is unit step function. With this excitation the relaxation current would be fractional integral of the input excitation that is from as depicted in above derivation i.e. $i(t)=\Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[\left(\left(\frac{V_{BB}}{R}\right)\delta(t)\right)\right]\right)$. We have fractional integration of delta function [6] as ${}_0\mathcal{I}_x^{\alpha}\left[\delta(x)\right]=\frac{1}{\Gamma(\alpha)}x^{\alpha-1}$; and using this formula we get $i(t)=\left(\frac{V_{BB}}{R}\right)\left(\frac{\Gamma(1+n)}{t^n}\right)$. This was what was derived in above as impulse response i.e. $g(t)=\frac{\Gamma(1+n)}{t^n}$.

If the excitation source is a step function as $v^{(1)}(t)=\left(\frac{V_{BB}}{R}\right)\left(u(t)\right)$; meaning $v(t)=\left(\frac{V_{BB}}{R}\right)t;\quad t\geq 0$ where the unit step function is, $u(t)=1,\quad t\geq 0;\quad u(t)=0,\quad t<0$ then the relaxation current is fractional integration of order $(1-n)$; that is $i(t)=\Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[\left(\frac{V_{BB}}{R}\right)u(t)\right]\right)$. Using the formula for fractional integration of a constant i.e. ${}_0\mathcal{I}_x^{\alpha}\mathrm{C}=\frac{\mathrm{C}}{\Gamma(1+\alpha)}x^{\alpha}$ [6], we have; the relaxation current

$$\begin{aligned}i(t)&=\left(\Gamma(1+n)\Gamma(1-n)\right)\left(\frac{V_{BB}}{R}\right)\frac{t^{1-n}}{\Gamma(2-n)}\\&=\frac{\Gamma(1+n)}{(1-n)}\left(\frac{V_{BB}}{R}\right)t^{1-n},\quad 0<n<1\end{aligned}$$

Fractional Derivative directly from Curie-von Schweidler Law

In experimental observations we find that capacitor has fractional order impedance [7]-[12], [22]-[26]. The impedance $Z(\omega)\sim\omega^{-n},\quad 0<n<1$ has implication of dissipation theory that we will not cover here. Practically on applying a step input voltage $v(t)=V_{BB}$ Volts at $t=0$ to a capacitor which is initially uncharged; we get a power-law decay of current given by Curie-von Schweidler as $i(t)\propto t^{-n}$. That we write in following way as suggested by experimental studies [12], [22]-[26].

$$i(t)=K_n\frac{V_{BB}}{t^n}\qquad t>0$$

The parameter K_n is constant. This is from observation and the evaluation of order of power-law function is $0.5<n<1$ [7]-[12]. Let the capacitor be excited by a step input of V_{BB} Volts, i.e. written as $v(t)=V_{BB}\left(u(t)\right)$, where $u(t)$ is unit step function. The Laplace of step input is $V(s)=\mathcal{L}\left\{v(t)\right\}=\mathcal{L}\left\{V_{BB}\left(u(t)\right)\right\}=V_{BB}/s$ and then taking Laplace of above power-law decay current, i.e. $I(s)=\mathcal{L}\left\{i(t)\right\}=\mathcal{L}\left\{K_nV_{BB}t^{-n}\right\}=K_nV_{BB}\left(\frac{(-n)!}{s^{-n+1}}\right)$, using the formula $(k-1)!=\Gamma(k)$ we get the following

$$\begin{aligned}I(s)&=K_n\frac{\Gamma(1-n)V_{BB}}{s^{1-n}}\\&=K_n\frac{\Gamma(1-n)}{s^{-n}}\left(\frac{V_{BB}}{s}\right)\end{aligned}$$

We get Transfer function of capacitor as following

$$H(s)=\frac{I(s)}{U(s)}=\frac{K_n\frac{\Gamma(1-n)}{s^{-n}}\left(\frac{U_0}{s}\right)}{\left(\frac{U_0}{s}\right)}$$

$$=K_n\left(\Gamma(1-n)\right)s^n=C_n s^n \qquad C_n=\frac{1}{K_n\Gamma(1-n)}$$

This is also admittance of capacitor or impedance equaling following

$$Z(s)=\frac{1}{C_n s^n},\quad 0<n<1$$

We write following expressions with understanding

$$Z(s)=\frac{U(s)}{I(s)} \qquad I(s)=\frac{1}{Z(s)}U(s); \qquad Z(s)=\frac{1}{C_n}s^{-n}, \qquad I(s)=C_n s^n U(s)$$

and get the following by Laplace inversion recognizing the $\mathcal{L}^{-1}\left\{s^nF(s)\right\}={}_0D_t^n\left[f(t)\right]$ i.e. fractional derivative operation [6]. Thus we have constituent relation governing fractional capacity i.e. following

$$i(t)=C_n\left({}_0D_t^n\left[v(t)\right]\right),\qquad 0<n<1$$

Same we derived in previous section too, but here the relation is coming directly from Curie-von Schweilder law where $i(t)\propto t^{-n},\quad 0<n<1$.

This fractional derivative expression gives a new capacitor theory [12], and we utilize this above formula to find characteristics of super-capacitors, variation of n with current excitation, and energy discharged to energy stored. Classically the expression of capacitor is $i(t)=C\left(D_t^{(1)}\left[v(t)\right]\right)$ i.e. with integer whole-one order derivative. Curie-von Schweidler law gives a different approach based on fractional calculus.

Conclusions

This note gave systematic approach to extract a histogram, describing the distribution function for relaxation rates from a relaxing function of time, i.e. $i(t)\sim t^{-n},\quad 0<n<1$. The empirical law Curie-von Schweidler law, which is a non-Debye relaxation, (which is also stated to be universal law of dielectric relaxation of current), when stressed with a constant voltage gave rate relaxation function as Zipf's power law distribution, the histogram we found out to be of a function of $H_\lambda(\lambda)\sim\lambda^{n-1},\quad 0<n<1$. We infer the simultaneous multi-body relaxations have a distribution i.e. right-skewed, with large number of relaxations with lower value of rate (slow rates), with long tail of small number of relaxations with faster relaxation rates. We noted that the possibility of having Zipfian distribution arises due to very-very large ratio maximum to minimum of spreads in the relaxation rates λ 's, and possibility of connected exponential distribution of many body simultaneous relaxations. The method we obtained for getting rate distributions of relaxation rates via formation of Laplace integral when extended for finding distribution of time constants, though mathematically correct yet gave contrary physical interpretation. Thus we carried out the entire discussion with rate distribution functions and not the time constant distribution function i.e. $H_{\tau}(\tau)$. We also showed that Curie-von Schweidler law gives constituent expression of current and voltage of capacitor via use of fractional derivative, i.e. $i(t) \propto {}_0D_t^n\left[v(t)\right]$, unlike classical capacitor relation i.e. $i(t) \propto D^{(1)}\left[v(t)\right]$. This we verified by using obtained by Zipf's distribution as power law for Curie-von Schwidler current relaxation law, assuming the scale dependence equivalent relaxation rate in the classical charging equation of capacitor with scale of relaxation varying from zero to infinity; i.e. $\lambda_{eq} \sim \lambda^{1/n}$. We also related the Curie-von Schweidler relaxation law gives a time varying rate i.e. $\lambda(t) = nt^{-1}$, indicating that the relaxation starts with very-very high rate, and becomes slower and slower with elapse of time. We have in this study formulated interesting integral representations for $i(t) = t^{-n}$; call it x^{-n} those are following

$$x^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty}\left(y^{(n-1)}\right)e^{-yx}\mathrm{d}y$$

$$x^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty} y^{-(n+1)}e^{-x/y}\mathrm{d}y$$

$$x^{-n} = 2\frac{\Gamma(1-n)}{\pi}\cos\left(\tfrac{(1-n)\pi}{2}\right)\int_0^{\infty} y^{(n-1)}\cos(yx)\left(\mathrm{d}y\right)$$

$$x^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} e^{-\left(y^{(1/n)}x\right)}\mathrm{d}y$$

The note gives a possible foundation for further studies in obtaining the rate relaxation distribution functions for other non-Debye type relaxation functions, and new type of explanation regarding reasons of Zipfian distributions.

Appendix

A.1 Berberan-Santo Method

Our aim is evaluate Laplace inverse $H_\lambda(\lambda)=\mathcal{L}^{-1}\{i(t)\}$ which is given as Laplace inversion integral expression i.e.

$$H_\lambda(\lambda)=\frac{1}{2\pi \mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda$$

Here we describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. First is change of variable i.e. from 'real time variable' to 'complex time variable' as $t=x+\mathrm{i}y$; with $\mathrm{i}=\sqrt{-1}$. Here the real part i.e. x is constant as a vertical line calls it $x=x_0$ a constant. The variable y is different from imaginary part of frequency in the usual Laplace variable ω in complex frequency parameter s.With this change we have the following expression for inverse Laplace transform

$$\begin{aligned}
H_\lambda(\lambda)&=\frac{1}{2\pi \mathrm{i}}\int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty}\left(i(t)\right)e^{\lambda t}\mathrm{d}t; \qquad t\equiv x_0+\mathrm{i}y\\
&=\frac{1}{2\pi \mathrm{i}}\int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty}\left(i(x_0+\mathrm{i}y)\right)\left(e^{\lambda(x_0+\mathrm{i}y)}\right)\left(\mathrm{d}(x_0+\mathrm{i}y)\right); \qquad \mathrm{d}x=0\\
&=\frac{1}{2\pi \mathrm{i}}\int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty}\left(i(x_0+\mathrm{i}y)\right)\left(e^{\lambda x_0}e^{\mathrm{i}\lambda y}\right)\left(\mathrm{i}\mathrm{d}y\right)\\
&=\frac{e^{x_0\lambda}}{2\pi}\int_{-\infty}^{+\infty}\left(i(x_0+\mathrm{i}y)\right)e^{\mathrm{i}\lambda y}\mathrm{d}y
\end{aligned}$$

Writing $e^{\mathrm{i}\lambda y}=\cos\lambda y+\mathrm{i}\sin\lambda y$ we get the following form

$$H_\lambda(\lambda)=\frac{e^{x_0\lambda}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(i(x_0+\mathrm{i}y)\right)\cos(\lambda y)\mathrm{d}y+\mathrm{i}\int_{-\infty}^{+\infty}\left(i(x_0+\mathrm{i}y)\right)\sin(\lambda y)\mathrm{d}y\right)$$

Write $i(x_0+\mathrm{i}y)=\mathrm{Re}\{i(x_0+\mathrm{i}y)\}+\mathrm{i}\,\mathrm{Im}\{i(x_0+\mathrm{i}y)\}$ and place in above expression to get the following expression

$$\begin{aligned}
H_\lambda(\lambda)&=\frac{e^{x_0\lambda}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\left(\cos(\lambda y)\right)\right)-\left(\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\left(\sin(\lambda y)\right)\right)\mathrm{d}y\right)\\
&+\mathrm{i}\frac{e^{x_0\lambda}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\left(\cos(\lambda y)\right)\right)+\left(\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\left(\sin(\lambda y)\right)\right)\mathrm{d}y\right)
\end{aligned}$$

Given that $H_\lambda(\lambda)$ is a real function, we get the following (i.e. equating the imaginary part to zero), we write the following

$$\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\left(\cos(\lambda y)\right)+\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\left(\sin(\lambda y)\right)\right)\mathrm{d}y\right)=0$$

Thus the above expression for $H_\lambda(\lambda)$ reduces to following (i.e. considering only real part)

$$H_\lambda(\lambda)=\frac{e^{x_0\lambda}}{2\pi}\left(\int_{-\infty}^{+\infty}\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)-\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y\right)$$

But we have $i(t)=\int_0^\infty\left(H_\lambda(\lambda)\right)e^{-\lambda t}\mathrm{d}\lambda$ and by putting $t=x_0+\mathrm{i}y$ we get following

$$i(x_0+\mathrm{i}y)=\int_0^\infty\left(H_\lambda(\lambda)\right)e^{-\lambda(x_0+\mathrm{i}y)}\mathrm{d}\lambda$$

$$=\int_0^\infty e^{-x_0\lambda}\left(H_\lambda(\lambda)\right)\cos(\lambda y)\mathrm{d}y-\mathrm{i}\int_0^\infty e^{-x_0\lambda}\left(H_\lambda(\lambda)\right)\sin(\lambda y)\mathrm{d}y$$

$$\mathrm{Re}\{i(x_0+\mathrm{i}y)\}=\int_0^\infty e^{-x_0\lambda}\left(H_\lambda(\lambda)\right)\cos(\lambda y)\mathrm{d}\lambda$$

$$\mathrm{Im}\{i(x_0+\mathrm{i}y)\}=-\int_0^\infty e^{-x_0\lambda}\left(H_\lambda(\lambda)\right)\sin(\lambda y)\mathrm{d}y$$

Using this in obtained expression for $H_\lambda(\lambda)$, we observe that integrand is even function for $\lambda>0$ therefore we re-write the formula as

$$H_\lambda(\lambda)=\frac{e^{x_0\lambda}}{\pi}\int_0^\infty\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)-\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y$$

Rewriting the above obtained expression i.e. in following form

$$\frac{\pi}{e^{x_0\lambda}}\left(H_\lambda(\lambda)\right)=\int_0^\infty\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)-\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y$$

and the relation obtained earlier i.e.

$$0=\int_0^{+\infty}\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)+\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y$$

and then adding and subtracting these above two expressions we get following

$$H_\lambda(\lambda) = \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y$$

$$H_\lambda(\lambda) = -\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\mathrm{d}y$$

Write in polar form as described below

$$i(x_0+\mathrm{i}y) = \rho(y)e^{\mathrm{i}\theta(y)} = \rho(y)\left(\cos\left(\theta(y)\right)+\mathrm{i}\sin\left(\theta(y)\right)\right) \qquad \rho(y) = \left|i(x_0+\mathrm{i}y)\right| \qquad \theta(y) = \angle i(x_0+\mathrm{i}y)$$

to get following formulas

$$\begin{aligned} H_\lambda(\lambda) &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y) - \mathrm{Im}\{i(x_0+\mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\rho(y)\left(\cos\theta(y)\right)\cos(\lambda y) - \rho(y)\left(\sin\theta(y)\right)\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\rho(y)\right)\left(\cos\left(\lambda y+\theta(y)\right)\right)\mathrm{d}y \end{aligned}$$

$$\begin{aligned} H_\lambda(\lambda) &= \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \rho(y)\left(\cos\left(\theta(y)\right)\right)\cos(\lambda y)\mathrm{d}y \\ &= -\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \rho(y)\left(\sin\left(\theta(y)\right)\right)\sin(\lambda y)\mathrm{d}y \end{aligned}$$

A.2 Few examples of Laplace inversion without contour integrations-by Berberan Santo Method

a. Consider a very simple case of decay function $i(t) = (t-a)^{-1}$ and converted to complex time as follows

$$i(t) = \frac{1}{t-a}; \qquad i(x_0+\mathrm{i}y) = \frac{1}{(x_0-a)+\mathrm{i}y}$$

We know from standard Laplace pair that is $\mathcal{L}^{-1}(s\pm a)^{-1} = e^{\mp at}$. Thus, for $i(t) = (t-a)^{-1}$ we should get via inverse Laplace the rate distribution function as $H_\lambda(\lambda) = e^{a\lambda}$. The application of the Berberan-Santro formula with $x_0 > a$ yields the following

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1}{(x_0-a)+\mathrm{i}y}\right\}\cos(\lambda y)\mathrm{d}y$$
$$=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \frac{(x_0-a)}{(x_0-a)^2+y^2}\cos(\lambda y)\mathrm{d}y$$
$$=\frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y=e^{a\lambda}$$

Here we say that $e^{a\lambda}$ has integral representation as $e^{a\lambda}=\frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y$.

Particularly for $a=-1$, we have $i(t)=(t+1)^{-1}$. The condition $x_0>-1$ enables us to choose $x_0=0$ we get following integral representation for $e^{-\lambda}$ which is also rate distribution function $H_\lambda(\lambda)$ is following

$$H_\lambda(\lambda)=e^{-\lambda}=\frac{2}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{1+y^2}\mathrm{d}y$$

b. Let the decay function be $i(t)=t(t^2+1)^{-1}$ and its complex time representation as follows

$$i(t)=\frac{t}{t^2+1}; \qquad i(x_0+\mathrm{i}y)=\frac{x_0+\mathrm{i}y}{(x_0+\mathrm{i}y)^2+1}$$

Well if $F(s)=(s)/(s^2+1)$ its inverse is $\cos(t)$. Thus we should have the distribution function $H_\lambda(\lambda)=\cos(\lambda)$. With use of above formula with setting $x_0=1$

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1+\mathrm{i}y}{(1+\mathrm{i}y)^2+1}\right\}\cos(\lambda y)\mathrm{d}y$$
$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1+\mathrm{i}y}{(2-y^2)+2\mathrm{i}y}\right\}\cos(\lambda y)\mathrm{d}y$$
$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{(1+\mathrm{i}y)(2-y^2-2\mathrm{i}y)}{(2-y^2)^2+4y^2}\right\}\cos(\lambda y)\mathrm{d}y$$
$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4}\mathrm{d}y=\cos(\lambda)$$

Thus integral representation of $\cos(\lambda)$ which is also rate distribution function $H_\lambda(\lambda)$ for relaxation function $i(t)=t(t^2+1)^{-1}$ is $\cos(\lambda)=\frac{2e^{\lambda}}{\pi}\int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4}\mathrm{d}y$.

These definite integrals as obtained above are difficult to solve in closed form, even in simple cases. But, they allow obtaining results that are not so direct with contour integration, and are suited for numerical integration. This method gives integral representations of various functions as we demonstrated above-can be useful for plotting the histogram $H_{\lambda}(\lambda)$ for any relaxation function $i(t)$.

In reality of decay functions, we can take $x_0 = 0$; as decay function will not expected to have singularity at time $t > 0$.

c. For a case of 'exponential-decay' i.e. $i(t) = e^{-t/\tau_0}$, obviously this function has only one decay rate i.e. $\lambda_0 = \frac{1}{\tau_0}$. As per procedure discussed above we do Laplace inversion by taking complex time with $t = 0 + \mathrm{i}y$; making it following

$$
\begin{aligned}
& i(t) = e^{-t/\tau_0} \qquad t = \mathrm{i}y \\
& i(\mathrm{i}y) = e^{-\mathrm{i}\left(y/\tau_0\right)} = \cos\left(\tfrac{y}{\tau_0}\right) - \mathrm{i}\sin\left(\tfrac{y}{\tau_0}\right) \\
& \mathrm{Re}\left\{i(\mathrm{i}y)\right\} = \cos\left(\tfrac{y}{\tau_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
H_{\lambda}(\lambda) &= \frac{e^{x_0\lambda}}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\left\{i(x_0+\mathrm{i}y)\right\}\cos(\lambda y) - \mathrm{Im}\left\{i(x_0+\mathrm{i}y)\right\}\sin(\lambda y)\Big)\mathrm{d}y \qquad x_0 = 0 \\
&= \frac{1}{\pi}\int_0^{\infty}\Big(\mathrm{Re}\left\{i(\mathrm{i}y)\right\}\cos(\lambda y) - \mathrm{Im}\left\{i(\mathrm{i}y)\right\}\sin(\lambda y)\Big)\mathrm{d}y \\
&= \frac{1}{\pi}\int_0^{\infty}\Big(\cos\left(\tfrac{y}{\tau_0}\right)\cos(\lambda y) + \sin\left(\tfrac{y}{\tau_0}\right)\sin(\lambda y)\Big)\mathrm{d}y \\
&= \frac{1}{\pi}\int_0^{\infty} \cos\left(y\left(\lambda - \tfrac{1}{\tau_0}\right)\right)\mathrm{d}y = \delta\left(\lambda - \tfrac{1}{\tau_0}\right); \qquad \tau_0^{-1} = \lambda_0 \\
H_{\lambda}(\lambda) &= \delta(\lambda - \lambda_0)
\end{aligned}
$$

The above relation is taken from Fourier Integral as described by Joseph Fourier [16], [28], [29] as following

$$
f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\, f(\alpha)\int_{-\infty}^{\infty}\mathrm{d}p\cos(px - p\alpha)
$$

Which tantamount to introduction of Dirac Delta function as [28], [29]

$$
\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\cos(px - p\alpha)
$$

d. For a decay function of stretched exponential type i.e.

$$
i(t) = e^{-(t/\tau_0)^{\beta}}
$$

in complex time variable $t = \mathrm{i}y$ we get the polar form as following

$$i(\mathrm{i}y) = e^{-(\mathrm{i}y/\tau_0)^\beta} = e^{-(y/\tau_0)^\beta (\mathrm{i})^\beta}$$
$$= e^{-(y/\tau_0)^\beta \left(\cos(\beta\pi/2) + \mathrm{i}\sin(\beta\pi/2)\right)}$$
$$= e^{\left(-(y/\tau_0)^\beta \cos(\beta\pi/2)\right)} e^{\left(-\mathrm{i}(y/\tau_0)^\beta \sin(\beta\pi/2)\right)}; \qquad \rho(y) = e^{\left(-(y/\tau_0)^\beta \cos(\beta\pi/2)\right)}, \quad \theta(y) = \left(\frac{y}{\tau_0}\right)^\beta \sin\left(\tfrac{\beta\pi}{2}\right)$$
$$= \left(\rho(y)\right) e^{\mathrm{i}(\theta(y))}$$

Therefore we write following

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{\pi} \int_0^\infty \rho(y) \cos\left(\lambda y + \theta(y)\right) \mathrm{d}y$$
$$= \frac{1}{\pi} \int_0^\infty e^{\left(-(y/\tau_0)^\beta \cos(\beta\pi/2)\right)} \left(\cos\left(\lambda y - \left(\tfrac{y}{\tau_0}\right)^\beta \sin\left(\tfrac{\beta\pi}{2}\right)\right)\right) \mathrm{d}y$$

Doing change of variable $u = y/\tau_0$ and $\tau_0 \mathrm{d}u = \mathrm{d}y$ we obtain the following

$$H_\lambda(\lambda) = \frac{\tau_0}{\pi} \int_0^\infty \left(e^{\left(-u^\beta \cos(\beta\pi/2)\right)} \cos\left(\lambda\tau_0 u - u^\beta \sin\left(\tfrac{\beta\pi}{2}\right)\right)\right) \mathrm{d}u$$

Using other formulas we will get

$$H_\lambda(\lambda) = \frac{2\tau_0}{\pi} \int_0^\infty (\mathrm{d}u) e^{\left(-u^\beta \cos(\beta\pi/2)\right)} \cos\left(u^\beta \sin\left(\tfrac{\beta\pi}{2}\right)\right) \cos(\lambda\tau_0 u)$$
$$H_\lambda(\lambda) = \frac{2\tau_0}{\pi} \int_0^\infty (\mathrm{d}u) e^{\left(-u^\beta \cos(\beta\pi/2)\right)} \sin\left(u^\beta \sin\left(\tfrac{\beta\pi}{2}\right)\right) \sin(\lambda\tau_0 u)$$

Any other linear combination is also valid for getting solution $H_\lambda(\lambda)$.

e. The radioactive decay we write as pure exponential decay, however, Becquerel used compressed hyperbola function to describe this as

$$i(t) = \frac{1}{\left(1 + (1-\beta)\left(\frac{t}{\tau_0}\right)\right)^{1/(1-\beta)}}$$

We have following steps

$$i(\mathrm{i}y) = \frac{1}{\left(1 + (1-\beta)\left(\frac{\mathrm{i}y}{\tau_0}\right)\right)^{1/(1-\beta)}}$$
$$|i(\mathrm{i}y)| = \rho(y) = \left(1 + \left(\frac{(1-\beta)y}{\tau_0}\right)^2\right)^{-1/(2(1-\beta))} ; \quad \angle i(\mathrm{i}y) = \theta(y) = -\frac{\tan^{-1}\left(\frac{(1-\beta)y}{\tau_0}\right)}{1-\beta}$$

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\rho(y)\right)\cos\left(\lambda y + \left(\theta(y)\right)\right) \mathrm{d}y ; \qquad x_0 = 0$$

$$= \frac{1}{\pi}\int_0^\infty \mathrm{d}y \left(1+\left(\tfrac{(1-\beta)y}{\tau_0}\right)^2\right)^{-1/(2(1-\beta))} \cos\left(\lambda y - \frac{\tan^{-1}\left(\frac{(1-\beta)y}{\tau_0}\right)}{1-\beta}\right)$$

With change of variable $u = \frac{(1-\beta)y}{\tau_0}$, $\frac{\tau_0}{1-\beta}\mathrm{d}u = \mathrm{d}y$, we get

$$H_\lambda(\lambda) = \frac{\tau_0}{\pi(1-\beta)}\int_0^\infty \left(\mathrm{d}u\right)(1+u^2)^{-1/(2(1-\beta))}\cos\left(\tfrac{\lambda\tau_0 u - \tan^{-1}u}{1-\beta}\right)$$

Using other formulas we get

$$H_\lambda(\lambda) = \frac{2\tau_0}{\pi(1-\beta)}\int_0^\infty \left(\mathrm{d}u\right)\left((1+u^2)^{-1/(2(1-\beta))}\right)\cos\left(\tfrac{\tan^{-1}u}{1-\beta}\right)\cos(\lambda\tau_0 u)$$

$$H_\lambda(\lambda) = \frac{2\tau_0}{\pi(1-\beta)}\int_0^\infty \left(\mathrm{d}u\right)\left((1+u^2)^{-1/(2(1-\beta))}\right)\sin\left(\tfrac{\tan^{-1}u}{1-\beta}\right)\sin(\lambda\tau_0 u)$$

f. The rate distribution function $H_\lambda(\lambda)$ for a simple power law as

$$i(t) = \left(1+\left(\tfrac{t}{\tau_0}\right)^\alpha\right); \quad 0<\alpha<1$$

will be expressed via same rule as above

$$i(\mathrm{i}y) = \frac{1}{1+\left(\frac{\mathrm{i}y}{\tau_0}\right)^\alpha} = \frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha (\mathrm{i})^\alpha}$$

$$= \frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha e^{\mathrm{i}(\alpha\pi/2)}} = \frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha \left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)}$$

The real part of the complex function is

$$\mathrm{Re}\left\{i(\mathrm{i}y)\right\} = \frac{\left(\frac{y}{\tau_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{y}{\tau_0}\right)^{2\alpha} + 2\left(\frac{y}{\tau_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}$$

$$H_\lambda(\lambda) = \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y; \qquad x_0 = 0$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\left(\frac{y}{\tau_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{y}{\tau_0}\right)^{2\alpha}+2\left(\frac{y}{\tau_0}\right)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda y)(\mathrm{d}y); \qquad u = \frac{y}{\tau_0}, \quad \tau_0\,\mathrm{d}u = \mathrm{d}y$$

$$= \frac{2\tau_0}{\pi}\int_0^\infty \frac{(u)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}{(u)^{2\alpha}+2(u)^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda\tau_0 u)(\mathrm{d}u)$$

g. The rate distribution function $H_\lambda(\lambda)$ for a simple power law as

$$i(t) = t^{-\alpha}; \quad 0 < \alpha < 1$$

will be expressed via same rule as above

$$i(\mathrm{i}y) = \frac{1}{(\mathrm{i}y)^\alpha} = \frac{1}{(y)^\alpha(\mathrm{i})^\alpha}$$

$$= \frac{1}{(y)^\alpha e^{\mathrm{i}(\alpha\pi/2)}} = \frac{1}{(y)^\alpha\left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)}$$

$$= \frac{y^\alpha\cos\left(\frac{\alpha\pi}{2}\right)-\mathrm{i}y^\alpha\sin\left(\frac{\alpha\pi}{2}\right)}{y^{2\alpha}}$$

The real part of the complex function is

$$\mathrm{Re}\{i(\mathrm{i}y)\} = \frac{\cos\left(\frac{\alpha\pi}{2}\right)}{(y)^\alpha}$$

$$H_\lambda(\lambda) = \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y; \qquad x_0 = 0$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{y^\alpha}\right)\cos(\lambda y)(\mathrm{d}y); \qquad u = y, \qquad \mathrm{d}u = \mathrm{d}y$$

$$= \frac{2}{\pi}\cos\left(\frac{\alpha\pi}{2}\right)\int_0^\infty u^{-\alpha}\cos(\lambda u)(\mathrm{d}u)$$

From Laplace Tables we have $H_\lambda(\lambda) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$ from Laplace inverse of $i(t) = t^{-\alpha}$, therefore we write following

$$H_\lambda(\lambda)=\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}=\frac{2}{\pi}\cos\left(\tfrac{\alpha\pi}{2}\right)\int_0^\infty u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

$$\lambda^{\alpha-1}=\frac{2}{\pi}\Gamma(\alpha)\left(\cos\left(\tfrac{\alpha\pi}{2}\right)\right)\int_0^\infty u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

h. The following power series is generalized power series solution for relaxation function

$$i(t)=A\left(\frac{t}{\tau}\right)^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k}$$

Putting, $\xi=(t/\tau)$ $\alpha=\beta=0.5$ gives

$$i(\xi)=A\left(\exp(\xi)\right)\left(\mathrm{erfc}(\xi^{1/2})\right)$$

where erfc is the 'complementary error function'. Putting $\alpha=\beta=1$ we obtain relaxation response of as Debye relaxation

$$i(\xi)=A\exp(-\xi)$$

The sum contained in the relaxation function is the generalized Mittag-Leffler (GML) function; denoted by $E_{\alpha,\beta}(\xi)$ which reads as follows

$$E_{\alpha,\beta}(\xi)=\sum_{k=0}^{\infty}\frac{(\xi)^k}{\Gamma(\alpha k+\beta)}\qquad \alpha,\beta>0\qquad E_{\alpha,1}(\xi)=E_\alpha(\xi)$$

For negative ξ, we have the following expressions for GML function

$$E_{\alpha,\beta}(-\xi)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\beta)}\xi^k\qquad E_{\alpha,\beta}(-\xi^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\beta)}(\xi)^{\alpha k}$$

The asymptotic expansion for the Mittag-Leffler function for negative arguments at $\xi\uparrow\infty$ is the following [28], [29]

$$E_{\alpha,\alpha}(-\xi)\sim\frac{\alpha}{\Gamma(1-\alpha)}\xi^{-2},\quad \alpha\neq1\qquad E_{\alpha,\gamma}(-\xi)\sim\frac{1}{\Gamma(\gamma-\alpha)}\xi^{-1},\quad \gamma\neq\alpha$$

$$E_{\alpha,\alpha}(-\xi^\alpha)\sim\frac{\alpha}{\Gamma(1-\alpha)}x^{-2\alpha},\quad \alpha\neq1\qquad E_{\alpha,\gamma}(-\xi^\alpha)\sim\frac{1}{\Gamma(\gamma-\alpha)}\xi^{-\alpha},\quad \gamma\neq\alpha$$

With these approximations we express the asymptotic behavior of the relaxation function for short and long times. The relaxation function is $i(\xi)=A\xi^{\alpha-\beta}E_{\alpha,\gamma}(-\xi^\alpha)$

$$i(\xi) = A\xi^{\alpha-1}E_{\alpha,\alpha}\left(-\xi^{\alpha}\right) = \begin{cases} A\dfrac{\xi^{\alpha-1}}{\Gamma(\alpha)} & \text{as} \quad \xi \downarrow 0 \\ A\dfrac{\alpha}{\Gamma(1-\alpha)}\xi^{-(1+\alpha)} & \text{as} \quad \xi \uparrow \infty \end{cases}$$

For the other case we have

$$i(\xi) = A\xi^{\alpha-\beta}E_{\alpha,\gamma}\left(-\xi^{\alpha}\right) = \begin{cases} A\dfrac{\xi^{\alpha-\beta}}{\Gamma(\gamma)} & \text{as} \quad \xi \downarrow 0 \\ A\dfrac{1}{\Gamma(1-\beta)}\xi^{-\beta} & \text{as} \quad \xi \uparrow \infty \end{cases} \qquad \gamma = \alpha - \beta + 1$$

We write some of the important properties of Mittag-Leffler function as

$$E_{\alpha}(-\xi) = E_{2\alpha}(\xi^2) - \xi E_{2\alpha,1+\alpha}(\xi^2)$$

$$E_{2\alpha}(\xi^2) = \frac{E_{\alpha}(\xi) + E_{\alpha}(-\xi)}{2}$$

$$E_{\alpha}(-\mathrm{i}y) = E_{2\alpha}(-y^2) - \mathrm{i}yE_{2\alpha,1+\alpha}(-y^2)$$
$$\mathrm{Re}\left\{E_{\alpha}(-\mathrm{i}y)\right\} = E_{2\alpha}(-y^2)$$

We can extract the rate distribution function i.e. $H_{\lambda,\alpha}(\lambda)$ for Mittag-Leffler decay $i(t) = E_{\alpha}(-\xi)$, $\xi = t/\tau$, with the Laplace inversion formula derived in earlier section, to expand it as Laplace transform, as follows:

$$E_{\alpha}(-\xi) = \int_0^{\infty}\left(H_{\lambda,\alpha}(\lambda)\right)e^{-\lambda\xi}\mathrm{d}\lambda$$

Put $\xi = \mathrm{i}y$, thus we have $\mathrm{Re}\left\{E_{\alpha}(-\mathrm{i}y)\right\} = E_{2\alpha}(-y^2)$; and write

$$H_{\lambda,\alpha}(\lambda) = \frac{2}{\pi}\int_0^{\infty}\mathrm{Re}\left\{i(\mathrm{i}y)\right\}\cos(\lambda y)\mathrm{d}y$$
$$= \frac{2}{\pi}\int_0^{\infty}E_{2\alpha}(-y^2)\cos(\lambda y)\mathrm{d}y$$

For various α, $0 < \alpha < 1$, $\lambda > 0$, we have following integral representations for $H_{\lambda,\alpha}(\lambda)$

For $\alpha = 1$ $E_{\alpha}(-\xi) = e^{-\xi}$, the rate distribution is

$$H_{\lambda,1}(\lambda)=\frac{2}{\pi}\int_0^\infty E_2(-y^2)\cos(\lambda y)\mathrm{d}y=\frac{2}{\pi}\int_0^\infty \cosh(\mathrm{i}y)\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2}{\pi}\int_0^\infty \cos(y)\cos(\lambda y)\mathrm{d}y$$

$$=\delta(\lambda-1)$$

For $\alpha=\frac{1}{2}$ i.e. $i(t)=E_{1/2}(-\xi)$ we have

$$H_{\lambda,1/2}(\lambda)=\frac{2}{\pi}\int_0^\infty E_1(-y^2)\cos(\lambda y)\mathrm{d}y=\frac{2}{\pi}\int_0^\infty e^{-y^2}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{1}{\sqrt{\pi}}e^{-\left(\lambda^2/4\right)}$$

For $\alpha=0$ i.e. $i(t)=E_0(-\xi)$ we have $E_0(-\xi)=\frac{1}{1+\xi}$ and $\mathrm{Re}\left\{E_0(-\mathrm{i}y)\right\}=E_0(-y^2)=\frac{1}{1+y^2}$

$$H_{\lambda,0}(\lambda)=\frac{2}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{1+y^2}\mathrm{d}y=e^{-\lambda}$$

References

1. Curie, Jaques, "Recherches sur le pouvoir inducteur specifique et la conductibilite des corps cristallises". Annales de Chimie et de Physique. 17 pp 384–434. 1889.
2. Schweidler, Ergon Ritter von ,"Studien über die Anomalien im Verhalten der Dielektrika (Studies on the anomalous behaviour of dielectrics)". Annalen der Physik. 329 (14): 711–770, pp 1907.
3. Jonscher, Andrzej Ka, Dielectric Relaxation in Solids, Chelsea Dielectrics Press Limited 1983.
4. Jameson, N. Jordan; Azarian, Michael H.; Pecht, Michael, Thermal Degradation of Polyimide Insulation and its Effect on Electromagnetic Coil Impedance. Proceedings of the Society for Machinery Failure Prevention Technology 2017 Annual Conference. 2017.
5. Shantanu Das et al. Micro-structural roughness of electrodes manifesting as temporal fractional order differential equation in super-capacitor transfer characteristics. International Journal of Mathematics and Computation Vol. 20, Issue 3, pp 94-113, 2013.
6. Shantanu Das, Functional Fractional Calculus, 2nd Edition Springer-Verlag, Germany, 2011.
7. Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, Frequency dependent piecewise fractional order modeling of ultra-capacitors using hybrid optimization and fuzzy clustering, Journal of Power Sources 335, pp 98-104, 2015.
8. Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, Charge discharge energy efficiency analysis of ultra-capacitor with fractional order dynamics using hybrid optimization and its experimental validation, International Journal of Electronics & Communications (AEU) 78, pp 2714-280 2017.
9. Elwakil AS, Allagui A, Maundy BJ, Psycchalinos CA, A low frequency oscillator using super-capacitor, AEU –Int. J Electron Commun. 70(7) pp970-3, 2016.
10. Freebon TJ, Maundy B, Elwakil AS, Fractional order models of super-capacitors, batteries, fuel-cell: a survey. Mater Renew Sustain Energy 4(3) pp 1-7, 2015.
11. Freebon TJ, Maundy B, Elwakil AS, Measurement of super-capacitor fractional order model parameters from voltage excited step response, IEEE J Emerging Sel Top Circuits Syst. 3(3), pp367-76, 2013.

12. Svante Westerlund, Lars Ekstam, Capacitor Theory, IEEE Trans on Dielectrics and Insulation, Vol. 1, No. 5, pp826, 1994.
13. Moutushi Dutta Choudhury, Subrata Chandra, Soma Nag, Shantanu Das, Sujata Tarafdar, Spreading of Non-linear and Newtonian Fluids on a Solid Substrate under Pressure, Journal of Physics: Conference Series 319, (2011), IOP Publishing 2011.
14. Moutushi Dutta Choudhury, Shantanu Das, Sujata Tarafdar, Effect of loading history on visco-elastic potato starch gel, Colloids and Surfaces A: Physicochemical and Engineering Aspects 492 pp 47-53 2016.
15. Moutushi Dutta Choudhury, Subrata Chandra, Soma Nag, Shantanu Das, Sujata Tarafdar, Forced spreading and rheology of starch gel: Viscoelastic modeling with fractional calculus, Colloids and Surfaces A: Physicochemical and Engineering Aspects, 407, pp 64-70 2012.
16. JB Fourier (1822), 'Théorie analytique de la chaleur' (The Analytical Theory of Heat- English translation by Alexander Freeman, ed.); The University Press. p. 408., cf p 449 and pp 546–551. 1878.
17. Berberan-Santos Mario. N, Analytical inversion of Laplace transforms without contour integration: application to luminescence decay laws and other relaxation functions, Journal of mathematical Chemistry, Vol. 38, No.2 August 2005.
18. Powers, David M W, Application and explanation of Zipf's Law. Association for Computational Linguistics: 151–160. 1998.
19. Per Bak, "How Nature Works: The science of self-organized criticality", Springer-Verlag, New York, 1996.
20. George K Zipf, Human Behavior and the Principle of Least Effort. Addison-Wesley, 1949.
21. Ramon Ferrer i Cancho & Ricard V. Sole, Least effort and the origins of scaling in human language, Proceedings of the National Academy of Sciences of USA, 100 (3): pp 788-791, 2003.
22. S. Chiniba, R Tobazeon, Long Term Break-down of Polypropylene Films in Different Media, Conference Record of the 1985 International Conference on Properties and Application of Dielectric Materials, Xian, China, IEEE, June 24-29, pp566-584, 1985.
23. J W Borough, W G Brammer, J Burnham, Degradation of PVDF Capacitors during Accelerated Tests, Report Huges Aircraft Company, Culver City, CA, pp 219-223, 1980.
24. K Thorborg, Power Electronics, Chalmer Tech. Univ, Goteborg, Sweeden, 1895.
25. A K Jonscher, Dielectric Relaxation in Solids, Chelsea dielectric Press, London 1983.
26. A R Blythe, Electrical Properties of Polymer, Cambridge University Press, 1979.
27. Somasri Hazra, Tapati Dutta, Shantanu Das, Sujata Tarafdar, Memory of Electric Field in Laponite and How It Affects Crack Formation: Modeling through Generalized Calculus, Langmuir, DOI:10.1021/acs.langmuir.7b02034, 2017.
28. A. D. Polyanin, A. V. Manzhirov. "Handbook of Integral Equations", 2nd Edition, Chapman & Hall/CRC, Boca Raton, FL (2008)
29. Abramowitz. M, Stegun. I.A. (eds.), Handbook of Mathematical Functions, Nat. Bur. Stand. Appl. Math. Ser 55. U.S. Govt. Printing Office Washington D C (1964).
30. Ramon Ferrer i Cancho, Richard V Sole, Least effort and origins of scaling in human language, PNAS, Vol. 100, No.3, pp 788-791 (2003).
31. Davis Easley, Jon Kleinberg, Power Laws and Rich Get Richer Phenomena, (Chapter-18): Networks Crowds and Markets: Reasoning about a Highly Connected World. Cambridge University Press, 2010.