A New Theory of Capacitor

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ABSTRACT

In this study, we revisit the concept of classical capacitor theory-and derive possible new explanations to definition of capacitance, charge stored in capacitor. We introduce the capacity function with respect to time to describe capacitor. Here we will describe that charge stored at any time in a capacitor is convolution integration of capacity function of capacitor and voltage stress across it. This approach however is different to conventional method where we multiply the capacity and voltage functions to obtain charge stored. This new concept is in line with observation of charge stored, relaxation current in form of impulse function i.e. for ideal geometrical constant capacity capacitor, and power-law decay current that is given by universal dielectric relaxation called as Curie von-Schweidler law, when an uncharged capacitor is impressed with a constant voltage stress. This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current, that we will discuss with this new concept of redefining the charge store definition i.e. via this convolution integral approach.

Keywords: convolution integral, fractional derivative, fractional integration, Curie-von Schweidler law, fractional capacity, geometrical capacity, time varying capacity function

Introduction

The classical geometric capacitor or a constant capacitor having constant value of Farad means that it has constant value at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used i.e. having relative permittivity ε_r is lossless and is constant (and is a purely real number) at all frequencies; and the capacity is given as $C_1 = \varepsilon_r A/d$ i.e. by using geometric factor of area to the electrode separation ratio. This we have learnt in textbooks. This ideal capacity is constant at all the frequencies is called geometric capacity. This constant in frequency domain is actually an impulse function in time domain. A general practical capacitor, which is not a constant in frequency domain, is having a function in time domain and we call it as capacity function in time. Therefore, we say that charge stored in capacitor, as a function of time is not usual multiplication operation of capacity function and voltage stress; instead, the charge is convolution integral of the two. However, the charge described in frequency domain as a function of frequency is multiplication operation of frequency domain functions of capacity-function and voltage-function. We will revise this concept of capacitor in the paper, and derive various concepts.

The Curie-von Schweidler law relates to relaxation current in dielectric when a step DC voltage is applied and is given by $i(t) \sim t^{-n}$, where $t > 0$ and the power (exponent) i.e. n is called relaxation constant or decay constant, where $0 < n < 1$ [1]-[4], [12], [35].We note that n is non-integer. This relaxation law is taken as universal law, at least for dielectric relaxations. The Curie-von Schweidler behavior has been observed in many instances, since late 19th Century, such as those shown in dielectric studies and experiments [3] [4], [12], [22]-[26], [35]. This power law relaxation of the non-Debye type i.e. $i(t) \sim t^{-n}$ has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of time constants τ or relaxation rates λ varying from near zero to infinity [4], [5], and [6]. The observations of power law relaxation are also made in the experiments and studies with super-capacitors [7]-[11]. These studies also indicate the fractional calculus is used as constituent expression to describe super-capacitors. The use of empirical power law i.e. Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12] [35], by taking the concept of charge stored at any time as usual product of capacity function and voltage stressed. We will revise the concept of capacitor in classical theory and apply the new concept of charge stored at ant time as convolution integral of capacity function and the voltage stress and also apply this concept in capacitors with observed Curie-von Schewdler relaxation current, and obtain same results as in [12] and [35]. We will also point out the differences with this new approach to the earlier approach in finding the capacity function.

A brief about ideal textbook capacitor

What we know about geometric capacitor or a constant capacitor of say value C_1 is a constant value of Farad at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used ε_r is lossless and is constant at all frequencies; and the capacity is given as $C_1 = \varepsilon_r A/d$ i.e. by using geometric factor of area to separation ratio. This ideal capacity is constant at all the frequencies is called geometric capacity. Therefore, if we say s as complex frequency (Laplace variable) then this constant capacity is given as following

$$\begin{aligned} C(s) &= C_1 \qquad s = \mathrm{i}\omega \\ C(\omega) &= C_1 \end{aligned} \tag{1}$$

The Laplace complex frequency is written in (1)as $s = \mathrm{i}\omega$ for writing sinusoidal or steady state frequency domain analysis [32]. Since the inverse Laplace transform of $F(s) = 1$ gives time function $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \delta(t)$ i.e. a Dirac delta function, we say the time varying capacity function of geometric capacitor is following

$$C(t) = C_1\delta(t) \tag{2}$$

Therefore, we say that a constant ideal capacitor has a capacity of Dirac delta function. Thus for example if the capacity of a capacitor is a function of frequency say as $C(s) = C_m s^{-m}$; then the time varying capacity function for this capacitor is

$$C(t) = \frac{C_m}{(m-1)!} t^{m-1}; \qquad t > 0 \tag{3}$$

The capacity function is constant $C(t) = C_0$ for $t > 0$ only if the frequency function is $C(s) = C_0 s^{-1}$.

Therefore, say when we apply a voltage function $v(t)$ to uncharged capacity we write the charge stored at any time as convolution integral as follows

$$\begin{aligned} q(t) &= C(t) * v(t) \\ &= \int_{-\infty}^{t} \big(C(t-x)\big)\big(v(x)\big)\mathrm{d}x = \int_{-\infty}^{t} \big(C(y)\big)\big(v(t-y)\big)\mathrm{d}y \end{aligned} \tag{4}$$

This is against conventional way of writing the charge i.e.

$$q(t) = C(t)v(t) \tag{5}$$

This argument we will explain in the subsequent section.

In reality the capacity of a capacitor, say of $1\mu\mathrm{F}$ means this value is at particular frequency of measurement standard is at $1\mathrm{kHz}$ (also depends on application). Practically due to losses in ε_r the value of capacity of capacitor is varying in frequency; therefore in reality we have time varying capacity $C(t)$.

Reviewing concept of charge in constant capacitor-the classical theory

We have standard expression of 'impedance of a capacitor' i.e. $Z(s)$ expressed in frequency domain as following

$$Z(s) = \frac{V(s)}{I(s)} = \frac{1}{C_1 s} \tag{6}$$

Thus from (6), we have the capacity function expressed in frequency domain as a function as

$$C_1 = \frac{s^{-1}I(s)}{V(s)} \tag{7}$$

We note that C_1 is Laplace transformed quantity, i.e. $C_1 = \mathcal{L}\{C(t)\}$; and in this case of 'constant capacity' the capacity function in time is $C(t) = C_1\delta(t)$ (2). Therefore, we have in frequency domain representation of capacitor as function of Laplace variable s, so we call it as $C(s) = \mathcal{L}\{C(t)\}$. Therefore, general relation of capacity in frequency domain we have

$$C(s) = \frac{s^{-1}I(s)}{V(s)} \tag{8}$$

The numerator term in (8) i.e. $s^{-1}I(s)$ in time domain is $\int_0^t i(x)\mathrm{d}x$ [32] that is charge the $q(t)$, i.e. $q(t) = \int_0^t i(x)\mathrm{d}x$ with its Laplace transform as $Q(s) = \mathcal{L}\{q(t)\}$. Therefore, we write charge in frequency domain as following

$$Q(s) = C(s)V(s) \tag{9}$$

This (9) is the expression in frequency domain. In the time domain, we write the charge equation as convolution integral [32], i.e. using $\mathcal{L}^{-1}\{(F_1(s))(F_2(s))\} = \left(\mathcal{L}^{-1}\{F_1(s)\}\right) * \left(\mathcal{L}^{-1}\{F_2(s)\}\right) = f_1(t) * f_2(t)$ i.e. the following

$$q(t) = C(t) * v(t) \\ = \int_{-\infty}^{t} C(t-x)\big(v(x)\big)\mathrm{d}x \tag{10}$$

Where convolution operation is denoted as $*$, and the convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as

$$f_1(t) * f_2(t) = \int_{-\infty}^{t} \big(f_1(t-x)\big)\big(f_2(x)\big)\mathrm{d}x \tag{11}$$

Let an uncharged capacitor of constant capacity at $t=0$, of value C_1 is charged with a step voltage applied at $t=0$ i.e. $v(t) = V_{BB}\big(u(t)\big); \quad t \geq 0$. We say charged stored at any time for $t>0$ is $q(t) = C_1 V_{BB}$, whereas the charge is $q(t)=0$ for $t<0$. Thus the charge in time domain is a step function, we denote that as $q(t) = C_1 V_{BB}\big(u(t)\big)$; with $u(t)$ as unit-step at $t=0$. Laplace transform of this step charge is

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{s} \qquad\qquad Q(s) = \mathcal{L}\left\{C_1 V_{BB} u(t)\right\} = \frac{C_1 V_{BB}}{s} \tag{12}$$

The first derivative of charge i.e. $q^{(1)}(t)$ gives the charging current i.e.

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ = \frac{\mathrm{d}}{\mathrm{d}t} C_1 V_{BB} u(t) = C_1 V_{BB}\big(\delta(t)\big) \tag{13}$$

This is classical result that we all know is as per classical capacitor-theory that is charging current is impulse function at the time of application of voltage step. This impulse current also comes from circuit equation i.e. $\frac{1}{C_1}\int i(t)\mathrm{d}t = V_{BB}\big(u(t)\big)$; and the classical theory deals with geometrical capacitor given by $C_1 = \varepsilon_r A/d$. Now let us look at convolution integral, for $q(t) = C(t) * v(t) = \int_{-\infty}^{t}\big(C(t-x)\big)\big(v(x)\big)\mathrm{d}x$ for $t>0$ i.e. where we have $v(x) = V_{BB}$, for $t>0$. Only if we define $C(t)$ as function of time as the capacity function i.e. $C(t) = C_1\big(\delta(t)\big)$ we will be getting $q(t) = C_1 V_{BB}$ for $t \geq 0$ demonstrated in following steps

$$q(t) = C(t) * \big(v(t)\big) = \int_{-\infty}^{t} C(t-x)\big(v(x)\big)\mathrm{d}x; \quad C(x) = C_1\big(\delta(x)\big), \quad v(x) = V_{BB}, \quad x \geq 0 \\ = \int_{0}^{t} C_1\big(\delta(t-x)\big)\big(V_{BB}\big)\mathrm{d}x; \qquad t \geq 0 \\ = C_1 V_{BB}; \qquad t \geq 0 \tag{14}$$

We have used identity $\int\big(\delta(x_0 - x)\big)\big(f(x)\big)\mathrm{d}x = f(x_0)$, i.e. property of delta function. Thus from above (14) we get charge as step function at $t=0$, given as

$$q(t) = C_1 V_{BB}\big(u(t)\big) \tag{15}$$

The meaning of capacity function $C(t)$ in time domain is $C(t)=C_1\left(\delta(t)\right)$ i.e. an impulse of height C_1 at the time of application of voltage excitation (i.e. $t=0$), refer Figure-1. Whereas in frequency domain, the definition of capacity is, i.e. for geometrical capacity is $C(s)=C_1$ i.e. $C(s)=\mathcal{L}\left\{C_1\delta(t)\right\}=C_1$ a constant value at all frequencies, that we have discussed in (1) earlier.

Therefore, with $V(s)=V_{BB}/s$ we get $Q(s)=C(s)V(s)=C_1V_{BB}/s$. Thus when we say a capacitor is having a constant value, it implies that it is an 'impulse' function at the time of application of voltage stress; in time domain, say at $t=0$. The constant capacity C_1, is written as capacity function of time as $C(t)=C_1\left(\delta(t)\right)$. For any other time say at time $t=t_0$ of application of voltage-stress the classical geometrical constant capacitor is expressed as capacity function $C(t)=C_1\left(\delta(t-t_0)\right)$, which is uncharged and applied with voltage stress at $t=t_0$.

Say we apply $v(t)=\cos at$ at $t=0$, for $t\geq 0$ (then Laplace transform is $V(s)=\frac{s}{s^2+a^2}$), to an uncharged constant capacitor $C(s)=C_1$. This gives $Q(s)=C_1\left(\frac{s}{s^2+a^2}\right)$ implying $q(t)=C_1\cos at;\quad t\geq 0$. Thus we observe for a constant capacitor, there is no phase difference between $v(t)$ and $q(t)$. We do following convolution integration. Also, refer Figure-1 for curves of $v(t)$ and $q(t)$ that have no delay implying no phase difference for a constant capacity case.

$$\begin{aligned} q(t)=C(t)*\left(v(t)\right)&=\int_{-\infty}^{t}C(t-x)\left(v(x)\right)\mathrm{d}x;\quad C(x)=C_1\left(\delta(x)\right),\quad v(x)=\cos ax;\quad x\geq 0\\ &=\int_0^t C_1\left(\delta(t-x)\right)\left(\cos ax\right)\mathrm{d}x;\qquad t\geq 0\\ &=C_1\cos at;\qquad t\geq 0 \end{aligned} \tag{16}$$

We have used identity $\int\left(\delta(x_0-x)\right)\left(f(x)\right)\mathrm{d}x=f(x_0)$, i.e. property of delta function. Thus, we have general expression for any time varying voltage $v(t)$ applied at uncharged capacitor with geometrical capacity given by capacity function as $C(t)=C_1\left(\delta(t)\right)$, will have charge $q(t)$ for $t\geq 0$ as following

$$\begin{aligned} q(t)=C(t)*\left(v(t)\right)&=\int_{-\infty}^{t}\left(C(t-x)\right)\left(v(x)\right)\mathrm{d}x;\quad C(x)=C_1\left(\delta(x)\right),\quad v(x);\quad x\geq 0\\ &=\int_0^t C_1\left(\delta(t-x)\right)\left(v(x)\right)\mathrm{d}x;\qquad t\geq 0\\ &=C_1\left(v(t)\right);\qquad t\geq 0 \end{aligned} \tag{17}$$

Now we differentiate the above expression of $q(t)$ to write following expression

$$
\begin{aligned}
i(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1\left(v(t)\right)\right), \qquad t \ge 0 \\
&= v(t)\frac{\mathrm{d}C_1}{\mathrm{d}t} + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\
&= \left(v(t)\right)\left(C_1\left(\delta(t)\right)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} = C_1\left(v(0)\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\
&= i(0) + i(t), \qquad t \ge 0
\end{aligned}
\tag{18}
$$

The first term at RHS indicate the value of current at $t = 0$. The constant function starting at $t = 0$ i.e. C_1 when differentiated gives $C_1\delta(t)$. This unit delta functions at $t = 0$, i.e. $\delta(t)$ when multiplied by $v(t)$ gives $v(0)\delta(t)$. This is from property $\int\left(\delta(x_0 - x)\right)\left(f(x)\right)\mathrm{d}x = f(x_0)$, differentiation of this gives $\left(\delta(x_0 - x)f(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x} f(x_0) = f(x_0)\delta(x)$. Thus at $t = 0$ we have $i(0) = C_1 v(0)$ and $i(0) = 0$ for $t > 0$. Compositely we write $i(0) = C_1 v(0)\left(\delta(t)\right)$, i.e. specifying its value at only $t = 0$. The second term is $i(t)$ for $t \neq 0$, that is $i(t) = C_1\left(v^{(1)}(t)\right)$ (Refer Figure-1).

$$
i(t) = C_1 v(0)\left(\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{19}
$$

As an example, we take $v(t) = V_{BB}u(t)$ a step input at time $t = 0$. We have $v^{(1)}(t) = 0$ for $t > 0$; and at $t = 0$ we have, $v(0) = V_{BB}$ so $i(0) = C_1 V_{BB}\left(\delta(t)\right)$; this makes $i(t) = C_1 V_{BB}\left(\delta(t)\right), \quad t \ge 0$. This is for geometrical capacity charging current is impulse function.

Generally the capacitance is not a constant parameter of the capacitor, it varies in frequency and therefore in time too. The constant capacitor concept is approximation when we assume the relative permittivity ε_r to be constant (note that geometrical capacity we define as $C_1 = \varepsilon_r A/d$). We note that only a loss free capacitor has a constant capacitance in frequency domain. Losses manifest themselves in frequency domain as a phase angle, ϕ by which $q(t)$ lags $v(t)$, or given as loss tangent i.e. $\tan\phi$ in the charge expression of capacitor i.e.

$$
\begin{aligned}
q(t) &= C(t) * v(t) \\
Q(s) &= \left(C(s)\right)\left(V(s)\right)
\end{aligned}
\tag{20}
$$

For $C(\omega) = C_1$, the constant geometrical capacitor with capacity function as $C(t) = C_1\delta(t)$ (1) (2), we have $\tan\phi = \frac{\mathrm{Im}\,C(\omega)}{\mathrm{Re}\,C(\omega)} = 0$; that is ideal lossless capacitor.

Therefore, we say that charge stored in capacitor, as a function of time is not multiplication operation of capacity and voltage; instead, the charge is convolution integral. However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage. A time varying capacity has a delay between $v(t)$ and $q(t)$ that is shown in Figure-1 (for time varying capacity case), where $q(t)$ lags $v(t)$.

From the expression, $C(s)=Q(s)/V(s)$ we write the time varying capacity $C(t)$ as by use of convolution integral in the following steps

$$\begin{aligned} C(s)&=\frac{Q(s)}{V(s)} \\ \mathcal{L}^{-1}\left\{C(s)\right\}&=\mathcal{L}^{-1}\left\{\left(Q(s)\left(V(s)\right)^{-1}\right)\right\} \\ C(t)&=\mathcal{L}^{-1}\left\{Q(s)\right\}*\mathcal{L}^{-1}\left\{(V(s)^{-1}\right\} \\ &=\left(q(t)\right)*\left(\left(v(t)\right)^{-1}\right) \end{aligned} \tag{21}$$

From above derivation (21), we say that capacity i.e. $C(t)\neq q(t)/v(t)$ i.e. not the usual ratio of charge to voltage in time domain, but it is given as convolution expression i.e.

$$\begin{aligned} C(t)&=q(t)*\left(v(t)\right)^{-1} \\ &=\int_{-\infty}^{t}\frac{q(t-x)}{v(x)}\mathrm{d}x=\int_{-\infty}^{t}\frac{q(x)}{v(t-x)}\mathrm{d}x \end{aligned} \tag{22}$$

Let us verify, with $q(t)=\left(C_1V_{BB}\right)\left(u(t)\right)$ i.e. at $t=0$ and $q(t)=0$ for $t<0$, and $v(t)=V_{BB}u(t)$ i.e. a step voltage at $t=0$, gives

$$\begin{aligned} C(t)&=q(t)*\left(v(t)\right)^{-1} \\ &=\int_{-\infty}^{t}\frac{C_1V_{BB}u(t-x)}{V_{BB}u(x)}\mathrm{d}x=\int_{-\infty}^{t}\frac{C_1V_{BB}u(x)}{V_{BB}u(t-x)}\mathrm{d}x \\ &=C_1\left(u(t)*u(t)^{-1}\right) \\ &=C_1\left(\delta(t)\right) \end{aligned} \tag{23}$$

We have used inverse identity i.e. $f*f^{-1}=\delta$.

Therefore, capacity at any time is history of ratio of charge to voltage given by convolution integral. We can verify with say $q(t)=\left(C_1\cos at\right)u(t)$ for $t\geq 0$ with $v(t)=\left(C_1\cos at\right)u(t)$ for $t\geq 0$ gives the following

$$\begin{aligned} C(t)&=q(t)*\left(v(t)\right)^{-1} \\ &=\int_{-\infty}^{t}\frac{C_1\cos a(t-x)}{\left(\cos ax\right)}\mathrm{d}x=\int_{-\infty}^{t}\frac{C_1\cos ax}{\cos a(t-x)}\mathrm{d}x \\ &=C_1\left(\cos(at)*\left(\cos(at)\right)^{-1}\right) \\ &=C_1\left(\delta(t)\right) \end{aligned} \tag{24}$$

We have used inverse identity i.e. $f*f^{-1}=\delta$. We note here the formula used in [12], [35] is $C(t)=q(t)/v(t)$.

Fractional Derivative directly from Curie-von Schweidler Law

Practically on applying a step input voltage $v(t) = V_{BB}$ Volts at $t = 0$ to a capacitor which is initially uncharged; we get a power-law decay of current given by empirical Curie-von Schweidler as $i(t) \sim t^{-n}; \quad 0 < n < 1$ [12], [35]. That we write in following way as indicated by experimental studies [12], [22]-[26].

$$i(t) = K_n \frac{V_{BB}}{t^n} \qquad t > 0 \tag{25}$$

The parameter K_n is proportionality constant, while [12], [35] use the proportionality constant as $1/h_1$. This is from observation and the evaluation of order of power-law function is $0.5 < n < 1$ [7]-[12]. Let the capacitor be excited by a step input of V_{BB} Volts, i.e. written as $v(t) = V_{BB}\left(u(t)\right)$, where $u(t)$ is unit step function. The Laplace transform of step input is following

$$V(s) = \mathcal{L}\left\{v(t)\right\} = \mathcal{L}\left\{V_{BB}\left(u(t)\right)\right\} = \frac{V_{BB}}{s} \tag{26}$$

and then taking Laplace transform of (25) power-law decay current using $\mathcal{L}\left\{t^m\right\} = m!s^{-(m+1)}$, [32] we write following

$$\begin{aligned} I(s) &= \mathcal{L}\left\{i(t)\right\} = \mathcal{L}\left\{K_n V_{BB} t^{-n}\right\} \\ &= K_n V_{BB}\left(\frac{(-n)!}{s^{-n+1}}\right) \end{aligned} \tag{27}$$

using the formula for generalization of factorial i.e. $(\alpha - 1)! = \Gamma(\alpha)$ [28], [29], [35]we get the following expressions

$$\begin{aligned} I(s) &= K_n \frac{\Gamma(1-n)V_{BB}}{s^{1-n}} \\ &= K_n \frac{\Gamma(1-n)}{s^{-n}}\left(\frac{V_{BB}}{s}\right) \end{aligned} \tag{28}$$

We get Transfer function [32] of capacitor as following expression

$$\begin{aligned} G(s) = \frac{I(s)}{V(s)} &= \frac{K_n \frac{\Gamma(1-n)}{s^{-n}}\left(\frac{V_{BB}}{s}\right)}{\left(\frac{V_{BB}}{s}\right)} \\ &= K_n\left(\Gamma(1-n)\right)s^n = C_n s^n \qquad C_n = K_n\left(\Gamma(1-n)\right) \end{aligned} \tag{29}$$

This expression i.e. $G(s)$ is admittance expression in complex frequency (s) domain of a capacitor. From here, we write impedance expression for capacitor as following

$$Z(s) = \frac{1}{C_n s^n}, \quad 0 < n < 1 \tag{30}$$

From the obtained expression (29) i.e. $I(s)=C_n s^n\left(V(s)\right)$ and by Laplace inversion by using the identity $\mathcal{L}^{-1}\left\{s^n F(s)\right\}={}_0D_t^n\left[f(t)\right]$ i.e. fractional derivative operation [6], [33], we get the constituent relation for capacity as following

$$i(t)=C_n\left({}_0D_t^n\left[v(t)\right]\right),\qquad 0<n<1 \tag{31}$$

The 'fractional capacity' C_n is in unit of $\mathrm{Farad}/\mathrm{sec}^{1-n}$; which is constant given by $C_n=K_n\left(\Gamma(1-n)\right)$. This fractional derivative expression gives a new capacitor theory [12], [35] and we utilize this above formula to find characteristics of super-capacitors, variation of n with current excitation, and energy discharged to energy stored [22]-[26]. Classically the expression of capacitor is $i(t)=C_1\left(D_t^{(1)}\left[v(t)\right]\right)$ i.e. with integer one-whole order derivative.

Curie-von Schweidler law gives a different approach for capacitor theory based on fractional calculus [12], [35]. In experimental observations, we find that capacitor has fractional order impedance [7]-[12], [22]-[26], [35]. This section gives us the understanding that this empirical law i.e. Curie-von Schweidler law gives a relation of voltage and current of capacitor by using fractional derivative. We will derive this (31) by the new approach of the definition of charge in the subsequent section.

Charge in a capacitor with Curie-von Schweidler relaxation current and its consequence to have time varying capacitor

For Curie-von Schweidler law we have relaxation current as noted earlier empirically expressed as $i(t)=K_nV_{BB}t^{-n},\quad 0<n<1$ for $t>0$, i.e. when uncharged capacitor is applied with a step voltage $v(t)=V_{BB}\left(u(t)\right)$ at $t=0$. This empirical expression of current relaxation gives a relation of incremental charge Δq (or $\mathrm{d}q$ in infinitesimal small limit) when 'pulse' of a voltage of magnitude V_{BB} is applied for a duration Δt (or in infinitesimal small limit $\mathrm{d}t$) given by following expressions

$$\Delta q=\frac{K_nV_{BB}\Delta t}{t^n}\qquad\qquad \mathrm{d}q=\frac{K_nV_{BB}\mathrm{d}t}{t^n} \tag{32}$$

With this above expression (or by $q(t)=\int_0^t\mathrm{d}q$) we write the charge accumulated for this power law decay current as following

$$\begin{aligned}q(t)&=\int_0^t\mathrm{d}q=\int_0^t\frac{K_nV_{BB}\mathrm{d}x}{x^n}\\&=\frac{K_nV_{BB}}{(1-n)}t^{1-n},\qquad 0<n<1\qquad t>0\end{aligned} \tag{33}$$

From the expression in frequency domain (8) i.e. $C(s)=\left(s^{-1}I(s)\right)/\left(V(s)\right)=\left(Q(s)\right)/\left(V(s)\right)$ we have for $i(t)=K_nV_{BB}t^{-n}$, $I(s)=K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}$, and $V(s)=V_{BB}/s$, this gives $C(s)$ as following

$$C(s)=\frac{\left(s^{-1}I(s)\right)}{V(s)}=\frac{s^{-1}\left(K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}\right)}{V_{BB}s^{-1}}$$
$$=\frac{K_n\left(\Gamma(1-n)\right)}{s^{1-n}};\qquad m!=\Gamma(1+m) \tag{34}$$
$$=K_n\frac{(-n)!}{s^{1+(-n)}}$$

Now doing inverse Laplace transform using $\mathcal{L}^{-1}\left\{(m!)/s^{(1+m)}\right\}=t^m$ of above we get time dependent capacity function $C(t)$ as following

$$C(t)=K_nt^{-n};\quad 0<n<1,\quad t>0 \tag{35}$$

Using the convolution integral with this time dependent capacity function (35) step voltage applied at time zero, i.e. we get following expression for charge stored as following

$$q(t)=\left(C(t)\right)*\left(v(t)\right)=\int_{-\infty}^{t}\left(C(t-x)\right)\left(v(x)\right)\mathrm{d}x,\qquad C(x)=K_nx^{-n},\quad v(x)=V_{BB};\quad t>0$$
$$=\int_0^t K_n\left((t-x)^{-n}\right)\left(V_{BB}\right)\mathrm{d}x,\qquad 0<n<1 \tag{36}$$
$$=-V_{BB}K_n\left.\frac{(t-x)^{1-n}}{1-n}\right|_{x=0}^{x=t}=\frac{V_{BB}K_n}{1-n}t^{1-n}$$

The above-obtained expression $q(t)=\frac{V_{BB}K_n}{1-n}t^{1-n}$ obtained via our formula $q(t)=C(t)*v(t)$ is same as we got via $q(t)=\int_0^t\mathrm{d}q$ above in (33).

Points about breakdown mechanism of capacitor and loss tangent and comparison with earlier theories

We note here that for $0<n<1$, the charge $q(t)\uparrow\infty$ as $t\uparrow\infty$, when the capacity function is $C(t)=K_nt^{-n}$, following Curie-von Schweidler decay current. Whereas for a classical capacity function i.e. given as $C(t)=C_1\delta(t)$, the charge at $t\uparrow\infty$ is $q(\infty)=C_1V_{BB}$. This observation i.e. $q(t)\uparrow\infty$ for $t\uparrow\infty$ in our derivation is with convolution formula i.e. $q(t)=C(t)*v(t)$ is in line with the observations in [12], [35], where the used expression for charge is $q(t)=\left(C(t)\right)\left(v(t)\right)$. This is the new idea of breakdown of capacitors due to accumulation of enough charge at a constant voltage (even though voltage is less than the breakdown limit of dielectric proposed in [12], [35].

In [12] and [35] the charge formula used is $C(t)=q(t)/v(t)$ and not via convolution approach that we discussed in this paper; and with this in [12] and [35] gets the time dependent capacity function as following (where the constant h_1 is used in Curie von-Schweidler relaxation current and $h_1\equiv\left(K_n\right)^{-1}$)

$$C(t)=\frac{t^{1-n}}{h_1(1-n)},\qquad t>0;\qquad 0<n<1 \tag{37}$$

The frequency domain representation for $C(t)$ obtained in [12] and [35] is following

$$C(s)=\frac{(1-n)!}{h_1(1-n)}s^n\,,\qquad 0<n<1,\quad s=\mathrm{i}\omega$$
$$C(\omega)=\left(\frac{\omega^n(1-n)!}{h_1(1-n)}\right)\left(\cos\tfrac{n\pi}{2}+\mathrm{i}\sin\tfrac{n\pi}{2}\right) \tag{38}$$

Here from (38) we expresses loss tangent as $\tan\phi=\frac{\operatorname{Im}C(\omega)}{\operatorname{Re}C(\omega)}=\tan\left(\frac{n\pi}{2}\right)$, which is not correct. This above expression (37) and (38) of [12] and [35] says that the time varying capacity function will be growing to infinity as time grows. Also in frequency domain, we will be getting infinite value at infinite frequency. This gives us notion of unrealistic property of capacity function.

Whereas we have from our new derivation (35) the following

$$C(t)=K_n t^{-n}\,;\qquad t>0,\quad 0<n<1$$
$$C(s)=K_n\left(\Gamma(1-n)\right)s^{-(1-n)}\,;\qquad s=\mathrm{i}\omega \tag{39}$$
$$C(\omega)=K_n\left(\Gamma(1-n)\right)\omega^{-(1-n)}\left(\cos\tfrac{(1-n)\pi}{2}-\mathrm{i}\sin\tfrac{(1-n)\pi}{2}\right)$$

where the capacity function tends towards zero for large time and frequency. From above we get loss tangent as

$$\tan\phi=\frac{\operatorname{Im}C(\omega)}{\operatorname{Re}C(\omega)}=\tan\left(\tfrac{(1-n)\pi}{2}\right) \tag{40}$$

which is also as reported in [12], [35].

However, [12] and [35] gives it other expressions, same as that we will derive and report subsequently.

Further derivations regarding fractional capacitor in conjugation to classical capacitor

Now we do the steps as we did for classical capacitor, from the impedance relation i.e.

$$Z(s)=s^{-n}\frac{1}{C_n};\qquad 0<n<1 \tag{41}$$

with $C_n(s)=K_n\left(\Gamma(1-n)\right)$ as obtained in earlier initial section, a constant in units of $\mathrm{Farad}/\sec^{1-n}$.We note that $C_n(s)=K_n\left(\Gamma(1-n)\right)$ is in units of $\mathrm{Farad}/\sec^{1-n}$; a "fractional form" of unit, defining a "fractional capacity" as constant in the frequency domain. Thus, we expect that in time domain the fractional capacity be given by delta function at $t=0$ i.e.

$$C_n(t)=\left(K_n\left(\Gamma(1-n)\right)\right)\left(\delta(t)\right) \tag{42}$$

From this (42) we write following steps, with $C_n(s)=\mathcal{L}\left\{C_n(t)\right\}$, $\mathcal{L}^{-1}\left\{s^{-n}F(s)\right\}={}_0\mathcal{I}_t^n f(t)$ defining fractional integration [6], [33] of fractional order $0<n<1$

$$C_n(s)=\frac{s^{-n}I(s)}{V(s)}=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n}\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad 0<n<1$$

$$=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\,{}_0\mathcal{I}_t^{1}\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\int_0^t i(x)\mathrm{d}x\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad q(t)=\int_0^t i(x)\mathrm{d}x$$

$$=\frac{\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad {}_0\mathcal{I}_t^{n-1}f(t)={}_0D_t^{1-n}f(t) \tag{43}$$

$$\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}=\left(\mathcal{L}\left\{v(t)\right\}\right)\left(\mathcal{L}\left\{C_n(t)\right\}\right)$$

$$\mathcal{L}\left\{C_n(t)\right\}=\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}\left(\mathcal{L}\left\{v(t)\right\}\right)^{-1}$$

$$C_n(t)=\left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}$$

Therefore, we write following formulas for fractional capacitor as with conjugation to classical capacitor theory as

$$\begin{aligned}
&C_n(t)=\left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1};\qquad 0<n<1\\
&{}_0D_t^{1-n}\left[q(t)\right]=\left(C_n(t)\right)*\left(v(t)\right)\\
&q(t)={}_0D_t^{n-1}\left[\left(C_n(t)\right)*\left(v(t)\right)\right]\\
&\qquad\quad=\left({}_0D_t^{n-1}\left[\left(C_n(t)\right)\right]\right)*\left(v(t)\right)\\
&C(t)={}_0D_t^{n-1}\left[\left(C_n(t)\right)\right]={}_0\mathcal{I}_t^{1-n}\left[\left(C_n(t)\right)\right]\\
&q(t)=\left(C(t)\right)*\left(v(t)\right)
\end{aligned} \tag{44}$$

Using (36) i.e. $q(t)=\frac{K_nV_{BB}}{1-n}t^{1-n}$ in (44), we get ${}_0D_t^{1-n}\left[q(t)\right]=\frac{K_nV_{BB}}{1-n}\left(\frac{\Gamma(1-n+1)}{\Gamma(1-n+1-1+n)}t^{1-n-1+n}\right)=K_nV_{BB}\left(\Gamma(1-n)\right)$ this we have got by formula ${}_0D_x^{\upsilon}x^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\upsilon)}x^{\beta-\upsilon}$ [6], [33]. Thus, we obtain the following

$$\begin{aligned}
C_n(t)&=\left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}\\
&=\left(K_nV_{BB}\left(\Gamma(1-n)\right)\left(u(t)\right)\right)*\left(V_{BB}\left(u(t)\right)\right)^{-1}\\
&=K_n\left(\Gamma(1-n)\right)\delta(t)
\end{aligned} \tag{45}$$

We used identity i.e. $f*f^{-1}=\delta$, the inverse relation in (45). We consider

$$C(t)={}_0D_t^{(n-1)}\left[C_n(t)\right];\quad t>0,\quad 0<n<1 \tag{46}$$

i.e. time varying capacity function as defined as fractional integral of the order $1-n$ for the fractional capacity function i.e. $C_n(t)$ i.e. in units of $\mathrm{Farad}/\sec^{1-n}$., which is constant in frequency domain as $C_n(\omega)=K_n\left(\Gamma(1-n)\right)$ i.e. a fractional capacitor. Using $C_n(t)=K_n\left(\Gamma(1-n)\right)\delta(t)$ as obtained above (42), we write following

$$
\begin{aligned}
C(t) &= {}_0D^{n-1}\left[C_n(t)\right] \\
&= {}_0\mathcal{I}_t^{1-n}\left[\left(K_n\left(\Gamma(1-n)\right)\left(\delta(t)\right)\right)\right] \\
&= K_n\left(\Gamma(1-n)\right)\left({}_0\mathcal{I}_t^{1-n}\left[\delta(t)\right]\right); \qquad {}_0\mathcal{I}_x^{\beta}\left[\delta(x)\right] = \frac{1}{\Gamma(\beta)}x^{\beta-1} \\
&= K_n\left(\Gamma(1-n)\right)\left(\frac{t^{1-n-1}}{\Gamma(1-n)}\right) = K_n t^{-n}
\end{aligned} \tag{47}
$$

The expression $C(t) = K_n t^{-n}$ we had obtained earlier too (35). We obtain a general expression of charge for Curie-von Schweidler relaxing current in a capacitor, when stressed with a time varying voltage $v(t)$ applied at $t = 0$; is $q(t) = \left(K_n t^{-n}\right) * \left(v(t)\right)$ elaborated below

$$
\begin{aligned}
& q(t) = \left(C(t)\right) * \left(v(t)\right) = \int_{-\infty}^{t}\left(C(t-x)\right)\left(v(x)\right)\mathrm{d}x \\
& C(x) = K_n x^{-n} \qquad x > 0
\end{aligned} \tag{48}
$$

The convolution integral is following

$$
q(t) = \int_0^t K_n \frac{v(x)}{(t-x)^n}\,\mathrm{d}x \tag{49}
$$

As we did for geometrical capacity in previous section, we differentiate (49) this $q(t)$ to get $i(t)$ and write following

$$
\begin{aligned}
i(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\
&= K_n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)}{(t-x)^n}\,\mathrm{d}x
\end{aligned} \tag{50}
$$

We apply formula of integration by parts i.e.

$$
\int_0^t \left(f_1(x)\right)\left(f_2(x)\right)\mathrm{d}x = \left[f_1(x)\int f_2(x)\mathrm{d}x\right]_{x=0}^{x=t} - \int_0^t \left(\left(f_1^{(1)}(x)\right)\int_0^t \left(f_2(x)\right)\mathrm{d}x\right)\mathrm{d}x \tag{51}
$$

to evaluate $\int_0^{\infty} \frac{v(x)}{(t-x)^n}\mathrm{d}x$ as in following steps

$$
\begin{aligned}
\int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} &= \left[v(x)\int \frac{\mathrm{d}x}{(t-x)^n}\right]_{x=0}^{x=t} - \int_0^t \left(v^{(1)}(x)\int \frac{\mathrm{d}x}{(t-x)^n}\right)\mathrm{d}x \\
&= v(x)\left(-\frac{(t-x)^{1-n}}{1-n}\right)\Bigg|_{x=0}^{x=t} - \int_0^t v^{(1)}(x)\left(\frac{(-1)(t-x)^{1-n}}{1-n}\right)\mathrm{d}x \\
&= \frac{v(0)}{1-n}t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\mathrm{d}x
\end{aligned} \tag{52}
$$

Now we differentiate and write the following

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(0)}{1-n}t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n}(t-x)^{1-n}\,\mathrm{d}x\right) \\
&= v(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{1-n}}{1-n}\right) - \int_0^t \frac{v^{(1)}(x)}{1-n}\frac{\mathrm{d}\left((-1)(t-x)^{1-n}\right)}{\mathrm{d}t}\mathrm{d}x \\
&= \frac{v(0)}{t^n} - \int_0^t \frac{v^{(1)}(x)}{1-n}\left((-1)(1-n)(t-x)^{1-n-1}\right)\mathrm{d}x \\
&= \frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)}{(t-x)^n}\mathrm{d}x
\end{aligned} \tag{53}
$$

This gives $i(t)$ as following

$$
\begin{aligned}
i(t) &= K_n \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{v(x)}{(t-x)^n}\mathrm{d}x \\
&= K_n\frac{v(0)}{t^n} + K_n\int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}; \quad K_n = \frac{C_n}{\Gamma(1-n)}; \qquad 0<n<1
\end{aligned} \tag{54}
$$

For $v(t) = V_{BB}\left(u(t)\right)$ i.e. a step voltage applied at time $t=0$ to a time varying capacity function given as $C(t) = K_n t^{-n}$ we have for $t>0$, $v^{(1)}(t)=0$ with $v(0)=V_{BB}$ demonstrated below

$$
\begin{aligned}
i(t) &= K_n\frac{v(0)}{t^n} + K_n\int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad v(0)=V_{BB}; \quad v^{(1)}(x)=0, \quad x>0 \\
&= K_n\frac{V_{BB}}{t^n} + K_n\int_0^t \frac{(0)\mathrm{d}x}{(t-x)^n} = K_n\frac{V_{BB}}{t^n}
\end{aligned} \tag{55}
$$

to get $i(t) = K_n V_{BB} t^{-n}$, for $t>0$.

We recover the Curie-von Schweidler law. For a constant capacitor case with capacity function as $C(t) = C_1\delta(t)$, we have the relation that we derived earlier (18) (19); i.e.

$$
i(t) = C_1 v(0)\left(\delta(t)\right) + C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}
$$

The Figure-1 gives summary of our discussion about a constant capacity to a time varying capacity.

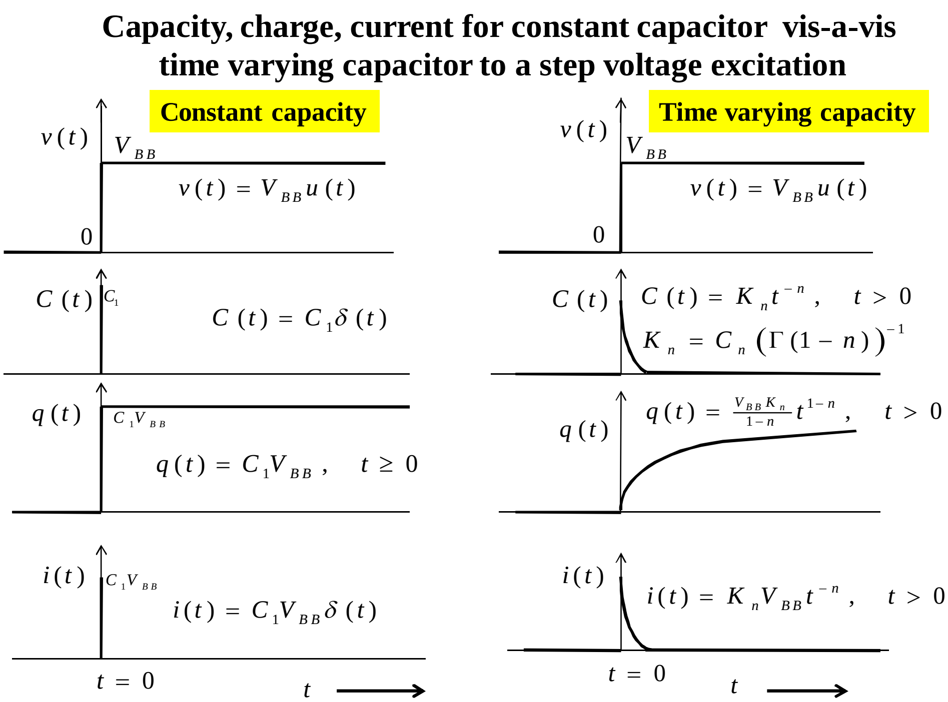

Figure-1: Summary of discussion about constant capacity vis-à-vis time varying capacity

Appearance of fractional derivative

We have formed a time varying capacitor with a dielectric whose relaxation to a step voltage at $t=0$ of constant magnitude follows a power law given by empirical expression of Curie-von Schweidler. We have got current and charge expression for any arbitrary voltage function $v(t)$ applied at $t=0$ in above section (54) as following

$$\begin{aligned} i(t) &= K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \\ q(t) &= (C(t)) * (v(t)) = (K_n t^{-n}) * (v(t)) \\ &= \int_0^t K_n \frac{v(x)}{(t-x)^n}\mathrm{d}x \end{aligned} \tag{56}$$

The fractional derivative for $0 < n < 1$ is defined as following two ways [6], [33]

$$\begin{aligned} {}_0D_t^n[f(t)] &= \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n}\mathrm{d}x \\ &= \frac{1}{\Gamma(1-n)} \left(\frac{f(0)}{t^n} + \int_0^t \frac{f^{(1)}(x)}{(t-x)^n}\mathrm{d}x \right); \quad t > 0 \end{aligned} \tag{57}$$

The first definition is of Riemann-Liouville type i.e. ${}_{0}^{RL}D_t^n\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x,\quad 0<n<1$ and in the second expression's second term i.e. $\frac{1}{\Gamma(1-n)}\int_0^t\frac{f^{(1)}(x)}{(t-x)^n}\mathrm{d}x$ is Caputo fractional derivative i.e. ${}_{0}^{C}D_t^n\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\int_0^t\frac{f^{(1)}(x)}{(t-x)^n}\mathrm{d}x;\quad 0<n<1$. Therefore, we have ${}^{RL}D_t^n\left[f(t)\right]={}^{C}D_t^n\left[f(t)\right]+\frac{f(0)}{\Gamma(1-n)}t^{-n}$, i.e. relation between the two definitions of fractional derivative [6], [33].

Integrating the expression ${}_0D_t^n\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x$, once we write the following

$$
\begin{aligned}
{}_0I_t^1\left({}_0D_t^n\left[f(t)\right]\right)&=\int_0^t\left(\frac{1}{\Gamma(1-n)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x\right)\right)\mathrm{d}x\\
&=\frac{1}{\Gamma(1-n)}\int_0^t\left(\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x\right)^{(1)}\mathrm{d}x\\
&=\frac{1}{\Gamma(1-n)}\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x
\end{aligned}
\tag{58}
$$

We have used in (58) $\mathcal{I}_t^{(1)}\left(f^{(1)}(t)\right)\equiv f(t)$. Using the composition rule [6], [33] i.e. ${}_0\mathcal{I}_t^1\left({}_0D_t^n\left[f(t)\right]\right)={}_0\mathcal{I}_t^{1-n}\left[f(t)\right]={}_0D_t^{n-1}\left[f(t)\right]$, we re-write (58) as following

$$
\begin{aligned}
{}_0D_t^{n-1}\left[f(t)\right]&={}_0\mathcal{I}_t^{1-n}\left[f(t)\right]=\frac{1}{\Gamma(1-n)}\int_0^t\frac{f(x)}{(t-x)^n}\mathrm{d}x;\quad 0<n<1;\qquad 1-n=\upsilon\\
{}_0D_t^{-\upsilon}\left[f(t)\right]&={}_0\mathcal{I}_t^{\upsilon}\left[f(t)\right]=\frac{1}{\Gamma(\upsilon)}\int_0^t\frac{f(x)}{(t-x)^{1-\upsilon}}\mathrm{d}x
\end{aligned}
\tag{59}
$$

Using the definitions of fractional derivative (57), we apply to current expression (54) in following steps

$$
\begin{aligned}
i(t)&=K_n\frac{v(0)}{t^n}+K_n\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}\qquad 0<n<1\\
&=K_n\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\left(\frac{v(0)}{t^n}+\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}\right)\right),\qquad K_n\left(\Gamma(1-n)\right)=C_n\\
&=C_n\left({}_0D_t^n\left[v(t)\right]\right),\qquad 0<n<1
\end{aligned}
\tag{60}
$$

Applying the expression for fractional integration (59) to the charge expression, we get following

$$
\begin{aligned}
q(t)&=\left(C(t)\right)*\left(v(t)\right)=\left(K_nt^{-n}\right)*\left(v(t)\right)\\
&=\int_0^tK_n\frac{v(x)}{(t-x)^n}\mathrm{d}x\\
&=K_n\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\int_0^t\frac{v(x)}{(t-x)^n}\mathrm{d}x\right);\qquad K_n\left(\Gamma(1-n)\right)=C_n\\
&=C_n\left({}_0\mathcal{I}_t^{(1-n)}\left[v(t)\right]\right)\\
&=C_n\left({}_0D_t^{n-1}\left[v(t)\right]\right)\qquad t>0\qquad 0<n<1
\end{aligned}
\tag{61}
$$

We apply a voltage $v(t)=V_{BB}$ at $t=0$ to an uncharged fractional capacitor with capacity function $C(t)=C_n \frac{1}{\Gamma(1-n)} t^{-n}$, applying the above formula (61) we get

$$\begin{aligned} q(t) &= C_n \left({}_0D_t^{n-1}\left[v(t)\right]\right) \qquad t>0 \qquad 0<n<1 \\ &= C_n\, {}_0D_t^{n-1}\left[V_{BB}\right] \qquad {}_0D_t^{\alpha}\left[\mathrm{C}\right] = \mathrm{C}\frac{\Gamma(1)}{\Gamma(1-\alpha)}t^{-\alpha} \\ &= C_n V_{BB}\frac{\Gamma(1)}{\Gamma(1+1-n)}t^{1-n} \\ &= \frac{C_n}{(1-n)\left(\Gamma(1-n)\right)}V_{BB}t^{1-n} = \frac{K_n V_{BB}}{(1-n)}t^{1-n} \end{aligned} \tag{62}$$

The same expression we showed earlier and in Figure-1.

Experimental Validation of range of relaxation exponent in Curie-von Schweidler law

The Curie-von Schweidler empirical law of power law relaxation, i.e. $i(t) \propto t^{-n}$ states that $0<n<1$ .This is validated via experiments on dielectric relaxations. A 100V step input applied to a completely discharged capacitor of $0.47\mu\mathrm{F}$ having metalized paper dielectric, and the current decay is recorded with time. The graphs of log-log plot i.e. $\log\left(i(t)\right)$ vs. $\log(t)$ show a straight line of average slope -0.86 [12], [22]-[26]. This experiment indicates a Curie-von Schweidler law, with $i(t) \propto t^{-n}$, having $n=0.86$. The exponent n is in the range of $0.85<n<1$ in several di-electric relaxation experiments [12], [22]-[26]. The experiments with super-capacitors [7], [8], show range as $0.5<n<1$. A very low value of exponent n is found in relaxation of Laponite studies averagely $n=0.09$ [27]. In this Laponite study [27] though the exponent n was obtained on 'self-discharge' curves with various charging time history-showing memory effect, the expression obtained for self-discharge decay of voltage assumes fractional capacity-that in turn assumes Curie-von Schweidler law as current relaxation function.

Summary

We say that charge stored in capacitor, as a function of time is not multiplication operation of capacity and voltage; instead, the charge is convolution integral. However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage. We say that capacity not the usual ratio of charge to voltage in time domain, but it is given as convolution expression. We note here that for a fractional capacitor, the charge goes to infinity for large times, when the fractional capacitor is placed on a constant voltage; whereas, for a classical capacitor function the charge at large time is a constant. This observation in our derivation is with convolution formula defining the charge stored in capacitor and is consistence with other fractional capacitor models. This is the new idea of breakdown of capacitors due to accumulation of enough charge at a constant voltage (even though voltage is less than the breakdown limit of dielectric. In tabular form (Table-1), we present the various concepts that we discussed with this new approach of charge store in classical capacitor and fractional capacitor.

S.No.	Description	Classical Geometrical (Constant) Capacity ($n=1$)	Fractional Capacity $0<n<1$
1	Relaxing current to a step voltage V_{BB} excitation to uncharged capacitor at $t=0$	$i(t)=C_1V_{BB}\delta(t)$	$i(t)=K_nV_{BB}t^{-n}\ , \quad t>0$
2	Relaxing Current in frequency domain	$I(s)=C_1V_{BB}$	$I(s)=K_nV_{BB}\left(\Gamma(1-n)\right)s^{n-1}$
3	Capacity in time domain and in frequency domain	$C(t)=C_1\delta(t)$ $C(s)=C_1$	$C_n(t)=K_n\left(\Gamma(1-n)\right)\delta(t)$ $C(t)={}_0D_t^{n-1}\left[C_n(t)\right]$ $C(t)=K_nt^{-n}$ $C(s)=K_n\left(\Gamma(1-n)\right)s^{n-1}$
4	Charge function to a step voltage V_{BB} applied at $t=0$	$q(t)=C(t)*v(t)$ $=C_1V_{BB}\ ; \quad t\geq 0$	$q(t)=C(t)*v(t)$ $=K_nV_{BB}\left(\frac{1}{1-n}\right)t^{1-n}\ ; \quad t>0$
5	Current to an arbitrary voltage function $v(t)$ applied at $t=0$	$i(t)=C_1v(0)\delta(t)+C_1\dfrac{\mathrm{d}v(t)}{\mathrm{d}t}$	$i(t)=K_nv(0)t^{-n}+K_n\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}$
6	Current voltage expression	$i(t)=C_1\left({}_0D_t^1v(t)\right)$	$i(t)=C_n\left({}_0D_t^nv(t)\right)$
7	Charge voltage expression for arbitrary voltage $v(t)$ function applied at $t=0$	$q(t)=\left(C_1\delta(t)\right)*v(t)$ $=C_1v(t); \quad t\geq 0$	$q(t)=C_n\left({}_0D_t^{n-1}v(t)\right), \quad t>0$

Table-1: Summary of the discussions regarding classical capacitor and fractional capacitor

References

1. Curie, Jaques, "**Recherches sur le pouvoir inducteur specifique et la conductibilite des corps cristallises**". Annales de Chimie et de Physique. 17 pp 384–434. 1889.
2. Schweidler, Ergon Ritter von,"**Studien über die Anomalien im Verhalten der Dielektrika (Studies on the anomalous behaviour of dielectrics)**". Annalen der Physik. 329 (14): 711–770, pp 1907.

3. Jonscher, Andrzej Ka, "**Dielectric Relaxation in Solids**", Chelsea Dielectrics Press Limited 1983.
4. Jameson, N. Jordan; Azarian, Michael H.; Pecht, Michael, "**Thermal Degradation of Polyimide Insulation and its Effect on Electromagnetic Coil Impedance**". Proceedings of the Society for Machinery Failure Prevention Technology 2017 Annual Conference. 2017.
5. Shantanu Das et al. "**Micro-structural roughness of electrodes manifesting as temporal fractional order differential equation in super-capacitor transfer characteristics**". International Journal of Mathematics and Computation Vol. 20, Issue 3, pp 94-113, 2013.
6. Shantanu Das, "**Functional Fractional Calculus**", 2nd Edition Springer-Verlag, Germany, 2011.
7. Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, "**Frequency dependent piecewise fractional order modeling of ultra-capacitors using hybrid optimization and fuzzy clustering**", Journal of Power Sources 335, pp 98-104, 2015.
8. Manoranjan Kumar, Subhojit Ghosh, Shantanu Das, "**Charge discharge energy efficiency analysis of ultra-capacitor with fractional order dynamics using hybrid optimization and its experimental validation**", International Journal of Electronics & Communications (AEU) 78, pp 2714-280 2017.
9. Elwakil AS, Allagui A, Maundy BJ, Psycchalinos CA, "**A low frequency oscillator using super-capacitor**", AEU –Int. J Electron Commun". 70(7) pp970-3, 2016.
10. Freebon TJ, Maundy B, Elwakil AS, "**Fractional order models of super-capacitors, batteries, fuel-cell: a survey**". Mater Renew Sustain Energy 4(3) pp 1-7, 2015.
11. Freebon TJ, Maundy B, Elwakil AS, "**Measurement of super-capacitor fractional order model parameters from voltage excited step response**", IEEE J Emerging Sel Top Circuits Syst. 3(3), pp367-76, 2013.
12. Svante Westerlund, Lars Ekstam, "**Capacitor Theory**", IEEE Trans on Dielectrics and Insulation, Vol. 1, No. 5, pp826, 1994.
13. Moutushi Dutta Choudhury, Subrata Chandra, Soma Nag, Shantanu Das, Sujata Tarafdar, "**Spreading of Non-linear and Newtonian Fluids on a Solid Substrate under Pressure**", Journal of Physics: Conference Series 319, (2011), IOP Publishing 2011.
14. Moutushi Dutta Choudhury, Shantanu Das, Sujata Tarafdar, "**Effect of loading history on visco-elastic potato starch gel**". Colloids and Surfaces A: Physicochemical and Engineering Aspects 492 pp 47-53 2016.
15. Moutushi Dutta Choudhury, Subrata Chandra, Soma Nag, Shantanu Das, Sujata Tarafdar, "**Forced spreading and rheology of starch gel: Viscoelastic modeling with fractional calculus**", Colloids and Surfaces A: Physicochemical and Engineering Aspects, 407, pp 64-70 2012.
16. JB Fourier (1822), "'**Théorie analytique de la chaleur' (The Analytical Theory of Heat- English translation by Alexander Freeman, ed.)**"; The University Press. p. 408. cf p 449 and pp 546–551. 1878.
17. Berberan-Santos Mario. N, "**Analytical inversion of Laplace transforms without contour integration: application to luminescence decay laws and other relaxation functions**", Journal of mathematical Chemistry, Vol. 38, No.2 August 2005.
18. Powers, David M W, "**Application and explanation of Zipf's Law**", Association for Computational Linguistics: 151–160. 1998.
19. Per Bak, "**How Nature Works: The science of self-organized criticality**", Springer-Verlag, New York, 1996.
20. George K Zipf, "**Human Behavior and the Principle of Least Effort**". Addison-Wesley, 1949.
21. D. Sornette, "**Mechanism of Power laws without self-organization**" International Journal of Modern Physics C13, pp 133-136; 2001

22. S. Chiniba, R Tobazeon, "**Long Term Break-down of Polypropylene Films in Different Media**", Conference Record of the 1985 International Conference on Properties and Application of Dielectric Materials, Xian, China, IEEE, June 24-29, pp566-584, 1985.
23. J W Borough, W G Brammer, J Burnham, "**Degradation of PVDF Capacitors during Accelerated Tests**", Report Huges Aircraft Company, Culver City, CA, pp 219-223, 1980.
24. K Thorborg, "**Power Electronics**", Chalmer Tech. Univ, Goteborg, Sweeden, 1985.
25. A K Jonscher, "**Dielectric Relaxation in Solids**", Chelsea dielectric Press, London 1983.
26. A R Blythe, "**Electrical Properties of Polymer**", Cambridge University Press, 1979.
27. Somasri Hazra, Tapati Dutta, Shantanu Das, Sujata Tarafdar, "**Memory of Electric Field in Laponite and How It Affects Crack Formation: Modeling through Generalized Calculus**", Langmuir, DOI:10.1021/acs.langmuir.7b02034, 2017.
28. A. D. Polyanin, A. V. Manzhirov. "**Handbook of Integral Equations**", 2nd Edition, Chapman & Hall/CRC, Boca Raton, FL (2008)
29. Abramowitz. M, Stegun. I.A. (eds.), "**Handbook of Mathematical Functions**", Nat. Bur. Stand. Appl. Math. Ser 55. U.S. Govt. Printing Office Washington D C (1964).
30. Ramon Ferrer i Cancho, Richard V Sole, "**Least effort and origins of scaling in human language**", PNAS, Proceedings of the National Academy of Sciences of USA, 100 (3): pp 788-791, 2003.
31. Davis Easley, Jon Kleinberg, "**Power Laws and Rich Get Richer Phenomena**", (Chapter-18): Networks Crowds and Markets: Reasoning about a Highly Connected World. Cambridge University Press, 2010.
32. Nagrath I J, Gopal M, "**Control System Engineering**" (Second Edition), Wiley Eastern Limited New Age International Limited, 1982.
33. Oldham K B, Spanier J. "**The Fractional Calculus**", Academic Press 1974.
34. Mario N Berberan-Santos, "**Properties of the Mittag-Leffler Relaxation Function**", Journal of Mathematical Chemistry, Vol. 38, No. 4, 2005.
35. Svante Westerlund, "**Dead Matter has Memory**", Physica Scripta, Vol. 43, pp 174-179, 1991.
36. Simantini Majumdar, Somasri Hazra, Moutushi Dutta Choudhury, Suparna Dutta Sinha, Shantanu Das, Tapas Ranjan Middya, Sujata Tarafdar, Tapati Dutta, "**A study of reheological properties of visco-elasic materials using fractional calculus**", Colloids & Surfaces A : Physicochem. Eng. Aspects 516, pp 181-189, 2017.