
Memorized Relaxation with Singular and Non-Singular Memory Kernels for basic Relaxation of Dielectric vis-à-vis Curie-von Schweidler & Kohlrausch Relaxation laws

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Abstract

We have constructed the basic dielectric relaxation's evolution expression for relaxing current as convolution operation of the chosen memory kernel and rate of change of applied voltage. We have studied types of memory kernels singular, non-singular and combination of singular and non-singular (mixed) decaying functions. With these, we form constitutive equations for relaxation dynamics of dielectrics; i.e. capacitor. We observe that though mathematically we can use non-singular kernels yet this does not give presently much useful practical or physically realizable results and interpretations. We relate our observations to relaxation currents given via Curie-von Schweidler and Kohlrausch laws. The Curie-von Schweidler law gives singular power law as basic relaxation current in dielectric relaxation; whereas the Kohlrausch law is Electric field relaxation in dielectric as non-Debye function taken as stretched exponential a non-singular function. These two laws are used since late nineteenth century for various dielectric relaxation characterization studies. Here we arrive at general capacitor constitutive equation and with each type of memory kernel we give corresponding impedance function and in some cases equivalent circuit representation for the capacitor element. We classify these systems as Curie-von Schweidler type for system with single singular memory kernel and Kohlrausch type for system evolved via using non-singular or mixed functions as memory kernel. We note that use of singular memory kernel gives constituent relations and impedance functions that are experimentally verified in large number of cases of dielectric studies. Therefore, we have a question, does natural relaxation dynamics for dielectrics have a singular memory kernel, and the relaxation current function is singular in nature? Is it the singular relaxation function for capacitor dynamics with singular memory kernel remains universal law for dielectric relaxation? However, we are not questioning researchers modeling relaxation of dielectric via non-singular functions, yet we are hinting about complexity and lack of interpretability of basic constituent equation of dielectric relaxation dynamics thus obtained via considering non-singular and mixed memory kernels- perhaps due to insufficient experimental evidences presently. However, the method employed in this study is general method. This method can be used to form memorized constituent equations for other systems (say Radioactive Decay/Growth, Diffusion and Wave phenomena) from basic evolution equation, i.e. effect function as convolution of memory kernel with cause function.

Keywords

Convolution, Memory Kernel, Fractional Derivative, Fractional Integration, power-law, Mittag-Leffler function, stretched exponential function, derivative of stretched exponential function, Curie-von Schweidler law, Kohlraush law

1. Introduction

In this paper, we give simple mathematical treatment to derive the dielectric relaxation laws (or constitutive equations) with several types of memory kernels to a relaxation law that we formulate as convolution operation i.e. $i(t) \propto k(t) * v^{(1)}(t)$. The function $k(t)$ is the memory kernel, $v(t)$ is the applied voltage stress function, with $v^{(1)}(t)$ as its first derivative and $i(t)$ as relaxing current. This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with ‘memory-kernel’, that is $k(t)$. Implying that the value of the process $i(t)$ at present instant is being influenced by all the states of $v^{(1)}(t)$ the system had been through in past.

Here in this discussion we take the memory kernel $k(t)$ as singular and non-singular functions like delta function $k(t) \propto \delta(t)$, singular power-law decay function $k(t) \propto t^{-\alpha}$, $0 < \alpha < 1$; non-singular power law $k(t) \propto (1 + \lambda t^\alpha)^{-1}$ Mittag-Leffler function $k(t) \propto E_\alpha(-\lambda t^\alpha)$, exponential decay function $k(t) \propto e^{-\lambda t}$, stretched exponential decay function $k(t) \propto e^{-(\lambda t)^\alpha}$ and derivative of stretched exponential decay function $k(t) \propto t^{\alpha-1} e^{-(\lambda t)^\alpha}$, and arrive at the constitutive relations of voltage and currents for dielectric relaxation of these systems. We also write corresponding impedance functions for all of these studied systems. In few cases, we try and give equivalent circuit for the capacitor expression that we obtained.

The empirically and experimentally derived law called ‘universal dielectric relaxation law’ also called as Curie-von Schweidler law, is observed since late 19th century; [1]-[4], [28]. This is classical power law for current decay i.e. $i(t) \propto t^{-\alpha}$; $0 < \alpha < 1$. Here relaxation of current is inverse of power of time for a constant step-voltage excitation to uncharged capacitor. This universal power law relaxation is described by a singular function. This empirical Curie-von Schweidler (1872) relaxation law is used to derive fractional differential equations describing constituent expression for capacitor i.e. $i(t) \propto {}_0D_t^\alpha v(t)$, or $i(t) \propto v^{(\alpha)}(t)$ that is ‘fractional capacitor’, [5]-[12], [17]-[20]. Where ${}_0D_t^\alpha$ is fractional differentiation operation. With $\alpha = 1$, we have classical capacitor i.e. $i(t) \propto D_t^1 v(t)$ or $i(t) \propto v^{(1)}(t)$, described with classical one-whole differentiation i.e. D_t^1 . We will see the classical case $i(t) \propto v^{(1)}(t)$ is zero-memory case.

The Electric Field relaxation in dielectric relaxations is described empirically by Kohlraush law (1854) i.e. $E(t) \sim e^{-(\lambda t)^\alpha}$, with $0 < \alpha < 1$; is stretched exponential decay function [30]-[34]. This is the electric field $E(t)$ relaxation inside dielectric when a constant field is applied. The current flowing inside the dielectric while the field is relaxing is (displacement current), is proportional to derivative of the decaying electric field, i.e. $i(t) \propto E^{(1)}(t)$. In this Kohlraush law relaxation, we find that current is

$i(t) \propto t^{\alpha-1} e^{-(\lambda t)^\alpha}$; and has singular power law decay component with a stretched exponential decay component. We will derive that this Kohlraush decay law for dielectric gives constituent expression as $i(t) \propto \sum_{n=0}^{\infty} w_n {}_0I_t^{(n+1)\alpha} [v^{(1)}(t)]$. Here ${}_0I_t^{(n+1)\alpha}$ is integration operation of order $(n+1)\alpha$ on function $v^{(1)}(t)$ with integration limits 0 to t . The w_n are weights for each term. The convergence of this type of series is discussed in [37]-[41]. This is infinite sums of weighted fractional integrals that is much away from usual constituent laws $i(t) \propto v^{(\alpha)}(t)$ or classical constituent expression as $i(t) \propto v^{(1)}(t)$ -that are experimentally verified.

Here we will show for classical case, the memory kernel is a delta-function, or the relaxing system is with ‘no-memory’ (zero-memory case). We will derive this law i.e. $i(t) \propto v^{(\alpha)}(t)$, that is fractional derivative of voltage, with memory kernel as singular kernel of power-law type, i.e. $k(t) \propto t^{-\alpha}$. We will also show that if the memory kernel were of nonsingular functions or of Kohlraush type or mixed type, then the constitutive equations of current and voltage of those capacitors are too complicated and does not give physical sense of interpretability though are mathematically doable. With this described method, we can derive various constitutive equations for various other types of memory kernels.

We generalize all the cases studied for chosen memory kernel $k(t)$, and write the constituent equation of capacitor as $i(t) \propto \sum_n w_n {}_0I_t^* [v^{(1)}(t)]$, where * denotes integer, fractional order for the integration operation (symboling as ${}_0I_t^*$). We note that for classical case that we call zero memory case the order of integration is zero-and the summation term has only single term. We get single term for summation in case for memory kernel as singular power law, with order of integration as fractional number. We classify these two cases as Curie-von Scheidler type capacitors. For all other cases of non-singular or mixed function as memory kernel we get infinite number of term in the above generalized constituent equation of capacitor. These types with infinite series we term them as Kohlraush type capacitors. For all the cases that we studied we give impedance function in the Laplace domain i.e. $Z(s) = V(s)/I(s)$ where $V(s) = \mathcal{L}\{v(t)\}$ and $I(s) = \mathcal{L}\{i(t)\}$. This function is studied for various impedance spectroscopy experiments by taking $s = i\omega$, that gives frequency dispersion curves.

Therefore $i(t) \propto \sum_n w_n {}_0I_t^* [v^{(1)}(t)]$ describes a general constituent equation of capacitor with relaxation current evolution expression is given as $i(t) \propto k(t) * v^{(1)}(t)$; with $k(t)$ as memory kernel (singular, non-singular or mixed) functions of the system. The different $k(t)$ will have different weights w_n and number of terms in the summation with order of integrations ${}_0I_t^*$ -as for case to case.

Several interesting previous pioneering work on using non-singular kernels exists. In electrical circuit theory [35] gives the basics of fractional electric circuits involving fractional time derivatives in the sense of Riemann–Liouville, Caputo (defined via singular kernel) and Caputo–Fabrizio (defined via non-singular kernel). The examples analyzed use mainly Caputo time-fractional derivative but comparative analyses with derivative based on different relaxation kernels are also provided. In [36], non-singular

kernel is employed to describe anomalous diffusion of fractional order and then derivation of wait time and jump length statistics are derived. The viscoelasticity modeling via non-singular kernel is discussed in [39], [40].

The paper is organized as sections, the Section-2 gives basics of evolution of system dynamics, i.e. “impulse Response of System”. Section 3 gives Preliminaries of Fractional Calculus (with singular kernel and non-singular kernel). This section includes Prabhakar’s Calculus and use of Prabhakar’s function to define fractional integration. Readers will note similarity with this section and our development of capacitor equations with various types on singular, non-singular and mixed memory kernels. The Section-4 takes the classical capacitor where we show that this is a zero memory case. The Section-5 deals with Memory Kernel as singular power law, with Section-6 comparing the memorized relaxation and zero-memory case for a capacitor. Section-7 to Section-11 deals with memory kernel as non-singular functions a. Non-singular Decaying Power law, b. Mittag-Leffler Function, c. Exponential Function, d. Stretched Exponential Function, e. Derivative of Stretched exponential Function (mixed singular and non-singular function). Simulation plots in MATLAB is done to get visualization of memorized relaxation of impulse response current for various memory-indexes. Section-12 is detailed Discussions with observations and inferences with references to practical results of various experiments in dielectric relaxation studies; then followed by Conclusion and References.

2. Basic Evolution Equation for a Process and Impulse Response of System a Review

2.1 Generalized Evolution Equation for cause and effect

The system will respond with its characteristic function call it $k(t)$; by universal law as $y(t) = k(t) * x(t)$; i.e. convolution operation [5], [6], [25]. This we term as general evolution equation, relating cause and effect. The convolution integral is $y(t) = \int_{-\infty}^t k(t-\tau)x(\tau)d\tau$ The characteristics function $k(t)$ we term as ‘memory-kernel’ in this evolution equation given as convolution. This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with ‘memory-kernel’, that is $k(t)$. Implying that the value of the process $y(t)$ at present instant is being influenced by all the states of $x(t)$ the system had been through in past.

We will study dielectric relaxation where cause is $v^{(1)}(t) \leftarrow x$ i.e. rate of change of applied voltage, and effect is dielectric relaxation current $i(t) \leftarrow y(t)$ i.e. the effect with various types of memory-kernel $k(t)$. From this Generalized Evolution Equation various other constituent laws can be formulated.

2.2. Formation of various constituent laws from General Evolution equation

From the evolution equation $y(t) = k(t) * x(t)$ by taking $y(t) = x^{(1)}(t)$, we can have constituent laws for other processes. With memory kernel as delta function $k(t) \propto \delta(t)$ (for a zero-memory case) we get classical growth or decay law given by classical constitutive equation i.e. $x^{(1)}(t) \propto x(t)$. This classical growth or decay law is modified if the memory kernel is other than

delta function, as $x^{(1)}(t) = k(t) * x(t)$. To this case, i.e. $x^{(1)}(t) = k(t) * x(t)$ we modify this to have $x^{(1)} = k * L[x]$, where L is a linear operator, say Laplacian in one dimensional, i.e. $L \equiv \frac{\partial^2}{\partial x^2}$ with $x \equiv c(x, t)$ and $x^{(1)} \equiv \frac{\partial}{\partial t}[c(x, t)] = c^{(1)}(x, t)$ i.e. say concentration variable in space-time coordinates; we get various diffusion equations with the considered memory kernels $k(t)$. Extending this case, with $x^{(2)} = k * L[x]$ as evolution equation, we formulate various wave equations with the considered memory kernels $k(t)$.

The physical laws are constitutive equations, relating cause and response (effect). The cause is input function of time t i.e. $x(t)$ and its response function (effect) is $y(t)$, also a function of time. For example, we have law for force $f(t)$ acting on a body of mass m is; $f = mv^{(1)}$ that is proportional to time rate of change of velocity v . The cause in this case is $v^{(1)} \leftarrow x$; and effect is $f \leftarrow y$. We have radioactive decay law as $N^{(1)} = -\lambda N$, we relate cause as number of decaying nuclei at a particular time i.e. $N \leftarrow x$ and effect as $N^{(1)} \leftarrow y$. The diffusion laws (without advection) in one-dimensional case is, $\frac{\partial}{\partial t}c = \mathbb{D} \frac{\partial^2}{\partial x^2}c$, with c denoting concentration or density of diffusing species. Here the cause and effect is $\frac{\partial^2}{\partial x^2}c \leftarrow x$ and $\frac{\partial}{\partial t}c \leftarrow y$ respectively. The one-dimensional wave mechanics is governed by constitutive equation $\frac{\partial^2}{\partial t^2}w = c^2 \frac{\partial^2}{\partial x^2}w$, with cause as $\frac{\partial^2}{\partial x^2}w \leftarrow x$ and effect as $\frac{\partial^2}{\partial t^2}w \leftarrow y$ respectively. All these examples of constituent laws are classical laws, and are for system without any memory; that is with $k(t) \propto \delta(t)$; that is zero memory case. All these constituent laws are modified by different memory kernels $k(t)$ other than delta function.

2.3. Impulse response of system a review

The output call it $y(t)$ a variable in time ($t \in \mathbb{R}$), of a system represented by function of time variable $k(t)$ to an input variable call it $x(t)$ acting at time $t=0$, is given by evolution equation [5], [6], [25] as follows

$$y(t) = \int_0^t k(t-\tau)x(\tau)d\tau; \quad t \geq 0 \quad (1)$$

If we take Laplace transform, with $\mathcal{L}\{x(t)\} = X(s)$, $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{k(t)\} = K(s)$, we get

$$Y(s) = (K(s))(X(s)) \quad (2)$$

Where variable s is Complex variable, i.e. $s \in \mathbb{C}$. The input $x(t)$ in case is delta-function (an impulse) at $t=0$; we have $y(t) = k(t)$. This $k(t)$ is called 'impulse response' of the system [6], [25]. The evolution equation (1) of $y(t)$ in time domain is convolution integral.

We have several physical systems that are proportional to rate of change of some other physical quantity that is acting as input. Say we have rate of change of a quantity call it $g(t)$ represented by first time derivative i.e. $g^{(1)}(t)$, then our input variable in (1) is

$x(t) = g^{(1)}(t)$; then we have evolution equation in terms of impulse response function of the system, $k(t)$ as

$$y(t) = \int_0^t (k(t-\tau))(g^{(1)}(\tau))d\tau \quad (3)$$

Some physical systems can be casted as (1) and (3). For example, current through a capacitor classically related to voltage given as $i(t) = Cv^{(1)}(t)$, force on a mass to rate of change of velocity as $f(t) = mv^{(1)}(t)$ and stress related to rate of change in Newtonian viscous element as $\sigma(t) = \eta\varepsilon^{(1)}(t)$.

2.3. System relaxation with Memory

Looking at the time evolution equation (1), if the input variable $x(t)$ acts only at time $t=0$ thereafter vanishes at $t>0$ and we observe $y(t)$ even at $t>0$ while ($x(t)=0$ for $t>0$); we may term that system is remembering its past input. In that case, we say system relaxes with ‘memory’ [6], [22], [27]. In ideal cases as described by the constitutive equations for capacitor $i(t)$, force function $f(t)$ and stress $\sigma(t)$ behave ‘without memory’. It can be seen when the ‘rate terms’ input in the RHS of these constitutive equations vanishes after application at $t=0$, we have no observation of the output at $t>0$. Simply if the rate terms in the RHS of all these constitutive equations is described by delta function, then the output is also delta function at $t=0$.

We have analogy with a real life situation. Say someone gives a pinch to our body. We have its effect linger after the cause. This is memorized relaxation. The pinch is impulse input that vanishes immediately after the application; while our body pain relaxes even after that and gradually, the effect of pinch dies out. The response of lingering pain from pinch definitely depends on the body’s memory-kernel. The index of memory gives the effect differently, shorter, longer, or even decaying with oscillations!

The convolution integral (1) can in general have lower terminal of integration as $t=-\infty$ or $a>-\infty$ as the case may be for application of input $x(t)$; that we depict as follows

$$y(t) = \int_{-\infty}^t k(t-\tau)x(\tau)d\tau; \quad y(t) = \int_a^t k(t-\tau)x(\tau)d\tau \quad (4)$$

The evolution equation we take especially for our study as evolution of current in a dielectric i.e. $i(t) \leftarrow y(t)$ when there is rate of change of input voltage i.e. $v^{(1)}(t) \leftarrow x(t)$ and we will call $k(t)$ the memory kernel for relaxing current and we will have convolution as $i(t) = k(t) * v^{(1)}(t)$.

This means the evolution of process (relaxation) is wrapped up (convoluted) in the convolution expression with ‘memory-kernel’, that is $k(t)$. Implying that the value of the process $i(t)$ at present instant is being influenced by all the states of $v^{(1)}(t)$ the system had been through in past. Our study will be to obtain constitutive equations i.e. relation of $i(t)$ and $v^{(1)}(t)$ for various types of memory kernels $k(t)$ given by decaying function of singular or non-singular or in combination.

3. Preliminaries of Fractional Calculus

3.1. Classical Fractional Calculus with Singular Kernel Function

For a function $f(t)$ for $t \geq 0$, the Riemann-Liouville fractional integration [6], [26], [29] of order $\nu \in \mathbb{R}^+$ is defined as

$${}_0I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau \quad (5)$$

Where $\Gamma(\nu)$ is Euler's Gamma function, is generalization of factorial function [23], [29], we have $\Gamma(\nu) = (\nu-1)!$. The formula (5) is ${}_0I_t^\nu [f(t)] = \left(\frac{t^{\nu-1}}{\Gamma(\nu)}\right) * f(t)$ is convolution operation, with power-law memory kernel. This is $k(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}$ and is singular function for case $0 < \nu < 1$. The formula (5) is appearing as generalization of Cauchy's multiple integration formula of m fold integration [6], [26], [29] where $m \in \mathbb{N}$ given as follows

$${}_0I_t^m [f(t)] = \frac{1}{(m-1)!} \int_0^t (t-\tau)^{m-1} f(\tau) d\tau; \quad m = 1, 2, 3, \dots \quad (6)$$

The fractional derivative of order β for $0 < \beta < 1$ by Riemann-Liouville (RL) formula [6], [26], [29] is

$${}_0D_t^\beta [f(t)] = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-\tau)^{-\beta} f(\tau) d\tau; \quad 0 < \beta < 1 \quad (7)$$

The (7) is fractionally integrating the function by order $(1-\beta)$ by formula (5) and then followed by one-whole differentiation. We note that (7) is also having convolution operation and with singular kernel. There is reverse operation called Caputo's fractional derivative, where we have a function $f(t)$ defined for $t \geq 0$ and is differentiable i.e. $f^{(1)}(t)$ exists for $t \geq 0$. The Caputo fractional derivative [6], [26], [29] for fractional order $0 < \beta < 1$ is given as

$${}_0^cD_t^\beta [f(t)] = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} f^{(1)}(\tau) d\tau; \quad 0 < \beta < 1 \quad (8)$$

Thus for (8) we need to get first the one-whole order derivative that is $f^{(1)}(t)$, and then carry out fractional integration for order β , by formula (5). The formula (8) also employs singular kernel. The Caputo and Riemann-Liouville (RL) fractional derivative are related [6], [26], [29] by

$${}_0D_t^\beta [f(t)] = {}_0^cD_t^\beta [f(t)] + \frac{f(0)}{\Gamma(1-\beta)} t^{-\beta}; \quad 0 < \beta < 1 \quad (9)$$

We mention that both the fractional derivatives are equal when initial value is zero i.e. $f(0) = 0$. We note that fractional derivative of constant is not zero in RL sense, but is a power function (and that is singular at start point) i.e. ${}_0D_t^\beta [K] = \frac{K}{\Gamma(1-\beta)} t^{-\beta}$. Whereas the Caputo's fractional derivative of a constant is zero, i.e. ${}_0^cD_t^\beta [K] = 0$, [6], [26], [29].

The fractional integration and fractional differentiation of delta function [6], [26], [29] is as follows

$${}_0I_t^\nu \delta(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}; \quad {}_0D_t^\nu \delta(t) = \frac{1}{\Gamma(-\nu)} t^{-\nu-1}, \quad 0 < \nu < 1 \quad (10)$$

Fractional derivative and fractional integration of power function $f(t) = Kt^p$ [6], [26], [29] is

$${}_0I_t^\nu Kt^p = K \frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} t^{p+\nu}, \quad {}_0D_t^\nu Kt^p = K \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} t^{p-\nu}, \quad p > -1 \quad (11)$$

The Laplace transform of fractional integral operation is

$$\mathcal{L}\left\{{}_0I_t^\nu f(t)\right\} = s^{-\nu} F(s) \quad (12)$$

Laplace transform of Caputo derivative for fractional order $0 < \nu < 1$ is

$$\mathcal{L}\left\{{}_0^C D_t^\nu f(t)\right\} = s^\nu F(s) - s^{\nu-1} f(0) \quad (13)$$

Like in classical calculus, we have exponential function; similarly, in fractional Calculus we have Mittag-Leffler function. The series definition Mittag Leffler function is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}; \quad \text{Re}[\alpha, \beta] > 0 \quad (14)$$

For $\beta = 1$ we have $E_{\alpha,1}(z) = E_\alpha(z)$; is called One-Parameter Mittag-Leffler function. The Laplace transformation of Mittag-Leffler function is

$$\mathcal{L}\left\{E_\alpha(\lambda t^\alpha)\right\} = \frac{s^{\alpha-1}}{s^\alpha - \lambda} \quad (15)$$

We observe that for $E_\alpha(-bt^\alpha)\big|_{\alpha=1} = e^{-bt}$, and $E_\alpha(-at^\alpha)\big|_{\alpha=2} = \cos\sqrt{at}$.

We point here that $f(t) = E_\alpha(\lambda t^\alpha)$ is eigen-function for fractional differential equation with Caputo derivative i.e. ${}_0^C D_t^\alpha f(t) = \lambda f(t)$; and $f(t) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)$ is eigen-function for fractional differential equation with RL fractional derivative, i.e. ${}_0D_t^\alpha f(t) = \lambda f(t)$ [6], [26], [29]. We will be using these concepts of fractional calculus in our discussion

3.2. Fractional Calculus with Non-Singular Kernel

3.2.a. Fractional Caputo Derivative with Non-Singular kernel

The new definition of Fractional derivative with non-singular kernel is proposed in [45], [46] that we describe in this section. From (8) we write the formulation of Caputo derivative, as ${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{(1)}(\tau) d\tau}{(t-\tau)^\alpha}$, for fractional order $0 < \alpha < 1$. This is applicable

if the function $f(t)$ has one-whole derivative $f^{(1)}(t)$ in the interval under consideration. As per our discussions, this Caputo derivative is fractional integration of order $1-\alpha$ for function $f^{(1)}(t)$ i.e. ${}_a^C D_t^\alpha f(t) = {}_a I_t^{1-\alpha} f^{(1)}(t)$. Here the kernel of integration is ‘power function’ and is a singular function at $t=0$ i.e. $k(t) \sim t^{-\alpha}$. In terms of the convolution integral ${}_a^C D_t^\alpha f(t)$ is ${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left((k(t)) * (f^{(1)}(t)) \right)$. The kernel if considered

as $k(t) \sim \exp\left(-\frac{\alpha}{1-\alpha} t\right)$, instead usual ‘power function’ then we have Caputo-Fabrizio (CF) definition given as

$${}_a^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha} (t-\tau)\right) f^{(1)}(\tau) d\tau; \quad 0 < \alpha < 1 \quad (16)$$

Where $M(\alpha)$ is normalization constant $M(0) = M(1) = 1$.

Now if we have a kernel as ‘higher transcendental function’ Mittag-Leffler $k(t) \sim E_\alpha\left(-\frac{\alpha}{1-\alpha}t^\alpha\right)$, $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ (14) then we have Atangana-Baleanu-Caputo (ABC) definition given as

$${}^{ABC}D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \left(E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\right) \left(f^{(1)}(\tau)\right) d\tau, \quad 0 < \alpha < 1 \quad (17)$$

Where $B(\alpha)$ is normalization constant $B(0) = B(1) = 1$.

We note that our classical definition of fractional derivative (RL or Caputo) is connected to Riemann-Liouville fractional integration, with kernel in convolution integration as ‘power function’. We observe that in those classical definitions, the kernel of fractional integration is $\sim t^{-\alpha}$ which is singular at $t=0$; whereas the definitions (16) and (17) have kernel of convolution integration as non-singular functions-described by exponential and Mittag-Leffler functions respectively.

One may verify, (16) by expanding $\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)$ in series and then using Cauchy’s formula for repeated integration, that this expression is weighted series sum of various orders of integer integrals. Similarly by expanding $E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)$ in (17) one may verify that this is actually is weighted series sum of various orders of fractional integrals of RL type.

3.2.b. Three Parameter Mittag-Leffler Function $f(z) = E_{\alpha,\beta}^\gamma(z)$ (Prabhakar Function)

This function is modification of Two-Parameter Mittag-Leffler function done by T R Prabhakar in 1971, [42] defined in series form as following

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta) k!} z^k \quad (18)$$

with $z \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{C}$ and $\text{Re}[\alpha] > 0$. In (18) we have $(\gamma)_k$ as Pochhammer Number, the rising factorial and is $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. This (18) is entire function. The Laplace Transformation of the Prabhakar function is following

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}^\gamma(-\lambda t^\alpha)\right\} = \frac{s^{\alpha\gamma-\beta}}{(s^\alpha + \lambda)^\gamma}; \quad \text{Re}[s] > |\lambda|^{1/\alpha} \quad (19)$$

From (18) and (19) we note that $E_{\alpha,\beta}^\gamma(z) = E_{\alpha,\beta}^1(z)$ and $E_\alpha(z) = E_{\alpha,1}^1(z)$.

3.2.c. Prabhakar Integral

Using the function $k_{\alpha,\beta}^\gamma(t) = t^{\beta-1} E_{\alpha,\beta}^\gamma(\lambda t^\alpha)$ as kernel of convolution integral, we can define Prabhakar Integral [42] as ${}_a\mathbf{I}_{(\alpha,\beta,\lambda);t}^\gamma[f(t)] = (k_{\alpha,\beta}^\gamma(t)) * (f(t))$, described as following expression

$${}_a\mathbf{I}_{(\alpha,\beta,\lambda);t}^\gamma[f(t)] = \int_a^t \left((t-\tau)^{\beta-1} E_{\alpha,\beta}^\gamma(\lambda(t-\tau)^\alpha)\right) (f(\tau)) d\tau \quad (20)$$

We note that Prabhakar integral (20) is having Kernel of integration different from power law function ($\sim t^{\nu-1}$) that we used as Kernel in Riemann-Liouville Fractional Integration formula.

3.2.d. Prabhakar Integral as series-sum of Riemann-Liouville fractional integrals

We know the Riemann-Liouville fractional integral as ${}_a I_t^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\tau)^{\nu-1} f(\tau) d\tau$

Let us expand (20) with inserting (18) as in following steps [37]-[42]

$$\begin{aligned}
{}_a \mathbf{I}_{(\alpha,\beta,\lambda);t}^\gamma [f(t)] &= \int_a^t \left((t-\tau)^{\beta-1} E_{\alpha,\beta}^\gamma (\lambda(t-\tau)^\alpha) \right) (f(\tau)) d\tau \\
&= \int_a^t \left((t-\tau)^{\beta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{(\lambda(t-\tau)^\alpha)^k}{k!} \right) (f(\tau)) d\tau \\
&= \sum_{k=0}^{\infty} \left(\frac{(\gamma)_k \lambda^k}{k!} \left(\frac{1}{\Gamma(\alpha k + \beta)} \right) \int_a^t (t-\tau)^{\alpha k + \beta - 1} f(\tau) d\tau \right) \\
&= \sum_{k=0}^{\infty} \frac{(\gamma)_k \lambda^k}{k!} \left({}_a I_t^{\alpha k + \beta} f(t) \right) = \sum_{k=0}^{\infty} \frac{\lambda^k \Gamma(\gamma + k)}{k! \Gamma(\gamma)} \left({}_a I_t^{\alpha k + \beta} f(t) \right)
\end{aligned} \tag{21}$$

We used $I_t^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-\tau)^{\nu-1} f(\tau) d\tau$ Using (21) we can write the following in expanded form

$$\begin{aligned}
{}_a \mathbf{I}_{(\alpha,\beta,\lambda);t}^\gamma [f(t)] &= \sum_{k=0}^{\infty} \frac{(\gamma)_k \lambda^k}{k!} \left({}_a I_t^{\alpha k + \beta} f(t) \right); \quad (\gamma)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)}; \quad \Gamma(\nu + 1) = \nu \Gamma(\nu) \\
&= {}_a I_t^\beta f(t) + \gamma \lambda \left({}_a I_t^{\alpha + \beta} f(t) \right) + \frac{\gamma(\gamma + 1) \lambda^2}{2!} \left({}_a I_t^{2\alpha + \beta} f(t) \right) \\
&\quad + \frac{\gamma(\gamma + 1)(\gamma + 2) \lambda^3}{3!} \left({}_a I_t^{3\alpha + \beta} f(t) \right) + \dots
\end{aligned} \tag{22}$$

In (22) putting, $\gamma = \beta = 1$ we have following

$$\begin{aligned}
{}_a \mathbf{I}_{(\alpha,1,\lambda);t}^1 [f(t)] &= \int_a^t \left(E_\alpha (\lambda(t-\tau)^\alpha) \right) (f(\tau)) d\tau \\
&= {}_a I_t^1 f(t) + \lambda \left({}_a I_t^{\alpha+1} f(t) \right) + \lambda^2 \left({}_a I_t^{2\alpha+1} f(t) \right) \\
&\quad + \lambda^3 \left({}_a I_t^{3\alpha+1} f(t) \right) + \dots
\end{aligned} \tag{23}$$

We note that Prabhakar integral defined via non-singular kernel $k_{\alpha,1}^1(t) = E_\alpha (\lambda t^\alpha)$ in (23) is sum of classical Riemann-Liouville fractional integration with power law kernel.

We will see the similarities of these fundamentals of Prabhakar Integrals described in (22) (23) when we deal with several types of non-singular memory kernels $k(t)$ in our dielectric relaxation evolution equation i.e. $i(t) = k(t) * v^{(1)}(t)$. This Fractional Calculus with non-singular memory kernel is developed and applied by many pioneers [35]-[46].

4.0. Classical Dielectric Relaxation case of Capacitor

4.1: Constitutive equation and evolution equation of classical capacitor

The response current flows through a dielectric (or capacitor) if and only if there is rate of change in the applied voltage across it. That is classically we have relaxation current $i(t)$ as $i(t) \propto v^{(1)}(t)$. The classical Capacitor constitutive expression relating time function of current through capacitor to voltage stress applied is following

$$i(t) = C \frac{dv(t)}{dt} = C(v^{(1)}(t)) \quad (24)$$

We can modify the above (24) expression i.e. $i(t) = Cv^{(1)}(t)$ or $i(t) = C(D_t^1 v(t))$ and write

$$\begin{aligned} i(t) &= C \int_{-\infty}^t (\delta(t-\tau)v^{(1)}(\tau))d\tau \\ &= \int_{-\infty}^t (C\delta(t-\tau)v^{(1)}(\tau))d\tau = (C\delta(t)) * (v^{(1)}(t)) \end{aligned} \quad (25)$$

This comes from property of delta function, i.e. $\int \delta(x-y)f(y)dy = f(x)$ [23]. In above (25) for the convolution integral, we have kernel of integration as delta function call it $k(t) = C\delta(t)$. With this we get

$$i(t) = (k(t)) * (v^{(1)}(t)) \quad (26)$$

The expression we have casted as (1) and (3). The kernel $k(t)$ we will now term as Memory Kernel. The above expression we term as evolution equation for dielectric relaxation-for classical capacitor.

Let us give a unit voltage step input, $v(t) = u(t)$ applied at $t = 0$. This means $v(t) = 1$ for $t \geq 0$ to an uncharged capacitor i.e. $v(t) = 0$ for $t < 0$; then we have $v^{(1)}(t) = \delta(t)$ i.e. differentiation of unit-step input. Placing this value in evolution expression above, we get

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \quad k(t) = C\delta(t) \\ &= \int_0^t (C\delta(t-\tau)v^{(1)}(\tau))d\tau = C \int_0^t (\delta(t-\tau)\delta(\tau))d\tau \\ &= C\delta(t) \end{aligned} \quad (27)$$

This above result is direct result of $i(t) = Cv^{(1)}(t)$; as the differentiation of unit step function is delta-function. This is classical 'impulse response' of ideal capacitor.

4.3. Impedance of classical capacitor

The Laplace transformed relations of $i(t) = Cv^{(1)}(t)$ is

$$I(s) = C(sV(s) - v(0)) \quad (28)$$

Doing Laplace transform of evolution equation (26), we get

$$\begin{aligned} \mathcal{L}\{i(t)\} &= \mathcal{L}\{(k(t)) * (v^{(1)}(t))\} \\ I(s) &= \mathcal{L}\{k(t)\} \mathcal{L}\{v^{(1)}(t)\} \\ &= C(K(s))(sV(s) - v(0)), \quad K(s) = \mathcal{L}\{k(t)\} = \mathcal{L}\{C\delta(t)\} = C \\ &= C(sV(s) - v(0)) \end{aligned} \quad (29)$$

We get the same result as earlier (28).

The impedance expression in Laplace domain is $Z(s) = V(s)/I(s)$ taking initial condition $v(0) = 0$, from (28) we write the following (with $s = i\omega$)

$$Z(s) = \frac{1}{sC} \quad Z(\omega) = \frac{1}{i\omega C} \quad (30)$$

These are standard expression of classical capacitor.

From the classical theory with Newtonian Calculus as the constitutive equation of capacitor $i(t) = Cv^{(1)}(t)$ we get a delta impulse current uncharged capacitor is impressed with a constant step voltage. This is a singular relaxation current function.

4.2. Constitutive equation of Classical capacitor with Memory Kernel-a zero memory case

From the classical law we have arrived at the equation, which is following

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \\ &= \int_{-\infty}^t (k(t-\tau))(v^{(1)}(\tau))d\tau \end{aligned} \quad (31)$$

It so happens that the classical capacitor equation $i(t) = Cv^{(1)}(t)$ is associated with Memory Kernel $k(t) = C\delta(t)$.

This physically implies that the system $i(t) = Cv^{(1)}(t)$ has zero-memory. That is just after the instance of application of voltage stress i.e. at $t = 0^+$ the memory kernel vanishes i.e. $k(t) = 0$ for $t > 0$. Whereas at $k(t) = \infty$ only at $t = 0$; and is singular function. This is a 'singular memory kernel'. Now we will study relaxation of currents to unit step input of capacitors for various kernels-singular and non-singular. We rewrite the classical capacitor equation as follows

$$i(t) = C \left({}_0I_t^0 v^{(1)}(t) \right) \quad (32)$$

This (32) implies the constituent expression for a zero-memory case relaxation where memory kernel as we saw is singular delta function $k(t) = C\delta(t)$.

5.0 Constitutive equation of Capacitor due to Singular Power Law Memory Kernel

Let us have the power law decay kernel described as

$$k(t) = Ct^{-\alpha}; \quad 0 < \alpha < 1 \quad (33)$$

In (33) above C is a positive constant. The above memory is singular at origin with its derivative as minus infinity. This means that we have memory kernel $k(t) = \infty$ at $t = 0$ and monotonically decaying after that i.e. $t > 0$, with $k^{(1)}(t)|_{t=0} = -\infty$. However, we say that this kernel is a singular kernel. This is some way mimicking the actual memory or forgetfulness. That is as the time goes the memory fades away.

With this we have following steps

$$\begin{aligned}
i(t) &= (k(t)) * (v^{(1)}(t)) \\
\mathcal{L}\{i(t)\} &= \mathcal{L}\{(k(t)) * (v^{(1)}(t))\} \\
I(s) &= (\mathcal{L}\{k(t)\})(\mathcal{L}\{v^{(1)}(t)\}), \quad \mathcal{L}\{k(t)\} = \mathcal{L}\{Ct^{-\alpha}\} = C \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} \\
&= \left(C \frac{\Gamma(1-\alpha)}{s^{1-\alpha}} \right) (sV(s) - v(0)) \\
&= C(\Gamma(1-\alpha))(s^\alpha V(s) - s^{\alpha-1}v(0)), \quad v(0) = 0, \quad V(s) = \frac{1}{s} \\
I(s) &= C \left(\frac{\Gamma(1-\alpha)}{s^{1-\alpha}} \right)
\end{aligned} \tag{34}$$

From (34) above we get $i(t) = \mathcal{L}^{-1}\{I(s)\}$ as

$$i(t) = Ct^{-\alpha}; \quad 0 < \alpha < 1 \tag{35}$$

The plot of this current (35) is shown in Figure-1, with $C = 1$

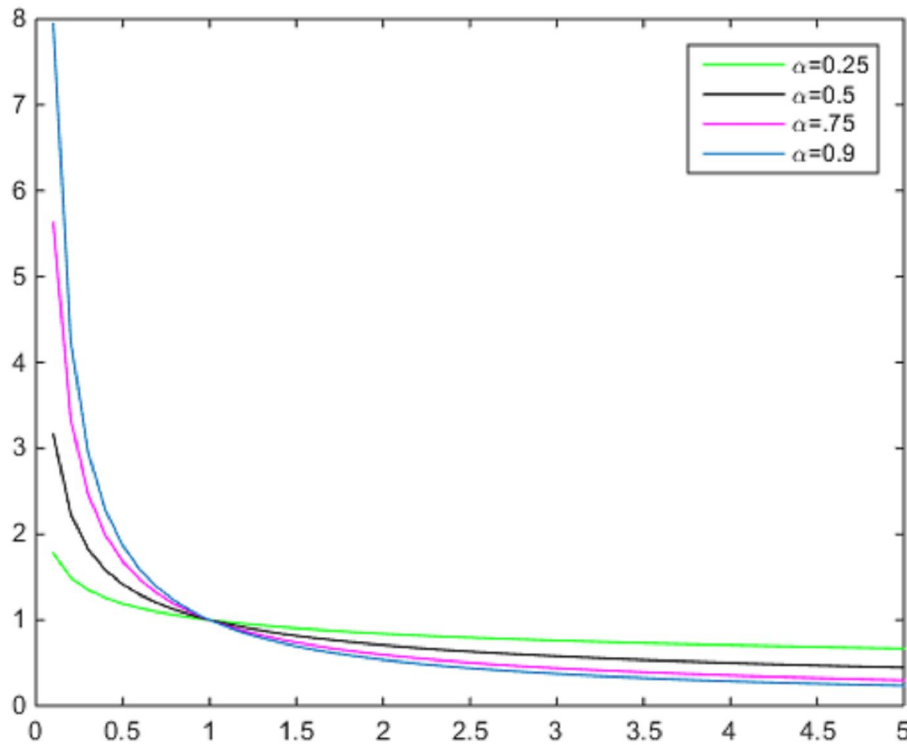


Figure-1: Response Current $i(t) = Ct^{-\alpha}$ to Unit Step Voltage Excitation for dielectric relaxation with singular power law memory kernel for various memory indexes: Y-Axis current $i(t)$ and X-Axis time t

This (35) is ‘impulse response’ of the above system. Therefore, we are getting a power law current decay $i(t) \sim t^{-\alpha}$ for the memory kernel in constitutive equation as a power

law $k(t) \sim t^{-\alpha}$; for an uncharged capacitor stressed with unit step voltage. From Figure-1, we see that as the index-of memory α is decreasing the current excitation lingers longer. Thus as the memory index decreases the forgetfulness is reduced. The memory or forgetfulness is described by the fractional number α in the memory kernel $k(t) = Ct^{-\alpha}$. This power law i.e. singular function, is also called Universal Dielectric relaxation law, observed since late 19th century [1]-[4], [28]. This is termed as power-law [13]-[16], [27], [28].

Now we obtain constitutive relation for capacitor with memory kernel that is singular and has no derivative at start point i.e. $k(t) = Ct^{-\alpha}$; $0 < \alpha < 1$. We write

$$\begin{aligned}
 i(t) &= (k(t)) * (v^{(1)}(t)) \\
 &= \int_{-\infty}^t (k(t-\tau))(v^{(1)}(\tau))d\tau; \quad k(t) = Ct^{-\alpha}; \quad t \geq 0 \\
 &= \int_0^t (C(t-\tau)^{-\alpha})(v^{(1)}(\tau))d\tau \\
 &= C(\Gamma(1-\alpha)) \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-\tau)^{-\alpha})(v^{(1)}(\tau))d\tau \right) \\
 &= C(\Gamma(1-\alpha)) \left({}^C_0D_t^\alpha v(t) \right)
 \end{aligned} \tag{36}$$

In above (36) we have used the definition of Caputo fractional derivative for fractional order $0 < \alpha < 1$ i.e. ${}^C_0D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-\tau)^{-\alpha})(f^{(1)}(\tau))d\tau$ (8).

Thus, our constitutive equation for a capacitor having Singular Power Law Memory kernel is given by fractional differential equation, and is changed from classical capacitor case $i(t) = Cv^{(1)}(t)$, i.e.

$$\begin{aligned}
 i(t) &= C(\Gamma(1-\alpha)) \left({}^C_0D_t^\alpha v(t) \right); \quad C_\alpha = C(\Gamma(1-\alpha)) \\
 i(t) &= C_\alpha \left({}^C_0D_t^\alpha v(t) \right); \quad 0 < \alpha < 1
 \end{aligned} \tag{37}$$

In above (37) putting $\alpha = 1$ we get classical relation $i(t) = Cv^{(1)}(t)$. The above expression is obtained and is used in [5], [7]-[12], [17]-[20].

The Laplace Transform of Caputo Fractional Derivative [6], [26], [29] for fractional order $0 < \alpha < 1$ is $\mathcal{L} \left\{ {}^C_0D_t^\alpha f(t) \right\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. Using this we write Laplace Transform of as

$$\begin{aligned}
 \mathcal{L} \{ i(t) \} &= C_\alpha \mathcal{L} \left\{ {}^C_0D_t^\alpha v(t) \right\}; \quad 0 < \alpha < 1 \\
 I(s) &= C_\alpha \left(s^\alpha V(s) - s^{\alpha-1} v(0) \right)
 \end{aligned} \tag{38}$$

We note that in (38) above putting $\alpha = 1$ we obtain the classical result i.e. $I(s) = C(sV(s) - v(0))$. This we call as ‘fractional capacitor’ having constituent relation $i(t) = C_\alpha v^{(\alpha)}(t)$. With $v(0) = 0$ we write impedance as $Z(s) = V(s) / I(s)$ i.e.

$$Z(s) = \frac{1}{s^\alpha C_\alpha} \quad Z(\omega) = \frac{1}{i^\alpha \omega^\alpha C_\alpha} \tag{39}$$

The expression $Z(\omega)$ is indicative of having Real part and Imaginary part unlike classical capacitor which is purely imaginary quantity. We have

$$Z(\omega) = \frac{1}{\omega^\alpha C_\alpha} \cos \frac{\alpha\pi}{2} - i \frac{1}{\omega^\alpha C_\alpha} \sin \frac{\alpha\pi}{2} \quad (40)$$

We verify the relaxation current with $V(s) = s^{-1}$ and $v(0) = 0$ i.e. for unit step input $v(t) = u(t)$; $t \geq 0$, applied to initially uncharged capacitor with $v(0) = 0$, $t < 0$ from (38)

$$\begin{aligned} I(s) &= C_\alpha (s^\alpha V(s) - s^{\alpha-1} v(0)); \quad V(s) = \frac{1}{s}, \quad v(0) = 0 \\ I(s) &= C_\alpha s^{\alpha-1}, \quad \mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}} \\ i(t) &= \mathcal{L}^{-1}\{I(s)\} = \mathcal{L}^{-1}\{C_\alpha s^{\alpha-1}\} \\ &= \frac{C_\alpha}{\Gamma(1-\alpha)} t^{-\alpha}, \quad 0 < \alpha < 1, \quad t \geq 0 \end{aligned} \quad (41)$$

With $C_\alpha = C(\Gamma(1-\alpha))$ we get from above $i(t) = Ct^{-\alpha}$; same as we got in earlier in (35).

We rewrite the expression of fractional capacitor i.e. $i(t) = C_\alpha v^{(\alpha)}(t)$ as follows

$$i(t) = C_{\alpha 0} I_t^{1-\alpha} v^{(1)}(t) \quad (42)$$

This (42) comes from intermediate steps of (36), the relaxation current is fractional integration of order $1-\alpha$ of rate of change of voltage input across a fractional capacitor. With $\alpha = 1$ we obtain classical case of zero-memory i.e. $i(t) = C_0 I_t^0 v^{(1)}(t)$.

6.0. Difference between zero-memory and memory based relaxation cases of Capacitor

We observed that for a classical case the relaxation current $i(t)$ is delta function at the start of application of unit step-voltage to an uncharged system. Therefore, as soon as the rate of change of voltage vanishes at $t > 0$ we have relaxing current as zero. This is zero memory case with memory kernel as $k(t) \propto \delta(t)$. Where we observe from above section i.e. $i(t) \propto t^{-\alpha}$ with a singular power-law memory kernel as $k(t) \propto t^{-\alpha}$ ($0 < \alpha < 1$), we have a finite current even the rate of change of voltage vanished at $t > 0$. Therefore, the capacitor is memorizing the excitation that once took place as a rate of change in voltage and capacitor is ‘relaxing with memory’. Well this $i(t) \propto t^{-\alpha}$ was the case with singular power law memory kernel. Now in subsequent sections we will discuss non-singular memory kernels and see the constitutive relations that we get for capacitor expression.

7.0. Constitutive equation for Capacitor due to Non-Singular power law Memory Kernel

We have seen earlier that the kernel of singular power-law i.e. $k(t) \propto t^{-\alpha}$; $0 < \alpha < 1$ gives a constitutive equation with fractional derivative i.e. $i(t) \propto v^{(\alpha)}(t)$. We modify the power-law to a non-singular type with following type

$$k(t) = C(1 + \lambda t)^{-\alpha}; \quad 0 < \alpha < 1, \quad \lambda > 0 \quad (43)$$

In above (43) C is a positive constant. In above we have $k(0) = C$ and $k^{(1)}(0) = -C\alpha$, unlike singular kernel $k(t) \propto t^{-\alpha}$. With this we do following calculations for obtaining constitutive equation

$$\begin{aligned}
 i(t) &= (k(t)) * (v^{(1)}(t)) \\
 &= \int_0^t (C(1 + \lambda(t - \tau))^{-\alpha} (v^{(1)}(\tau))) d\tau \\
 &= C \int_0^t \left(\binom{-\alpha}{0} (\lambda(t - \tau))^0 + \binom{-\alpha}{1} (\lambda(t - \tau)) + \binom{-\alpha}{2} (\lambda(t - \tau))^2 + \dots \right) (v^{(1)}(\tau)) d\tau \\
 &= C \int_0^t \left(1 + (-\alpha)(\lambda(t - \tau)) + \frac{(-\alpha)(-\alpha-1)}{2!} (\lambda(t - \tau))^2 + \dots \right) (v^{(1)}(\tau)) d\tau \\
 &= C \left(\int_0^t (v^{(1)}(\tau)) + \frac{(-\alpha)}{1!} \int_0^t (\lambda(t - \tau)) (v^{(1)}(\tau)) d\tau \right. \\
 &\quad \left. + \frac{(-\alpha)(-\alpha-1)}{2!} \int_0^t (\lambda(t - \tau))^2 (v^{(1)}(\tau)) d\tau \right. \\
 &\quad \left. + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t (\lambda(t - \tau))^3 (v^{(1)}(\tau)) d\tau \dots \dots \dots \right) \tag{44}
 \end{aligned}$$

In above (44) we used binomial expansion [23] for $(1 + z)^{-\alpha}$. We use repeated integration formula of Cauchy's (6) i.e. ${}_0I_t^m f(t) = \frac{1}{(m-1)!} \int_0^t (t - \tau)^{m-1} f(\tau) d\tau$, [6], [26], [29] and get the following expression

$$\begin{aligned}
 i(t) &= C \left(\int_0^t (v^{(1)}(\tau)) + \frac{(-\alpha)}{1!} \int_0^t (\lambda(t - \tau)) (v^{(1)}(\tau)) d\tau \dots \right. \\
 &\quad \left. \dots + \frac{(-\alpha)(-\alpha-1)}{2!} \int_0^t (\lambda(t - \tau))^2 (v^{(1)}(\tau)) d\tau + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!} \int_0^t (\lambda(t - \tau))^3 (v^{(1)}(\tau)) d\tau \dots \right) \\
 &= C \left(\int_0^t (v^{(1)}(\tau)) + (-\alpha)\lambda \left(\frac{1}{(2-1)!} \int_0^t (t - \tau)^{2-1} (v^{(1)}(\tau)) d\tau \right) + \right. \\
 &\quad \left((-\alpha)(-\alpha - 1)\lambda^2 \left(\frac{1}{(3-1)!} \int_0^t (t - \tau)^{3-1} (v^{(1)}(\tau)) d\tau \right) + \dots \right) \\
 &= C \left({}_0I_t^1 v^{(1)}(t) + (-\alpha)\lambda \left({}_0I_t^2 v^{(1)}(t) \right) + (-\alpha)(-\alpha - 1)\lambda^2 \left({}_0I_t^3 v^{(1)}(t) \right) + \dots \dots \dots \right) \tag{45}
 \end{aligned}$$

It so happens that for this kernel $k(t) = C(1 + \lambda t)^{-\alpha}$ which is non-singular power-law the constitutive equation is for $i(t)$ is weighted sum of integrals (one whole, two whole, three whole; and so on) of $v^{(1)}(t)$.

With excitation $v^{(1)}(t) = \delta(t)$ to an uncharged capacitor, that is applying a unit step input i.e. $v(t) = u(t)$ we obtain following from above

$$\begin{aligned}
i(t) &= C \left({}_0I_t^1 v^{(1)}(t) + (-\alpha)\lambda \left({}_0I_t^2 v^{(1)}(t) \right) + (-\alpha)(-\alpha-1)\lambda^2 \left({}_0I_t^3 v^{(1)}(t) \right) + \dots \right) \\
&= C \left({}_0I_t^1 \delta(t) + (-\alpha)\lambda \left({}_0I_t^2 \delta(t) \right) + (-\alpha)(-\alpha-1)\lambda^2 \left({}_0I_t^3 \delta(t) \right) + \dots \right) \\
&= C \left(1 - \alpha\lambda t + \alpha(\alpha+1)\lambda^2 \left(\frac{t^2}{2} \right) - \alpha(\alpha+1)(\alpha+2)\lambda^3 \left(\frac{t^3}{(2)(3)} \right) + \dots \right) \\
&= C(1 + \lambda t)^{-\alpha}
\end{aligned} \tag{46}$$

This series in (46) we plot in Figure-2 with various values of α , for $C = \lambda = 1$

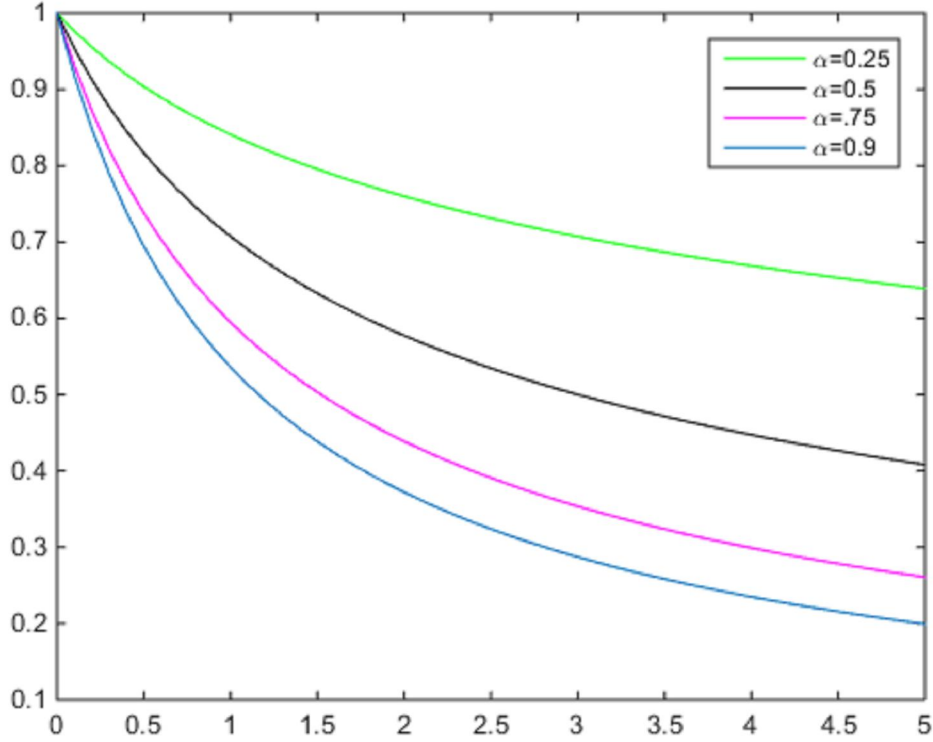


Figure-2: Response Current $i(t) = C(1 + \lambda t)^{-\alpha}$ to Unit Step Voltage Excitation for dielectric relaxation with non-singular power law memory kernel for various memory indexes: Y-Axis current $i(t)$ and X-Axis time t

Here in Figure-2 we observe that as α is decreasing the relaxing current lingers longer thus it is index of memory or forgetfulness in the kernel $k(t) = C(1 + \lambda t)^{-\alpha}$. The forgetfulness reduces as memory index decreases.

The above (46) relaxation expression says that the current lingers in a capacitor while the rate of change of voltage vanishes at $t > 0$; giving memorized relaxation of current.

This $i(t) = C(1 + \lambda t)^{-\alpha}$ is 'impulse response' of the system. This was also observed with singular power law memory kernel. However, the constitutive equation for capacitor in this case is with $k(t) = C(1 + \lambda t)^{-\alpha}$

$$i(t) = C \left({}_0I_t^1 v^{(1)}(t) + (-\alpha)\lambda \left({}_0I_t^2 v^{(1)}(t) \right) + (-\alpha)(-\alpha-1)\lambda^2 \left({}_0I_t^3 v^{(1)}(t) \right) + \dots \right) \tag{47}$$

This is very different from $i(t) = Cv^{(1)}(t)$ and $i(t) = C_\alpha v^{(\alpha)}(t)$; the classical case and case with fractional derivative respectively. Here in above (47) we are getting a series sum of weighted repeated integration i.e.

$$i(t) = C \sum_{n=1}^{\infty} w_n \left({}_0I_t^n v^{(1)}(t) \right) \quad (48)$$

With weights in above as $w_1 = 1$, $w_2 = -\alpha\lambda$, $w_3 = (\alpha)(\alpha+1)\lambda^2 \dots$

Taking Laplace transform of $i(t) = C \sum_{n=1}^{\infty} w_n \left({}_0I_t^n v^{(1)}(t) \right)$ we write impedance expression in Laplace domain $Z(s) = V(s) / I(s)$ as

$$Z(s) = \frac{1}{C \left(\sum_{n=1}^{\infty} w_n s^{1-n} \right)} \quad (49)$$

Though mathematically we have constructed the constitutive expression with non singular power law memory kernel, $k(t) = C(1 + \lambda t)^{-\alpha}$, the impedance expression as indicated above (49), is not verified for any dielectric system.

8.0. Constitutive equation of Capacitor due to Mittag-Leffler function as Non-Singular Memory Kernel

Here we take Memory Kernel as following for $t \geq 0$

$$k(t) = CE_\alpha(-\lambda t^\alpha); \quad 0 < \alpha < 1 \quad (50)$$

In above (50) C and λ are a positive real constants. For the constitutive equation with this Memory Kernel (50), we write the following

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \\ &= C \int_0^t (E_\alpha(-\lambda(t-\tau)^\alpha)) (v^{(1)}(\tau)) d\tau \\ &= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda(t-\tau)^\alpha)^n}{\Gamma(\alpha n + 1)} \right) (v^{(1)}(\tau)) d\tau \\ &= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^n}{\Gamma(\alpha n + 1)} \right) \int_0^t (t-\tau)^{\alpha n} v^{(1)}(\tau) d\tau \\ &= C \left(\sum_{n=0}^{\infty} (-1)^n \lambda^n \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} v^{(1)}(\tau) d\tau \right) \\ &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n + 1} [v^{(1)}(t)] \right) \end{aligned} \quad (51)$$

Where in above (51) we used the operator ${}_0I_t^\nu$, $\nu = \alpha n + 1$, which is Riemann-Liouville fractional integration (5) of order ν defined as ${}_0I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$. We write above (51) as series sum of weighted fractional integration

$$\begin{aligned}
i(t) &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n \left({}_0I_t^{\alpha n+1} \left[v^{(1)}(t) \right] \right) \\
&= C \left({}_0I_t^1 v^{(1)}(t) - \lambda \left({}_0I_t^{\alpha+1} v^{(1)}(t) \right) + \lambda^2 \left({}_0I_t^{2\alpha+1} v^{(1)}(t) \right) - \lambda^3 \left({}_0I_t^{3\alpha+1} v^{(1)}(t) \right) + \dots \right) \quad (52) \\
&= C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} v^{(1)}(t) \right)
\end{aligned}$$

With weights in this case as $w_0 = 1$, $w_1 = -\lambda$; ... $w_n = (-1)^n \lambda^n$ We get similar (but not the same) result that of earlier case with non-singular power law memory kernel and this too is very different from singular kernels that gave of $i(t) = Cv^{(1)}(t)$ and $i(t) = C_\alpha v^{(\alpha)}(t)$.

Thus, a Memory Kernel i.e. $k(t) = C - \frac{\lambda C t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 C t^{2\alpha}}{\Gamma(2\alpha+1)} - \dots$; $0 < \alpha < 1$ i.e. series-sum of power laws acting on derivative of voltage function $v^{(1)}(t)$, gives a relaxing current $i(t)$ with series sum of fractional integrations of various orders acting on rate of change of voltage i.e. $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} v^{(1)}(t) \right)$. We note that Memory Kernel in this case i.e. $k(t) = CE_\alpha(-\lambda t^\alpha)$ is not singular function at $k(0) = C$, and its derivative is not defined i.e. $k^{(1)}(t) \Big|_{t=0} = -\infty$.

Now we give a unit step input to this system so we have $v(t) = 1$, $t \geq 0$; with $v^{(1)}(t) = \delta(t)$. Placing this in derived constituent equation, i.e.

$i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} v^{(1)}(t) \right)$, we write the following

$$\begin{aligned}
i(t) &= Cv(t) - \lambda C \left({}_0I_t^{\alpha+1} v^{(1)}(t) \right) + \lambda^2 C \left({}_0I_t^{2\alpha+1} v^{(1)}(t) \right) - \lambda^3 C \left({}_0I_t^{3\alpha+1} v^{(1)}(t) \right) + \dots \\
&= C - \lambda C \left({}_0I_t^{\alpha+1} \delta(t) \right) + \lambda^2 C \left({}_0I_t^{2\alpha+1} \delta(t) \right) - \lambda^3 C \left({}_0I_t^{3\alpha+1} \delta(t) \right) + \dots \\
&= C - \lambda C \left(\frac{t^\alpha}{\Gamma(\alpha+1)} \right) + \lambda^2 C \left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) - \lambda^3 C \left(\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \right) + \dots \quad (53) \\
&= C \left(1 + \frac{(-\lambda)t^\alpha}{\Gamma(\alpha+1)} + \frac{(-\lambda)^2(t^\alpha)^2}{\Gamma(2\alpha+1)} + \frac{(-\lambda)^3(t^\alpha)^3}{\Gamma(3\alpha+1)} + \dots \right) \\
&= C \left(\sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(\alpha n+1)} \right) = CE_\alpha(-\lambda t^\alpha), \quad t \geq 0
\end{aligned}$$

In Figure-3 we plot the above (53) series (with $C = \lambda = 1$), to give various relaxation currents for various values of memory index α .

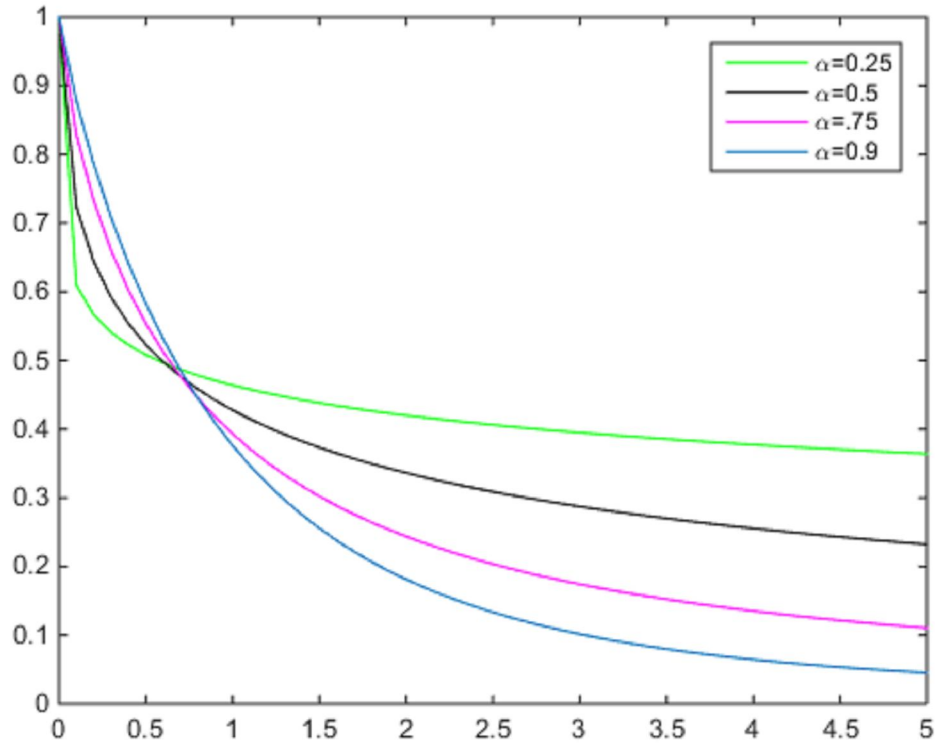


Figure-3: Response Current $i(t) = CE_{\alpha}(-\lambda t^{\alpha})$ to Unit Step Voltage Excitation for dielectric relaxation with non-singular Mittag-Leffler kernel for various memory indexes: Y-Axis current $i(t)$ and X-Axis time t

In Figure-3 we observe that as the index of memory α decreases in the memory kernel $k(t) = CE_{\alpha}(-\lambda t^{\alpha})$, the forgetfulness too decreases, i.e. the current response lingers for longer times.

In above (53) derivation we have used formula for fractional integration of delta function i.e. ${}_0I_t^{\nu} \delta(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ (11). What we observe that the relaxation current $i(t)$ to uncharged capacitor excited by unit-step voltage input; relaxes in proportional to the memory kernel function i.e. $i(t) \propto k(t)$ in this case $k(t) \sim E_{\alpha}(-\lambda t^{\alpha})$. Here in above $i(t) = CE_{\alpha}(-\lambda t^{\alpha})$ the current relaxes at $t > 0$ even while the rate of change of voltage has vanished; therefore memorizing the past excitation. We note that by placing $\alpha = 1$ we are not getting classical case i.e. $i(t) = Cv^{(1)}(t)$.

Let us do Laplace Transformation for evolution equation with Mittag-Leffler function as memory kernel, as depicted as follows

$$\begin{aligned}
i(t) &= (k(t)) * (v^{(1)}(t)) \\
\mathcal{L}\{i(t)\} &= \mathcal{L}\{(k(t)) * (v^{(1)}(t))\} \\
I(s) &= (\mathcal{L}\{k(t)\})(\mathcal{L}\{v^{(1)}(t)\}), \quad \mathcal{L}\{k(t)\} = \mathcal{L}\{CE_\alpha(-\lambda t^\alpha)\} = C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) \\
&= C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) (sV(s) - v(0)) \\
&= C \left(\left(\frac{s^\alpha}{s^\alpha + \lambda} \right) V(s) - \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) v(0) \right), \quad v(0) = 0, \quad V(s) = \frac{1}{s} \\
I(s) &= C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) \\
i(t) &= C \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{s^\alpha + \lambda} \right\} \\
&= CE_\alpha(-\lambda t^\alpha)
\end{aligned} \tag{54}$$

The same that we got in earlier via time domain analysis.

We write using constituent equation derived i.e. $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{\alpha n+1} v^{(1)}(t))$ the impedance in Laplace domain $Z(s) = V(s) / I(s)$ as

$$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-\alpha n}} \tag{55}$$

The other way to get closed form expression is using intermediate steps in Laplace Transform derivation above (55) i.e. $I(s) = C \left(\frac{s^{\alpha-1}}{s^\alpha + \lambda} \right) (sV(s) - v(0))$. This gives with $v(0) = 0$ the following expressions for $Z(s)$ and with $s = i\omega$, $Z(\omega)$

$$\begin{aligned}
Z(s) &= \frac{s^\alpha + \lambda}{Cs^\alpha} \\
Z(\omega) &= \left(\frac{1}{C} + \frac{\lambda}{C\omega^\alpha} \cos \frac{\alpha\pi}{2} \right) - i \left(\frac{\lambda}{C\omega^\alpha} \sin \frac{\alpha\pi}{2} \right)
\end{aligned} \tag{56}$$

Though we have constructed constitutive equation i.e. $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{\alpha n+1} v^{(1)}(t))$ with considering non-singular memory kernel as $k(t) = CE_\alpha(-\lambda t^\alpha)$, the impedance structures as expressed in above expressions are not yet observed in any experiment to best of our knowledge for any dielectric experiments.

With the above (56) closed form expression for a capacitor formed for non-singular memory kernel $k(t) = CE_\alpha(-\lambda t^\alpha)$ we can construct the capacitor impedance as composed of a lossy (resistive) element of value C^{-1} Ohms, in series with a ‘fractional capacitor’ of order $0 < \alpha < 1$ having constituent expression $i(t) = C_\alpha v^{(\alpha)}(t)$, with $C_\alpha \equiv C / \lambda$, formed via singular power law kernel, $k(t) \propto t^{-\alpha}$. The series impedance we express as follows

$$Z(s) = \frac{1}{C} + \frac{1}{\left(\frac{C}{\lambda}\right) s^\alpha} \tag{57}$$

Doing inverse Laplace of above (57) i.e. series connected impedances with $Z(s) = V(s)/I(s)$ the equivalent circuit equation is

$$v(t) = \frac{1}{C}i(t) + \frac{\lambda}{C}({}_0I_t^\alpha i(t)) \quad (58)$$

The above (58) equivalent constitutive equation for two series connected impedances is equivalent to our obtained constitutive equation $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{\alpha n+1} v^{(1)}(t))$, for this system with memory kernel as Mittag-Leffler decay function $k(t) = CE_\alpha(-\lambda t^\alpha)$.

9.0. Constitutive equation of Capacitor due to exponential function as Non-Singular Memory Kernel

Here we take Memory kernel as

$$k(t) = Ce^{-\lambda t}, \quad t \geq 0, \quad \lambda > 0; \quad C > 0 \quad (59)$$

The constitutive equation with Memory Kernel as $k(t) = Ce^{-\lambda t}$ gives the following

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \\ &= C \int_0^t (e^{-\lambda(t-\tau)}) (v^{(1)}(\tau)) d\tau \\ &= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{(-\lambda(t-\tau))^n}{n!} \right) (v^{(1)}(\tau)) d\tau \\ &= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n (\lambda)^n}{n!} \right) \int_0^t (t-\tau)^n v^{(1)}(\tau) d\tau \\ &= C \left(\sum_{n=0}^{\infty} (-1)^n \lambda^n \right) \left(\frac{1}{n!} \int_0^t (t-\tau)^n v^{(1)}(\tau) d\tau \right) \\ &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n ({}_0I_t^{n+1} [v^{(1)}(t)]) \end{aligned} \quad (60)$$

Thus, the memory Kernel which is pure exponential function i.e. $k(t) = Ce^{-\lambda t}$ gives a relaxation current which is weighted series sum of integer order multiple integration of rate of change of voltage function. Like in case of, $k(t) = CE_\alpha(-\lambda t^\alpha)$ we write the following

$$\begin{aligned} i(t) &= C \sum_{n=0}^{\infty} (-1)^n \lambda^n ({}_0I_t^{n+1} [v^{(1)}(t)]) \\ &= C ({}_0I_t^1 v^{(1)}(t)) - \lambda C ({}_0I_t^2 v^{(1)}(t)) + \lambda^2 C ({}_0I_t^3 v^{(1)}(t)) - \lambda^3 C ({}_0I_t^4 v^{(1)}(t)) + \dots \\ &= C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} [v^{(1)}(t)]) \end{aligned} \quad (61)$$

$$w_0 = 1, \quad w_1 = -\lambda, \quad w_2 = \lambda^2, \quad \dots w_n = (-1)^n \lambda^n$$

In the constitutive equation with memory kernel as non-singular Mittag-Leffler function, i.e. $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{\alpha n+1} v^{(1)}(t))$ if we place $\alpha = 1$ we get above (61)

$$\text{i.e. } i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} [v^{(1)}(t)]).$$

We give a step input to system having Memory kernel (59) and observe the following

$$\begin{aligned}
i(t) &= Cv(t) - \lambda C({}_0I_t^2 v^{(1)}(t)) + \lambda^2 C({}_0I_t^3 v^{(1)}(t)) - \lambda^3 C({}_0I_t^4 v^{(1)}(t)) + \dots \\
&= C - \lambda C({}_0I_t^2 \delta(t)) + \lambda^2 C({}_0I_t^3 \delta(t)) - \lambda^3 C({}_0I_t^4 \delta(t)) + \dots \\
&= C - \lambda Ct + \lambda^2 C\left(\frac{t^2}{2!}\right) - \lambda^3 C\left(\frac{t^3}{3!}\right) + \dots \\
&= C\left(1 + \frac{(-\lambda)t}{1!} + \frac{(-\lambda)^2(t)^2}{2!} + \frac{(-\lambda)^3(t)^3}{3!} + \dots\right) \\
&= C\left(\sum_{n=0}^{\infty} \frac{(-\lambda t)^n}{n!}\right) = Ce^{-\lambda t}, \quad t \geq 0
\end{aligned} \tag{62}$$

In above (62) we have used ${}_0I_t^m \delta(t) = \frac{1}{(m-1)!} t^{m-1}$, $m = 1, 2, 3, \dots$; that is integration of delta-function [23]. In addition, we assumed $v(0) = 0$, that is voltage stress applied to uncharged capacitor, thus we wrote ${}_0I_t^1 v^{(1)}(t) = v(t)$ in the above (62) derivation.

That the relaxation current to unit step voltage input to an uncharged capacitor having the memory kernel as exponential decay function $k(t) \sim e^{-\lambda t}$ has relaxation current $i(t) \propto k(t)$. this is $i(t) = Ce^{-\lambda t}$ ‘impulse response’ of this system.

We note that the Memory Kernel $k(t) \sim e^{-\lambda t}$ is non-singular function and has derivative every-where. Let us apply Laplace Transformation as depicted below

$$\begin{aligned}
i(t) &= (k(t)) * (v^{(1)}(t)) \\
\mathcal{L}\{i(t)\} &= \mathcal{L}\{(k(t)) * (v^{(1)}(t))\} \\
I(s) &= (\mathcal{L}\{k(t)\})(\mathcal{L}\{v^{(1)}(t)\}), \quad \mathcal{L}\{k(t)\} = \mathcal{L}\{Ce^{-\lambda t}\} = C\left(\frac{1}{s+\lambda}\right) \\
&= \left(\frac{C}{s+\lambda}\right)(sV(s) - v(0)) \\
&= C\left(\left(\frac{s}{s+\lambda}\right)V(s) - \left(\frac{1}{s+\lambda}\right)v(0)\right), \quad v(0) = 0, \quad V(s) = \frac{1}{s} \\
I(s) &= C\left(\frac{1}{s+\lambda}\right) \\
i(t) &= C\mathcal{L}^{-1}\left\{\frac{1}{s+\lambda}\right\} \\
&= Ce^{-\lambda t}
\end{aligned} \tag{63}$$

We get same result of as earlier (62).

By using the obtained constitutive equation, i.e. $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{n+1} [v^{(1)}(t)] \right)$ we write the impedance expression in this case in Laplace domain as

$$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-n}} \tag{64}$$

In this case, we can also get a closed form formula for impedance as we did for Mittag-Leffler case in earlier section and write

$$Z(s) = \frac{s + \lambda}{sC} \quad (65)$$

$$Z(\omega) = \frac{1}{C} - i \frac{\lambda}{\omega C}$$

This implies this system is equivalent to series connected impedances one as pure resistive element with value C^{-1} Ohms, connected to ideal capacitor with capacitor value C / λ . From this, we have equivalent constitutive equation as

$$v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} \int_0^t i(\tau) d\tau \quad (66)$$

The above (66) $v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} ({}_0I_t^1 i(t))$ is equivalent to our obtained constituent equation i.e. following

$$i(t) = C \int_0^t v^{(1)}(\tau) d\tau - \lambda C \int_0^t \int_0^t v^{(1)}(\tau) d\tau d\tau + \lambda^2 C \int_0^t \int_0^t \int_0^t v^{(1)}(\tau) d\tau d\tau d\tau - \dots \quad (67)$$

The above (67) is expanded form of expression $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} [v^{(1)}(t)])$.

Though we have constructed constitutive equation i.e. $i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} v^{(1)}(t))$ with considering non-singular memory kernel as $k(t) = Ce^{-\lambda t}$, the impedance structures as expressed in above expressions (65) are not yet observed in any experiment to best of our knowledge for any dielectric experiments.

10.0. Constitutive equation of Capacitor due to stretched exponential Non-Singular Memory Kernel

Here we take Memory Kernel as stretched exponential function

$$k(t) = Ce^{-(\lambda t)^\alpha}, \quad t \geq 0, \quad \lambda > 0; \quad 0 < \alpha < 1; \quad C > 0 \quad (68)$$

With $\alpha = 1$ the situation is same as for the case of pure exponential kernel for memory. We now proceed in following steps

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \\ &= C \int_0^t \left(e^{-(\lambda(t-\tau))^\alpha} \right) (v^{(1)}(\tau)) d\tau \\ &= C \int_0^t \left(\sum_{n=0}^{\infty} \frac{((-\lambda(t-\tau))^\alpha)^n}{n!} \right) (v^{(1)}(\tau)) d\tau \\ &= C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^{\alpha n}}{n!} \right) \int_0^t (t-\tau)^{\alpha n} v^{(1)}(\tau) d\tau \\ &= C \left(\sum_{n=0}^{\infty} \Gamma(\alpha n + 1) \left(\frac{(-1)^n \lambda^{\alpha n}}{n!} \right) \right) \left(\frac{1}{\Gamma(\alpha n + 1)} \int_0^t (t-\tau)^{\alpha n} v^{(1)}(\tau) d\tau \right) \\ &= C \sum_{n=0}^{\infty} (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma(\alpha n + 1)}{n!} \right) ({}_0I_t^{\alpha n + 1} [v^{(1)}(t)]) \end{aligned} \quad (69)$$

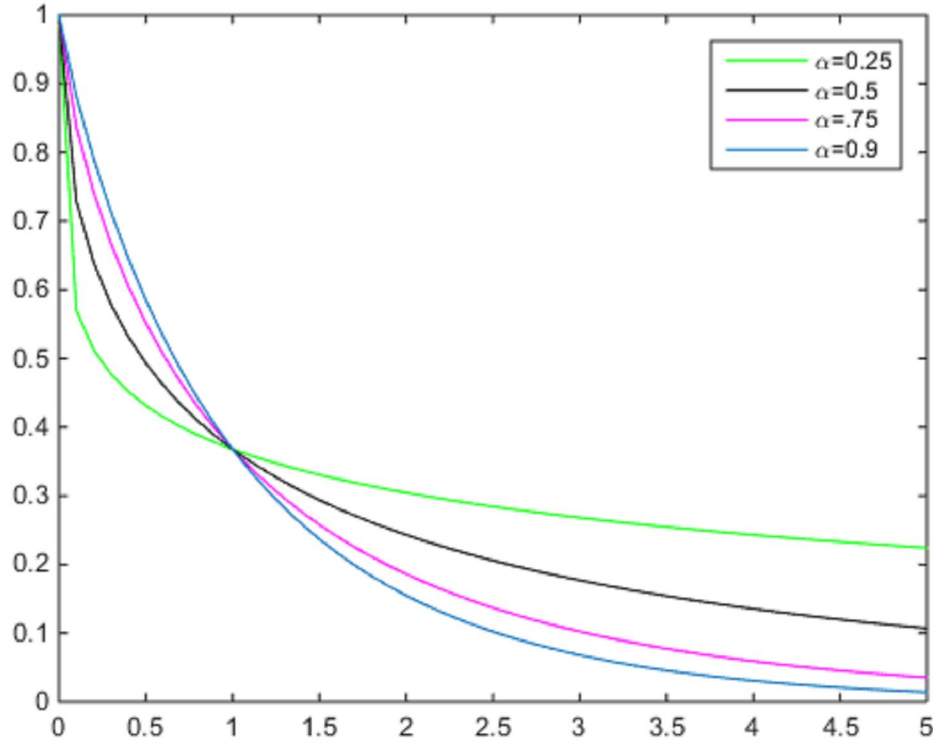


Figure-4: Response Current to Unit Step Voltage Excitation for dielectric relaxation with non-singular Stretched Exponential kernel for various memory indexes: Y-Axis current $i(t)$ and X-Axis time t

We plot the relaxation current to unit step input voltage with $C = \lambda = 1$ in Figure-4, for various memory index α . There is similarity of this with Figure-3. Here also we note that as the memory index is decreasing the forgetfulness also decreases, as the impulse response lingers for longer time, with memory kernel $k(t) = Ce^{-(\lambda t)^\alpha}$.

This gives the constitutive equation for $i(t)$ with series weighted sum of fractional integration of various orders of input $v^{(1)}(t)$; similar to the case with Mittag-Leffler function as Memory kernel.

$$i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} \left[v^{(1)}(t) \right] \right); \quad w_n = (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma(\alpha n+1)}{n!} \right) \quad (70)$$

We note that with $\alpha = 1$ we obtain the exact case for memory kernel with exponential function i.e. $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{n+1} \left[v^{(1)}(t) \right] \right)$.

In this case we get impedance in Laplace domain as

$$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-\alpha n}} \quad (71)$$

Though we have constructed constitutive equation i.e. $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} v^{(1)}(t) \right)$ with considering non-singular memory kernel as $k(t) = C e^{-(\lambda t)^\alpha}$, the impedance structures as expressed in above expression (71) are not yet observed in any experiment to best of our knowledge for any dielectric experiments.

11.0. Derivative of the Stretched Exponential Function

We take the memory kernel as derivative of stretched exponential function i.e

$$k(t) = -\frac{d}{dt} C e^{-(\lambda t)^\alpha} = C \alpha \lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha} \quad (72)$$

So we have memory kernel in following form

$$k(t) = C \alpha \lambda (\lambda t)^{-(1-\alpha)} e^{-(\lambda t)^\alpha}, \quad C, \lambda > 0, \quad 0 < \alpha < 1 \quad (73)$$

We note that this memory kernel has a singular component i.e. $t^{-(1-\alpha)}$ and stretched exponential component i.e. $e^{-(\lambda t)^\alpha}$. With this memory kernel we write following steps

$$\begin{aligned} i(t) &= (k(t)) * (v^{(1)}(t)) \\ &= C \int_0^t \left(\alpha \lambda (\lambda \tau)^{-(1-\alpha)} e^{-(\lambda \tau)^\alpha} \right) (v^{(1)}(\tau)) d\tau \\ &= \alpha \lambda^\alpha C \int_0^t \left((t-\tau)^{-(1-\alpha)} \sum_{n=0}^{\infty} \frac{((-\lambda(t-\tau))^\alpha)^n}{n!} \right) (v^{(1)}(\tau)) d\tau \\ &= \alpha \lambda^\alpha C \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^{\alpha n}}{n!} \right) \int_0^t (t-\tau)^{(n+1)\alpha-1} v^{(1)}(\tau) d\tau \\ &= \alpha \lambda^\alpha C \left(\sum_{n=0}^{\infty} \Gamma((n+1)\alpha) \left(\frac{(-1)^n \lambda^{\alpha n}}{n!} \right) \right) \left(\frac{1}{\Gamma((n+1)\alpha)} \int_0^t (t-\tau)^{(n+1)\alpha-1} v^{(1)}(\tau) d\tau \right) \\ &= C_\lambda \sum_{n=0}^{\infty} (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right) \left({}_0I_t^{(n+1)\alpha} [v^{(1)}(t)] \right); \quad C_\lambda = \alpha \lambda^\alpha C \end{aligned} \quad (74)$$

We write the above as $i(t) = C_\lambda \sum_{n=0}^{\infty} w_n \left({}_0I_t^{(n+1)\alpha} v^{(1)}(t) \right)$ with $w_n = (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right)$, and $C_\lambda = \alpha \lambda^\alpha C$. In expanded form we write

$$\begin{aligned} i(t) &= C_\lambda \sum_{n=0}^{\infty} w_n \left({}_0I_t^{(n+1)\alpha} v^{(1)}(t) \right); \quad C_\lambda = \alpha \lambda^\alpha C \\ &= C_\lambda \left(w_0 \left({}_0I_t^\alpha v^{(1)}(t) \right) + w_1 \left({}_0I_t^{2\alpha} v^{(1)}(t) \right) + w_2 \left({}_0I_t^{3\alpha} v^{(1)}(t) \right) + \dots \right) \end{aligned} \quad (75)$$

With weights as $w_0 = \Gamma(\alpha)$, $w_1 = -\lambda^\alpha \Gamma(2\alpha)$, $w_2 = \frac{\lambda^{2\alpha}}{2} \Gamma(3\alpha)$, $w_3 = -\frac{\lambda^{3\alpha}}{3!} \Gamma(4\alpha)$.

We used ${}_0I_t^\nu [f(t)] = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$ in above (74) derivation. With the similar procedures as demonstrated in earlier results with input as $v^{(1)}(t) = \delta(t)$ we write the following for obtaining ‘impulse response’

$$\begin{aligned}
i(t) &= C_\lambda \sum_{n=0}^{\infty} (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right) \left({}_0I_t^{(n+1)\alpha} [v^{(1)}(t)] \right) \\
&= C_\lambda \sum_{n=0}^{\infty} (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right) \left({}_0I_t^{(n+1)\alpha} [\delta(t)] \right) \\
&= C_\lambda \sum_{n=0}^{\infty} (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right) \left(\frac{t^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} \right) \\
&= C_\lambda \sum_{n=0}^{\infty} \left(\frac{(-1)^n \lambda^{\alpha n} t^{\alpha n} t^{\alpha-1}}{n!} \right) = C_\lambda t^{\alpha-1} e^{-(\lambda t)^\alpha}
\end{aligned} \tag{76}$$

We plot the series expression of (76) in Figure-5, to get relaxation current to unit step voltage input excitation for various memory index α , with $C = \lambda = 1$

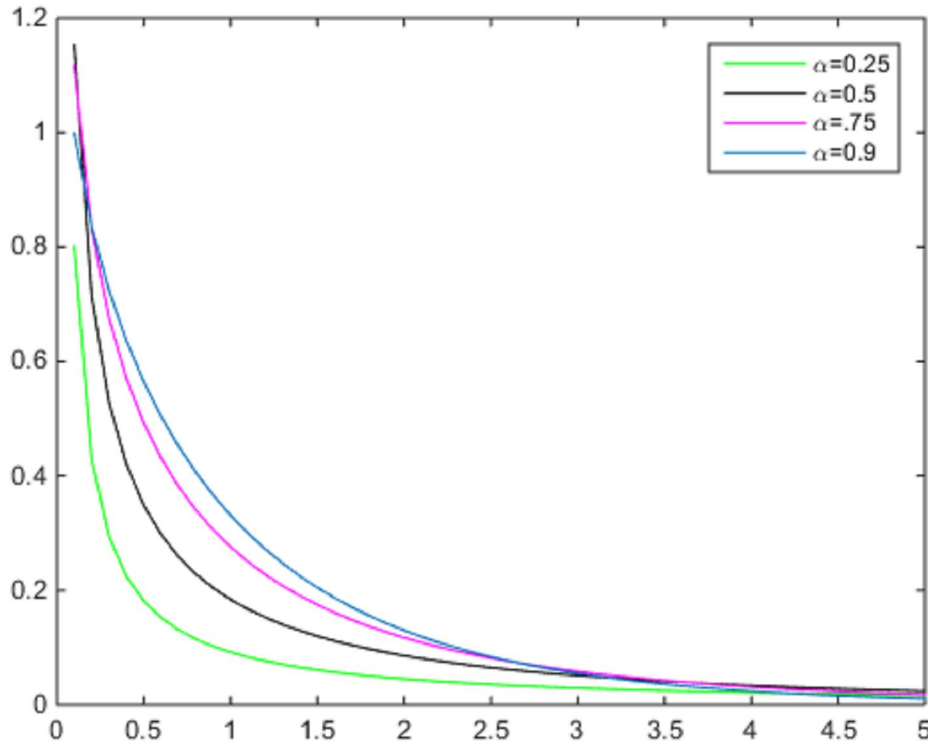


Figure-5: Response Current to Unit Step Voltage Excitation for dielectric relaxation with singular kernel Derivative of Stretched Exponential for various memory indexes: Y-Axis current $i(t)$ and X-Axis time t

We observe from Figure-5 that the memory index α , in $k(t) = C\alpha\lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha}$ behaves opposite to all other previous cases. Here as the α index of memory decreases, we get greater forgetfulness i.e. the relaxation decreases quickly as the index of memory decrease.

We used the formula i.e. ${}_0I_t^\nu \delta(t) = \frac{1}{\Gamma(\nu)} t^{\nu-1}$ in above (76) derivation.

The impedance in Laplace domain for system is

$$Z(s) = \frac{1}{C_\lambda \sum_{n=0}^{\infty} w_n s^{1-(n+1)\alpha}}, \quad C_\lambda = \alpha \lambda^\alpha C \quad (77)$$

Though we have constructed constitutive equation i.e. $i(t) = C_\lambda \sum_{n=0}^{\infty} w_n \left({}_0 I_t^{(n+1)\alpha} v^{(1)}(t) \right)$ with considering non-singular memory kernel as $k(t) = C_\lambda t^{\alpha-1} e^{-(\lambda t)^\alpha}$, the impedance structures as expressed in above expression (77) are not yet observed in any experiment to best of our knowledge for any dielectric experiments.

12.0. Discussions

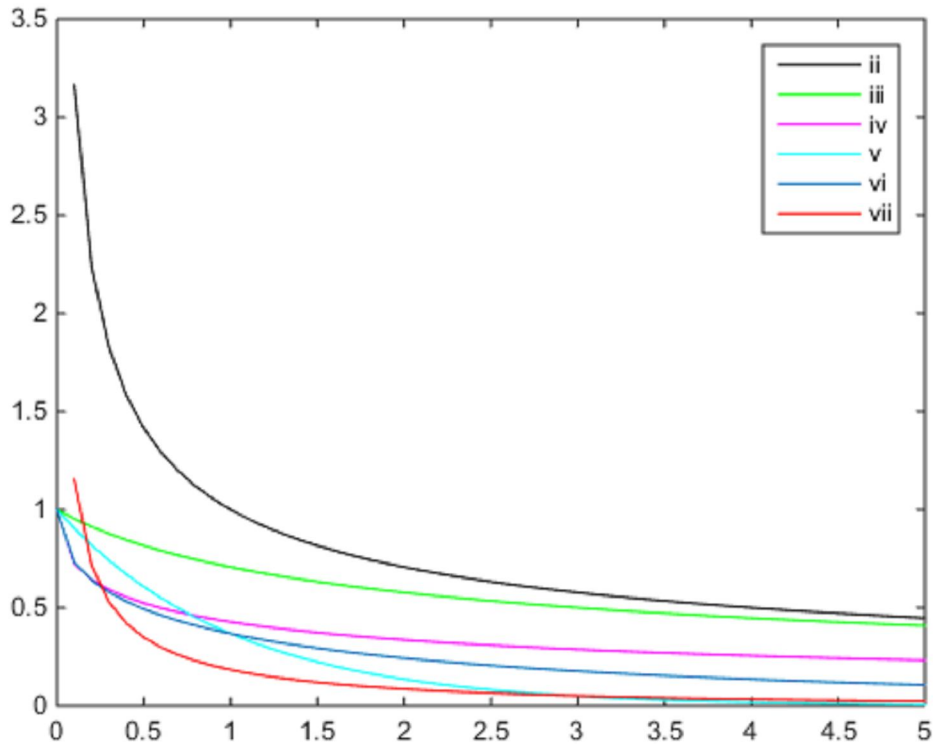


Figure-6: Response Current to Unit Step Voltage Excitation for dielectric relaxation with non-singular and singular memory kernels with memory index as 0.5: Y-Axis current $i(t)$ and X-Axis time t

S.No	Function of Memory Kernel $k(t)$ $0 < \alpha < 1$	Type	Constitutive Equation of Capacitor	Impedance in Laplace Domain
1	Delta Function $C\delta(t)$	Singular	$i(t) = Cv^{(1)}(t)$ $i(t) = C({}_0I_t^0 v^{(1)}(t))$	$Z(s) = \frac{1}{sC}$
2	Power Law $Ct^{-\alpha}$	Singular	$i(t) = C_\alpha v^{(\alpha)}(t)$ $C_\alpha = C(\Gamma(1-\alpha))$ $i(t) = C_\alpha({}_0I_t^{1-\alpha} v^{(1)}(t))$	$Z(s) = \frac{1}{s^\alpha C_\alpha}$
3	Non-singular Power Law $C(1+\lambda t)^{-\alpha}$	Non-Singular	$i(t) = C \sum_{n=1}^{\infty} w_n ({}_0I_t^n v^{(1)}(t))$ $w_1 = 1, w_2 = -\alpha\lambda,$ $w_3 = (\alpha)(\alpha+1)\lambda^2 \dots$	$Z(s) = \frac{1}{C \left(\sum_{n=1}^{\infty} w_n s^{1-n} \right)}$
4	Mittag-Leffler $CE_\alpha(-\lambda t^\alpha)$	Non-Singular	$i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} v^{(1)}(t))$ $w_n = (-1)^n \lambda^n$ $v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} ({}_0I_t^\alpha i(t))$	$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-\alpha n}}$ $Z(s) = \frac{1}{C} + \frac{1}{\left(\frac{C}{\lambda}\right) s^\alpha}$
5	Exponential $Ce^{-\lambda t}$	Non-Singular	$i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{n+1} v^{(1)}(t))$ $w_n = (-1)^n \lambda^n$ $v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} \int_0^t i(\tau) d\tau$	$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-n}}$ $Z(s) = \frac{1}{C} + \frac{\lambda}{sC}$
6	Stretched-Exponential $Ce^{-(\lambda t)^\alpha}$	Non-Singular	$i(t) = C \sum_{n=0}^{\infty} w_n ({}_0I_t^{\alpha n+1} [v^{(1)}(t)])$ $w_n = (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma(\alpha n+1)}{n!} \right)$	$Z(s) = \frac{1}{C \sum_{n=0}^{\infty} w_n s^{-\alpha n}}$
7	Derivative of Stretched Exponential $C\alpha\lambda^\alpha t^{\alpha-1} e^{-(\lambda t)^\alpha}$	Singular	$i(t) = C_\lambda \sum_{n=0}^{\infty} w_n ({}_0I_t^{(n+1)\alpha} v^{(1)}(t))$ $C_\lambda = \alpha\lambda^\alpha C$ $w_n = (-1)^n \left(\frac{\lambda^{\alpha n} \Gamma((n+1)\alpha)}{n!} \right)$	$Z(s) = \frac{1}{C_\lambda \sum_{n=0}^{\infty} w_n s^{1-(n+1)\alpha}}$

Table-1: Summary of results of various singular and Non-singular Memory Kernels

12.1. Observations

In Figure-6, we represent various relaxation currents $i(t)$ for memory index $\alpha = 0.5$, for unit step voltage excitation. The curves with various colors with corresponding numbers

are for memory kernels, with $C = \lambda = 1$ (ii)- $k(t) = Ct^{-\alpha}$, (iii)- $k(t) = C(1 + \lambda t)^{-\alpha}$, (iv)- $k(t) = CE_{\alpha}(-\lambda t^{\alpha})$, (v)- $k(t) = Ce^{-\lambda t}$, (vi)- $k(t) = Ce^{-(\lambda t)^{\alpha}}$ and (vii)- $k(t) = C\alpha\lambda^{\alpha}t^{\alpha-1}e^{-(\lambda t)^{\alpha}}$.

The Table-1 summarizes the result of constituent equation for dielectric relaxation of capacitor and lists its impedance function in Laplace domain, for various types singular or non-singular memory kernels considered in our current evolution equation for relaxing current defined via convolution operation of memory kernel and rate of change of applied voltage. We can summarize that the constituent equation as a general capacitor law i.e.

$$i(t) \propto \sum_n w_n \left({}_0I_t^* v^{(1)}(t) \right) \quad (78)$$

for all the cases, discussed. That is weighted infinite sum of integral operations (integer or fractional order-denoted as*in operator ${}_0I_t^*$) on the rate of change of voltage.

For a zero memory case, with $k(t) = C\delta(t)$ the classical constituent expression is standard textbook capacitor relation i.e. $i(t) = Cv^{(1)}(t)$, that we write in above generalized form as $i(t) = C \left({}_0I_t^0 v^{(1)}(t) \right)$. This is zero memory case with order of integration acting as zero, on the input $v^{(1)}(t)$. The impedance of this zero memory case is $Z(s) = (Cs)^{-1}$, that is ideal textbook capacitors.

For a singular decaying power law as memory kernel, i.e. $k(t) = Ct^{-\alpha}$, our constituent equation for capacitor is via fractional derivative and we write that as $i(t) = C_{\alpha} v^{(\alpha)}(t)$. Casting this in standard formulation as above, we write as fractional integration of order $1-\alpha$ on the input change $v^{(1)}(t)$, i.e. $i(t) = C_{\alpha} \left({}_0I_t^{1-\alpha} v^{(1)}(t) \right)$, with fractional order impedance of this system as $Z(s) = (s^{\alpha} C_{\alpha})^{-1}$.

We note that in these two cases, we are having only one term in summation expression of (78). Here we point out that this system is experimentally verified via several approaches in dielectric relaxation studies. We call this as fractional capacitor-appears in real life observations. The entire experiments and analysis on dielectric studies as done in [1]-[5], [6]-[12], [17]-[22], [27]-[28]; confirm that this constituent equation, $i(t) = C_{\alpha} v^{(\alpha)}(t)$, and fractional order impedance, $Z(s) = (s^{\alpha} C_{\alpha})^{-1}$ is valid. The basis of dielectric relaxation current is Curie-von Schweidler law i.e. when a constant step voltage excites uncharged system is given as singular power law $i(t) \propto t^{-\alpha}$. This expression of ‘fractional capacitor’ also explains the memory effect that is observed in these systems. The memory effect is that if we keep afloat a capacitor on a constant voltage for certain time and then we disconnect it, then the self-discharge phenomena is depending on time that system has been placed on float charge. In short, capacitors (with fractional order) memorizes its charging history. This is verified in [12], [22] experimentally and then via constituent equation i.e. $i(t) = C_{\alpha} v^{(\alpha)}(t)$, and fractional impedance $Z(s) = (s^{\alpha} C_{\alpha})^{-1}$. The power-law is observed in various systems as in [13]-[16], [28]. We do not observe the capacitor memorizing its charging history for ideal text book capacitor, i.e. capacitor with zero-memory.

The next system we analyzed is kernel as non-singular power law $k(t) = C(1 + \lambda t)^{-\alpha}$ that gave constituent expression for capacitor as $i(t) = C \sum_{n=1}^{\infty} w_n \left({}_0I_t^n v^{(1)}(t) \right)$. This is infinite

series of weighted sum of repeated integrations operating on $v^{(1)}(t)$. The impedance function for this type of system is $Z(s) = C^{-1} \left(\sum_{n=1}^{\infty} w_n s^{1-n} \right)^{-1}$. This though derived mathematically this lacks experimental validation in dielectric studies.

Then our analysis followed for memory kernel as Mittag-Leffler function i.e. $k(t) = CE_{\alpha}(-\lambda t^{\alpha})$ that is non-singular. We derived constituent equation as $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} v^{(1)}(t) \right)$. Here we are able to write a compact equation from this series as $v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} \left({}_0I_t^{\alpha} i(t) \right)$ that gives voltage drop across a loop comprising of resistance of value C^{-1} Ohms and a ‘fractional capacitor’ of value C/λ . The impedance relation that we obtained for this system is $Z(s) = C^{-1} \left(\sum_{n=0}^{\infty} w_n s^{-\alpha n} \right)^{-1}$ giving series connected impedances as equivalent circuit given as $Z(s) = C^{-1} + \left(\frac{C}{\lambda} \right)^{-1} s^{-\alpha}$. Though mathematically we derived these expressions yet experimentally this phenomena is not observed yet. This lacks experimental validation as was done for singular power law case.

Next our memory kernel was non-singular a pure exponential function, $k(t) = Ce^{-\lambda t}$, that gave constituent equation as $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{n+1} v^{(1)}(t) \right)$. Here too we are able to write equivalent expression as $v(t) = \frac{1}{C} i(t) + \frac{\lambda}{C} \left({}_0I_t^1 i(t) \right)$. The impedance we obtain as $Z(s) = C^{-1} \left(\sum_{n=0}^{\infty} w_n s^{-n} \right)^{-1}$ an equivalent circuit as $Z(s) = C^{-1} + \left(\frac{C}{\lambda} \right)^{-1} s^{-1}$. For this type of system the experimental validity for dielectric experiments are lacking.

The stretched exponential decay functions are studied since 1854 as Kohlraush postulated the electric field in dielectric relaxes as $\sim e^{-(\lambda t)^{\alpha}}$. Many studies use this to study dispersion of Electric Modulus and dielectric constants [30]-[34]. This singular relaxation function we take as memory kernel $k(t) = Ce^{-(\lambda t)^{\alpha}}$ and obtained the constituent expression as $i(t) = C \sum_{n=0}^{\infty} w_n \left({}_0I_t^{\alpha n+1} \left[v^{(1)}(t) \right] \right)$. We note similarity with the Mittag-Leffler type memory kernel’s constituent equation that we obtained, but weights are different. We did not proceed to get equivalent circuit in this case. The impedance obtained is $Z(s) = C^{-1} \left(\sum_{n=0}^{\infty} w_n s^{-\alpha n} \right)^{-1}$; and we did not reduce this to equivalent circuit. That is due to lack of close form simple expression of Laplace Transform of the stretched exponential function. This impedance expression also lacks experimental validity.

Then we used memory kernel as derivative of stretched exponential function as $k(t) = C_{\lambda} t^{\alpha-1} e^{-(\lambda t)^{\alpha}}$; this has singular part and stretched exponential part. With this we obtained constituent expression for capacitor as $i(t) = C_{\lambda} \sum_{n=0}^{\infty} w_n \left({}_0I_t^{(n+1)\alpha} v^{(1)}(t) \right)$, and impedance as $Z(s) = C_{\lambda}^{-1} \left(\sum_{n=0}^{\infty} w_n s^{1-(n+1)\alpha} \right)^{-1}$. We observe that this memory kernel though singular has infinite fractional order integrations in constituent equation. Here too we have no evidence of experimental validity for dielectric relaxation studies.

12.2. Inference

From the above discussion, we are at universal constituent law for dielectric relaxation as noted in general $i(t) \propto \sum_n w_n \left({}_0I_t^* v^{(1)}(t) \right)$. We note that the singular memory kernels i.e. delta function and singular power function, has only one term in the constituent equation, i.e. zero memory case with integration order as zero and fractional capacitor case with integration order as fractional order of $0 < (1-\alpha) < 1$. Thus for singular function case we have $i(t) \propto {}_0I_t^* v^{(1)}(t)$.

We point out the singular function memory kernel with stretched exponential function also included does not have single term, but is like all other constituent equations that we got for non-singular kernel as infinite terms in summation and thus is $i(t) \propto \sum_n w_n \left({}_0I_t^* v^{(1)}(t) \right)$. We also observe the non-singular memory kernels do give the expressions with infinite series $i(t) \propto \sum_n w_n \left({}_0I_t^* v^{(1)}(t) \right)$, with integer order or fractional order integrations (contained in * for operator ${}_0I_t^*$).

Summarily we can thus classify the memorized relaxation dynamics for dielectric as (1) Curie-von Schweidler type with memory kernel as singular. Here the constituent equation is having only one integration term operating on rate of change of voltage; and (2) Kohlraush type mainly described by non singular memory kernel, or via combination of singular and decay function, returns constitutive equation as infinite series of sum of weighted integrations. In some cases, we demonstrated the simplification to get a simpler constituent equation. However, Curie-von Schweidler types are experimentally validated for capacitor cases, the Kohlraush type are not observed in experiments of dielectric relaxation studies.

Conclusions

A very relevant question that is: if we have in reality a singular memory kernel or a non-singular memory kernel, for dielectric relaxation dynamics, that constitutes basic capacitor expression. This study shows if we are having a singular memory kernel (alone and not with combination with other decay function), then we observe the reality better, for dielectric relaxation mechanism. That is we get classical capacitor formula and fractional capacitor formula; which have been experimentally observed and validated. With single singular memory kernels in the evolution equation of the system, we see that the constituent expression of capacitor for current is first derivative of voltage or fractional derivative of voltage applied, or we say that capacitor current is zero order integration or fractional order integration of rate of change of applied voltage. This capacitor behavior we have classified as Curie-von Schweidler type. Though mathematically non-singular memory kernel is possible, yet the constitutive equation for capacitor relaxation dynamics does not give the useful information. Here we get infinite weighted sum series of multiple integrations or fractional integrations applied to rate of change of applied voltage. This type of capacitor relaxation we classified as Kohlraush type. The universal dielectric relaxation law of Curie-von Schweidler still holds with single singular power-law memory kernel and not via non-singular memory kernels, or via combination of singular and decay function. This maybe because we are unable presently assigns physical sense to mathematically correct constitutive equations for capacitor relaxation, due to non-singular memory kernel, or via combination of singular

and decay function, perhaps due to lack of experimental evidences for the derived constituent laws and impedance functions. Here we are not questioning researchers modeling relaxation of dielectric via non-singular functions, yet we are hinting about validity of basic constituent equation of capacitor dynamics and the impedance functions thus obtained via considering non-singular memory kernel-that we obtain as weighted series sum of several orders (fractional or integer order) integrals operating on rate of change of voltage. We are not presently having experimental validation to make such conclusions. Therefore dichotomy still exists-the single singular memory kernel, giving singular current relaxation function as ‘impulse-response’ of capacitor and its corresponding impedance function-though experimentally validated; yet physically we are unable to give ‘physical sense’ to the singularity. Whereas if we assume that memory kernel is of non-singular type, we get current relaxation function that is too non-singular via derived constitutive equations that are difficult to visualize and impedance functions are yet not experimentally validated.

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