# Theory of capacitors with a new formulation of charge storage concept and observed relaxation current as per Curie-von Schweidler law & determining its rate histogram function as Zipf's distribution


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## ABSTRACT


The classical power law relaxation, i.e. relaxation of current with inverse of power of time for a step-voltage excitation to dielectric-as popularly known as Curie-von Schweidler law is empirically derived and is observed in several relaxation experiments on various dielectrics studies since late 19$^{th}$ Century. This relaxation law is also regarded as 'universal-law' for dielectric relaxations; and is also termed as power law. This empirical Curie-von Schewidler relaxation law is then used to derive fractional differential equations describing constituent expression for capacitor. In this write-up, we give simple mathematical treatment to derive the distribution of relaxation rates of this Curie-von Schweidler law, and show that the histogram of relaxation rate follows Zipf's power law distribution. We also show the method developed here give Zipfian power law distribution for relaxing time constants. Then we will show however mathematically correct this may be, but physical interpretation from the obtained time constants distribution are contradictory to the Zipfian rate relaxation distribution. In this write-up, we develop possible explanation that as to why Zipfian distribution of relaxation rates appears for Curie-von Schweidler Law, and relate this law to time variant rate of relaxation. We also note down several integral representation formulations derived in this write-up for the Curie-von Schweidler relaxation law. In this note we derive appearance of fractional derivative while using Zipfian power law distribution that gives notion of scale dependent relaxation rate function for Curie-von Schweidler relaxation phenomena. This note gives analytical approach to get insight of a non-Debye relaxation and gives a new treatment especially much used empirical Curie-von Schweidler (universal) relaxation law. In this study, we revisit the concept of classical capacitor theory-and derive possible new explanations to the definition of capacitance, charge stored in capacitor. We introduce the capacity function with respect to time to describe the charge storage in a classical capacitor and fractional capacitor. Here we will describe that charge stored at any time in a capacitor is 'convolution integral' of defined capacity function of a capacitor and voltage stress across it. This approach however is different to the conventional method where we multiply the capacity and voltage functions to obtain charge stored. This new concept is in line with the observation of charge stored, relaxation current in form of impulse function for ideal geometrical capacitor of constant capacity, and power-law decay current that is given by universal dielectric relaxation law called as Curie von-Schweidler law, when an uncharged capacitor is impressed with a constant voltage stress. This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current that is formulation of 'fractional capacitor'. A 'fractional capacitor' we will discuss with this new concept of redefining the charge store definition i.e. via this convolution integral approach, and obtain the loss tangent value. We will also show for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in terms of fractional units of Farads per second to the power a fractional number, to units of Farads. From the defined capacity function, we will also derive integrated capacity of capacitor. We will also give possible physical explanation by taking example of porous and non-porous pitchers of constant volume holding water and thus, explaining the various aspects of classical capacitor and fractional capacitor that we arrive with this new formulation. We note that circuit theory with classical calculus and fractional calculus remains unaltered.

**Keywords:** Power law, relaxation rate distribution function, fractional derivative, fractional integration, Curie-von Schweidler law, time-constants, Laplace integral, Zipf's Law, integral representation, time dependent relaxation rate, scale dependent relaxation rate, non-Debye relaxation, time dependent capacitance, convolution integral, fractional capacity, geometrical capacity, time varying capacity function, integrated capacity, loss tangent, fractional unit.

## Introduction

The Curie-von Schweidler law relates to relaxation current in dielectric when a step DC voltage is applied and is given by $i(t) \sim t^{-n}$, where $t > 0$ and the power (exponent) i.e. $n$ is called relaxation constant or decay constant, where $0 < n < 1$ [1]-[4]. We note that $n$ is non-integer. This relaxation law is taken as universal law, at least for dielectric relaxations. Whereas we are used to Debye type of relaxation i.e. exponential decay law given by $i(t) \sim e^{-t/\tau_0}$ or $i(t) \sim e^{-\lambda_0 t}$ where $\tau_0$ is the relaxation time constant while $\lambda_0$ denotes the relaxation rate of the process with $\lambda_0 = \tau_0^{-1}$ . The radioactive decay is example of ideal Debye law where the exponential decay is governed by 'one-lumped' decay constant i.e. $\lambda_0$ . The Curie-von Schweidler behavior has been observed in many instances, since late 19$^{th}$ Century, such as those shown in dielectric studies and experiments [3] [4], [12], [22]-[26].

This power law relaxation of the non-Debye type i.e. $i(t) \sim t^{-n}$ has been interpreted as a many-body problem but can also be formulated as an infinite number of independent relaxing bodies meaning infinite number of time constants $\tau$ or relaxation rates $\lambda$ varying from near zero to infinity [4], [5], and [6]. The observations of power law relaxation are also made in the experiments and studies with super-capacitors [7]-[11]. These studies also indicate the fractional calculus is used as constituent expression to describe super-capacitors. The use of empirical power law i.e. Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12]. Apart from relaxation of current decay in dielectrics and super-capacitors, the power law type or non-Debye relaxation is observed in visco-elastic experiments strain relaxation in [13], [14], [15], and [36].

In this write-up, we are giving the derivation of the distribution of relaxation rates ( $\lambda$ ) particularly for Curie-von Schweidler law and we observe the distribution nature as Zipf's distribution. We try to reason out as to why this distribution of relaxation rates takes Zipfian nature. We also show that Curie-von Schweidler law has time varying rate of relaxation. This note will not deal with the mathematics of Zipfian distribution (or power law distribution) like probability density, cumulative probability density function, and the conditions of finding finite mean, variance or standard deviation for power law distribution. This write-up describes finding the distribution function of relaxation rates (or histogram) by formulating Laplace integral, and show that the distribution thus obtained is a Zipf's power law.

We extend this mathematical approach to get the distribution function for time constants ( $\tau$ ). We get that time constants are also distributed as Zipf's power law; but the observation points to a contrary physical interpretation derived from this obtained power law distribution for the time constants. Thus we can conclude that this method developed by Laplace integral approach is restricted to get only distribution of relaxation rates i.e. $\lambda$ and not to get the distribution of time constants i.e. $\tau$ . Though we are discussing especially Curie-von Schweidler law, yet we will tabulate relaxation rate distributions for obtained for some other relaxation functions which are obtained via this Laplace integral method.

We shall demonstrate the formation of fractional derivative in the expression relating current and voltage considering the relaxation rates as Zifian distribution; and thus forming a scale dependent power law for relaxation rates as the scale varies from zero to infinity. Though by experiments one cannot make histogram directly for the rates of relaxation for any non-Debye processes, yet this mathematical procedure that we develop helps in extracting this information from the observations relaxation function. This is new treatment, and much more research is required, across various dynamic processes.

The classical geometric capacitor or a constant capacitor having constant value of Farad means that it has constant value at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used i.e. having loss less relative permittivity $\varepsilon_r$ and is constant (and is a purely real number with loss tangent value as zero) at all frequencies. The capacity in this classical sense is given as $C_1 = \varepsilon_r A / d$ i.e. by using geometric factor of ratio of area to the electrode separation. This we have learnt in textbooks. This ideal capacity is constant at all the frequencies is called geometric capacity. This constant value in frequency domain is actually an impulse function in time domain. A general practical capacitor, which is not a constant in frequency domain, is having a function in time domain and we call it as capacity function in time. Therefore, we say that charge stored in capacitor, as a function of time is not usual multiplication operation of capacity function and voltage stress; instead, the charge is convolution integral of the two. However, the charge described in frequency domain as a function of frequency is multiplication operation of frequency domain functions of capacity-function and voltage-function. We will revise this concept of capacitor in the write-up, and derive various concepts.

Curie-von Schweidler Law of relaxation of current to a step input of voltage to get constituent relation with fractional derivative was proposed in [12] [35], by taking the concept of charge stored at any time as usual product of capacity function and voltage stressed. We will revise the concept of capacitor in classical theory and apply the new concept of charge stored at any time as convolution integral of capacity function and the voltage stress and also apply this concept in capacitors with observed Curie-von Schewdler relaxation current, and obtain same results as in [12] and [35]. We will also point out the differences with this new approach to the earlier approach in finding the capacity function.

## The Zipf's power law distribution and probable hypothesis for its mechanism

The Zipf's law is widely referred in linguistic studies, economics studies, population studies [18]-[21], [30], and [31]. We use this for a dielectric relaxation law (i.e. Curie-von Schweidler law), which is observed as $i(t) \sim t^{-n}$, $\quad 0 < n < 1$, since late 19th century. We derived histogram of relaxation rates for relaxation function $i(t) \sim t^{-n}$ and show that it follows Zipf's power law. We try to give possible reasons as to why Zipfian distribution is observed for the distribution of relaxation rates. The histogram function of Zipf's law is $H(x) \sim x^{-m}$ , a power law type. In this section we assume that relaxation rates $\lambda$ 's follow a Zipfian histogram say $H_\lambda(\lambda) \sim \lambda^{-m}$. This we will derive in subsequent section. The $\lambda$ 's are relaxation rates of infinite number of relaxing bodies, simultaneously relaxing as per Debye law i.e. $\sim e^{-\lambda t}$.

Zipf's Law is an empirical law formulated using mathematical statistics that refers to the fact that many types of data studied in the physical and social sciences can be approximated with a Zipfian distribution. This distribution is one of a family of related discrete power law probability distributions [18]-[21], [30], and [31]. This power law distribution helps to describe phenomena where large events are rare, but small ones are quite common. For example, there are few large earthquakes but many small ones. There are a few mega-cities, but many small towns. There are few words, such as 'and' and 'the' that occur very frequently, but many which occur rarely.

The emergence of a complex language is one of the fundamental events of human evolution, and several remarkable features suggest the presence of fundamental principles of organization. These principles seem to be common to all languages. The best known is the so-called Zipf's law, which states that the frequency of a word decays as a (universal) power law of its rank. The possible origins of this law have been controversial, and its meaningfulness is still an open question. One of the early hypotheses of Zipf of a principle of least effort for explaining the law is shown to be sound [30], [31]. But still the exact mechanism how the Zipf's distribution manifests is debated.

Many of the things that we measure have a typical size or 'scale'. We ask ourselves why the relaxation rates $\lambda$ cannot be arranged as simple 'normal distribution'. Like while we plot the height of person in X-axis and percentage of occurrence of that particular height in Y-axis, we get a 'normal distribution' peaked around mean height with a spread both ways, a histogram (Figure-1). We find that ratio of maximum height and minimum height of a person is finite (or relatively low value). For example, as per Guinness book of records tallest person was having height 272cm and shortest person was having the height of 57cm, making this ratio 4.8. This ratio is relatively low value. We see the most adults are about 180cm tall-there is some variations around this figure notably depending on sex, but we never measure persons having height of 10cm or 500cm.

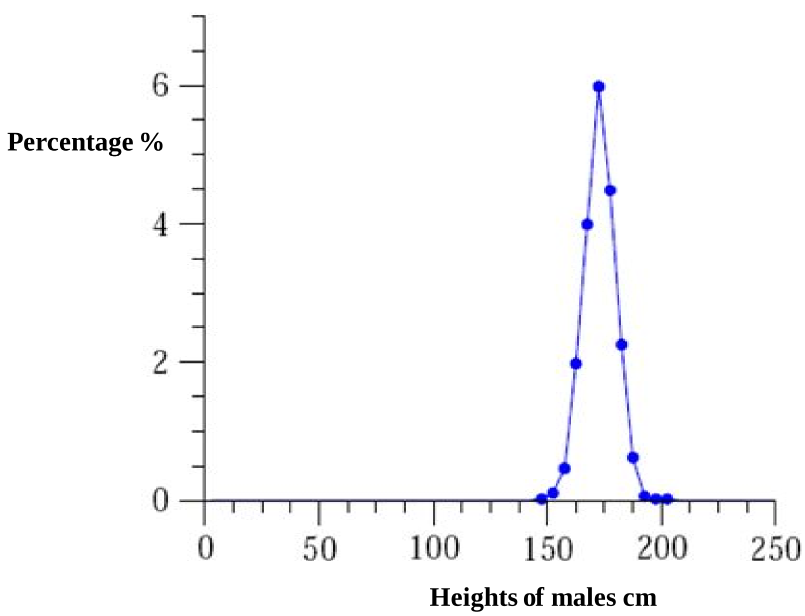


**Figure-1: Histogram of height of males**

But not all things we measure are peaked around a typical value. Some may over a very large dynamic range, sometimes many orders of magnitude. For example the ratio of population of largest town to population of smallest town is about 1, 50,000. The histogram if plotted for X-axis with population of cities and Y-axis with percentage of cities having that population; the distribution will not show the 'normal-distribution'. The histogram of cities & population is highly 'right-skewed', meaning that while the bulk of distribution occurs for fairly small sizes-i.e. most cities have small population-there is small number of cities with population much higher than a said typical value, producing the long tail to the right of histogram (Figure-2). This right skewed form is qualitatively quite different from histogram of person's height. That is because we know that there is large dynamic range from smallest to largest city sizes, we can immediately infer that there can only be a small number of very large cities. The histogram of this sort is like a function i.e. $H_x(x) \sim x^{-m}$ . The distribution of this nature is called Zip's power law distribution.

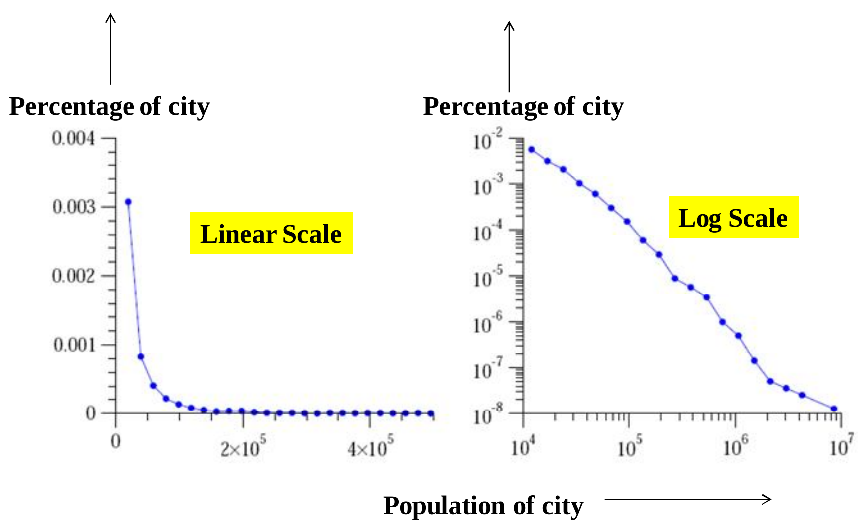


**Figure-2: Histogram of population of all cities with population of 10,000 or more in linear and log scale**

The same we observe when relaxation rates call them $\lambda$ having a large (ideally infinite) spreads follow Zipfian distribution, we call that $H_\lambda(\lambda)$ and will show that the histogram follows the function i.e. $H_\lambda(\lambda) \sim \lambda^{-m}$ . Thus one reason that this non-Debye relaxation of Curie-von Schweidler Law i.e. $i(t) \sim t^{-n}$ in dielectric is having infinite spread of relaxation rates of $\lambda$ 's-thus forming a Zipfian power law.

## Zipfian power law distribution manifests due to connected exponential processes-a probable hypothesis

A much more common distribution than power law is the exponential distribution. In this complex relaxation mechanism i.e. $i(t) \propto t^{-n}$ that we are discussing we consider infinite bodies relaxing simultaneously, in different time scales ($\mathrm{T}$). We consider that a complex relaxation mechanism and a quantity $\mathrm{T}$ say survival time of a relaxing body, has exponential distribution of probability $p(\mathrm{T}) \sim e^{-a\mathrm{T}}$. This means that a probability for a body having very large survival time (age) is very low; and vice-versa. Then $p(\mathrm{T})\mathrm{dT}$ indicates the fraction of survival numbers of bodies between survival time $\mathrm{T}$ and $\mathrm{T}+\mathrm{dT}$. Now suppose that the real quantity we are interested is not $\mathrm{T}$ but other quantity $\lambda$, say the relaxation rate of discharge which is exponentially related to $\mathrm{T}$; thus $\lambda \sim e^{-b\mathrm{T}}$. That implies the surviving bodies with very large time of survival (age) have a very low rate of relaxation. This also states that $\mathrm{d}\lambda = -\mathrm{dT}$. Then if probability distribution of $\lambda$ is $p(\lambda)$; then we have $p(\lambda)\mathrm{d}\lambda = -p(\mathrm{T})\,\mathrm{d}\,\mathrm{T}$ (statement about conservation of probability [21]). The negative sign indicates opposite movement, as $\mathrm{T}$ is increased from $\mathrm{T}$ to $\mathrm{T}+\mathrm{dT}$, and then $\lambda$ is decreased from $\lambda$ to $\lambda+(-\mathrm{d}\lambda)$. This means that number of simultaneously discharging units having relaxation rates between $\lambda$ and $\lambda+\mathrm{d}\lambda$ is equal to number of surviving bodies having survival time between $\mathrm{T}$ and $\mathrm{T}+\mathrm{dT}$. Thus, we write following steps

$$
\begin{aligned}
p(\lambda) = -p(\mathrm{T})\frac{\mathrm{dT}}{\mathrm{d}\lambda} = -\frac{p(\mathrm{T})}{\left(\frac{\mathrm{d}\lambda}{\mathrm{dT}}\right)} \sim \frac{e^{-a\mathrm{T}}}{be^{-b\mathrm{T}}} = \frac{e^{\frac{-a}{-b}(-b\mathrm{T})}e^{b\mathrm{T}}}{b} \\
= \frac{\lambda^{\frac{a}{b}}\lambda^{-1}}{b} \sim \lambda^{-\left(1-\frac{a}{b}\right)} \\
\sim \lambda^{-m}; \qquad m = 1-\tfrac{a}{b}
\end{aligned} \tag{1}
$$

The above discussion gives a power law distribution for relaxation rates $\lambda$'s where there is combination of exponential processes. Thus, we expect that in our complex relaxation process governed by Curie-von Scweidler Law $i(t) \propto t^{-n}$, which is having infinite number of simultaneously discharging bodies that will have a power law distribution for relaxation rates as a histogram $H_\lambda(\lambda) \sim \lambda^{-m}$. This we will derive subsequently. We proceed with this explanation and hypothesis. This could be one explanation in physical sense, in line with exponential distribution in the Boltzmann distribution of energies in statistical mechanics.

## Complex non-Debye relaxation composed with several exponential types Debye decay functions

We call the Curie-von Shweidler relaxation law $i(t) \sim t^{-n}; \quad 0 < n < 1$ as complex process, of non-Debye type. In this section, we formulate the method to extract the histogram of the relaxation rates call it $H_\lambda(\lambda)$, for a complex non-Debye relaxation process $i(t)$, which we assume to be composed of several Debye type relaxations $e^{-\lambda t}$, with $\lambda$ varying from zero to infinity. The complex decay is expressed as following with several rate constants $\lambda_1, \lambda_2, \lambda_3, \ldots$ with weights $a_1, a_2, a_3, \ldots$, where $\lambda$ is having units in $\sec^{-1}$ i.e. 'per second', and is equal to inverse of time constant i.e. $\lambda_k = (\tau_k)^{-1}; \quad k = 1, 2, 3\ldots$. We write following composite relaxation expression as sum of several 'discrete' relaxations of Debye type i.e.

$$
\begin{aligned}
i(t) &= a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + \ldots \\
&= \sum a_k e^{-\lambda_k t} \\
i(0) &= a_1 + a_2 + a_3 + \ldots
\end{aligned} \tag{2}
$$

In continuum limit we may write the above as following

$$i(t) = \int_0^\infty \left( H_\lambda(\lambda) \right) e^{-\lambda t} \mathrm{d}\lambda \tag{3}$$

Where the function i.e. $H_\lambda(\lambda)$ is the distribution-function of the rate of the relaxation ($\lambda$) of the process, or we may call it as histogram of relaxation rates.

While for the case with discrete set of relaxation rates i.e. $\lambda_1, \lambda_2, \lambda_3, \ldots$ the rate distribution function would be having discrete delta functions ( $\delta(t-\lambda_k), \quad k = 1,2,3,\ldots$ ) at points $\lambda_1, \lambda_2, \lambda_3, \ldots$; which we write like following expression

$$\begin{aligned} H_\lambda(\lambda) &= a_1\delta(\lambda-\lambda_1) + a_2\delta(\lambda-\lambda_2) + a_3\delta(\lambda-\lambda_3) + \ldots \\ &= \sum a_k \delta(\lambda-\lambda_k) \end{aligned} \tag{4}$$

From above formulation if we have only one single Debye relaxation i.e. having only one rate constant say $\lambda_0$ i.e. $i(t) = e^{-\lambda_0 t}$ then $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$. This is verified in the following expression

$$\begin{aligned} i(t) &= \int_0^\infty \left( (H_\lambda \lambda) \right) e^{-\lambda t} \mathrm{d}\lambda \\ &= \int_0^\infty \left( \delta(\lambda-\lambda_0) \right) e^{-\lambda t} \mathrm{d}\lambda = e^{-\lambda_0 t} \end{aligned} \tag{5}$$

In above we used the property of delta function [16], [28], [29] i.e. $\int \left( \delta(x-x_0) \right) f(x) \mathrm{d}x = f(x_0)$.

## Extraction of relaxation-rate distribution function or histogram function by inverse Laplace transformation of formulated Laplace integral

In this section we formulate Laplace integral and getting inverse Laplace transform of time domain response i.e. $\mathcal{L}^{-1}\{i(t)\}$ to get relaxation rate distribution function i.e. $H_\lambda(\lambda)$. Conventionally we are used to get inverse Laplace transform of a frequency domain function to time domain function; we note here we will be inverting a time domain function.

The Laplace transform $F(s)$ of a function in time domain $f(t)$ is defined as following integral transform relation [28], [29], [32], i.e. called Laplace integral

$$\begin{aligned} & F(s) \overset{\text{def}}{=} \int_0^\infty \left( f(t) \right) e^{-st} \mathrm{d}t \qquad F(s) = 0 \quad \text{for} \quad s < 0 \\ & F(s) = \mathcal{L}\{ f(t) \} \end{aligned} \tag{6}$$

This is standard integral transform of a function $f(t)$ from a time domain ($t$) to a complex frequency domain i.e. $s = \mathrm{Re}\{s\} + \mathrm{i}\omega; \quad \mathrm{i} = \sqrt{-1}$; where real part is significant in the transient response and the imaginary part of the frequency corresponds to 'steady-state' response; in classical 'Control Science'

[32]. Here $f(t)$ is 'inverse Laplace transform' of $F(s)$, and we write $\mathcal{L}^{-1}\{F(s)\}=f(t)$ and $\mathcal{L}\{f(t)\}=F(s)$ .

We have in earlier section derived $i(t)=\int_0^{\infty}\left(H_{\lambda}(\lambda)\right)e^{-\lambda t}\mathrm{d}\lambda$ . Compare this with defined Laplace integral expression as follows

$$i(t)=\int_0^{\infty}\left(H_{\lambda}(\lambda)\right)e^{-t\lambda}\mathrm{d}\lambda \qquad F(s)=\int_0^{\infty}\left(f(t)\right)e^{-st}\mathrm{d}t \tag{7}$$

Both above are Laplace transform expressions, (or Laplace integrals). The first expression is transforming the function $H_{\lambda}(\lambda)$ from $\lambda$ domain to 'complex' $t$ time domain; while the second one is transforming $f(t)$ from $t$ domain to 'complex' $s$ frequency domain. Thus, both are Laplace integral expressions with change of variable and symbol. Therefore we can say $H_{\lambda}(\lambda)$ is inverse Laplace Transform of $i(t)$ in the first expression, i.e. $H_{\lambda}(\lambda)=\mathcal{L}^{-1}\{i(t)\}$ ; as $f(t)$ is inverse Laplace of $F(s)$ in the second expression, i.e. $f(t)=\mathcal{L}^{-1}\{F(s)\}$ .

Therefore in order to get the rate distribution-function $H_{\lambda}(\lambda)$ from the decay curve (or relaxation-function $i(t)$), we need to perform inverse Laplace Transform of the time function $i(t)$ . The definition of inverse Laplace Transform is described as following integral expressions

$$f(t)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(F(s)\right)e^{st}\mathrm{d}s \qquad H_{\lambda}(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda \tag{8}$$

In the above expression $x$ is real number larger than $x_0$ , where $x_0$ being such that $i(t)$ has some form of singularity on the real line $\mathrm{Re}\{t\}=x_0$ but is analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\{t\}>x_0$ , [28], [29], and [32]. Thus in the formulation we treat time variable as complex quantity say $t=x+\mathrm{i}y$ in the expression of inverse Laplace Transform i.e. $H_{\lambda}(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda$ . Though we cannot explain presently physical meaning of concept an 'imaginary time' in the expression of complex time quantity $t=x+\mathrm{i}y$ , yet mathematically there is no restriction in assuming time to be complex number. We thus proceed in mathematical sense to invert a function in complex time variable, by techniques of Laplace inversion.

Table-1 and Table-2 lists several types' relaxation functions $i(t)$ and its inverse Laplace $H_{\lambda}(\lambda)$ describing the rate distribution function; mostly got from standard Laplace transform tables [32]. The integral representations of few functions of $H_{\lambda}(\lambda)$ , shown in Table-1 i.e. for entries 12 to 16 we get via Berberan-Santos method [17], [34], [38] (Appendix). The entry 12 is stretched exponential decay function and entry 13 is Becquerel's compressed hyperbolic radioactive decay function; the entry 15 and 16 is for Mittag-Leffler function and entry 14 is general power law relaxation. These integral representations of $H_{\lambda}(\lambda)$ are difficult to solve but are easy to plot via use of numerical integration techniques. The Table-2 gives the Laplace inversion of $i(t)$ to get $H_{\lambda}(\lambda)$ in integral representation only.

We have observed in the previous section that a Debye relaxation of $i(t) \sim e^{-\lambda_0 t}$ has rate distribution as $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$ i.e. it is given by a delta function at point $\lambda = \lambda_0$. This we verify with known Laplace relation i.e. $\mathcal{L}\{f(t-t_0)\} = e^{-st_0}F(s)$, [32] where $\mathcal{L}\{f(t)\} = F(s)$. In addition, we have $\mathcal{L}\{\delta(t)\} = 1$; thus, we can write $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$. From here, we can write with change of variable for $i(t) = e^{-\lambda_0 t}$ the inverse Laplace of this time domain function in $\lambda$ domain we get as $H_\lambda(\lambda) = \delta(\lambda - \lambda_0)$, i.e. the rate distribution function.

If there is no decay then say $i(t) = 1 = e^{-\lambda_0 t}; \quad \lambda_0 = 0$; the rate distribution function is delta function at origin i.e. $H_\lambda(\lambda) = \delta(t)$.

| **S No.** | **Relaxation function** $i(t), \quad t > 0$ | **Rate Distribution function** $H_\lambda(\lambda), \quad \lambda > 0$ $H_\lambda(\lambda) = \mathcal{L}^{-1}\{i(t)\}$ |
|---|---|---|
| 1 | $i(t) = A$:Constant Function | $H_\lambda(\lambda) = A(\delta(\lambda)) = \frac{A}{\pi}\int_0^\infty \cos(\lambda y)\mathrm{d}y$ |
| 2 | $i(t) = t^{-1}$ | $H_\lambda(\lambda) = 1 = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda y}{y}\mathrm{d}y, \quad \lambda > 0$ |
| 3 | $i(t) = t^{-n}$ | $H_\lambda(\lambda) = \frac{1}{(n-1)!}\lambda^{n-1} = \frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^\infty \frac{\cos(\lambda y)}{y^n}\mathrm{d}y$ |
| 4 | $i(t) = (t+a)^{-1}$ | $H_\lambda(\lambda) = e^{-a\lambda} = \frac{2a}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{y^2+a^2}\mathrm{d}y$ |
| 5 | $i(t) = (t+a)^{-n}$ | $H_\lambda(\lambda) = \frac{1}{(n-1)!}\lambda^{n-1}e^{-a\lambda}$ |
| 6 | $i(t) = at^{-1}(t+a)^{-1}$ | $H_\lambda(\lambda) = 1 - e^{-a\lambda}$ |
| 7 | $i(t) = (t+a)^{-1}(t+b)^{-1}$ | $H_\lambda(\lambda) = \frac{1}{b-a}\left(e^{-a\lambda} - e^{-b\lambda}\right)$ |
| 8 | $i(t) = e^{-\lambda_0 t}$ | $H_\lambda(\lambda) = \delta(\lambda - \lambda_0) = \frac{1}{\pi}\int_0^\infty \cos\left(y(\lambda - \lambda_0)\right)\mathrm{d}y$ |
| 9 | $i(t) = t(t^2+a)^{-1}$ | $H_\lambda(\lambda) = \cos\left(\lambda\sqrt{a}\right)$ <br> $a = 1, \quad H_\lambda(\lambda) = \cos\lambda = \frac{2e^\lambda}{\pi}\int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4}\mathrm{d}y$ |
| 10 | $\sqrt{a}(t^2+a)^{-1}$ | $H_\lambda(\lambda) = \sin\left(\lambda\sqrt{a}\right)$ |
| 11 | $i(t) = \sqrt{a}\left((t+b)^2+a\right)^{-1}$ | $H_\lambda(\lambda) = e^{-b\lambda}\sin\left(\lambda\sqrt{a}\right)$ |
| 12 | $i(t) = e^{-(t/\tau_0)^\beta}$ | $H_\lambda(\lambda) = \frac{\tau_0}{\pi}\int_0^\infty \left(e^{\left(-u^\beta\cos(\beta\pi/2)\right)}\cos\left(\lambda\tau_0 u - u^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\right)\mathrm{d}u$ |
| 13 | $i(t) = \left(1+(1-\beta)\left(\frac{t}{\tau_0}\right)\right)^{-1/(1-\beta)}$ | $H_\lambda(\lambda) = \frac{\tau_0}{\pi(1-\beta)}\int_0^\infty (\mathrm{d}u)(1+u^2)^{-1/(2(1-\beta))}\cos\left(\frac{\lambda\tau_0 u - \tan^{-1}u}{1-\beta}\right)$ |
| 14 | $i(t) = \left(1+\left(\frac{t}{\tau_0}\right)^\alpha\right); \quad 0 < \alpha < 1$ | $H_\lambda(\lambda) = \frac{2\tau_0}{\pi}\int_0^\infty \mathrm{d}u \frac{u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{u^{2\alpha}+2u^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda\tau_0 u)$ |

| | | |
|---|---|---|
| 15 | $i(t) = E_\alpha\left(-t/\tau_0\right)$ <br> $E_\alpha(-\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+1)} \xi^k$ | $H_\lambda(\lambda) = \frac{2}{\pi} \int_0^\infty E_{2\alpha}(-y^2)\cos(\lambda y)\mathrm{d}y$ |
| 16 | $i(t) = E_{\frac{1}{2}}\left(t/\tau_0\right)$ | $H_\lambda(\lambda) = \frac{1}{\sqrt{\pi}} e^{-\left(\lambda^2/4\right)}$ |

**Table-1: Several relaxation functions and corresponding rate distribution functions**

If the relaxation function is of say $i(t) = 1 - e^{-\lambda_0 t}$; then we have $H_\lambda(\lambda) = \mathcal{L}^{-1}\left\{1 - e^{-\lambda_0 t}\right\}$; giving $H_\lambda(\lambda) = \delta(\lambda) - \delta(\lambda - \lambda_0)$.

From these observations, we say that for our earlier expression i.e. $H_\lambda(\lambda) = \sum a_k \delta(\lambda - \lambda_k)$, the coefficients $a_k$ 's can have negative values as well for some type of relaxation function.

| **S. No.** | $i(t)$ | $H_\lambda(\lambda) = \mathcal{L}^{-1}\{i(t)\}$ <br> **Rate distribution function in integral representation of function in time domain by inverse Laplace transform by Berberan-Santos method** |
|---|---|---|
| 1 | $\frac{1}{t+a}$ | $H_\lambda(\lambda) = \frac{2a}{\pi} \int_0^\infty \frac{\cos(\lambda y)}{y^2+a^2} \mathrm{d}y$ |
| 2 | $\frac{t}{t^2+1}$ | $H_\lambda(\lambda) = \frac{2e^\lambda}{\pi} \int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4} \mathrm{d}y$ |
| 3 | $e^{-\lambda_0 t}$ | $H_\lambda(\lambda) = \frac{1}{\pi} \int_0^\infty \cos\left(y(\lambda - \lambda_0)\right)\mathrm{d}y$ |
| 4 | 1 | $H_\lambda(\lambda) = \frac{1}{\pi} \int_0^\infty \cos(\lambda y)\mathrm{d}y$ |
| 5 | $\frac{1}{t}$ | $H_\lambda(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{\sin\lambda y}{y} \mathrm{d}y$ |
| 6 | $t^{-\alpha}$ | $H_\lambda(\lambda) = \frac{2}{\pi} \cos\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{\cos(\lambda y)}{y^\alpha} \mathrm{d}y$ |
| 7 | $\left(1 + (1-\beta)\left(\frac{t}{a}\right)\right)^{-\frac{1}{(1-\beta)}}$ | $H_\lambda(\lambda) = \frac{a}{\pi(1-\beta)} \int_0^\infty \left(1+u^2\right)^{-\frac{1}{2(1-\beta)}} \cos\left(\frac{a\lambda u - \tan^{-1} u}{(1-\beta)}\right)\mathrm{d}u; \quad u = \frac{1-\beta}{a} y$ |
| 8 | $e^{-(t/a)^\beta}$ | $H_\lambda(\lambda) = \frac{a}{\pi} \int_0^\infty \left(e^{-u^\beta \cos(\beta\pi/2)}\right) \cos\left(a\lambda u - u^\beta \sin\left(\frac{\beta\pi}{2}\right)\right)\mathrm{d}u; \quad u = \frac{y}{a}$ |
| 9 | $\frac{a^\alpha}{a^\alpha + t^\alpha}$ | $H_\lambda(\lambda) = \frac{2a}{\pi} \int_0^\infty \frac{u^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1}{u^{2\alpha} + 2u^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+1} \cos(a\lambda u)\mathrm{d}u; \quad u = \frac{y}{a}$ |
| 10 | $\frac{t^\alpha}{t(1+t^\alpha)}$ | $H_\lambda(\lambda) = \frac{2}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^\infty \frac{y^{\alpha-1}}{1+2y^\alpha \cos\left(\frac{\alpha\pi}{2}\right)+y^{2\alpha}} \cos\left(y\lambda^{1/\alpha}\right)\mathrm{d}y$ |
| 11 | $\frac{k}{t(t^\alpha+k)}$ | $H_\lambda(\lambda) = \frac{2}{\pi} \int_0^\infty \frac{y^{\alpha-1} k \cos\left(\frac{(\alpha+1)\pi}{2}\right)}{y^{2\alpha} + 2y^\alpha k \sin\left(\frac{(\alpha+1)\pi}{2}\right) + k^2} \cos(y\lambda)\mathrm{d}y$ |

**Table-2: Few examples of Laplace inverted function by using Berberan-Santos method represented by integral representation only**

For example if $i(t)=t(t+1)^{-1}$ is a relaxation function that initially grows to a maximum value and then starts falling as time increases, it has rate distribution function as $H_\lambda(\lambda)=\mathcal{L}^{-1}\left\{t(t+1)^{-1}\right\}=\cos\lambda$, a oscillatory one. Thus, in this case the distribution function i.e. $H_\lambda(\lambda)$ can take positive as well as negative values.

One interesting observation is for a relaxation function $i(t)=t^{-1}$ the relaxation distribution function is $H_\lambda(\lambda)=1$ -a 'uniform distribution', for $\lambda\geq 0$. All these are listed in Table-1.

The Laplace inversion is usually carried out by contour integration. However, the very modern technique of Berberan-Santos [17], [34], [38] method is the analytical Laplace inversion without the usual contour integration. We describe this now briefly.

Our aim is evaluate Laplace inverse $H_\lambda(\lambda)=\mathcal{L}^{-1}\left\{i(t)\right\}$, which is given as Laplace inversion integral expression i.e.

$$H_\lambda(\lambda)=\frac{1}{2\pi\mathrm{i}}\int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty}\left(i(t)\right)e^{t\lambda}\mathrm{d}\lambda \tag{9}$$

Here we describe Berberan-Santos method formulas for evaluation of the Laplace inversion without going for contour integration. First is change of variable i.e. from 'real time variable' to 'complex time variable' as $t=x+\mathrm{i}y$; with $\mathrm{i}=\sqrt{-1}$. Here the real part i.e. $x$ is constant as a vertical line calls it $x=x_0$ a constant. The formulas are following [17], [34], [38] (Refer Appendix)

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\mathrm{Re}\left\{i(x_0+\mathrm{i}y)\right\}\cos(\lambda y)\mathrm{d}y \qquad H_\lambda(\lambda)=-\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\mathrm{Im}\left\{i(x_0+\mathrm{i}y)\right\}\sin(\lambda y)\mathrm{d}y \tag{10}$$

Consider a very simple case of decay function $i(t)=(t-a)^{-1}$ and converted to complex time as follows

$$i(t)=\frac{1}{t-a}; \qquad i(x_0+\mathrm{i}y)=\frac{1}{(x_0-a)+\mathrm{i}y} \tag{11}$$

We know from standard Laplace pair that is $\mathcal{L}^{-1}(s\pm a)^{-1}=e^{\mp at}$. Thus, for $i(t)=(t-a)^{-1}$ we should get via inverse Laplace the rate distribution function as $H_\lambda(\lambda)=e^{a\lambda}$. The application of the Berberan-Santros formula with $x_0>a$ yields the following

$$\begin{aligned}H_\lambda(\lambda)&=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\mathrm{Re}\left\{i(x_0+\mathrm{i}y)\right\}\cos(\lambda y)\mathrm{d}y=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\mathrm{Re}\left\{\tfrac{1}{(x_0-a)+\mathrm{i}y}\right\}\cos(\lambda y)\mathrm{d}y\\&=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\frac{(x_0-a)}{(x_0-a)^2+y^2}\cos(\lambda y)\mathrm{d}y\\&=\frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^\infty\frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y=e^{a\lambda}\end{aligned} \tag{12}$$

Here we say that $e^{a\lambda}$ has integral representation as $e^{a\lambda} = \frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^{\infty}\frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y$.

Particularly for $a=-1$, we have $i(t)=(t+1)^{-1}$ . The condition $x_0>-1$ enables us to choose $x_0=0$ we get following integral representation for $e^{-\lambda}$ which is also rate distribution function $H_\lambda(\lambda)$ is following

$$H_\lambda(\lambda)=e^{-\lambda} = \frac{2}{\pi}\int_0^{\infty}\frac{\cos(\lambda y)}{1+y^2}\mathrm{d}y \tag{13}$$

## Derivation of rate distribution function for Curie-von Schweidler relaxation law

For the Curie-von Schweidler relaxation of type i.e. $i(t)\sim t^{-n}$ then rate distribution function is $H_\lambda(\lambda)=\mathcal{L}^{-1}\left\{t^{-n}\right\}$. With the known Laplace pair i.e. $\mathcal{L}^{-1}\left\{s^{-(\alpha+1)}\right\}=\frac{1}{\alpha!}t^{\alpha}$ , we can write the following steps

$$\begin{aligned}H_\lambda(\lambda)&=\mathcal{L}^{-1}\left\{t^{-n}\right\}\\&=\frac{1}{(n-1)!}\lambda^{(n-1)}=\frac{1}{m!}\lambda^{m};\quad m=n-1;\quad \Gamma(\alpha)=(\alpha-1)!,\quad \alpha\in\mathbb{R}\\&=\frac{1}{\Gamma(n)}\lambda^{n-1}=\frac{1}{\Gamma(m+1)}\lambda^{m};\quad \lambda>0\end{aligned} \tag{14}$$

Therefore, above discussion suggests that for a power law type relaxation, i.e. Curie-von Schweidler law i.e. $i(t)\propto t^{-n};\quad 0<n<1$, the relaxation rates $\lambda$ 's are also having a power law distribution of type i.e. $H_\lambda(\lambda)\sim\lambda^{m},\quad m=n-1,\quad -1<m<0,\quad \lambda>0$. This is Zipf's power law with $m<0$.

For dielectric relaxation as observed that $0<n<1$ in Curie-von Schweidler relaxation $i(t)\sim t^{-n}$, the rate relaxation distribution function $H_\lambda(\lambda)\sim\lambda^{m}$ has exponent in power $-1<m<0$. Considering graph of $H_\lambda(\lambda)=\lambda^{m};\quad m<0$ as histogram, we infer that for Curie-von Schweidler relaxation function, i.e. $i(t)\sim t^{-n};\quad 0<n<1$ there are very large number of relaxations with small $\lambda$ i.e. large number of slower decay takes place, compared to fewer faster decay rates-and the histogram $H_\lambda(\lambda)=\lambda^{m};\quad m<0$ is highly right skewed with long tail (Figure-2).

From the above discussion and using our Laplace integral i.e. $i(t)=\int_0^{\infty}\left(H_\lambda(\lambda)\right)e^{-t\lambda}\mathrm{d}\lambda$ we write for Curie-von Schweidler relaxation function the following

$$t^{-n}=\frac{1}{\Gamma(n)}\int_0^{\infty}\left(\lambda^{(n-1)}\right)e^{-t\lambda}\mathrm{d}\lambda \tag{15}$$

The above expression is integral representation of the $t^{-n}$ shows weighted averaging of infinite Debye relaxations i.e. $e^{-\lambda t}$ with weight $\lambda^{(n-1)}$ applied for all $\lambda$ from zero to infinity.

Using Berberan-Santos method [17], [34], [38] (Refer Appendix) we get the Laplace inversion of $i(t)=t^{-n}$. In reality of decay functions, we can take $x_0=0$; in complex time variable i.e. $t=x_0+\mathrm{i}y$ as the decay function is not expected to have singularity at time $t>0$. Choosing $x_0=0$ in $t=x_0+\mathrm{i}y$ we have $i(\mathrm{i}y)=(\mathrm{i}y)^{-n}$, that is following

$$i(\mathrm{i}y)=\frac{1}{(\mathrm{i}y)^n}=\frac{y^{-n}}{\left(\cos\left(\frac{n\pi}{2}\right)+\mathrm{i}\sin\left(\frac{n\pi}{2}\right)\right)};\quad i^n=e^{(in\pi/2)}=\cos\left(\frac{n\pi}{2}\right)+\mathrm{i}\sin\left(\frac{n\pi}{2}\right) \tag{16}$$

The real part of the complex function is $\mathrm{Re}\{i(\mathrm{i}y)\}=y^{-n}\cos\left(\frac{n\pi}{2}\right)$. Now using the Berberan–Snatos formula we get following steps

$$\begin{aligned}H_\lambda(\lambda)&=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty\left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\right)\cos(\lambda y)\mathrm{d}y;\qquad x_0=0\\&=\frac{2}{\pi}\int_0^\infty\left(y^{-n}\cos\left(\frac{n\pi}{2}\right)\right)\cos(\lambda y)(\mathrm{d}y);\quad u=y,\quad \mathrm{d}u=\mathrm{d}y\\&=\frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^\infty u^{-n}\cos(\lambda u)(\mathrm{d}u)\end{aligned} \tag{17}$$

From Laplace Tables we have $H_\lambda(\lambda)=\frac{1}{\Gamma(n)}\lambda^{n-1}$ i.e. from Laplace inverse of $i(t)=t^{-n}$, therefore we write following

$$H_\lambda(\lambda)=\frac{1}{\Gamma(n)}\lambda^{n-1}=\frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^\infty u^{-n}\cos(\lambda u)(\mathrm{d}u) \tag{18}$$

So we have from above $\mathcal{L}^{-1}\{t^{-n}\}=\frac{2}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^\infty u^{-n}\cos(\lambda u)(\mathrm{d}u)$, this is $H_\lambda(\lambda)$, which is also $H_\lambda(\lambda)=\frac{1}{\Gamma(n)}\lambda^{n-1}$ as per our above discussion. So we write the following from above steps as

$$\lambda^{n-1}=\frac{2(\Gamma(n))}{\pi}\cos\left(\frac{n\pi}{2}\right)\int_0^\infty u^{-n}\cos(\lambda u)(\mathrm{d}u) \tag{19}$$

Now we do trick of changing variable $\lambda$ to $t$, $n-1$ to $-n$, and rearrange above expression to get $t^{-n}=\frac{2\Gamma(1-n)}{\pi}\cos\left(\frac{(1-n)\pi}{2}\right)\int_0^\infty u^{(n-1)}\cos(tu)(\mathrm{d}u)$. Considering now $u$ as $\lambda$ we write another integral representation of $t^{-n}$ as follows

$$t^{-n}=2\frac{\Gamma(1-n)}{\pi}\cos\left(\frac{(1-n)\pi}{2}\right)\int_0^\infty \lambda^{(n-1)}\cos(\lambda t)(\mathrm{d}\lambda) \tag{20}$$

This means that if we choose basic relaxation function as $\cos(\lambda t)$, then Curie-von Schweidler relaxation $i(t)=t^{-n}$ is weighted sum of all $\cos(\lambda t)$'s with weights $\lambda^{n-1}$, as $\lambda$ is varied from zero to infinity.

## Zipf's power law distribution function for distribution of relaxation time constants for Curie-von Schweidler relaxation law-a physical contradiction

Now converting to $\tau = \lambda^{-1}$, we assume the Distribution of time-constants call it $H_{\tau}(\tau) \sim \tau^{-m}$ , is the Zipf's power law distribution. However direct taking of reciprocal of obtained inversion of $H_{\lambda}(\lambda)$ i.e. the rate distribution function got via Laplace inversion of $i(t)$ is not possible. This we demonstrate in this section.

As we have formulated Laplace integral i.e. $i(t) = \int_0^{\infty} \left(H_{\lambda}(\lambda)\right) e^{-t\lambda} \mathrm{d}\lambda$, just by replacing, $\lambda = \tau^{-1}$, $\mathrm{d}\lambda = -(\tau)^{-2} \mathrm{d}\tau$ we will get $i(t) = \int_{\infty}^{0} \left(H_{\lambda}(\lambda)\right)(-\tau)^{-2} e^{-t/\tau} \mathrm{d}\tau$ that is

$$i(t) = \int_0^{\infty} \left(H_{\lambda}(\lambda)\right)(\tau)^{-2} e^{-t/\tau} \mathrm{d}\tau \tag{21}$$

This is not Laplace integral. Now we do the following steps, for $i(t) = t^{-n}$ and with obtained $H_{\lambda}(\lambda) = \frac{1}{\Gamma(n)} \lambda^{n-1}$

$$\begin{aligned} t^{-n} &= \int_0^{\infty} \left(\tfrac{1}{\Gamma(n)} \lambda^{n-1}\right)(\tau)^{-2} e^{-t/\tau} \mathrm{d}\tau; \qquad \lambda = \tau^{-1} \\ &= \frac{1}{\Gamma(n)} \int_0^{\infty} \tau^{1-n} \tau^{-2} e^{-t/\tau} \mathrm{d}\tau \\ &= \frac{1}{\Gamma(n)} \int_0^{\infty} \tau^{-(n+1)} e^{-t/\tau} \mathrm{d}\tau \end{aligned} \tag{22}$$

We write the two representations of $t^{-n}$ as following integrals

$$t^{-n} = \frac{1}{\Gamma(n)} \int_0^{\infty} \left(\lambda^{(n-1)}\right) e^{-t\lambda} \mathrm{d}\lambda \qquad t^{-n} = \frac{1}{\Gamma(n)} \int_0^{\infty} \left(\tau^{-(n+1)}\right) e^{-t/\tau} \mathrm{d}\tau \tag{23}$$

Thus, we have $H_{\tau}(\tau) \sim \tau^{-(n+1)}$, as we have $H_{\lambda}(\lambda) \sim \lambda^{n-1}$. Now we verify the above obtained result in following discussion.

By the logic that we had constructed $i(t) = \int_0^{\infty} \left(H_{\lambda}(\lambda)\right) e^{-t\lambda} \mathrm{d}\lambda$ which is Laplace integral; we will similarly get the integral $i(t) = \int_0^{\infty} \left(H_{\tau}(\tau)\right) e^{-t/\tau} \mathrm{d}\tau$ which is not a direct Laplace Transform formula. Following steps will convert this expression into the Laplace Transform formula, and from there we will extract $H_{\tau}(\tau)$.

$$
\begin{aligned}
i(t) &= \int_0^\infty \left(H_\tau(\tau)\right) e^{-t/\tau} \mathrm{d}\tau; \quad \tau = \lambda^{-1}, \quad \mathrm{d}\tau = -\lambda^{-2}\mathrm{d}\lambda \\
&= \int_\infty^0 \left(-\lambda^{-2}\left(H_\tau(\tau)\right)\right)\left(e^{-t\lambda}\right)\mathrm{d}\lambda; \qquad \mathrm{F} = \lambda^{-2}\left(H_\tau(\tau)\right) \\
&= \int_0^\infty \left(\mathrm{F}\right)\left(e^{-t\lambda}\right)\mathrm{d}\lambda
\end{aligned}
\tag{24}
$$

We proceed further as follows

$$
\begin{aligned}
\mathrm{F} &= \mathcal{L}^{-1}\left\{i(t)\right\} = \frac{1}{\Gamma(n)}\lambda^{(n-1)} \\
\lambda^{-2}\left(H_\tau(\tau)\right) &= \frac{1}{\Gamma(n)}\lambda^{(n-1)} \\
H_\tau(\tau) &= \frac{\lambda^{n-1}\lambda^2}{\Gamma(n)}; \qquad \lambda = \tau^{-1} \\
&= \frac{\tau^{-(n+1)}}{\Gamma(n)}
\end{aligned}
\tag{25}
$$

Now we take different approach to verify the above obtained expression for $H_\tau(\tau)$ . Let us have set of relaxation functions with various time constants $\tau$ ranging from $0$ to infinity that is $\left\{e^{-t/\tau_1}, e^{-t/\tau_2}, e^{-t/\tau_3}, \ldots\right\}$, comprising of infinite number of functions, in continuum in $\tau$ . The relaxation function varies from very-very quick decay (when $\tau \approx 0$) to very-very slow decay curve (when $\tau \approx \infty$). We construct a weighted decay function as $\tau^{-m}e^{-t/\tau}$ . This shows that we are multiplying by weight $\tau^{-m}; \quad m > 0$ the decay function $e^{-t/\tau}$ . We are assuming Zipf's type distribution of $\tau$ , in form of $H_\tau(\tau) \sim \tau^{-m}$ , meaning the lowest time constant i.e. fastest decay occurs more frequent than slow decay i.e. large time constant. The time constant parameter $\tau$ let vary from $0$ to infinity and construct the following integral '$\mathrm{I}$', i.e.

$$
\mathrm{I} = \int_0^\infty \tau^{-m} e^{-t/\tau} \mathrm{d}\tau
\tag{26}
$$

The integral $\mathrm{I}$ gives notion of weighted average of infinite relaxation functions. We do the substitution i.e. $\frac{t}{\tau} = y$ , i.e. $\tau = \frac{t}{y}$ and $\mathrm{d}\tau = \left(-y\right)^{-2} t\left(\mathrm{d}y\right)$ in the above integral to get following steps

$$
\begin{aligned}
\mathrm{I} &= \int_0^\infty \tau^{-m} e^{-t/\tau} \mathrm{d}\tau \\
&= \int_\infty^0 \left(\tfrac{t}{y}\right)^{-m} e^{-y} \left(-y\right)^{-2} t\left(\mathrm{d}y\right) \\
&= t^{-m+1} \int_0^\infty e^{-y} y^{m-2} \mathrm{d}y
\end{aligned}
\tag{27}
$$

By using the definition of the Gamma function in integral form i.e. $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} \mathrm{d}y$ , we write the above integral as

$$\begin{aligned} \mathrm{I} &= t^{-(m-1)}\int_0^\infty e^{-y}y^{m-2}\mathrm{d}y \\ &= t^{-(m-1)}\left(\Gamma(m-1)\right) \\ \int_0^\infty \tau^{-m}e^{-t/\tau}\mathrm{d}\tau &= \frac{\Gamma(m-1)}{t^{m-1}} \end{aligned} \tag{28}$$

Putting $m-1=n$ in above we get integral representation of the power law $t^{-n}$ and we represent this by time constant distribution function $H_\tau(\tau)$ in following expressions

$$\begin{aligned} t^{-n} &= \frac{1}{\Gamma(n)}\int_0^\infty \tau^{-(n+1)}e^{-t/\tau}\mathrm{d}\tau \\ t^{-n} &= \int_0^\infty \left(H_\tau(\tau)\right)e^{-t/\tau}\mathrm{d}\tau \end{aligned} \tag{29}$$

Earlier we have obtained $t^{-n}=\frac{1}{\Gamma(n)}\int_0^\infty\left(\lambda^{(n-1)}\right)e^{-t\lambda}\mathrm{d}\lambda$ ; where we called rate distribution function as $H_\lambda(\lambda)=\frac{1}{\Gamma(n)}\lambda^{(n-1)}$. Now from above weighted average logic we get $H_\tau(\tau)=\frac{1}{\Gamma(n)}\tau^{-(n+1)}$ ; these two are not reciprocal of each other.

Therefore we can conclude that Curie-von Schweidler law ($\sim t^{-n}$) relates to weighted averaging of several classical Debye relaxations (of type $e^{-t/\tau}$) over several time constants from zero to infinity, that is having Zipf's power-law with time constant distribution as $H_\tau(\tau)\sim\tau^{-(n+1)}$. What does it say for $0<n<1$, that $H_\tau(\tau)\sim\tau^{-(n+1)}$ is also a right-skewed distribution, where the lower time constants (faster decay) appear more than larger time constant (slower decay). This is contradiction to what we inferred for $H_\lambda(\lambda)\sim\lambda^{n-1}$ .

This contradiction we demonstrate. As for $i(t)=e^{-\lambda_0 t}$ we got $H_\lambda(\lambda)=\delta(\lambda-\lambda_0)$ ; we expect $H_\tau(\tau)=\delta(\tau-\tau_0)$ for $i(t)=e^{-t/\tau_0}$ ; let us see what happens in following steps

$$\begin{aligned} & i(t)=e^{-\lambda_0 t}; \quad \mathrm{F}=\mathcal{L}^{-1}\{i(t)\} \\ & \mathrm{F}=\mathcal{L}^{-1}\left\{e^{-\lambda_0 t}\right\}=\delta(\lambda-\lambda_0); \quad \mathrm{F}=\lambda^{-2}\left(H_\tau(\tau)\right) \\ & \lambda^{-2}\left(H_\tau(\tau)\right)=\delta(\lambda-\lambda_0) \\ & H_\tau(\tau)=\lambda^2\left(\delta(\lambda-\lambda_0)\right); \qquad \lambda=\tau^{-1} \\ & \qquad =\frac{\delta(\tau-\tau_0)}{\tau^2}\neq\delta(\tau-\tau_0) \end{aligned} \tag{30}$$

However, mathematically we can get integral representation for any $i(t)=\int_0^\infty\left(H_\tau(\tau)\right)e^{-t/\tau}\mathrm{d}\tau$ but physically it will be contradictory to Laplace integral $i(t)=\int_0^\infty\left(H_\lambda(\lambda)\right)e^{-\lambda t}\mathrm{d}\lambda$ . Hence we will deal with the relaxation rate distribution function that we extracted as $H_\lambda(\lambda)$ from $i(t)$ via our devised method of Laplace inversion.

## Time dependent relaxation rate for a complex non-Debye relaxation process

Any decay function $i(t)$ is written as a general formulation in following way

$$i(t) = \exp\left(-\int_0^t \left(\lambda(\xi)\right)\mathrm{d}\xi\right) \tag{31}$$

where $\lambda(\xi)$ is the time ($\xi$) dependent rate coefficient. When the relaxation is pure exponential, one has $\lambda(\xi)$ as constant say $\lambda_0$ described as $\lambda(t) = \lambda_0$.

$$\begin{aligned} i(t) &= \exp\left(-\int_0^t \left(\lambda(\xi)\right)\mathrm{d}\xi\right) \\ &= \exp\left(-\int_0^t \left(\lambda_0\right)\mathrm{d}\xi\right) = \exp\left(-\lambda_0\, \xi\big|_0^t\right) \\ &= e^{-\lambda_0 t} \end{aligned} \tag{32}$$

Thus, we get a Debye relaxation for a system having constant rate of relaxation. To extract $\lambda(t)$ that is time dependent rate coefficient we have to follow the following steps

$$\begin{aligned} -\int_0^t \left(\lambda(\xi)\right)\mathrm{d}\xi &= \ln\left(i(t)\right) \\ \lambda(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(i(t)\right) \end{aligned} \tag{33}$$

We use the above rule for Mittag-Leffler relaxation function i.e. $i(t) = E_\alpha(-t)$, this is defined as $E_\alpha(-t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\beta)} t^k$ . For $i(t) = E_\alpha(-t)$ the $\lambda(t)$ is extracted as in following steps

$$\begin{aligned} \lambda(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(E_\alpha(-t)\right) \\ &= -\frac{1}{E_\alpha(-t)}\frac{\mathrm{d}E_\alpha(-t)}{\mathrm{d}t} = -\frac{1}{E_\alpha(-t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty}\frac{(-1)^k t^k}{\Gamma(\alpha k+1)}\right) \\ &= -\frac{1}{E_\alpha(-t)}\left(\sum_{k=0}^{\infty}\frac{(k+1)(-t)^k}{\Gamma(1+\alpha+\alpha k)}\right) \end{aligned} \tag{34}$$

For Curie-von Schweilder relaxation law $i(t) = t^{-n}$ we have time dependent rate relaxation rate as

$$
\begin{aligned}
\lambda(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(t^{-n}\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\left(-n\ln t\right) = n\left(t^{-1}\right) \qquad \tau(t) = \frac{1}{\lambda(t)} \\
\tau(t) &= \frac{t}{n}
\end{aligned} \tag{35}
$$

Therefore, we have two observations that Curie-von Schweidler relaxation $i(t) \sim t^{-n}$ has time constant distributed as Zipfian power law $H_{\lambda} \sim \lambda^{n-1}; \quad 0 < n < 1$, while the relaxation rate constant is variable in time as a function $\lambda(t) = n/t$. Thus implying that relaxation starts with very-very fast relaxation at a very-very high rate and as the time goes the rate constant decreases indicating slow rate of current decay.

## Scale dependence of relaxation rates give capacitors charging current as per Curie-von Schweidler law

Let a capacitor $c$ be connected to a voltage source $v(t)$ Volts, at time $t = 0$; obviously, this capacitor will get charged to the battery voltage. Let this capacitor is uncharged at $t < 0$, thus there is no charge held by it, therefore the voltage across the capacitor is zero at $t < 0$, and the circuit current is $i(t) = 0; \quad t < 0$ The voltage balance equation assuming $r$ be the total resistance of the circuit (including internal resistance of Capacitor) at $t > 0$ is the following

$$
\left(\frac{1}{c}\int_{0}^{t} i(x)\mathrm{d}x\right) + ri(t) = v(t) \tag{36}
$$

Where $i(t)$ is the charging current flowing into the capacitor. The above integral equation may be differentiated and is put as following, for $t > 0$

$$
\begin{aligned}
&\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \lambda_0 i(t) = \left(\frac{1}{r}\right)\frac{\mathrm{d}v(t)}{\mathrm{d}t} \qquad \lambda_0 = (rc)^{-1} \qquad \tau_0 = rc \\
&i^{(1)}(t) + \lambda_0 i(t) = f(t)
\end{aligned} \tag{37}
$$

The RHS of above first order system indicates 'forcing function', which is $f(t)$. The forcing function is $f(t) = \frac{1}{r}v^{(1)}(t)$ in this case. If we take $v(t) = V_{BB}$ a constant, then considering $u(t) = 1; \quad t \geq 0$ and $u(t) = 0; \quad t < 0$, i.e. 'unit-step function', we have for RHS of the above equation as following

$$
\begin{aligned}
\frac{\mathrm{d}v(t)}{\mathrm{d}t} &= v^{(1)}(t) \\
v^{(1)}(t) &= V_{BB}\frac{\mathrm{d}\left(u(t)\right)}{\mathrm{d}t} \\
&= V_{BB}\left(\delta(t)\right)
\end{aligned} \tag{38}
$$

Thus substituting the above, we get following equation for $v(t)=V_{BB}$ for $t\geq 0$ the following

$$\begin{aligned}&\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\lambda_0 i(t)=\frac{V_{BB}}{r}\delta(t) \qquad \lambda_0=(rc)^{-1} \qquad \tau_0=rc\\&\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\lambda_0 i(t)=I_0\delta(t) \qquad I_0=\frac{V_{BB}}{r}\end{aligned} \tag{38}$$

We see that forcing function of above first order equation is by delta function $f(t)=I_0\delta(t)$ . The solution to the above equation gives Debye relaxation function i.e.

$$i(t)=I_0 e^{-\lambda_0 t} \tag{39}$$

This solution $i(t)\sim e^{-\lambda_0 t}$ is the 'impulse response' of the circuit equation. The relaxation current of the above system follows Debye's relaxation, with one relaxation rate $\lambda_0$ (also termed as Debye law). The rate distribution function is $H_\lambda(\lambda)=\delta(\lambda-\lambda_0)$ ; that we discussed in previous sections. With this $i(t)=e^{-\lambda_0 t}$ as Green's function i.e. $g(t)=e^{-\lambda_0 t}$ i.e. solution of differential equation with 'unit' impulse excitation ($I_0=1$) or say Homogeneous solution, i.e.

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\lambda_0 i(t)=\delta(t) \qquad i(t)=g(t)=e^{-\lambda_0 t} \tag{40}$$

Now we find if the input is say step function, call it $I_0u(t)$, where $u(t)=1$ for $t\geq 0$ and $u(t)=0$ for $t<0$ then we get relaxation function for current as convolution integral, i.e. depicted as in following steps

$$\begin{aligned}&\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\lambda_0 i(t)=I_0u(t) \qquad g(t)=e^{-\lambda_0 t}\\&i(t)=\left(I_0u(t)\right)*\left(g(t)\right)=\int_{-\infty}^{t}\left(I_0u(x)\right)\left(g(t-x)\right)\mathrm{d}x; \qquad g(t-x)=e^{-\lambda_0(t-x)}\\&\qquad=\int_{-\infty}^{t}\left(I_0u(x)\right)\left(e^{-\lambda_0(t-x)}\right)\mathrm{d}x=I_0\int_0^t e^{-\lambda_0(t-x)}\mathrm{d}x\\&i(t)=\frac{I_0}{\lambda_0}\left(1-e^{-\lambda_0 t}\right)\end{aligned} \tag{41}$$

We saw in earlier sections the relaxation rates ($\lambda$) distribution, for a Curie-von Schweidler relaxation law, i.e. $i(t)\sim t^{-n}$ is $H_\lambda(\lambda)\sim\lambda^{n-1}; \quad 0<n<1$; for relaxations in dielectrics. This is histogram of rates; it says that the relaxation of current is with several relaxation rates, which are distributed as discussed in Zipf's law fashion with right-skewed-histogram.

Thus if we represent the equivalent relaxation rate say $\lambda_{eq}\sim\lambda^{1/\alpha}; \quad 0<\alpha<1$ as scale of relaxation $\lambda$ varies from zero to infinity; we will not be incorrect in assuming this. That is as we slide from a low scale $\lambda$ to high scale $\lambda$ the equivalent relaxation rate $\lambda_{eq}$ will be different at different scales of relaxation. If the index parameter i.e. $\alpha=1$ then we have single rate constant system given by $\lambda_{eq}=\lambda$ always at all scales of relaxation i.e. $\lambda=\lambda_0=\tau_0^{-1}=(rc)^{-1}$, and with solution as $i(t)=e^{-\lambda_0 t}$.

We thus modify the capacitor discharge current equation, with $\lambda_{eq}=\lambda=\lambda_0$ i.e. with one relaxation rate at any scale of relaxation ( $\lambda$ ), i.e. $i^{(1)}(t)+\lambda_0 i(t)=\delta(t)$ to following i.e. variable $\lambda_{eq}=\lambda^{1/\alpha}$ at any scale of relaxation rate ( $\lambda$ )

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda_{eq}\right)i(t)=\delta(t) \qquad \frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda\right)^{1/\alpha}i(t)=\delta(t) \tag{42}$$

The initial condition is given as $i(t)=0$ for $t<0$. The above equation is having a free 'scale' parameter $\lambda$ varying from zero to infinity. The solution of the above is $i(t)=e^{-\lambda_{eq}t}$. We call this $i(t)=\exp(-\lambda_{eq}t)=\exp\left(-\lambda^{1/\alpha}t\right)$ as 'impulse response function' at a particular scale $\lambda$, i.e. we call it $h(\lambda,t);\quad \lambda\in\left(0,\infty\right)$

$$i(t)=h(\lambda,t)=e^{-\left(\lambda^{(1/\alpha)}t\right)};\qquad 0<\alpha<1 \tag{43}$$

The above expression actually is valid for all scale $\lambda$ varying from zero to infinity. Thus, on integrating, this 'impulse response function' on the free variable $(\lambda)$ from $0$ to $\infty$, we get the function of time and that is called 'impulse response' or the Green's function $g(t)$

$$\begin{aligned} g(t)&=\int_0^\infty h(\lambda,t)\mathrm{d}\lambda \\ &=\int_0^\infty e^{-\left(\lambda^{(1/\alpha)}t\right)}\mathrm{d}\lambda=\frac{\Gamma(1+\alpha)}{t^\alpha} \end{aligned} \tag{44}$$

To get above expression we substitute in $g(t)=\int_0^\infty e^{-\left(\lambda^{(1/\alpha)}t\right)}\mathrm{d}\lambda$ , $\lambda^{1/\alpha}t=x$ that makes following changes

$$\begin{aligned} &\lambda=\left(\frac{x}{t}\right)^\alpha;\qquad \left(\frac{x}{t}\right)=\lambda^{1/\alpha} \\ &\mathrm{d}\lambda=\alpha x^{\alpha-1}\left(\tfrac{1}{t}\right)^\alpha\mathrm{d}x \\ &\quad=\left(\frac{\alpha}{t}\right)\left(\frac{x}{t}\right)^{\alpha-1}\mathrm{d}x=\left(\frac{\alpha}{t}\right)\left(\lambda^{1/\alpha}\right)^{\alpha-1}\mathrm{d}x \\ &\mathrm{d}\lambda=\lambda^{1-(1/\alpha)}\left(\frac{\alpha}{t}\right)\mathrm{d}x \end{aligned} \tag{45}$$

Then by using definition of Gamma function i.e. $\Gamma(\upsilon)=\int_0^\infty e^{-y}y^{\upsilon-1}\mathrm{d}y$ , and its property $\upsilon\left(\Gamma(\upsilon)\right)=\Gamma(1+\upsilon)$ the following steps are followed to get the desired expression i.e. $g(t)=\frac{\Gamma(1+\alpha)}{t^\alpha}$

$$\begin{aligned}
g(t) &= \int_0^\infty e^{-\left(\lambda^{(1/\alpha)}t\right)}\mathrm{d}\lambda = \int_0^\infty e^{-x}\lambda^{1-\left(\frac{1}{\alpha}\right)}\left(\frac{\alpha}{t}\right)\mathrm{d}x \\
&= \int_0^\infty e^{-x}\left(\frac{\alpha}{t}\right)\lambda\lambda^{-\left(\frac{1}{\alpha}\right)}\mathrm{d}x, \qquad \lambda = \left(\frac{x}{t}\right)^\alpha \\
&= \int_0^\infty e^{-x}\left(\frac{\alpha}{t}\right)\left(\frac{\alpha}{t}\right)^\alpha\left(\frac{x}{t}\right)^{-1}\mathrm{d}x \\
&= \left(\frac{\alpha}{t}\right)\int_0^\infty e^{-x}\left(\frac{x}{t}\right)^\alpha\left(\frac{x}{t}\right)^{-1}\mathrm{d}x \\
&= \left(\frac{\alpha}{t}\right)\int_0^\infty e^{-x}\frac{x^{\alpha-1}}{t^{\alpha-1}}\mathrm{d}x = \left(\frac{\alpha}{t^\alpha}\right)\int_0^\infty e^{-x}x^{\alpha-1}\mathrm{d}x \\
&= \frac{\alpha\left(\Gamma(\alpha)\right)}{t^\alpha} = \frac{\Gamma(1+\alpha)}{t^\alpha}
\end{aligned} \tag{46}$$

By changing $\alpha$ to, $n$ we get integral representation of $t^{-n}$ as following

$$t^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^\infty e^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda \tag{47}$$

For $n=1$ case we have scale invariance $\lambda$ thus $i(t)=h(\lambda_0,t)=\exp\left(-\lambda_0 t\right); \quad n=1$ , where $\lambda_{eq}=\lambda_0$ at all scales. For this case $n=1$ 'impulse response' or Green's function is $g(t)=h(\lambda_0,t)=e^{-\lambda_0 t}$ same as 'impulse response function' i.e. $h(\lambda_0,t)$.

We find that $i(t)\sim t^{-n}; \quad 0<n<1$ for a system where the equivalent relaxation rate is $\lambda_{eq}=\lambda^{\frac{1}{n}}$; similar to a distribution function that we obtained as $H_\lambda(\lambda)\sim\lambda^{n-1}$ gives current as $i(t)\sim t^{-n}$. We write the two currents expressions $i(t)\sim t^{-n}$ obtained as following for $0<n<1$

$$t^{-n} = \frac{1}{\Gamma(n)}\int_0^\infty \lambda^{n-1}e^{-t\lambda}\mathrm{d}\lambda \qquad\qquad t^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^\infty e^{-\left(\lambda^{(1/n)}t\right)}\mathrm{d}\lambda \tag{48}$$

Therefore, we infer that the Curie-von Schweidler relaxation current for dielectric excited by a step voltage that follows the relation $i(t)\sim t^{-n}; \quad 0<n<1$ has distribution function $H_\lambda(\lambda)\sim\lambda^{n-1}$ a power law or Zipfian distribution, with scale variable relaxation rate described as $\lambda_{eq}=\lambda^{\frac{1}{n}}$.

## Appearance of fractional derivative-in the system having Zipfian power law distribution in relaxation rates, where the equivalent relaxation rate is scale dependent

The delta-function for excitation as shown in above section gives homogeneous system with solution as $g(t)=t^{-n}\left(\Gamma(1+n)\right)$ i.e.

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda\right)^{\frac{1}{n}}i(t)=\delta(t);\quad 0<n<1;\qquad i(t)=g(t)=\frac{\Gamma(1+n)}{t^{n}} \tag{49}$$

Now let the system described above be excited by a signal proportional to $f(t)\sim v^{(1)}(t)$, a derivative of voltage excitation function $v(t)$; so we write this as following

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t}+\left(\lambda\right)^{\frac{1}{n}}i(t)=v^{(1)}(t) \tag{50}$$

Note that if $v(t)=u(t)$, that is unit-step-function then $v^{(1)}(t)=\delta(t)$, we recover the above homogeneous differential equation. Then the response to this new excitation function $v^{(1)}(t)$ is convolution of Green's function obtained i.e. $g(t)=\frac{\Gamma(1+n)}{t^{n}}$ above, with the forcing function $f(t)$ i.e. now $\sim v^{(1)}(t)$. We write the following steps to get $i(t)$ for a forcing function $f(t)$

$$\begin{aligned}
i(t)&=\left(g(t)\right)*\left(f(t)\right)=\left(g(t)\right)*\left(v^{(1)}(t)\right)\\
&=\int_0^t\left(g(t-x)\right)\left(v^{(1)}(x)\right)\mathrm{d}x;\qquad g(t)=\frac{\Gamma(1+n)}{t^{n}}\\
&=\Gamma(1+n)\int_0^t\frac{v^{(1)}(x)}{(t-x)^{n}}\mathrm{d}x\qquad 0<n<1
\end{aligned} \tag{51}$$

Multiplying and dividing the above expression with $\Gamma(1-n)$ and using the definition of fractional integral [6], [33], [38] that is

$${}_0\mathcal{I}_t^{\upsilon}\left(f(t)\right)={}_0D_t^{-\upsilon}\left(f(t)\right)=\int_0^t\frac{1}{\Gamma(\upsilon)}(t-x)^{\upsilon-1}f(x)\mathrm{d}x\qquad \upsilon>0 \tag{52}$$

we get the following derivation

$$\begin{aligned}
i(t)&=\left(\Gamma(1+n)\right)\left(\Gamma(1-n)\right)\int_0^t\frac{(t-x)^{(-n)}}{\Gamma(1-n)}v^{(1)}(x)\mathrm{d}x\\
&=\Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[v^{(1)}(t)\right]\right);\qquad (1-n)>0\\
&=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{-(1-n)}\left[v^{(1)}(t)\right]\right)\\
&=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\,{}_0D_t^{-1}\left[v^{(1)}(t)\right]\right)\\
&=\Gamma(1+n)\Gamma(1-n)\left({}_0D_t^{n}\left[v(t)\right]\right),\qquad n<1
\end{aligned} \tag{53}$$

In above derivation we have used ${}_0\mathcal{I}_t^{\upsilon}={}_0D_t^{-\upsilon}$; $\upsilon>0$ and ${}_0D_t^{\alpha+\beta}={}_0D_t^{\alpha}\,{}_0D_t^{\beta}$; $\alpha>0;\beta<0$ [6], [33], [38]. This derivation implies the appearance of fractional derivative for cases where several relaxation rates (ideally infinite of them) define a relaxation process; which are having a scale dependence behavior, i.e. $\lambda_{eq}=\lambda^{\frac{1}{n}}$ with histogram distributed as Zipf's power law i.e. $H_{\lambda}(\lambda)\sim\lambda^{n-1}$, and the relaxation is by Curie-von Schweidler law i.e. $i(t)\sim t^{-n}$, $0<n<1$. Thus we have current through a system (having a complex relaxation process with several rate distributed as power law excited by a voltage $v(t)$ as fractional derivative of it, i.e. $i(t) \sim {}_0D_t^n\left[v(t)\right]$.

Let this system i.e. $\frac{\mathrm{d}}{\mathrm{d}t} i(t) + \lambda^{1/n} i(t) = v^{(1)}(t); \quad 0 < n < 1$ be excited by a source which is a delta function say $v^{(1)}(t) = \left(\left(I_0\right)\delta(t)\right)$; at $t = 0$ that is

$$\frac{\mathrm{d}}{\mathrm{d}t} i(t) + \lambda^{1/n} i(t) = I_0\left(\delta(t)\right); \quad 0 < n < 1 \tag{54}$$

This means $v(t) = \left(I_0\right)u(t)$; where $u(t)$ is unit step function at $t = 0$. With this excitation the relaxation current would be fractional integral of the input excitation that is from as depicted in above derivation i.e. $i(t) = \Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[I_0\delta(t)\right]\right)$. We have fractional integration of delta function [6], [33], [38] as ${}_0\mathcal{I}_x^{\upsilon}\left[\delta(x)\right] = \frac{1}{\Gamma(\upsilon)} x^{\upsilon-1}$; and using this formula we get $i(t) = I_0\left(\frac{\Gamma(1+n)}{t^n}\right)$. This was what was derived in above as impulse response (where $I_0 = 1$) i.e. $g(t) = t^{-n}\left(\Gamma(1+n)\right)$.

If the excitation source is a step function as $v^{(1)}(t) = I_0\left(u(t)\right)$ at $t = 0$; where the unit step function is $u(t) = 1, \quad t \geq 0\ ; \quad u(t) = 0, \quad t < 0$ that is

$$\frac{\mathrm{d}}{\mathrm{d}t} i(t) + \lambda^{1/n} i(t) = I_0\left(u(t)\right); \quad 0 < n < 1 \tag{55}$$

This is meaning that we have ramp voltage excitation as $v(t) = \left(I_0\right)t; \quad t \geq 0$ then the relaxation current is fractional integration of order $(1-n)$; that is $i(t) = \Gamma(1+n)\Gamma(1-n)\left({}_0\mathcal{I}_t^{(1-n)}\left[I_0 u(t)\right]\right)$. Using the formula for fractional integration of a constant i.e. ${}_0\mathcal{I}_x^{\upsilon}\mathrm{C} = \frac{\mathrm{C}}{\Gamma(1+\upsilon)} x^{\upsilon}$ [6], [33], [38] we have; the relaxation current as

$$\begin{aligned} i(t) &= \left(\Gamma(1+n)\Gamma(1-n)\right)\left(I_0\right)\frac{t^{1-n}}{\Gamma(2-n)} \\ &= \frac{\Gamma(1+n)}{(1-n)}\left(I_0\right)t^{1-n}, \quad 0 < n < 1 \end{aligned} \tag{56}$$

## Recalling classical capacitor and defining time dependent capacity function

What we know about geometric capacitor or a constant capacitor of say value $C_1$ is a constant value of $\mathrm{Farad}$ at all the frequencies from DC value of zero Hertz to infinite Hertz. This is ideal capacitor as though the dielectric used $\varepsilon_r$ is lossless and is constant at all frequencies; and the capacity is given as $C_1 = \varepsilon_r A / d$ i.e. by using geometric factor of area to electrode separation ratio. This ideal capacity is constant at all the frequencies is called geometric capacity. Therefore, if we say $s$ as complex frequency (Laplace variable) then this constant capacity is given as following function

$$\begin{aligned} &C(s) = C_1 \qquad s = \mathrm{i}\omega \\ &C(\omega) = C_1 \end{aligned} \tag{57}$$

The Laplace complex frequency is written in (57) as $s = \mathrm{i}\omega$ for writing sinusoidal or steady state frequency domain analysis [32]. From (57) we see that $C(\omega) = \mathrm{Re}\{C(\omega)\} - \mathrm{i}\,\mathrm{Im}\{C(\omega)\} = C_1 - \mathrm{i}(0)$ has only real part with imaginary part as zero at all frequencies; that gives loss tangent as $\tan\phi = \frac{\mathrm{Im}\{C(\omega)\}}{\mathrm{Re}\{C(\omega)\}} = 0$ . Thus, ideal capacitor (57) is a loss less capacitor. The dielectric loss is expressed as loss tangent for a complex dielectric quantity given as $\varepsilon_r(\omega) = \mathrm{Re}\,\varepsilon_r - \mathrm{i}\,\mathrm{Im}\,\varepsilon_r$ where loss tangent is given as $\tan\phi = \frac{\mathrm{Im}\,\varepsilon_r}{\mathrm{Re}\,\varepsilon_r}$.

Since the inverse Laplace transform of function $F(s) = 1$ gives time function i.e. $f(t) = \mathcal{L}^{-1}\{F(s)\} = \delta(t)$ i.e. a Dirac delta function at $t = 0$ , we say the 'time varying capacity function' call it $c(t)$ of geometric capacitor (ideal-capacitor) is following by taking inverse Laplace transform of (57), i.e. $c(t) = \mathcal{L}^{-1}\{C(s)\}$ we write the following expression

$$c(t) = C_1\delta(t) \tag{58}$$

Therefore, we say that a constant ideal capacitor has a 'capacity function' $c(t)$ as Dirac delta function. For example if the capacity of a capacitor is a function of frequency say as $C(s) = C_m s^{-m}$ ; then the time varying capacity function $c(t)$ for this capacitor $c(t) = \mathcal{L}^{-1}\{C(s)\}$ is following

$$c(t) = \frac{C_m}{(m-1)!}t^{m-1}; \qquad t > 0 \tag{59}$$

The capacity function is constant $c(t) = C_0$ for $t \geq 0$ only if the frequency function is $C(s) = C_0 s^{-1}$. Therefore, we say $c(t) = C_0$ is not a constant capacitor or a lossless capacitor. This capacitor with capacity function $c(t) = C_0$ in frequency domain in complex notation is $C(\omega) = 0 - \mathrm{i}\omega^{-1}C_0$, $\mathrm{i} = \sqrt{-1}$ with loss tangent as infinity.

Therefore, say when we apply a voltage function $v(t)$ to an uncharged capacity we write the charge stored at any time as convolution integral as follows

$$\begin{aligned} q(t) &= c(t) * v(t) \\ &= \int_{-\infty}^{t} \big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x = \int_{-\infty}^{t} \big(c(y)\big)\big(v(t-y)\big)\mathrm{d}y \end{aligned} \tag{60}$$

This is against conventional way of writing the charge i.e.

$$q(t) = c(t)v(t); \qquad c(t) = \frac{q(t)}{v(t)} \tag{61}$$

This argument we will explain in the subsequent section.

In reality the capacity of a capacitor, say of $1\mu\mathrm{F}$ means this value is at particular frequency of measurement standard is at $1\mathrm{kHz}$ (also depends on application) [12]. Practically due to losses in $\varepsilon_r$ the value of capacity of capacitor is varying in frequency; therefore in reality we have time varying capacity function $c(t)$ describing a capacitor.

## Reviewing concept of charge storage in constant capacitor-the classical theory

We have standard expression of 'impedance of a capacitor' i.e. $Z(s)$ expressed in frequency domain as following with $V(s)=\mathcal{L}\{v(t)\}$, $I(s)=\mathcal{L}\{i(t)\}$

$$Z(s)=\frac{V(s)}{I(s)}=\frac{1}{C_1 s} \qquad Z(\omega)=\frac{V(\omega)}{I(\omega)}=-\mathrm{i}\frac{1}{\omega C_1} \tag{62}$$

Thus from (62), we have the capacity function expressed in Laplace frequency domain as a function as

$$C_1=\frac{s^{-1}I(s)}{V(s)} \tag{63}$$

We note that $C_1$ is Laplace transformed quantity, i.e. $C_1=\mathcal{L}\{c(t)\}$; and in this case of 'constant capacity' the capacity function in time is $c(t)=C_1\delta(t)$ (58). Therefore, we have in frequency domain representation of capacitor as function of Laplace variable $s$, so we call it as $C(s)=\mathcal{L}\{c(t)\}$. Therefore, for a general relation of capacity in frequency domain we have following

$$C(s)=\frac{s^{-1}I(s)}{V(s)} \tag{64}$$

The numerator term in (64) i.e. $s^{-1}I(s)$ in time domain is $\int_0^t i(x)\mathrm{d}x$ [32] that is charge the $q(t)$, i.e. $q(t)=\int_0^t i(x)\mathrm{d}x$ with its Laplace transform as $Q(s)=\mathcal{L}\{q(t)\}$. Therefore, we write charge in frequency domain as following expression

$$Q(s)=C(s)V(s) \tag{65}$$

This (65) is the expression in frequency domain. In the time domain, we write the charge equation as convolution integral [32], i.e. using $\mathcal{L}^{-1}\{(F_1(s))(F_2(s))\}=\left(\mathcal{L}^{-1}\{F_1(s)\}\right)*\left(\mathcal{L}^{-1}\{F_2(s)\}\right)=f_1(t)*f_2(t)$ where $F_j(s)=\mathcal{L}\{f_j(t)\}$, $j=1,2$ i.e. the following

$$\begin{aligned} q(t)&=c(t)*v(t)\\ &=\int_{-\infty}^{t}\big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x \end{aligned} \tag{66}$$

Where in (66) convolution operation is denoted as $(*)$ and the convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as following convolution integral

$$f_1(t)*f_2(t)=\int_{-\infty}^{t}\big(f_1(t-x)\big)\big(f_2(x)\big)\mathrm{d}x=\int_{-\infty}^{t}\big(f_1(x)\big)\big(f_2(t-x)\big)\mathrm{d}x \tag{67}$$

Let an uncharged capacitor of constant capacity at $t=0$, of value $C_1$ is charged with a constant step voltage $V_{BB}$ applied at $t=0$ i.e. $v(t)=V_{BB}\big(u(t)\big);\quad t\ge 0$. We say charged stored at any time for $t>0$ is $q(t)=C_1V_{BB}$, whereas the charge is $q(t)=0$ for $t<0$. Thus the charge in time domain is a step function, we denote that as $q(t) = C_1 V_{BB}\left(u(t)\right)$; with $u(t)$ as unit-step at $t=0$. Laplace transform of this step charge is following

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{s} \qquad\qquad Q(s) = \mathcal{L}\left\{C_1 V_{BB} u(t)\right\} = \frac{C_1 V_{BB}}{s} \tag{68}$$

The first derivative of charge i.e. $q^{(1)}(t)$ gives the charging current i.e.

$$\begin{aligned} i(t) = q^{(1)}(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} C_1 V_{BB} u(t) = C_1 V_{BB}\left(\delta(t)\right) \end{aligned} \tag{69}$$

This is classical result that we all know is as per classical capacitor-theory that is charging current is impulse function at the time of application of voltage step. This impulse current also comes from circuit equation i.e. $\frac{1}{C_1}\int i(t)\mathrm{d}t = V_{BB}\left(u(t)\right)$; and the classical theory deals with geometrical capacitor given by $C_1 = \varepsilon_r A/d$. Now let us look at convolution integral, for $q(t) = c(t) * v(t) = \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x$ for $t>0$ i.e. where we have $v(x) = V_{BB}$, for $t \geq 0$. Only if we define $c(t)$ as function of time as the capacity function i.e. $c(t) = C_1\left(\delta(t)\right)$ we will be getting $q(t) = C_1 V_{BB}$ for $t>0$ demonstrated in following steps

$$\begin{aligned} q(t) = c(t) * v(t) &= \int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x; \quad c(x) = C_1\left(\delta(x)\right), \quad v(x) = V_{BB}, \quad x \geq 0 \\ &= \int_0^t C_1\left(\delta(t-x)\right)\left(V_{BB}\right)\mathrm{d}x = C_1 V_{BB}\int_0^t\left(\delta(t-x)\right)\mathrm{d}x; \qquad t \geq 0 \\ &= C_1 V_{BB}; \qquad t \geq 0 \end{aligned} \tag{70}$$

We have used identity $\int\left(\delta(x_0 - x)\right)\mathrm{d}x = 1$, i.e. property of delta function. Thus from above (70) we get charge as step function at $t=0$, given as

$$q(t) = C_1 V_{BB}\left(u(t)\right) \tag{71}$$

The meaning of capacity function $c(t)$ in time domain is $c(t) = C_1\left(\delta(t)\right)$ i.e. an impulse of height $C_1$ (in units $\mathrm{Farad}$) at the time of application of voltage excitation (i.e. $t=0$), refer Figure-3. Whereas, in the frequency domain, the definition of capacity i.e. for geometrical capacity is, $C(s) = C_1$ i.e. $C(s) = \mathcal{L}\left\{C_1\delta(t)\right\} = C_1$ a constant (in unit of $\mathrm{Farad}$) value at all frequencies that we have discussed in (57) earlier.

Therefore, with $V(s) = V_{BB}/s$ we get $Q(s) = C(s)V(s) = C_1 V_{BB}/s$. Thus when we say a capacitor is having a constant value, it implies that its capacity function is an 'impulse' function at the time of application of voltage stress; in time domain, say at $t=0$. The constant capacity $C_1$ is written as capacity function of time as $c(t) = C_1\left(\delta(t)\right)$. For any other time say at time, say $t = t_0$ of application of voltage-stress the classical geometrical constant capacitor is expressed as capacity function $c(\tau) = C_1\left(\delta(\tau)\right)$, with $\tau = t - t_0$. From now on we will state $t=0$ as time of application of voltage stress to uncharged capacitor with capacity function as $c(t)$.

Say we apply $v(t)=\cos at$ at $t=0$, for $t\geq 0$ (then Laplace transform of $v(t)$ is $V(s)=s/(s^2+a^2)$ ), to an uncharged constant capacitor $C(s)=C_1$. This gives $Q(s)=C_1\left(s/(s^2+a^2)\right)$ implying $q(t)=C_1\cos at;\quad t\geq 0$ . Thus we observe for a constant capacitor, there is no phase difference between $v(t)$ and $q(t)$. We do following convolution integration. Also, refer Figure-3 for curves of $v(t)$ and $q(t)$ that have no delay implying no phase difference for a constant capacity case.

$$\begin{aligned} q(t)=c(t)*v(t) &= \int_{-\infty}^{t}\big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x;\quad c(x)=C_1\left(\delta(x)\right),\quad v(x)=\cos ax;\quad x\geq 0 \\ &= \int_{0}^{t}C_1\left(\delta(t-x)\right)\left(\cos ax\right)\mathrm{d}x;\qquad t\geq 0 \\ &= C_1\cos at;\qquad t\geq 0 \end{aligned} \tag{72}$$

We have used identity $\int\left(\delta(x_0-x)\right)\left(f(x)\right)\mathrm{d}x=f(x_0)$ , i.e. property of delta function. Thus, we have general expression for any time varying voltage $v(t)$ applied at uncharged capacitor with geometrical capacity given by capacity function as $c(t)=C_1\left(\delta(t)\right)$, will have charge $q(t)$ for $t\geq 0$ as following convolution integral

$$\begin{aligned} q(t)=c(t)*v(t) &= \int_{-\infty}^{t}\big(c(t-x)\big)\big(v(x)\big)\mathrm{d}x;\quad c(x)=C_1\left(\delta(x)\right),\quad v(x);\quad x\geq 0 \\ &= \int_{0}^{t}C_1\left(\delta(t-x)\right)\left(v(x)\right)\mathrm{d}x;\qquad t\geq 0 \\ &= C_1\left(v(t)\right);\qquad t\geq 0 \end{aligned} \tag{73}$$

We have used identity $\int\left(\delta(x_0-x)\right)\left(f(x)\right)\mathrm{d}x=f(x_0)$, i.e. property of delta function. Now we differentiate the expression above (73) of $q(t)$ to write following expression

$$\begin{aligned} i(t)=q^{(1)}(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(C_1\left(v(t)\right)\right),\qquad t\geq 0 \\ &= v(t)\frac{\mathrm{d}C_1}{\mathrm{d}t}+C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\ &= \left(v(t)\right)\left(C_1\left(\delta(t)\right)\right)+C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}=C_1\left(v(0)\delta(t)\right)+C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} \\ &= i(0)+i(t),\qquad t\geq 0 \end{aligned} \tag{74}$$

The first term at RHS of (74), indicate the value of current at $t=0$. The constant function starting at $t=0$ i.e. $C_1$ when differentiated gives $C_1\delta(t)$. This unit delta functions at $t=0$, i.e. $\delta(t)$ when multiplied by $v(t)$ gives $v(0)\delta(t)$. This is from property $\int\left(\delta(x_0-x)\right)\left(f(x)\right)\mathrm{d}x=f(x_0)$, differentiation of this gives $\left(\delta(x_0-x)f(x)\right)=\frac{\mathrm{d}}{\mathrm{d}x}f(x_0)=f(x_0)\delta(x)$. Thus at $t=0$ we have $i(0)=C_1v(0)$ and $i(0)=0$ for $t>0$. Compositely we write $i(0)=C_1v(0)\left(\delta(t)\right)$, i.e. specifying its value at only $t=0$. The second term is $i(t)$ for $t \neq 0$, that is $i(t) = C_1\left(v^{(1)}(t)\right)$ (refer Figure-1). We write the following expression for $i(t)$ as

$$i(t) = C_1 v(0)\left(\delta(t)\right) + C_1 \frac{\mathrm{d}v(t)}{\mathrm{d}t} \tag{75}$$

The obtained expression (75) via the formulation $q(t) = c(t) * v(t)$ is consistent with expression obtained in [35].

As an example, we take $v(t) = V_{BB}u(t)$ a step input at time $t = 0$, to an uncharged capacitor. We have $v^{(1)}(t) = 0$ for $t > 0$; and at $t = 0$ we have, $v(0) = V_{BB}$ so $i(0) = C_1 V_{BB}\left(\delta(t)\right)$; this makes $i(t) = C_1 V_{BB}\left(\delta(t)\right), \quad t \geq 0$ .This is for geometrical capacity charging current is impulse function.

Generally the capacitance is not a constant parameter of the capacitor, it varies in frequency and therefore in time too. The constant capacitor concept is approximation when we assume the relative permittivity $\varepsilon_r$ to be constant (note that geometrical capacity we define as $C_1 = \varepsilon_r A / d$ ) [12], [35]. We note that only a loss free capacitor has a constant capacitance in frequency domain. Losses manifest themselves in frequency domain as a phase angle, $\phi$ by which $q(t)$ lags $v(t)$, or given as loss tangent i.e. $\tan\phi$ in the charge expression of capacitor i.e.

$$\begin{aligned} q(t) &= C(t) * v(t) \\ Q(s) &= \left(C(s)\right)\left(V(s)\right) \end{aligned} \tag{76}$$

For $C(\omega) = C_1$, the constant geometrical capacitor with capacity function as $c(t) = C_1\delta(t)$ (1) (2), we have $\tan\phi = \frac{\operatorname{Im} C(\omega)}{\operatorname{Re} C(\omega)} = 0$; that is ideal lossless capacitor.

Therefore, we say that charge stored in capacitor, as a function of time is not multiplication operation of capacity and voltage; instead, the charge is convolution integral. However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage. A time varying capacity has a delay between $v(t)$ and $q(t)$ that is shown in Figure-3 (for time varying capacity case), where $q(t)$ lags $v(t)$.

From the expression (76), $C(s) = Q(s)/V(s)$ we write the time varying capacity $c(t)$ as by use of convolution integral in the following steps

$$\begin{aligned} C(s) &= \frac{Q(s)}{V(s)} \\ \mathcal{L}^{-1}\left\{C(s)\right\} &= \mathcal{L}^{-1}\left\{\left(Q(s)\left(V(s)\right)^{-1}\right)\right\} \\ c(t) &= \mathcal{L}^{-1}\left\{Q(s)\right\} * \mathcal{L}^{-1}\left\{(V(s)^{-1}\right\} \\ &= \left(q(t)\right) * \left(v(t)\right)^{-1} \end{aligned} \tag{77}$$

From above derivation (77), we say that capacity i.e. $c(t) \neq q(t)/v(t)$ i.e. not the usual ratio of charge to voltage in time domain, but it is given as convolution expression i.e.

$$c(t) = \left(q(t)\right) * \left(v(t)\right)^{-1}$$
$$= \int_{-\infty}^{t} \frac{q(t-x)}{v(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{q(x)}{v(t-x)} \mathrm{d}x \tag{78}$$

Let us verify, with $q(t) = \left(C_1 V_{BB}\right)\left(u(t)\right)$ i.e. at $t = 0$ and $q(t) = 0$ for $t < 0$, and $v(t) = V_{BB} u(t)$ i.e. a step voltage at $t = 0$, gives following steps

$$c(t) = \left(q(t)\right) * \left(v(t)\right)^{-1}$$
$$= \int_{-\infty}^{t} \frac{C_1 V_{BB} u(t-x)}{V_{BB} u(x)} \mathrm{d}x = \int_{-\infty}^{t} \frac{C_1 V_{BB} u(x)}{V_{BB} u(t-x)} \mathrm{d}x$$
$$= C_1 \left(u(t) * u(t)^{-1}\right)$$
$$= C_1 \left(\delta(t)\right) \tag{79}$$

We have used inverse identity i.e. $f * f^{-1} = \delta$ in (79).

Therefore, capacity at any time is the history of ratio of charge to voltage given by convolution integral (78). We can verify with say $q(t) = \left(C_1 \cos at\right) u(t)$ for $t \geq 0$ with $v(t) = \left(C_1 \cos at\right) u(t)$ for $t \geq 0$ gives the following

$$c(t) = \left(q(t)\right) * \left(v(t)\right)^{-1}$$
$$= \int_{-\infty}^{t} \frac{C_1 \cos a(t-x)}{\left(\cos ax\right)} \mathrm{d}x = \int_{-\infty}^{t} \frac{C_1 \cos ax}{\cos a(t-x)} \mathrm{d}x$$
$$= C_1 \left(\left(\cos(at)\right) * \left(\cos(at)\right)^{-1}\right)$$
$$= C_1 \left(\delta(t)\right) \tag{80}$$

We have used inverse identity i.e. $f * f^{-1} = \delta$ in (80). We note here the formula used in [12], [35] is $c(t) = q(t) / v(t)$.

## Fractional derivative obtained directly from Curie-von Schweidler Law-formation of fractional capacitor

Practically on applying a step input voltage $v(t) = V_{BB}$ Volts at $t = 0$ to a capacitor which is initially uncharged; we get a power-law decay of current given by empirical Curie-von Schweidler as $i(t) \sim t^{-n}$; $0 < n < 1$ [12], [35]. That we write in following way as indicated by experimental studies [12], [22]-[26].

$$i(t) = K_n \frac{V_{BB}}{t^n} \qquad t > 0 \tag{81}$$

The parameter $K_n$ is proportionality constant, while in [12], [35] the proportionality constant as $1/h_1$. This is from observation and the evaluation of order of power-law function is $0.5 < n < 1$ [7]-[12], [35].

Let the capacitor be excited by a constant step input of $V_{BB}$ Volts, i.e. written as $v(t)=V_{BB}\left(u(t)\right)$, where $u(t)$ is unit step function. The Laplace transform of step input is following

$$V(s)=\mathcal{L}\left\{v(t)\right\}=\mathcal{L}\left\{V_{BB}\left(u(t)\right)\right\}=\frac{V_{BB}}{s} \tag{82}$$

and then taking Laplace transform of (25) power-law decay current using $\mathcal{L}\left\{t^m\right\}=m!s^{-(m+1)}$, [32] we write following expression for $I(s)=\mathcal{L}\left\{i(t)\right\}$

$$\begin{aligned} I(s)&=\mathcal{L}\left\{i(t)\right\}=\mathcal{L}\left\{K_nV_{BB}t^{-n}\right\} \\ &=K_nV_{BB}\left(\frac{(-n)!}{s^{-n+1}}\right) \end{aligned} \tag{83}$$

Using the formula for generalization of factorial i.e. $(\alpha-1)!=\Gamma(\alpha)$ [28], [29], [35], [38] we get the following expressions

$$\begin{aligned} I(s)&=K_n\frac{\Gamma(1-n)V_{BB}}{s^{1-n}} \\ &=K_n\frac{\Gamma(1-n)}{s^{-n}}\left(\frac{V_{BB}}{s}\right) \end{aligned} \tag{84}$$

We get Transfer function [32] of capacitor as following expression

$$\begin{aligned} G(s)=\frac{I(s)}{V(s)}&=\frac{K_n\dfrac{\Gamma(1-n)}{s^{-n}}\left(\dfrac{V_{BB}}{s}\right)}{\left(\dfrac{V_{BB}}{s}\right)} \\ &=K_n\left(\Gamma(1-n)\right)s^n=C_ns^n \qquad C_n=K_n\left(\Gamma(1-n)\right) \end{aligned} \tag{85}$$

This expression i.e. $G(s)$ is admittance expression in complex frequency ($s$) domain of a capacitor. Putting, $s=\mathrm{i}\omega$ in (85) we get $I(\omega)=\left(\cos\frac{n\pi}{2}+\mathrm{i}\sin\frac{n\pi}{2}\right)\omega^nC_nV(\omega)$. This means current leads voltage in fractional capacitor by angle $\frac{n\pi}{2}$. For $n=1$, i.e. for a classical geometrical ideal capacitor we have $I(\omega)=\mathrm{i}\omega C_1V(\omega)$, that is current leading voltage by angle of $90^0$. From here (85), we write impedance expression $Z(s)=V(s)/I(s)$ for fractional-capacitor as following

$$Z(s)=\frac{1}{C_ns^n}, \quad 0<n<1 \tag{86}$$

From the obtained expression (85) i.e. $I(s)=C_ns^n\left(V(s)\right)$ and by Laplace inversion by using the identity $\mathcal{L}^{-1}\left\{s^nF(s)\right\}={}_0D_t^n\left[f(t)\right]$ i.e. fractional derivative operation [6], [33], we get the constituent relation for capacity as following

$$i(t)=C_n\left({}_0D_t^n\left[v(t)\right]\right), \qquad 0<n<1 \tag{87}$$

The ‘fractional capacity’ $C_n$ is in unit of $\mathrm{Farad}/\sec^{1-n}$; [12], [35] which is constant given by $C_n = K_n\left(\Gamma(1-n)\right)$. This fractional derivative expression gives a new capacitor theory [12], [35] and we utilize this above formula to find characteristics of super-capacitors, variation of $n$ with current excitation, and energy discharged to energy stored [22]-[26]. Classically the expression of capacitor is $i(t) = C_1\left(D_t^{(1)}\left[v(t)\right]\right)$ i.e. with one-whole order (classical) derivative.

Curie-von Schweidler law gives a different approach for capacitor theory based on fractional calculus [12], [35], [37]. In experimental observations, we find that capacitor has fractional order impedance [7]-[12], [22]-[26], [35]. This section gives us the understanding that this empirical law i.e. Curie-von Schweidler law gives a relation of voltage and current of capacitor by using fractional derivative. We will derive this (87) by the new approach of the definition of charge in the subsequent section.

## Charge stored in a fractional capacitor using convolution integral of time varying capacity function due to Curie-von Schweidler relaxation current

For Curie-von Schweidler law we have relaxation current as noted earlier (81) empirically expressed as $i(t) = K_n V_{BB} t^{-n}$, $0 < n < 1$ for $t > 0$, i.e. when uncharged capacitor is applied with a step voltage $v(t) = V_{BB}\left(u(t)\right)$ at $t = 0$. This empirical expression of current relaxation gives a relation of incremental charge $\Delta q$ (or $\mathrm{d}q$ in infinitesimal small limit) when ‘pulse’ of a voltage of magnitude $V_{BB}$ is applied for a duration $\Delta t$ (or in infinitesimal small limit $\mathrm{d}t$) given by following expressions

$$\Delta q = \frac{K_n V_{BB} \Delta t}{t^n} \qquad \mathrm{d}q = \frac{K_n V_{BB} \mathrm{d}t}{t^n} \tag{88}$$

With this above (88) expression (or by $q(t) = \int_0^t \mathrm{d}q$ ) we write the charge accumulated for this power law decay current as following

$$\begin{aligned} q(t) &= \int_0^t \mathrm{d}q = \int_0^t \frac{K_n V_{BB} \mathrm{d}x}{x^n} \\ &= \frac{K_n V_{BB}}{(1-n)} t^{1-n}, \qquad 0 < n < 1 \qquad t > 0 \end{aligned} \tag{89}$$

From the expression in frequency domain (64) i.e. $C(s) = \left(s^{-1}I(s)\right)/\left(V(s)\right) = \left(Q(s)\right)/\left(V(s)\right)$ we have for $i(t) = K_n V_{BB} t^{-n}$ with $I(s) = K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}$, and $V(s) = V_{BB}/s$, this gives $C(s)$ as following

$$\begin{aligned} C(s) &= \frac{\left(s^{-1}I(s)\right)}{V(s)} = \frac{s^{-1}\left(K_n\left(\Gamma(1-n)\right)V_{BB}s^{n-1}\right)}{V_{BB}s^{-1}} \\ &= \frac{K_n\left(\Gamma(1-n)\right)}{s^{1-n}}; \qquad m! = \Gamma(1+m) \\ &= K_n \frac{(-n)!}{s^{1+(-n)}} \end{aligned} \tag{90}$$

Now doing inverse Laplace transform by using $\mathcal{L}^{-1}\left\{(m!)/s^{(1+m)}\right\}=t^{m}$ of above we get time dependent capacity function $c(t)$ as following

$$c(t)=K_{n}t^{-n}; \quad 0<n<1, \quad t>0 \tag{91}$$

Using the convolution integral with this time dependent capacity function (91) step voltage applied at time zero, i.e. we get following expression for charge stored as following

$$\begin{aligned} q(t)=\left(c(t)\right)*\left(v(t)\right)&=\int_{-\infty}^{t}\left(c(t-x)\right)\left(v(x)\right)\mathrm{d}x, \qquad c(x)=K_{n}x^{-n}, \quad v(x)=V_{BB}; \quad t>0 \\ &=\int_{0}^{t}K_{n}\left((t-x)^{-n}\right)\left(V_{BB}\right)\mathrm{d}x, \qquad 0<n<1 \\ &=-V_{BB}K_{n}\left.\frac{(t-x)^{1-n}}{1-n}\right|_{x=0}^{x=t}=\frac{V_{BB}K_{n}}{1-n}t^{1-n} \end{aligned} \tag{92}$$

The expression above (92) obtained expression $q(t)=\frac{V_{BB}K_{n}}{1-n}t^{1-n}$ obtained via our formula $q(t)=c(t)*v(t)$ is same as we got via $q(t)=\int_{0}^{t}\mathrm{d}q$ above in (89).

## Observations on breakdown mechanism of a fractional capacitor and loss tangent and comparison with earlier theory

We note here from (92) that for $0<n<1$, the charge $\lim_{t\uparrow\infty}q(t)=\infty$ when the capacity function is $c(t)=K_{n}t^{-n}$, following Curie-von Schweidler decay current. Whereas for a classical capacity function i.e. given as $c(t)=C_{1}\delta(t)$ , the charge at large times is $\lim_{t\uparrow\infty}q(t)=C_{1}V_{BB}$ (71). This observation i.e. $\lim_{t\uparrow\infty}q(t)=\infty$ in our derivation is with convolution formula i.e. $q(t)=c(t)*v(t)$ is in line with the observations in [12], [35], where the used expression for charge is $q(t)=\left(c(t)\right)\left(v(t)\right)$. This is the new idea of breakdown of capacitors due to accumulation of enough charge (electrostatic breakdown) at a constant voltage even though voltage is less than the breakdown limit of dielectric proposed in [12], [35].

In [12] and [35] the charge formula used is $c(t)=q(t)/v(t)$ and not via convolution approach that we discussed in this paper. And with this formula $c(t)=q(t)/v(t)$ in [12] and [35] gets the time dependent capacity function as following where the constant $h_{1}$ is used in Curie von-Schweidler relaxation current and $h_{1}\equiv\left(K_{n}\right)^{-1}$

$$c(t)=\frac{t^{1-n}}{h_{1}(1-n)}, \qquad t>0; \qquad 0<n<1 \tag{93}$$

The frequency domain representation for $c(t)$ obtained in [12] and [35] is following

$$C(s)=\frac{(1-n)!}{h_1(1-n)}s^n\,,\qquad 0<n<1,\quad s=\mathrm{i}\omega$$
$$C(\omega)=\left(\frac{\omega^n(1-n)!}{h_1(1-n)}\right)\left(\cos\tfrac{n\pi}{2}+\mathrm{i}\sin\tfrac{n\pi}{2}\right) \tag{94}$$

Here from (94) if we expresses loss tangent as $\tan\phi=\frac{\operatorname{Im}C(\omega)}{\operatorname{Re}C(\omega)}=\tan\left(\frac{n\pi}{2}\right)$, which is not correct, as the loss tangent is $\tan(1-n)\frac{\pi}{2}$ for a fractional capacitor. Therefore, in [12] and [35], the loss tangent is not calculated by the capacity function $c(t)$ (37), instead, phase difference $\psi$ is calculated between current $I(\omega)$ and voltage $V(\omega)$ by using admittance expression $G(s)$ (29) and then doing steady state (sinusoidal) analysis, and then writing loss tangent as $\tan\left(\frac{\pi}{2}-\psi\right)$, which is $\tan\left(\frac{(1-n)\pi}{2}\right)$.

This above expression (93) and (94) of [12] and [35] says that the time varying capacity function will be growing to infinity as time grows. Also in frequency domain, we will be getting infinite value at infinite frequency. This gives us notion of unrealistic property of capacity function, which is unstable.

Whereas we have from our new derivation (92) the following for a fractional capacitor

$$c(t)=K_n t^{-n}\,;\qquad t>0,\quad 0<n<1$$
$$C(s)=K_n\left(\Gamma(1-n)\right)s^{-(1-n)}\,;\qquad s=\mathrm{i}\omega \tag{95}$$
$$C(\omega)=K_n\left(\Gamma(1-n)\right)\omega^{-(1-n)}\left(\cos\tfrac{(1-n)\pi}{2}-\mathrm{i}\sin\tfrac{(1-n)\pi}{2}\right)$$

where the capacity function tends towards zero for large time and large frequency. From above (94) we get loss tangent as

$$\tan\phi=\frac{\operatorname{Im}\{C(\omega)\}}{\operatorname{Re}\{C(\omega)\}}=\tan\left(\tfrac{(1-n)\pi}{2}\right) \tag{96}$$

which is also as reported in [12], [35]; obtained differently than demonstrated in (95). However, [12] and [35] gives other expressions, same as that we will derive and report subsequently.

## Further derivations regarding fractional capacitor in conjugation to classical capacitor

Now we do the steps as we did for classical capacitor, from the obtained impedance relation of fractional capacitor i.e.

$$Z(s)=s^{-n}\frac{1}{C_n};\qquad 0<n<1\qquad C_n=K_n\Gamma(1-n) \tag{97}$$

with $C_n(s)=\mathcal{L}\{c_n(t)\}=C_n=K_n\left(\Gamma(1-n)\right)$ as obtained in earlier section, a constant in units of $\mathrm{Farad}/\sec^{1-n}$ We note that $C_n=K_n\left(\Gamma(1-n)\right)$ is in units of $\mathrm{Farad}/\sec^{1-n}$ ; a “fractional form” of unit [12], [35], defining a “fractional capacity” as constant in the frequency domain. Thus, we expect that in time domain the fractional capacity call it $c_n(t)$ be given by delta function at $t=0$ i.e. following

$$c_n(t)=\left(K_n\left(\Gamma(1-n)\right)\right)\left(\delta(t)\right)=C_n\delta(t) \tag{98}$$

The meaning of capacity function $c_n(t)$ in time domain is $c_n(t)=C_n\left(\delta(t)\right)$ i.e. an impulse of height $C_n$ (in units $\mathrm{Farad}/\sec^{1-n}$) at the time of application of voltage excitation (i.e. $t=0$). Whereas, in the frequency domain, the definition of fractional capacity is $C_n(s)=C_n$ i.e. $C_n(s)=\mathcal{L}\left\{C_n\delta(t)\right\}=C_n$ that is a constant (in unit of $\mathrm{Farad}/\sec^{1-n}$) value at all frequencies.

We say here that classical geometrical capacitor presents a $\mathrm{Farad}$ value as impulse function at the time of application of voltage stress, while the fractional capacitor presents a $\mathrm{Farad}/\sec^{1-n}$ value at the time of application of voltage.

From this (98) we write following steps, with $C_n(s)=\mathcal{L}\left\{c_n(t)\right\}$, $\mathcal{L}^{-1}\left\{s^{-n}F(s)\right\}={}_0\mathcal{I}_t^n f(t)$ defining fractional integration [6], [33] of fractional order $0<n<1$

$$
\begin{aligned}
C_n(s)&=\frac{s^{-n}I(s)}{V(s)}=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^n\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad 0<n<1\\
&=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\,{}_0\mathcal{I}_t^1\left[i(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}}=\frac{\mathcal{L}\left\{{}_0\mathcal{I}_t^{n-1}\int_0^t i(x)\mathrm{d}x\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad q(t)=\int_0^t i(x)\mathrm{d}x\\
&=\frac{\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}}{\mathcal{L}\left\{v(t)\right\}};\qquad {}_0\mathcal{I}_t^{n-1}f(t)={}_0D_t^{1-n}f(t)\\
\mathcal{L}&\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}=\left(\mathcal{L}\left\{v(t)\right\}\right)\left(\mathcal{L}\left\{c_n(t)\right\}\right)\\
\mathcal{L}&\left\{c_n(t)\right\}=\mathcal{L}\left\{{}_0D_t^{1-n}\left[q(t)\right]\right\}\left(\mathcal{L}\left\{v(t)\right\}\right)^{-1}\\
c_n(t)&=\left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1}
\end{aligned}
\tag{99}
$$

Therefore, we write following formulas for fractional capacitor in with conjugation to classical capacitor theory

$$
\begin{aligned}
c_n(t)&=\left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1};\qquad 0<n<1\\
{}_0D_t^{1-n}\left[q(t)\right]&=\left(c_n(t)\right)*\left(v(t)\right)\\
q(t)&={}_0D_t^{n-1}\left[\left(c_n(t)\right)*\left(v(t)\right)\right]\\
&=\left({}_0D_t^{n-1}\left[\left(c_n(t)\right)\right]\right)*\left(v(t)\right)\\
c(t)&={}_0D_t^{n-1}\left[\left(c_n(t)\right)\right]={}_0\mathcal{I}_t^{1-n}\left[\left(c_n(t)\right)\right]\\
q(t)&=\left(c(t)\right)*\left(v(t)\right)
\end{aligned}
\tag{100}
$$

Using obtained relation i.e. $q(t)=\frac{K_nV_{BB}}{1-n}t^{1-n}$ in (100), we get ${}_0D_t^{1-n}\left[q(t)\right]=\frac{K_nV_{BB}}{1-n}\left(\frac{\Gamma(1-n+1)}{\Gamma(1-n+1-1+n)}t^{1-n-1+n}\right)=K_nV_{BB}\left(\Gamma(1-n)\right)$ that is a constant function for $t>0$. This we have got by formula ${}_0D_x^{\upsilon}x^{\beta}=\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\upsilon)}x^{\beta-\upsilon}$ [6], [33], [38]. Thus, we write

${}_0D_t^{1-n}\left[q(t)\right] = K_n V_{BB}\left(\Gamma(1-n)\right)u(t)$ where $u(t)$ a unit-step function at $t=0$. We write the following for $c_n(t)$ as described in (100).

$$\begin{aligned} c_n(t) &= \left({}_0D_t^{1-n}\left[q(t)\right]\right)*\left(v(t)\right)^{-1} \\ &= \left(K_n V_{BB}\left(\Gamma(1-n)\right)\left(u(t)\right)\right)*\left(V_{BB}\left(u(t)\right)\right)^{-1} \\ &= K_n\left(\Gamma(1-n)\right)\delta(t) \end{aligned} \qquad (101)$$

We used identity i.e. $f * f^{-1} = \delta$ , the inverse relation in (101).

We consider the following relation (100) for time varying capacity function $c(t)$ from $c_n(t)$

$$c(t) = {}_0D_t^{(n-1)}\left[c_n(t)\right]; \quad t>0, \quad 0<n<1; \qquad {}_0D_t^{(n-1)} \equiv {}_0\mathcal{I}_t^{(1-n)} \qquad (102)$$

i.e. time varying capacity function defined as fractional integral of the order $1-n$ for the fractional capacity function i.e. $c_n(t)$ i.e. in units of $\text{Farad}/\sec^{1-n}$, which is constant in frequency domain as $C_n(\omega) = K_n\left(\Gamma(1-n)\right)$ i.e. a fractional capacitor. Using $c_n(t) = K_n\left(\Gamma(1-n)\right)\delta(t)$ as obtained above we write following

$$\begin{aligned} c(t) &= {}_0D^{n-1}\left[c_n(t)\right] \\ &= {}_0\mathcal{I}_t^{1-n}\left[\left(K_n\left(\Gamma(1-n)\right)\left(\delta(t)\right)\right)\right] \\ &= K_n\left(\Gamma(1-n)\right)\left({}_0\mathcal{I}_t^{1-n}\left[\delta(t)\right]\right); \qquad {}_0\mathcal{I}_x^{\beta}\left[\delta(x)\right] = \frac{1}{\Gamma(\beta)}x^{\beta-1} \\ &= K_n\left(\Gamma(1-n)\right)\left(\frac{t^{1-n-1}}{\Gamma(1-n)}\right) = K_n t^{-n} \end{aligned} \qquad (103)$$

The expression $c(t) = K_n t^{-n}$ we had obtained earlier too.

We note that the fractional integration operation in $c(t) = {}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]$ in (99), (100) is converting units in $\text{Farad}/\sec^{1-n}$ for $c_n(t)$ into units of $\text{Farad}$ for $c(t)$. This is because the fractional integration is integration with respect to fractional differential element $(\mathrm{d}t)^{1-n}$ i.e. ${}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right] = \int_0^t\left(c_n(t)\right)\mathrm{d}t^{(1-n)}$ .
Therefore, the capacity function $c(t) = K_n t^{-n}$ that we get for fractional capacitor is in units of $\text{Farad}$. This show for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in units of fractional units of Farads per second to the power a fractional number, to units of Farads, by formula $c(t) = {}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]$.

We obtain a general expression of charge $q(t)$ for Curie-von Schweidler relaxing current in a capacitor, that is having capacity function as $c(t) = K_n t^{-n}$ (103) when stressed with a time varying voltage $v(t)$ applied at $t=0$ is by convolution process as $q(t) = \left(K_n t^{-n}\right)*\left(v(t)\right)$ elaborated below

$$q(t) = (c(t)) * (v(t)) = \int_{-\infty}^{t} (c(t-x))(v(x))\mathrm{d}x$$
$$c(x) = K_n x^{-n} \qquad x > 0 \tag{104}$$

The convolution integral from (104), with $x = 0$ is following

$$q(t) = \int_0^t K_n \frac{v(x)}{(t-x)^n} \mathrm{d}x \tag{105}$$

As we did for geometrical capacity in previous section, we differentiate (105) this $q(t)$ to get $i(t)$ and write following

$$\begin{aligned} i(t) = q^{(1)}(t) &= \frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= K_n \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{v(x)}{(t-x)^n} \mathrm{d}x \end{aligned} \tag{106}$$

We apply formula of integration by parts i.e.

$$\int_0^t (f_1(x))(f_2(x))\mathrm{d}x = \left[ f_1(x) \int f_2(x)\mathrm{d}x \right]_{x=0}^{x=t} - \int_0^t \left( (f_1^{(1)}(x)) \int_0^t (f_2(x))\mathrm{d}x \right) \mathrm{d}x \tag{107}$$

to evaluate $\int_0^\infty \frac{v(x)}{(t-x)^n} \mathrm{d}x$ as written in following steps

$$\begin{aligned} \int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} &= \left[ v(x) \int \frac{\mathrm{d}x}{(t-x)^n} \right]_{x=0}^{x=t} - \int_0^t \left( v^{(1)}(x) \int \frac{\mathrm{d}x}{(t-x)^n} \right) \mathrm{d}x \\ &= v(x) \left( -\frac{(t-x)^{1-n}}{1-n} \right) \Bigg|_{x=0}^{x=t} - \int_0^t v^{(1)}(x) \left( \frac{(-1)(t-x)^{1-n}}{1-n} \right) \mathrm{d}x \\ &= \frac{v(0)}{1-n} t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n} (t-x)^{1-n} \mathrm{d}x \end{aligned} \tag{108}$$

Now we differentiate (108) and write the following

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{v(x)\mathrm{d}x}{(t-x)^n} &= \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{v(0)}{1-n} t^{1-n} - \int_0^t \frac{v^{(1)}(x)}{1-n} (t-x)^{1-n} \mathrm{d}x \right) \\ &= v(0) \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{t^{1-n}}{1-n} \right) - \int_0^t \frac{v^{(1)}(x)}{1-n} \frac{\mathrm{d}\left( (-1)(t-x)^{1-n} \right)}{\mathrm{d}t} \mathrm{d}x \\ &= \frac{v(0)}{t^n} - \int_0^t \frac{v^{(1)}(x)}{1-n} \left( (-1)(1-n)(t-x)^{1-n-1} \right) \mathrm{d}x \\ &= \frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)}{(t-x)^n} \mathrm{d}x \end{aligned} \tag{109}$$

This gives $i(t)$ as following

$$i(t) = K_n \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{v(x)}{(t-x)^n} \mathrm{d}x$$
$$= K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}; \quad K_n = \frac{C_n}{\Gamma(1-n)}; \qquad 0 < n < 1 \tag{110}$$

The expression (110) obtained with the formula $q(t) = \left(c(t)\right) * \left(v(t)\right)$, with $c(t) = K_n t^{-n}$ is consistent with obtained expression in [35].

For $v(t) = V_{BB}\left(u(t)\right)$ i.e. a constant step voltage applied at time $t = 0$ to a time varying capacity function given as $c(t) = K_n t^{-n}$ we have for $t > 0$, $v^{(1)}(t) = 0$ with $v(0) = V_{BB}$, the evaluation of $i(t)$ demonstrated below

$$i(t) = K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad v(0) = V_{BB}; \quad v^{(1)}(x) = 0, \quad x > 0$$
$$= K_n \frac{V_{BB}}{t^n} + K_n \int_0^t \frac{\left(0\right)\mathrm{d}x}{(t-x)^n} = K_n \frac{V_{BB}}{t^n} \tag{111}$$

to get $i(t) = K_n V_{BB} t^{-n}$, for $t > 0$. We recover the Curie-von Schweidler law. For a constant capacitor case with capacity function as $c(t) = C_1 \delta(t)$, we have the relation that we derived earlier i.e. $i(t) = C_1 v(0)\left(\delta(t)\right) + C_1\left(v^{(1)}(t)\right)$. The Figure-3 gives summary of our discussion about a constant capacity and a time varying capacity function.

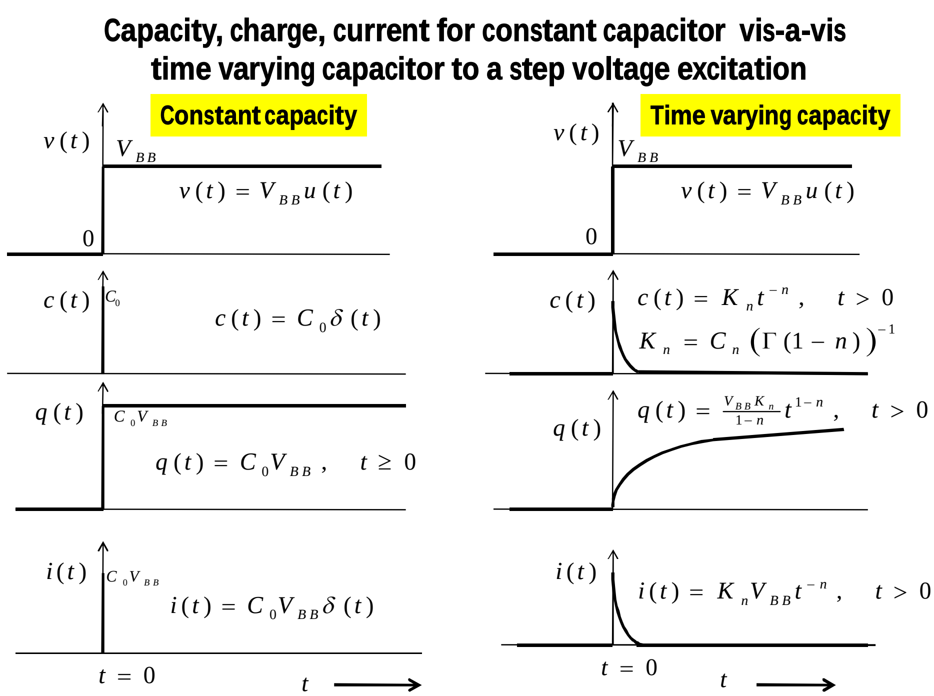


**Figure-3: Summary of discussion about constant capacity vis-à-vis time varying capacity**

## Appearance of fractional derivative in Fractional Capacitor

We have formed a time varying capacity function with a dielectric whose relaxation to a step voltage at $t=0$ of constant magnitude follows a power law given by empirical expression of Curie-von Schweidler law. We have got current and charge expression for any arbitrary voltage function $v(t)$ applied at $t=0$ in above section (110) as following

$$\begin{aligned} i(t) &= K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \\ q(t) &= \left(c(t)\right) * \left(v(t)\right) = \left(K_n t^{-n}\right) * \left(v(t)\right) \\ &= \int_0^t K_n \frac{v(x)}{(t-x)^n} \mathrm{d}x \end{aligned} \tag{112}$$

The fractional derivative for $0<n<1$ is defined as following two ways [6], [33], [38]

$$\begin{aligned} {}_0D_t^n\left[f(t)\right] &= \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x \\ &= \frac{1}{\Gamma(1-n)} \left( \frac{f(0)}{t^n} + \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} \mathrm{d}x \right); \quad t>0 \end{aligned} \tag{113}$$

The first definition is of Riemann-Liouville type i.e. ${}_0D_t^n\left[f(t)\right] = \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x, \quad 0<n<1$ and in the second expression's second term i.e. $\frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} \mathrm{d}x$ is Caputo fractional derivative i.e. ${}_0^CD_t^n\left[f(t)\right] = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f^{(1)}(x)}{(t-x)^n} \mathrm{d}x; \quad 0<n<1$. Therefore, we have ${}_0D_t^n\left[f(t)\right] = {}_0^CD_t^n\left[f(t)\right] + \frac{f(0)}{\Gamma(1-n)} t^{-n}$, i.e. relation between the two definitions of fractional derivative [6], [33], [38].

Integrating the expression ${}_0D_t^n\left[f(t)\right] = \frac{1}{\Gamma(1-n)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x$, once we write the following

$$\begin{aligned} {}_0\mathcal{I}_t^1\left({}_0D_t^n\left[f(t)\right]\right) &= \int_0^t \left( \frac{1}{\Gamma(1-n)} \left( \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x \right) \right) \mathrm{d}x \\ &= \frac{1}{\Gamma(1-n)} \int_0^t \left( \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x \right)^{(1)} \mathrm{d}x \\ &= \frac{1}{\Gamma(1-n)} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x \end{aligned} \tag{114}$$

We have used in (114) $\mathcal{I}_t^{(1)}\left(g^{(1)}(t)\right) \equiv g(t)$ . Using the composition rule [6], [33], [38] i.e. ${}_0\mathcal{I}_t^1\left({}_0D_t^n\left[f(t)\right]\right) = {}_0\mathcal{I}_t^{1-n}\left[f(t)\right] = {}_0D_t^{n-1}\left[f(t)\right]$, we re-write (114) as following

$$\begin{aligned} {}_0D_t^{n-1}\left[f(t)\right] &= {}_0\mathcal{I}_t^{1-n}\left[f(t)\right] = \frac{1}{\Gamma(1-n)} \int_0^t \frac{f(x)}{(t-x)^n} \mathrm{d}x; \quad 0<n<1; \quad 1-n=\upsilon \\ {}_0D_t^{-\upsilon}\left[f(t)\right] &= {}_0\mathcal{I}_t^{\upsilon}\left[f(t)\right] = \frac{1}{\Gamma(\upsilon)} \int_0^t \frac{f(x)}{(t-x)^{1-\upsilon}} \mathrm{d}x \end{aligned} \tag{115}$$

Using the definitions of fractional derivative (113), we apply to current expression (112) in following steps

$$
\begin{aligned}
i(t) &= K_n \frac{v(0)}{t^n} + K_n \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n} \qquad 0<n<1 \\
&= K_n\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\left(\frac{v(0)}{t^n} + \int_0^t \frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}\right)\right), \qquad K_n\left(\Gamma(1-n)\right) = C_n \\
&= C_n\left({}_0D_t^n\left[v(t)\right]\right), \qquad 0<n<1
\end{aligned}
\tag{116}
$$

Applying the expression for fractional integration (115) to the charge expression, we get following

$$
\begin{aligned}
q(t) &= \left(c(t)\right)*\left(v(t)\right) = \left(K_n t^{-n}\right)*\left(v(t)\right) \\
&= \int_0^t K_n \frac{v(x)}{(t-x)^n}\mathrm{d}x \\
&= K_n\left(\Gamma(1-n)\right)\left(\frac{1}{\Gamma(1-n)}\int_0^t \frac{v(x)}{(t-x)^n}\mathrm{d}x\right); \qquad K_n\left(\Gamma(1-n)\right) = C_n \\
&= C_n\left({}_0\mathcal{I}_t^{(1-n)}\left[v(t)\right]\right) \\
&= C_n\left({}_0D_t^{n-1}\left[v(t)\right]\right) \qquad t>0 \qquad 0<n<1
\end{aligned}
\tag{117}
$$

We apply a constant step voltage $v(t)=V_{BB}$ at $t=0$ to an uncharged fractional capacitor with capacity function $c(t)=\frac{C_n}{\Gamma(1-n)}t^{-n}$, applying the above formula (117) we get

$$
\begin{aligned}
q(t) &= C_n\left({}_0D_t^{n-1}\left[v(t)\right]\right) \qquad t>0 \qquad 0<n<1 \\
&= C_n\left({}_0D_t^{n-1}\left[V_{BB}\right]\right) \qquad {}_0D_t^{\alpha}\left[\mathrm{C}\right] = \mathrm{C}\frac{\Gamma(1)}{\Gamma(1-\alpha)}t^{-\alpha} \\
&= C_n V_{BB}\frac{\Gamma(1)}{\Gamma\left(1+(1-n)\right)}t^{1-n} = \frac{C_n}{(1-n)\left(\Gamma(1-n)\right)}V_{BB}t^{1-n}, \qquad K_n\left(\Gamma(1-n)\right) = C_n \\
&= \frac{K_n V_{BB}}{(1-n)}t^{1-n}
\end{aligned}
\tag{118}
$$

The same expression we showed earlier and in Figure-3.

## Integrated capacity defined from capacity function of a capacitor and explanation vis-à-vis a pitcher holding water

We take example of a pitcher, which holds water, of volume V. Let the pitcher be made of metal walls so that there are no pores. It is fully filled with water from empty state, hence once full it has no capacity left. This is like ideal capacitor, where the volume of water V remains fixed as constant after filling, with no left over capacity. Thus, an ideal capacitor described by capacity function $c(t)=C_1\delta(t)$, after it is charged at $t = 0$ with a constant voltage holds the constant charge $q(t) = C_1 V_{BB}$ at times $t > 0$ and at time, $t > 0$ this capacitor has zero capacity function, i.e. $c(t) = 0$ that is like no more capacity left to fill, like pitcher. Thus, we have maximum charge holding capacity in this case as $q_{\max} = \lim_{t\uparrow\infty} q(t) = C_1 V_{BB}$ . Therefore we can say the capacity function $c(t)$ at $t > 0$ indicates the left over capacity to fill from maximum charge say $q_{\max} = \lim_{t\uparrow\infty} q(t)$ .

Now let the walls of the pitcher be made of clay with an in-finitely porous material. As the pitcher gets the water volume V the pitcher walls too starts seepage of water into its pores. Thus, extra water keeps entering pores of the porous pitcher walls. This water filling process in the porous walls we call fractional capacity. Now due to infinite nature of these pores, we have a situation, that infinite amount of water keeps seeping into the walls. This is analogous to charging porous walls with water as charging a fractional capacitor where we derived $q_{\max} = \lim_{t\uparrow\infty} q(t) = \infty$ . Yet as we go on with charging process, the remaining capacity of holding the charge from maximum value (in this case infinity) keeps on decreasing but will never be going to zero, and thus we got the capacity function for a fractional capacitor as, $c(t) = K_n t^{-n}$ where $\lim_{t\uparrow\infty} c(t) = 0$ . The charge of a fractional capacitor as in the case of filling the porous walls gets the form that we derived as in derived i.e. $q(t) = \frac{K_n V_{BB}}{(1-n)} t^{1-n}$ for $t > 0$ increasing with time. This phenomena leads to electrostatic break down of capacitors [12], [35], even if the constant voltage $V_{BB}$ is lower than dielectric breakdown limit. Thus a fractional capacitor with $c(t) = K_n t^{-n}$ will break down when the electrostatic forces are enough due to large accumulation of charge at large times, even if $V_{BB}$ is lower than dielectric breakdown limit. While the ideal geometric capacitor with $c(t) = C_1 \delta(t)$ will have $\lim_{t\uparrow\infty} q(t) = C_1 V_{BB}$ and will never breakdown when $V_{BB}$ is less than dielectric breakdown limit.

We define integral capacity as following from the capacity function $c(t)$

$$c_{\text{int}}(t) = \int_0^t c(x)\mathrm{d}x; \quad t > 0 \tag{118}$$

The above (118) in integration of the capacity function w.r.t time from time of application of voltage excitation (in our case is $t = 0$). Thus for a classical capacitor with capacity function defined as $c(t) = C_1 \delta(t)$ we get integrated capacity as

$$c_{\text{int}}(t) = \int_0^t \left(C_1 \delta(x)\right)\mathrm{d}x = C_1 , \qquad t > 0 \tag{119}$$

We observe $\lim_{t\uparrow\infty} c_{\text{int}}(t) = C_1$ a constant value. This integrated capacity is what is discussed in classical theory that we derived from capacity function. Now for the case of fractional capacitor where the capacity function is $c(t) = K_n t^{-n}$ , the integrated capacity is

$$c_{\text{int}}(t) = \int_0^t K_n x^{-n}\mathrm{d}x = \frac{K_n}{(1-n)} t^{1-n}; \qquad t > 0 \tag{120}$$

This is same as (120) used in [12], [35]. We note in (120) $\lim_{t\uparrow\infty} c_{\text{int}}(t) = \infty$.

Thus, the term 'integrated capacity' $c_{\text{int}}(t)$ of capacitor is analogous to 'total' water holding capacity of pitcher. The total water holding capacity of pitcher with metal walls is constant is equivalent to classical capacitor case, while the total water holding capacity of walls of porous pitcher is infinity is equivalent to the fractional capacitor case. We mention here the expressions for $C_{\text{int}}(\omega) = \mathcal{L}\{c_{\text{int}}(t)\}\big|_{s=\text{i}\omega}$ cannot be used to determine the loss tangent, while from capacity function with $C(\omega) = \mathcal{L}\{c(t)\}\big|_{s=\text{i}\omega}$ is used to determine loss tangent value.

## Experimental results showing fractional capacitor

The Curie-von Schweidler empirical law of power law relaxation, i.e. $i(t) \propto t^{-n}$ states that $0 < n < 1$ .This is validated via experiments on dielectric relaxations. A 100V step input applied to a completely discharged capacitor of $0.47\mu\text{F}$ having metalized paper dielectric, and the current decay is recorded with time. The graphs of log-log plot i.e. $\log\left(i(t)\right)$ vs. $\log(t)$ show a straight line of average slope $-0.86$ [12], [22]-[26]. This experiment indicates a Curie-von Schweidler law, with $i(t) \propto t^{-n}$ , having $n = 0.86$. The exponent $n$ is in the range of $0.85 < n < 1$ in several dielectric relaxation experiments [12], [22]-[26]. The experiments with super-capacitors [7], [8], show range as $0.5 < n < 1$. A very low value of exponent $n$ is found in relaxation of Laponite studies averagely $n = 0.09$ [27]. In this Laponite study [27] though the exponent $n$ was obtained on 'self-discharge' curves with various charging time history-showing memory effect, the expression obtained for self-discharge decay of voltage assumes fractional capacity-that in turn assumes Curie-von Schweidler law as current relaxation function.

## Summary

In the tabular form (Table-3), we present the various concepts (formulas) that we discussed with this new approach of charge store in classical capacitor and fractional capacitor.

| S. No. | Parameter | Classical Geometrical (Constant) Capacity ($n=1$) | Fractional Capacity $0 < n < 1$ |
|---|---|---|---|
| 1 | Relaxing current to constant step voltage $V_{BB}$ applied to an un-charge capacitor at $t = 0$ | $i(t) = C_1 V_{BB}\delta(t)$<br>$C_1 \equiv \text{Farad}$ | $i(t) = K_n V_{BB} t^{-n} = \frac{C_n V_{BB}}{\Gamma(1-n)} t^{-n}, \quad t > 0$<br>$K_n \Gamma(1-n) = C_n, \quad C_n \equiv \text{Farad}/\text{sec}^{1-n}$ |

| | | | |
|---|---|---|---|
| 2 | Relaxing Current in frequency domain | $I(s)=C_1V_{BB}$<br>$I(\omega)=C_1V_{BB}$ | $I(s)=K_nV_{BB}\left(\Gamma(1-n)\right)s^{n-1}=C_nV_{BB}s^{n-1}$<br>$I(\omega)=\frac{C_nV_{BB}}{\omega^{1-n}}\left(\cos\left(\frac{(1-n)\pi}{2}\right)-\mathrm{i}\sin\left(\frac{(1-n)\pi}{2}\right)\right)$<br>$K_n\Gamma(1-n)=C_n\,, \quad C_n\equiv \mathrm{Farad/sec}^{1-n}$ |
| 3 | Capacity function in time domain and frequency domain with loss tangent | $c(t)=C_1\delta(t) \quad \mathrm{Farad}$<br>$C(s)=C_1 \quad \mathrm{Farad}$<br>$C(\omega)=C_1-\mathrm{i}(0)$<br>$\mathrm{Loss\text{-}tangent} \quad \tan\phi=0$ | $c_n(t)=K_n\left(\Gamma(1-n)\right)\delta(t)=C_n\delta(t) \quad \mathrm{Farad/sec}^{1-n}$<br>$C_n(s)=C_n=K_n\left(\Gamma(1-n)\right) \quad \mathrm{Farad/sec}^{1-n}$<br>$c(t)={}_0D_t^{n-1}\left[c_n(t)\right]={}_0\mathcal{I}_t^{1-n}\left[c_n(t)\right]; \quad \mathrm{Farad}$<br>$c(t)=K_nt^{-n}=\frac{C_n}{\Gamma(1-n)}t^{-n} \quad \mathrm{Farad}$<br>$C(s)=K_n\left(\Gamma(1-n)\right)s^{n-1}=C_ns^{n-1} \quad \mathrm{Farad}$<br>$C(\omega)=\frac{C_n}{\omega^{1-n}}\left(\cos\left(\frac{(1-n)\pi}{2}\right)-\mathrm{i}\sin\left(\frac{(1-n)\pi}{2}\right)\right)$<br>$\mathrm{Loss\text{-}tangent} \quad \tan\phi=\tan\left(\frac{(1-n)\pi}{2}\right)$<br>$K_n\Gamma(1-n)=C_n\,, \quad C_n\equiv \mathrm{Farad/sec}^{1-n}$ |
| 4 | Charge function to a constant step voltage $V_{BB}$ applied at $t=0$ | $q(t)=c(t)*v(t)$<br>$=C_1V_{BB}\,; \quad t\geq 0$ | $q(t)=c(t)*v(t)$<br>$=\frac{K_nV_{BB}}{1-n}t^{1-n}=\frac{C_nV_{BB}}{(1-n)\Gamma(1-n)}t^{1-n}\,; \quad t>0$<br>$K_n\Gamma(1-n)=C_n\,, \quad \mathrm{Farad/sec}^{1-n}$ |
| 5 | Current to an arbitrary voltage function $v(t)$ applied to uncharged capacitor at $t=0$ | $i(t)=C_1v(0)\delta(t)+C_1\frac{\mathrm{d}v(t)}{\mathrm{d}t}$ | $i(t)=K_nv(0)t^{-n}+K_n\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}$<br>$=\frac{C_nv(0)}{\Gamma(1-n)}t^{-n}+\frac{C_n}{\Gamma(1-n)}\int_0^t\frac{v^{(1)}(x)\mathrm{d}x}{(t-x)^n}$<br>$K_n\Gamma(1-n)=C_n \quad \mathrm{Farad/sec}^{1-n}$ |
| 6 | Current voltage relation | $i(t)=C_1\left({}_0D_t^1v(t)\right)$<br>$C_1\equiv\mathrm{Farad}$ | $i(t)=C_n\left({}_0D_t^nv(t)\right)$<br>$K_n\Gamma(1-n)=C_n \quad \mathrm{Farad/sec}^{1-n}$ |

| | | | |
|---|---|---|---|
| 7 | Charge voltage relation for arbitrary voltage | $q(t)=\left(c(t)\right)*\left(v(t)\right)$ $=\left(C_1\delta(t)\right)*v(t)$ $=C_1 v(t);\quad t\geq 0$ $C_1\equiv \text{Farad}$ | $q(t)=\left(c(t)\right)*\left(v(t)\right)$ $=\left(K_n t^{-n}\right)*\left(v(t)\right)$ $=C_n\left({}_0D_t^{n-1}v(t)\right)=C_n\left({}_0\mathcal{I}_t^{1-n}v(t)\right),\quad t>0$ $K_n\Gamma(1-n)=C_n\qquad \text{Farad / sec}^{1-n}$ |

**Table-3: Summary of the discussions regarding formulas for classical capacitor and fractional capacitor**

## Conclusions

This note gave systematic approach to extract a histogram, describing the distribution function for relaxation rates from a relaxing function of time, i.e. $i(t)\sim t^{-n}\,,\quad 0<n<1$. The empirical law Curie-von Schweidler law, which is a non-Debye relaxation, (which is also stated to be universal law of dielectric relaxation of current), when stressed with a constant voltage gave rate relaxation function as Zipf's power law distribution, the histogram we found out to be of a function of $H_\lambda(\lambda)\sim\lambda^{n-1},\quad 0<n<1$. We infer the simultaneous multi-body relaxations have a distribution i.e. right-skewed, with large number of relaxations with lower value of rate (slow rates), with long tail of small number of relaxations with faster relaxation rates. We noted that the possibility of having Zipfian distribution arises due to very-very large ratio maximum to minimum of spreads in the relaxation rates $\lambda$ 's, and possibility of connected exponential distribution of many body simultaneous relaxations. The method we obtained for getting rate distributions of relaxation rates via formation of Laplace integral when extended for finding distribution of time constants, though mathematically correct yet gave contrary physical interpretation. Thus, we carried out the entire discussion with rate distribution functions and not with the time constant distribution function i.e. $H_\tau(\tau)$. We also showed that Curie-von Schweidler law gives constituent expression of current and voltage of capacitor via use of fractional derivative, i.e. $i(t)\propto {}_0D_t^n\left[v(t)\right]$, unlike classical capacitor relation i.e. $i(t)\propto D^{(1)}\left[v(t)\right]$. This we verified by using obtained by Zipf's distribution as power law for Curie-von Schwidler current relaxation law, assuming the scale dependence equivalent relaxation rate in the classical charging equation of capacitor with scale of relaxation varying from zero to infinity; i.e. $\lambda_{eq}\sim\lambda^{1/n}$. We also related the Curie-von Schweidler relaxation law gives a time varying rate i.e. $\lambda(t)=nt^{-1}$, indicating that the relaxation starts with very-very high rate, and becomes slower and slower with elapse of time. We have in this study formulated interesting integral representations for $i(t)=t^{-n}$

$$t^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty}\left(y^{(n-1)}\right)e^{-yt}\mathrm{d}y$$

$$t^{-n} = \frac{1}{\Gamma(n)}\int_0^{\infty} y^{-(n+1)}e^{-t/y}\mathrm{d}y$$

$$t^{-n} = 2\frac{\Gamma(1-n)}{\pi}\cos\left(\tfrac{(1-n)\pi}{2}\right)\int_0^{\infty} y^{(n-1)}\cos(yt)\left(\mathrm{d}y\right)$$

$$t^{-n} = \frac{1}{n\left(\Gamma(n)\right)}\int_0^{\infty} e^{-\left(y^{(1/n)}t\right)}\mathrm{d}y$$

The note gives a possible foundation for further studies in obtaining the rate relaxation distribution functions for other non-Debye type relaxation functions, and new type of explanation regarding reasons of Zipfian distributions.

In this note we discussed that charge stored in capacitor, as a function of time is not the usual multiplication operation of capacity and voltage; instead, the charge is convolution integral of capacity function and voltage stressed across capacitor. However, the charge as a function of frequency is multiplication operation of frequency domain functions of capacity and voltage. We say that capacity not the usual ratio of charge to voltage in time domain, but it is given as convolution expression. We discussed in this note that for a fractional capacitor, the charge goes to infinity for large times, when the fractional capacitor is placed on a constant voltage; whereas, for a classical capacitor function the charge at large time is a constant value. This observation in our derivation is with convolution formula defining the charge stored in capacitor and is consistence with other fractional capacitor models. This new concept is in line with the observation of charge stored, relaxation current in form of impulse function for ideal geometrical capacitor of constant capacity, and power-law decay current that is given by universal dielectric relaxation law called as Curie von-Schweidler law, when an uncharged capacitor is impressed with a constant voltage stress. This universal dielectric relaxation law gives rise to fractional derivative relating voltage stress and relaxation current that is formulation of 'fractional capacitor'. We also showed for a fractional capacitor by the use of time varying capacity function we can convert the fractional capacity constant that is in fractional units of Farads per second to the power a fractional number, to units of Farads. A 'fractional capacitor' we discussed is with this new concept of redefining the charge store definition i.e. via this convolution integral approach, and we obtain the loss tangent value, from the described capacity function.

# Appendix

## A.1 Berberan-Santo Method

Our aim is evaluate Laplace inverse $H_\lambda(\lambda) = \mathcal{L}^{-1}\{i(t)\}$, which is given as Laplace inversion integral expression i.e.

$$H_\lambda(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} \left(i(t)\right) e^{t\lambda} \mathrm{d}\lambda$$

Here we describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. First is change of variable i.e. from 'real time variable' to 'complex time variable' as $t = x + \mathrm{i}y$; with $\mathrm{i} = \sqrt{-1}$. Here the real part i.e. $x$ is constant as a vertical line calls it $x = x_0$ a constant. The variable $y$ is different from imaginary part of frequency in the usual Laplace variable $\omega$ in complex frequency parameter $s$ .With this change we have the following expression for inverse Laplace transform

$$\begin{aligned}
H_\lambda(\lambda) &= \frac{1}{2\pi \mathrm{i}} \int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty} \left(i(t)\right) e^{\lambda t} \mathrm{d}t; \qquad t \equiv x_0 + \mathrm{i}y \\
&= \frac{1}{2\pi \mathrm{i}} \int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty} \left(i(x_0+\mathrm{i}y)\right)\left(e^{\lambda(x_0+\mathrm{i}y)}\right)\left(\mathrm{d}(x_0+\mathrm{i}y)\right); \qquad \mathrm{d}x = 0 \\
&= \frac{1}{2\pi \mathrm{i}} \int_{x_0-\mathrm{i}\infty}^{x_0+\mathrm{i}\infty} \left(i(x_0+\mathrm{i}y)\right)\left(e^{\lambda x_0} e^{\mathrm{i}\lambda y}\right)\left(\mathrm{i}\mathrm{d}y\right) \\
&= \frac{e^{x_0\lambda}}{2\pi} \int_{-\infty}^{+\infty} \left(i(x_0+\mathrm{i}y)\right) e^{\mathrm{i}\lambda y} \mathrm{d}y
\end{aligned}$$

Writing $e^{\mathrm{i}\lambda y} = \cos \lambda y + \mathrm{i} \sin \lambda y$ we get the following form

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{2\pi}\left( \int_{-\infty}^{+\infty} \left(i(x_0+\mathrm{i}y)\right)\cos(\lambda y)\mathrm{d}y + \mathrm{i}\int_{-\infty}^{+\infty} \left(i(x_0+\mathrm{i}y)\right)\sin(\lambda y)\mathrm{d}y \right)$$

Write $i(x_0+\mathrm{i}y) = \mathrm{Re}\{i(x_0+\mathrm{i}y)\} + \mathrm{i}\,\mathrm{Im}\{i(x_0+\mathrm{i}y)\}$ and place in above expression to get the following expression

$$\begin{aligned}
H_\lambda(\lambda) &= \frac{e^{x_0\lambda}}{2\pi}\left( \int_{-\infty}^{+\infty} \left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\left(\cos(\lambda y)\right)\right) - \left(\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\left(\sin(\lambda y)\right)\right)\mathrm{d}y \right) \\
&+ \mathrm{i}\frac{e^{x_0\lambda}}{2\pi}\left( \int_{-\infty}^{+\infty} \left(\mathrm{Re}\{i(x_0+\mathrm{i}y)\}\left(\cos(\lambda y)\right)\right) + \left(\mathrm{Im}\{i(x_0+\mathrm{i}y)\}\left(\sin(\lambda y)\right)\right)\mathrm{d}y \right)
\end{aligned}$$

Given that $H_\lambda(\lambda)$ is a real function, we get the following (i.e. equating the imaginary part to zero), we write the following

$$\left( \int_{-\infty}^{+\infty} \left( \mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} \left( \cos(\lambda y) \right) + \mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} \left( \sin(\lambda y) \right) \right) \mathrm{d}y \right) = 0$$

Thus the above expression for $H_\lambda(\lambda)$ reduces to following (i.e. considering only real part)

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{2\pi} \left( \int_{-\infty}^{+\infty} \left( \mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} \cos(\lambda y) - \mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} \sin(\lambda y) \right) \mathrm{d}y \right)$$

But we have $i(t) = \int_0^\infty \left( H_\lambda(\lambda) \right) e^{-\lambda t} \mathrm{d}\lambda$ and by putting $t = x_0 + \mathrm{i}y$ we get following

$$i(x_0 + \mathrm{i}y) = \int_0^\infty \left( H_\lambda(\lambda) \right) e^{-\lambda(x_0 + \mathrm{i}y)} \mathrm{d}\lambda$$

$$= \int_0^\infty e^{-x_0\lambda} \left( H_\lambda(\lambda) \right) \cos(\lambda y) \mathrm{d}y - \mathrm{i} \int_0^\infty e^{-x_0\lambda} \left( H_\lambda(\lambda) \right) \sin(\lambda y) \mathrm{d}y$$

$$\mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} = \int_0^\infty e^{-x_0\lambda} \left( H_\lambda(\lambda) \right) \cos(\lambda y) \mathrm{d}\lambda$$

$$\mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} = -\int_0^\infty e^{-x_0\lambda} \left( H_\lambda(\lambda) \right) \sin(\lambda y) \mathrm{d}y$$

Using this in obtained expression for $H_\lambda(\lambda)$, we observe that integrand is even function for $\lambda > 0$ therefore we re-write the formula as

$$H_\lambda(\lambda) = \frac{e^{x_0\lambda}}{\pi} \int_0^\infty \left( \mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} \cos(\lambda y) - \mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} \sin(\lambda y) \right) \mathrm{d}y$$

Rewriting the above obtained expression i.e. in following form

$$\frac{\pi}{e^{x_0\lambda}} \left( H_\lambda(\lambda) \right) = \int_0^\infty \left( \mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} \cos(\lambda y) - \mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} \sin(\lambda y) \right) \mathrm{d}y$$

and the relation obtained earlier i.e.

$$0 = \int_0^{+\infty} \left( \mathrm{Re}\left\{ i(x_0 + \mathrm{i}y) \right\} \cos(\lambda y) + \mathrm{Im}\left\{ i(x_0 + \mathrm{i}y) \right\} \sin(\lambda y) \right) \mathrm{d}y$$

and then adding and subtracting these above two expressions we get following

$$H_\lambda(\lambda) = \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0 + \mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y$$

$$H_\lambda(\lambda) = -\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Im}\{i(x_0 + \mathrm{i}y)\}\sin(\lambda y)\mathrm{d}y$$

Write in polar form as described below

$$i(x_0 + \mathrm{i}y) = \rho(y)e^{\mathrm{i}\theta(y)} = \rho(y)\left(\cos\left(\theta(y)\right) + \mathrm{i}\sin\left(\theta(y)\right)\right) \qquad \rho(y) = \left|i(x_0 + \mathrm{i}y)\right| \qquad \theta(y) = \angle i(x_0 + \mathrm{i}y)$$

to get following formulas

$$\begin{aligned} H_\lambda(\lambda) &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\{i(x_0 + \mathrm{i}y)\}\cos(\lambda y) - \mathrm{Im}\{i(x_0 + \mathrm{i}y)\}\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\rho(y)\left(\cos\theta(y)\right)\cos(\lambda y) - \rho(y)\left(\sin\theta(y)\right)\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\rho(y)\right)\left(\cos\left(\lambda y + \theta(y)\right)\right)\mathrm{d}y \end{aligned}$$

$$\begin{aligned} H_\lambda(\lambda) &= \frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \rho(y)\left(\cos\left(\theta(y)\right)\right)\cos(\lambda y)\mathrm{d}y \\ &= -\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \rho(y)\left(\sin\left(\theta(y)\right)\right)\sin(\lambda y)\mathrm{d}y \end{aligned}$$

## A.2 Few examples of Laplace inversion without contour integrations-by Berberan Santo Method

**a.** Consider a very simple case of decay function $i(t) = (t-a)^{-1}$ and converted to complex time as follows

$$i(t) = \frac{1}{t-a}; \qquad i(x_0 + \mathrm{i}y) = \frac{1}{(x_0 - a) + \mathrm{i}y}$$

We know from standard Laplace pair that is $\mathcal{L}^{-1}(s \pm a)^{-1} = e^{\mp at}$. Thus, for $i(t) = (t-a)^{-1}$ we should get via inverse Laplace the rate distribution function as $H_\lambda(\lambda) = e^{a\lambda}$. The application of the Berberan-Santro formula with $x_0 > a$ yields the following

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1}{(x_0-a)+\mathrm{i}y}\right\}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \frac{(x_0-a)}{(x_0-a)^2+y^2}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y=e^{a\lambda}$$

Here we say that $e^{a\lambda}$ has integral representation as $e^{a\lambda}=\frac{2(x_0-a)e^{x_0\lambda}}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{(x_0-a)^2+y^2}\mathrm{d}y$.

Particularly for $a=-1$, we have $i(t)=(t+1)^{-1}$. The condition $x_0>-1$ enables us to choose $x_0=0$ we get following integral representation for $e^{-\lambda}$ which is also rate distribution function $H_\lambda(\lambda)$ is following

$$H_\lambda(\lambda)=e^{-\lambda}=\frac{2}{\pi}\int_0^\infty \frac{\cos(\lambda y)}{1+y^2}\mathrm{d}y$$

**b.** Let the decay function be $i(t)=t(t^2+1)^{-1}$ and its complex time representation as follows

$$i(t)=\frac{t}{t^2+1};\qquad i(x_0+\mathrm{i}y)=\frac{x_0+\mathrm{i}y}{(x_0+\mathrm{i}y)^2+1}$$

Well if $F(s)=(s)/(s^2+1)$ its inverse is $\cos(t)$. Thus we should have the distribution function $H_\lambda(\lambda)=\cos(\lambda)$. With use of above formula with setting $x_0=1$

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \mathrm{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1+\mathrm{i}y}{(1+\mathrm{i}y)^2+1}\right\}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{1+\mathrm{i}y}{(2-y^2)+2\mathrm{i}y}\right\}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \mathrm{Re}\left\{\tfrac{(1+\mathrm{i}y)(2-y^2-2\mathrm{i}y)}{(2-y^2)^2+4y^2}\right\}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2e^{\lambda}}{\pi}\int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4}\mathrm{d}y=\cos(\lambda)$$

Thus integral representation of $\cos(\lambda)$ which is also rate distribution function $H_\lambda(\lambda)$ for relaxation function $i(t)=t(t^2+1)^{-1}$ is $\cos(\lambda)=\frac{2e^{\lambda}}{\pi}\int_0^\infty \frac{(y^2+2)\cos(\lambda y)}{y^4+4}\mathrm{d}y$.

These definite integrals as obtained above are difficult to solve in closed form, even in simple cases. But, they allow obtaining results that are not so direct with contour integration, and are suited for numerical integration. This method gives integral representations of various functions as we demonstrated above- can be useful for plotting the histogram $H_\lambda(\lambda)$ for any relaxation function $i(t)$.

In reality of decay functions, we can take $x_0 = 0$; as decay function will not expected to have singularity at time $t > 0$.

**c.** For a case of 'exponential-decay' i.e. $i(t) = e^{-t/\tau_0}$, obviously this function has only one decay rate i.e. $\lambda_0 = \frac{1}{\tau_0}$. As per procedure discussed above we do Laplace inversion by taking complex time with $t = 0 + \mathrm{i}y$; making it following

$$\begin{aligned} & i(t) = e^{-t/\tau_0} \qquad t = \mathrm{i}y \\ & i(\mathrm{i}y) = e^{-\mathrm{i}\left(y/\tau_0\right)} = \cos\left(\tfrac{y}{\tau_0}\right) - \mathrm{i}\sin\left(\tfrac{y}{\tau_0}\right) \\ & \mathrm{Re}\left\{i(\mathrm{i}y)\right\} = \cos\left(\tfrac{y}{\tau_0}\right) \end{aligned}$$

$$\begin{aligned} H_\lambda(\lambda) &= \frac{e^{x_0\lambda}}{\pi}\int_0^\infty \left(\mathrm{Re}\left\{i(x_0+\mathrm{i}y)\right\}\cos(\lambda y) - \mathrm{Im}\left\{i(x_0+\mathrm{i}y)\right\}\sin(\lambda y)\right)\mathrm{d}y \qquad x_0 = 0 \\ &= \frac{1}{\pi}\int_0^\infty \left(\mathrm{Re}\left\{i(\mathrm{i}y)\right\}\cos(\lambda y) - \mathrm{Im}\left\{i(\mathrm{i}y)\right\}\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{1}{\pi}\int_0^\infty \left(\cos\left(\tfrac{y}{\tau_0}\right)\cos(\lambda y) + \sin\left(\tfrac{y}{\tau_0}\right)\sin(\lambda y)\right)\mathrm{d}y \\ &= \frac{1}{\pi}\int_0^\infty \cos\left(y\left(\lambda - \tfrac{1}{\tau_0}\right)\right)\mathrm{d}y = \delta\left(\lambda - \tfrac{1}{\tau_0}\right); \qquad \tau_0^{-1} = \lambda_0 \\ H_\lambda(\lambda) &= \delta(\lambda - \lambda_0) \end{aligned}$$

The above relation is taken from Fourier Integral as described by Joseph Fourier [16], [28], [29] as following

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\, f(\alpha)\int_{-\infty}^{\infty}\mathrm{d}p\cos(px - p\alpha)$$

Which tantamount to introduction of Dirac Delta function as [28], [29]

$$\delta(x-\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p\cos(px - p\alpha)$$

**d.** For a decay function of stretched exponential type i.e.

$$i(t) = e^{-(t/\tau_0)^\beta}$$

in complex time variable $t = \mathrm{i}y$ we get the polar form as following

$$i(\mathrm{i}y)=e^{-(\mathrm{i}y/\tau_0)^\beta}=e^{-(y/\tau_0)^\beta(\mathrm{i})^\beta}$$
$$=e^{-(y/\tau_0)^\beta(\cos(\beta\pi/2)+\mathrm{i}\sin(\beta\pi/2))}$$
$$=e^{\left(-(y/\tau_0)^\beta\cos(\beta\pi/2)\right)}e^{\left(-\mathrm{i}(y/\tau_0)^\beta\sin(\beta\pi/2)\right)};\qquad \rho(y)=e^{\left(-(y/\tau_0)^\beta\cos(\beta\pi/2)\right)},\quad \theta(y)=\left(\frac{y}{\tau_0}\right)^\beta\sin\left(\tfrac{\beta\pi}{2}\right)$$
$$=\left(\rho(y)\right)e^{\mathrm{i}(\theta(y))}$$

Therefore we write following

$$H_\lambda(\lambda)=\frac{e^{x_0\lambda}}{\pi}\int_0^\infty\rho(y)\cos\left(\lambda y+\theta(y)\right)\mathrm{d}y$$
$$=\frac{1}{\pi}\int_0^\infty e^{\left(-(y/\tau_0)^\beta\cos(\beta\pi/2)\right)}\left(\cos\left(\lambda y-\left(\tfrac{y}{\tau_0}\right)^\beta\sin\left(\tfrac{\beta\pi}{2}\right)\right)\right)\mathrm{d}y$$

Doing change of variable $u=y/\tau_0$ and $\tau_0\mathrm{d}u=\mathrm{d}y$ we obtain the following

$$H_\lambda(\lambda)=\frac{\tau_0}{\pi}\int_0^\infty\left(e^{\left(-u^\beta\cos(\beta\pi/2)\right)}\cos\left(\lambda\tau_0u-u^\beta\sin\left(\tfrac{\beta\pi}{2}\right)\right)\right)\mathrm{d}u$$

Using other formulas we will get

$$H_\lambda(\lambda)=\frac{2\tau_0}{\pi}\int_0^\infty(\mathrm{d}u)e^{\left(-u^\beta\cos(\beta\pi/2)\right)}\cos\left(u^\beta\sin\left(\tfrac{\beta\pi}{2}\right)\right)\cos(\lambda\tau_0u)$$
$$H_\lambda(\lambda)=\frac{2\tau_0}{\pi}\int_0^\infty(\mathrm{d}u)e^{\left(-u^\beta\cos(\beta\pi/2)\right)}\sin\left(u^\beta\sin\left(\tfrac{\beta\pi}{2}\right)\right)\sin(\lambda\tau_0u)$$

Any other linear combination is also valid for getting solution $H_\lambda(\lambda)$.

**e.** The radioactive decay we write as pure exponential decay, however, Becquerel used compressed hyperbola function to describe this as

$$i(t)=\frac{1}{\left(1+(1-\beta)\left(\frac{t}{\tau_0}\right)\right)^{1/(1-\beta)}}$$

We have following steps

$$i(\mathrm{i}y)=\frac{1}{\left(1+(1-\beta)\left(\frac{\mathrm{i}y}{\tau_0}\right)\right)^{1/(1-\beta)}}$$
$$\left|i(\mathrm{i}y)\right|=\rho(y)=\left(1+\left(\frac{(1-\beta)y}{\tau_0}\right)^2\right)^{-1/(2(1-\beta))};\quad \angle i(\mathrm{i}y)=\theta(y)=-\frac{\tan^{-1}\left(\frac{(1-\beta)y}{\tau_0}\right)}{1-\beta}$$

$$H_\lambda(\lambda)=\frac{e^{x_0\lambda}}{\pi}\int_0^\infty\left(\rho(y)\right)\cos\left(\lambda y+\left(\theta(y)\right)\right)\mathrm{d}y;\qquad x_0=0$$

$$=\frac{1}{\pi}\int_0^\infty \mathrm{d}y\left(1+\left(\tfrac{(1-\beta)y}{\tau_0}\right)^2\right)^{-1/(2(1-\beta))}\cos\left(\lambda y-\frac{\tan^{-1}\left(\frac{(1-\beta)y}{\tau_0}\right)}{1-\beta}\right)$$

With change of variable $u=\frac{(1-\beta)y}{\tau_0},\quad \frac{\tau_0}{1-\beta}\mathrm{d}u=\mathrm{d}y$ , we get

$$H_\lambda(\lambda)=\frac{\tau_0}{\pi(1-\beta)}\int_0^\infty\left(\mathrm{d}u\right)(1+u^2)^{-1/(2(1-\beta))}\cos\left(\tfrac{\lambda\tau_0 u-\tan^{-1}u}{1-\beta}\right)$$

Using other formulas we get

$$H_\lambda(\lambda)=\frac{2\tau_0}{\pi(1-\beta)}\int_0^\infty\left(\mathrm{d}u\right)\left((1+u^2)^{-1/(2(1-\beta))}\right)\cos\left(\tfrac{\tan^{-1}u}{1-\beta}\right)\cos(\lambda\tau_0 u)$$

$$H_\lambda(\lambda)=\frac{2\tau_0}{\pi(1-\beta)}\int_0^\infty\left(\mathrm{d}u\right)\left((1+u^2)^{-1/(2(1-\beta))}\right)\sin\left(\tfrac{\tan^{-1}u}{1-\beta}\right)\sin(\lambda\tau_0 u)$$

**f.** The rate distribution function $H_\lambda(\lambda)$ for a simple power law as

$$i(t)=\left(1+\left(\tfrac{t}{\tau_0}\right)^\alpha\right);\quad 0<\alpha<1$$

will be expressed via same rule as above

$$i(\mathrm{i}y)=\frac{1}{1+\left(\frac{\mathrm{i}y}{\tau_0}\right)^\alpha}=\frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha(\mathrm{i})^\alpha}$$

$$=\frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha e^{\mathrm{i}(\alpha\pi/2)}}=\frac{1}{1+\left(\frac{y}{\tau_0}\right)^\alpha\left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)}$$

The real part of the complex function is

$$\mathrm{Re}\left\{i(\mathrm{i}y)\right\}=\frac{\left(\frac{y}{\tau_0}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{y}{\tau_0}\right)^{2\alpha}+2\left(\frac{y}{\tau_0}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}$$

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \operatorname{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y; \qquad x_0=0$$

$$=\frac{2}{\pi}\int_0^\infty \frac{\left(\frac{y}{\tau_0}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{y}{\tau_0}\right)^{2\alpha}+2\left(\frac{y}{\tau_0}\right)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda y)\left(\mathrm{d}y\right); \quad u=\frac{y}{\tau_0}, \quad \tau_0\,\mathrm{d}u=\mathrm{d}y$$

$$=\frac{2\tau_0}{\pi}\int_0^\infty \frac{(u)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}{(u)^{2\alpha}+2(u)^\alpha\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(\lambda\tau_0 u)\left(\mathrm{d}u\right)$$

**g.** The rate distribution function $H_\lambda(\lambda)$ for a simple power law as

$$i(t)=t^{-\alpha}; \quad 0<\alpha<1$$

will be expressed via same rule as above

$$i(\mathrm{i}y)=\frac{1}{(\mathrm{i}y)^\alpha}=\frac{1}{(y)^\alpha(\mathrm{i})^\alpha}$$

$$=\frac{1}{(y)^\alpha e^{\mathrm{i}(\alpha\pi/2)}}=\frac{1}{(y)^\alpha\left(\cos\left(\frac{\alpha\pi}{2}\right)+\mathrm{i}\sin\left(\frac{\alpha\pi}{2}\right)\right)}$$

$$=\frac{y^\alpha\cos\left(\frac{\alpha\pi}{2}\right)-\mathrm{i}y^\alpha\sin\left(\frac{\alpha\pi}{2}\right)}{y^{2\alpha}}$$

The real part of the complex function is

$$\operatorname{Re}\{i(\mathrm{i}y)\}=\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{(y)^\alpha}$$

$$H_\lambda(\lambda)=\frac{2e^{x_0\lambda}}{\pi}\int_0^\infty \operatorname{Re}\{i(x_0+\mathrm{i}y)\}\cos(\lambda y)\mathrm{d}y; \qquad x_0=0$$

$$=\frac{2}{\pi}\int_0^\infty\left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{y^\alpha}\right)\cos(\lambda y)\left(\mathrm{d}y\right); \quad u=y, \quad \mathrm{d}u=\mathrm{d}y$$

$$=\frac{2}{\pi}\cos\left(\frac{\alpha\pi}{2}\right)\int_0^\infty u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

From Laplace Tables we have $H_\lambda(\lambda)=\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}$ from Laplace inverse of $i(t)=t^{-\alpha}$, therefore we write following

$$H_{\lambda}(\lambda)=\frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}=\frac{2}{\pi}\cos\left(\tfrac{\alpha\pi}{2}\right)\int_{0}^{\infty}u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

$$\lambda^{\alpha-1}=\frac{2}{\pi}\Gamma(\alpha)\left(\cos\left(\tfrac{\alpha\pi}{2}\right)\right)\int_{0}^{\infty}u^{-\alpha}\cos(\lambda u)\left(\mathrm{d}u\right)$$

**h.** The following power series is generalized power series solution for relaxation function

$$i(t)=A\left(\frac{t}{\tau}\right)^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k}$$

Putting, $\xi=(t/\tau)$ $\alpha=\beta=0.5$ gives

$$i(\xi)=A\left(\exp(\xi)\right)\left(\mathrm{erfc}(\xi^{1/2})\right)$$

where erfc is the 'complementary error function'. Putting $\alpha=\beta=1$ we obtain relaxation response of as Debye relaxation

$$i(\xi)=A\exp(-\xi)$$

The sum contained in the relaxation function is the generalized Mittag-Leffler (GML) function; denoted by $E_{\alpha,\beta}(\xi)$ which reads as follows

$$E_{\alpha,\beta}(\xi)=\sum_{k=0}^{\infty}\frac{(\xi)^{k}}{\Gamma(\alpha k+\beta)}\qquad \alpha,\beta>0\qquad E_{\alpha,1}(\xi)=E_{\alpha}(\xi)$$

For negative $\xi$, we have the following expressions for GML function

$$E_{\alpha,\beta}(-\xi)=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{\Gamma(\alpha k+\beta)}\xi^{k}\qquad E_{\alpha,\beta}(-\xi^{\alpha})=\sum_{k=0}^{\infty}\frac{(-1)^{k}}{\Gamma(\alpha k+\beta)}(\xi)^{\alpha k}$$

The asymptotic expansion for the Mittag-Leffler function for negative arguments at $\xi\uparrow\infty$ is the following [28], [29]

$$E_{\alpha,\alpha}(-\xi)\sim\frac{\alpha}{\Gamma(1-\alpha)}\xi^{-2},\quad \alpha\neq 1\qquad E_{\alpha,\gamma}(-\xi)\sim\frac{1}{\Gamma(\gamma-\alpha)}\xi^{-1},\quad \gamma\neq\alpha$$

$$E_{\alpha,\alpha}(-\xi^{\alpha})\sim\frac{\alpha}{\Gamma(1-\alpha)}x^{-2\alpha},\quad \alpha\neq 1\qquad E_{\alpha,\gamma}(-\xi^{\alpha})\sim\frac{1}{\Gamma(\gamma-\alpha)}\xi^{-\alpha},\quad \gamma\neq\alpha$$

With these approximations we express the asymptotic behavior of the relaxation function for short and long times. The relaxation function is $i(\xi)=A\xi^{\alpha-\beta}E_{\alpha,\gamma}(-\xi^{\alpha})$

$$i(\xi) = A\xi^{\alpha-1}E_{\alpha,\alpha}\left(-\xi^{\alpha}\right) = \begin{cases} A\dfrac{\xi^{\alpha-1}}{\Gamma(\alpha)} & \text{as} \quad \xi \downarrow 0 \\ A\dfrac{\alpha}{\Gamma(1-\alpha)}\xi^{-(1+\alpha)} & \text{as} \quad \xi \uparrow \infty \end{cases}$$

For the other case we have

$$i(\xi) = A\xi^{\alpha-\beta}E_{\alpha,\gamma}\left(-\xi^{\alpha}\right) = \begin{cases} A\dfrac{\xi^{\alpha-\beta}}{\Gamma(\gamma)} & \text{as} \quad \xi \downarrow 0 \\ A\dfrac{1}{\Gamma(1-\beta)}\xi^{-\beta} & \text{as} \quad \xi \uparrow \infty \end{cases} \qquad \gamma = \alpha - \beta + 1$$

We write some of the important properties of Mittag-Leffler function as

$$E_{\alpha}(-\xi) = E_{2\alpha}(\xi^2) - \xi E_{2\alpha,1+\alpha}(\xi^2)$$

$$E_{2\alpha}(\xi^2) = \frac{E_{\alpha}(\xi) + E_{\alpha}(-\xi)}{2}$$

$$E_{\alpha}(-\mathrm{i}y) = E_{2\alpha}(-y^2) - \mathrm{i}yE_{2\alpha,1+\alpha}(-y^2)$$
$$\mathrm{Re}\left\{E_{\alpha}(-\mathrm{i}y)\right\} = E_{2\alpha}(-y^2)$$

We can extract the rate distribution function i.e. $H_{\lambda,\alpha}(\lambda)$ for Mittag-Leffler decay $i(t) = E_{\alpha}(-\xi)$, $\xi = t/\tau$, with the Laplace inversion formula derived in earlier section, to expand it as Laplace transform, as follows:

$$E_{\alpha}(-\xi) = \int_0^{\infty}\left(H_{\lambda,\alpha}(\lambda)\right)e^{-\lambda\xi}\mathrm{d}\lambda$$

Put $\xi = \mathrm{i}y$, thus we have $\mathrm{Re}\left\{E_{\alpha}(-\mathrm{i}y)\right\} = E_{2\alpha}(-y^2)$; and write

$$H_{\lambda,\alpha}(\lambda) = \frac{2}{\pi}\int_0^{\infty}\mathrm{Re}\left\{i(\mathrm{i}y)\right\}\cos(\lambda y)\mathrm{d}y$$
$$= \frac{2}{\pi}\int_0^{\infty}E_{2\alpha}(-y^2)\cos(\lambda y)\mathrm{d}y$$

For various $\alpha$, $0 < \alpha < 1$, $\lambda > 0$, we have following integral representations for $H_{\lambda,\alpha}(\lambda)$

For $\alpha = 1$ $E_{\alpha}(-\xi) = e^{-\xi}$, the rate distribution is

$$H_{\lambda,1}(\lambda)=\frac{2}{\pi}\int_0^{\infty}E_2(-y^2)\cos(\lambda y)\mathrm{d}y=\frac{2}{\pi}\int_0^{\infty}\cosh(\mathrm{i}y)\cos(\lambda y)\mathrm{d}y$$

$$=\frac{2}{\pi}\int_0^{\infty}\cos(y)\cos(\lambda y)\mathrm{d}y$$

$$=\delta(\lambda-1)$$

For $\alpha=\frac{1}{2}$ i.e. $i(t)=E_{1/2}(-\xi)$ we have

$$H_{\lambda,1/2}(\lambda)=\frac{2}{\pi}\int_0^{\infty}E_1(-y^2)\cos(\lambda y)\mathrm{d}y=\frac{2}{\pi}\int_0^{\infty}e^{-y^2}\cos(\lambda y)\mathrm{d}y$$

$$=\frac{1}{\sqrt{\pi}}e^{-\left(\lambda^2/4\right)}$$

For $\alpha=0$ i.e. $i(t)=E_0(-\xi)$ we have $E_0(-\xi)=\frac{1}{1+\xi}$ and $\mathrm{Re}\left\{E_0(-\mathrm{i}y)\right\}=E_0(-y^2)=\frac{1}{1+y^2}$

$$H_{\lambda,0}(\lambda)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos(\lambda y)}{1+y^2}\mathrm{d}y=e^{-\lambda}$$

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