

# **Mathematical Foundation for Resonance in Composite Medium for Structural Inclusions in Meta-Materials**

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## **Key words**

Resonance, effective permittivity, permeability, Structural inclusions, characteristics function, Meta-material, analyticity

## **Abstract**

Here in this note, general mathematical foundation is developed, for effective properties (permittivity, permeability, conductivity) for a composite material. To have designed properties for meta-material by the regular structural inclusions, in the host material, is the key, to have negative permittivity, negative permeability, thereby forming negative refractive index. The inclusion's spacing and dimensions being much smaller to the wavelength of the input Electromagnetic wave, exciting the media, makes us to extract effective properties of the media; considering homogenization. This note exactly shows the resonance formations in the mix of inclusions on host material, at a particular value of negative value of host's properties; giving poles in the characteristic function of the composite system. We form the basic theory on effective di-electric permittivity, but the theory is general for application in any other property, say permeability or conductivity. This mathematical foundation gives insight into resonant behavior of theory of meta-material.

## 1. Introduction

Consider a composite material made with inclusion permittivity  $\varepsilon_i$  embedded in host material having permittivity  $\varepsilon_h$ . What is the effective dielectric permittivity  $\varepsilon_e$ ? That is an inhomogeneous material made in the form of macroscopic mixture of several homogeneous components. This problem is addressed by Clausius-Mossotti, Maxwell-Garnet, and Lorenz-Lorentz relation [43]. In the case of two phase three dimensional medium, these theories yield the following expression for  $\varepsilon_e$  in terms of  $\varepsilon_i$ ,  $\varepsilon_h$  and the volume fractions  $\rho_i$ ,  $\rho_h$

$$\frac{\varepsilon_e - \varepsilon_h}{\varepsilon_e + 2\varepsilon_h} = \rho_i \frac{\varepsilon_i - \varepsilon_h}{\varepsilon_i + 2\varepsilon_h} \quad (1)$$

The (1) is approximate assumes only one grain of  $\varepsilon_i$  is considered explicitly, the surrounding material being replaced by an effective homogeneous and isotropic medium. Furthermore, the pure inclusions in the hosts are considered as spherical balls. In spite of these approximations, (1) works well gives exact effective values with small  $\rho_i$ . For larger values of  $\rho_i$ , this formulation (1) is not good approximation. Effective Medium Theory [44] gives estimate as (2); in those cases; that is (2)

$$\rho_i \frac{\varepsilon_i - \varepsilon_e}{\varepsilon_i + 2\varepsilon_e} + \rho_h \frac{\varepsilon_h - \varepsilon_e}{\varepsilon_h + 2\varepsilon_e} = 0 \quad (2)$$

The effective properties are thus found from the root of (2), that is (3)

$$\varepsilon_e = \frac{1}{4} \left\{ 3(\rho_i \varepsilon_i + \rho_h \varepsilon_h) - \varepsilon_i - \varepsilon_h \pm \sqrt{[3(\rho_i \varepsilon_i + \rho_h \varepsilon_h) - \varepsilon_i - \varepsilon_h]^2 + 8\varepsilon_i \varepsilon_h} \right\} \quad (3)$$

The expression (1) and (2) gives us not only effective dielectric constant, but also effective magnetic permeability, electrical or thermal conductivity, diffusivity, and several (mathematically) similar processes.

Expression (2) is considered better especially in case of dielectric heating experiments, where the volume fractions of the inclusion and hosts are comparable. However, in the meta-material case the (1) is good representation, as the inclusions are generally sparse. It will be proved also that (1) predicts resonance in  $\varepsilon_e$  for some negative value of ratio of  $\varepsilon_i / \varepsilon_h$ ; while no such resonance is observed for (2) and (3). This dielectric resonance we have seen in our experiments with near field for drilling with microwave experiments, where illuminated plasmoid is formed; due to phenomena of electrostatic resonances. We have made plasmonic meta-material, where the dielectric permeability and permittivity have resonances at designated frequency of design [14]-[21] (designed for Ka-band, designed for X-band).

## 2. Effective permeability and characteristic function qualitatively defined and phenomena of resonance

Let us consider cases where inclusion structures are simple, as depicted in figure-1, the a) is composite of plane parallel layers parallel to the excitation electric field  $E_0$ , the configuration b) is plane parallel layers perpendicular to the excitation electric field  $E_0$ , and third is a sparse

distribution of spherical inclusions (figure-2) , in a homogeneous host medium. The following are the relations which give the effective permittivity, those are, for parallel inclusions, parallel to excitation field, coming from equivalent circuit of parallel capacitors (figure-1a)

$$\varepsilon_e = \sum_1^n \rho_i \varepsilon_i \quad (4)$$

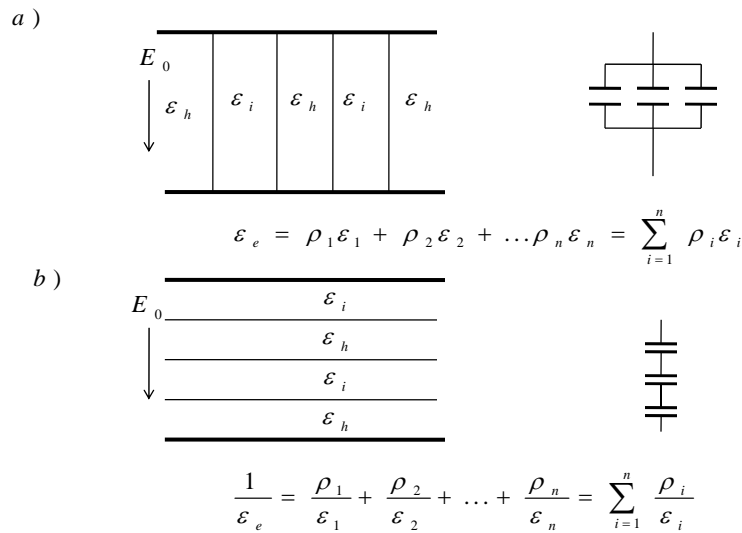
For the second configuration the effective parameter of permittivity, coming from series connection of capacitor (figure-1b) system is

$$\frac{1}{\varepsilon_e} = \sum_1^n \frac{\rho_i}{\varepsilon_i} \quad (5)$$

For spherical inclusions (figure-2), we have from (1), with algebraic manipulation the following, where  $n - 1$  spheres are embedded in host medium  $\varepsilon_n$ ; that is (6)

$$\frac{1}{\left(\frac{1}{3\varepsilon_n} + \frac{1}{\varepsilon_e - \varepsilon_n}\right)} = \sum_{i=1}^{n-1} \frac{\rho_i}{\left(\frac{1}{3\varepsilon_n} - \frac{1}{\varepsilon_i - \varepsilon_n}\right)} \quad (6)$$

The above are qualitative definition of effective permeability of simple inclusions.



**Figure-1 Simple geometry of structural inclusions defining effective parameter**

Restricting to simpler case, of two composites  $n=2$ , with  $\varepsilon_1$  in host  $\varepsilon_n$  and defining the characteristic function as  $\zeta = \varepsilon_e / \varepsilon_n$ , of  $v = \varepsilon_1 / \varepsilon_n$  we have after algebraic manipulation of (4) (5) and (6)

$$\zeta_a(v) = \rho_1 v + \rho_2 \quad (7)$$

$$\zeta_b(v) = v / (\rho_1 + \rho_2 v) \quad (8)$$

$$\zeta_c(v) = \frac{2\rho_1(v-1) + v + 2}{2 + \rho_1 + v(1 - \rho_1)} \quad (9)$$

In all cases a, b and c as in (7) (8) and (9), the characteristic function  $\zeta$  is a rational function with pole ( $v_{pole}$ ) at  $\infty$ ,  $-\rho_1/\rho_2$  and  $-(\rho_1 + 2)/(1 - \rho_1)$ .

A composite mixture of pure dielectric of different types, with real positive dielectric constants can be simulated by a network of lumped capacitances. The composite if is near one of its poles or zeros must have at least one of the  $\varepsilon_i$ 's as negative, that is what is indicated by qualitative analysis of (7) (8) (9);  $v = \varepsilon_i/\varepsilon_n < 0$  says  $\varepsilon_i < 0$ . Thus, to simulate the singularity (poles/zeros) by a network of lumped component, we would require both the capacitance as well as inductances, and we would require to apply a fixed (near resonant) frequency alternating voltage source to this network. Every pole of the  $\zeta(v)$  then would corresponds to a series resonance of the network where total admittance diverges, while every zero of  $\zeta(v)$  corresponds to parallel resonant where at resonance, the admittance goes to zero. In summary we have indicated at certain composition the effective permittivity of composite is resonating, where one of the permittivity values is negative. This is essence of making meta-material for negative permittivity and negative permeability, phenomena of resonance.

We now consider 'single sphere' of radius  $a$  with dielectric constant (permittivity)  $\varepsilon_1$  embedded in a much larger host material with dielectric constant  $\varepsilon_2$ . It is well known if uniform electric field  $E_0$  parallel to say  $z$ -axis, then the field distribution near the spherical inclusion is [30].

$$\begin{aligned} E(\mathbf{r}) &= E_0 - a^3 \frac{3(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_2} |E_0| \nabla \frac{\cos\theta}{r^2}; \quad r > a \\ E(\mathbf{r}) &= \frac{3\varepsilon_2 E_0}{\varepsilon_1 + 2\varepsilon_2}; \quad r < a \end{aligned} \quad (10)$$

Here, the origin of  $\mathbf{r}$  is centre of spherical inclusion. The  $\theta$ , is angle measured with respect to  $z$ -axis. We assume that the spherical inclusion lies in the middle of two equi-potential capacitor plates whose distance  $L$  is much larger than  $a$ . The expression (10) thus is modified to satisfy equipotential boundary conditions. When this is carried out by introduction of image charge inclusions [30], we calculate the volume averages of  $E$  and displacement  $D$  and thus get expression for  $\varepsilon_e$  as [30];

$$\varepsilon_e = \varepsilon_2 + \frac{4}{3} \frac{\pi a^3}{V} \varepsilon_2 \frac{3(\varepsilon_1 - \varepsilon_2)}{\varepsilon_1 + 2\varepsilon_1} \quad (11)$$

Where  $V$  is the total volume of the system-say capacitor. Clearly (11) diverges when

$$\varepsilon_1 = -2\varepsilon_2 \quad (12)$$

In which case we cannot satisfy the boundary condition at the surface of spherical inclusion with any finite value of the distorted field as (10) also diverges. It is also shown that when this critical value of  $\varepsilon_1$  is approached, it becomes more and more difficult to satisfy these conditions, and one encounters increasingly larger values of internal field and the induced dipole moment

of the inclusion. Here at the point of resonance, the giant surface fields are produced at the interface of the inclusion.

By comparing (11) with (1), it is also clear that in (1) all the poles coincide with the pole obtained for a single sphere. This is the result of neglecting the spatial fluctuations of the interactions between different inclusions, which is good approximation for sparsely distributed inclusions or when  $|\varepsilon_1 - \varepsilon_2| \ll \varepsilon_2$ . In actual cases, the poles are always distinct but narrowly distributed around (12). As the  $\rho_1$ , the density of spherical inclusion is enhanced (from sparse case); and the spherical inclusions begin to interact strongly, we expect broadening of the distribution of poles; also in a disordered case where shapes of inclusions vary in random way the broadening should be more. For these cases, to have robust theory of resonances is difficult to achieve though one can verify the results with numerical techniques. This advancement in mathematics is a good and important field of research.

### 3. Defining exactly the effective permittivity of composite material formally

Consider Maxwell equation in the composite material, for quasi-static case that is zero frequency case, when the electric field is curl free and satisfies (13)

$$\nabla \cdot (\varepsilon E) = 0 \quad (13)$$

In the (13) the;  $\varepsilon$  is a function of position vector  $\mathbf{r}$ , that is  $\varepsilon(\mathbf{r})$ , is the local value of the permittivity, which varies from one pure inclusion to next pure inclusion. First let us define exactly the meaning of effective dielectric permittivity. Assume that this effective phenomena description is on length scale large compared to length scale of in-homogeneities. The external electric field shining the composite has wave length very large compared to the length scales of in-homogeneities; is the case in our meta-material experiments [14]-[21]. Consider a parallel plate capacitor where the dielectric between the parallel plate is a composite having structural inclusions (figure-2). The electrostatic energy in capacitor of figure-2 when a given potential difference  $\phi$  Volts is applied across the plates, we define effective permittivity dielectric constant of an equivalent fictitious homogeneous filler, that would lead to same value of energy

$$\varepsilon_e E_0^2 = \frac{1}{V} \int \varepsilon(\mathbf{r}) E^2(\mathbf{r}) dV \quad (14)$$

Here  $V$  is the total volume of the capacitor, while  $E_0$  is the average Electric field defined as

$$E_0 = \frac{1}{V} \int E(\mathbf{r}) dV \quad (15)$$

The average electric field as in (15) depends on the applied potential,  $\phi$ . Since both the energy and the total charge stored on the capacitor plates depend only on the total Farads (the capacity of capacitor), the value of effective permittivity  $\varepsilon_e$  determined by the (14) automatically ensures that correct total charge is also reproduced by the effective fictitious homogeneous dielectric filler. Consider the composite of  $n-1$  different pure inclusions, then we can represent  $\varepsilon(\mathbf{r})$  with help of Heaviside step functions ( $H$ ) as

$$\varepsilon(\mathbf{r}) = \varepsilon_1 H_1(\mathbf{r}) + \varepsilon_2 H_2(\mathbf{r}) + \dots + \varepsilon_n H_n(\mathbf{r}) = \sum_{i=1}^n \varepsilon_i H_i(\mathbf{r}) \quad (16)$$

The step function  $H_i(\mathbf{r}) = 1$ , if  $\mathbf{r}$  is inside the  $i$ -th inclusion, while  $H_i(\mathbf{r}) = 0$  if  $\mathbf{r}$  is outside the  $i$ -th inclusion (phase); and  $\varepsilon_i$  permittivity of the  $i$ -th phase. Note (16) is generalization of two-phase system as described by (1) and (2), for  $n-1$  inclusions.

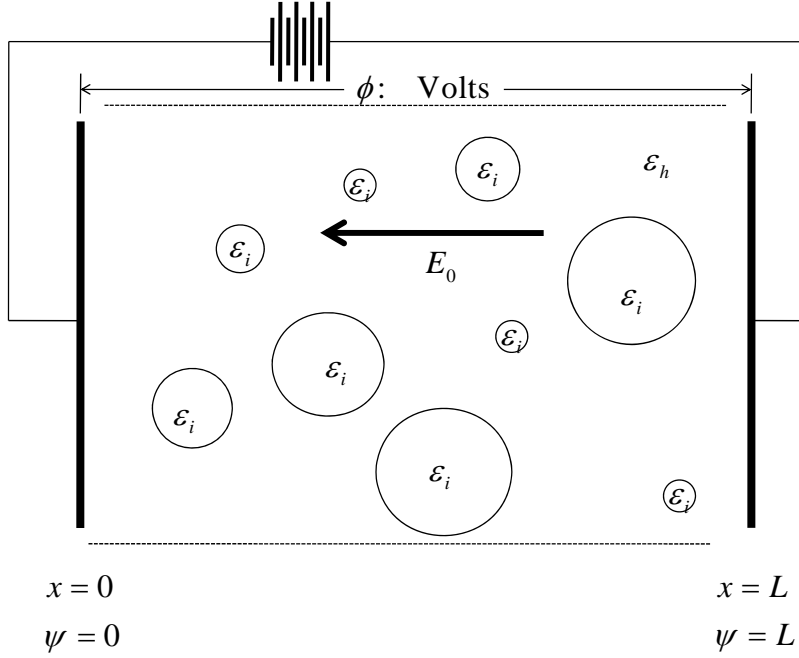


Figure-2 Capacitor filled with composite material dielectric

#### 4. Defining formally the characteristic function for a composite material

We now construct a composite's characteristic function, by ratio of  $v_i = \varepsilon_i / \varepsilon_n$ , where  $n$  is the host material with  $n-1$ , structural inclusions. Consider (15) and use (17), we have

$$\begin{aligned} \varepsilon_c E_0^2 &= \frac{1}{V} \int \varepsilon(\mathbf{r}) E^2(\mathbf{r}) dV \\ &= \frac{1}{V} \int [\varepsilon_1 H_1(\mathbf{r}) + \varepsilon_2 H_2(\mathbf{r}) + \dots + \varepsilon_n H_n(\mathbf{r})] E^2(\mathbf{r}) dV \end{aligned} \quad (18)$$

From (18) we re-write after simple algebraic manipulation

$$\frac{\varepsilon_c}{\varepsilon_n} = \frac{1}{V} \int \left( \frac{\varepsilon_1}{\varepsilon_n} H_1(\mathbf{r}) + \frac{\varepsilon_2}{\varepsilon_n} H_2(\mathbf{r}) + \dots + \frac{\varepsilon_{n-1}}{\varepsilon_n} H_{n-1}(\mathbf{r}) + H_n(\mathbf{r}) \right) \frac{E^2(\mathbf{r})}{E_0^2} dV \quad (19)$$

Define, the ratios of composite to host permittivity as

$$v_1 = \frac{\varepsilon_1}{\varepsilon_n}; \quad v_2 = \frac{\varepsilon_2}{\varepsilon_n}, \dots, v_{n-1} = \frac{\varepsilon_{n-1}}{\varepsilon_n}; \quad v_n = 1 \quad (20)$$

Define

$$H_v = v_1 H_1 + v_2 H_2 + \dots v_{n-1} H_{n-1} + H_n = \sum_{i=1}^n v_i H_i = \frac{\varepsilon(\mathbf{r})}{\varepsilon_n} \quad (21)$$

Define normalized scalar potential field as

$$\frac{E(\mathbf{r})}{|E_0|} = -\nabla \psi \quad (22)$$

Using the definitions of (20), (21) and (22), and substituting the same in (19), we form a characteristic equation for a composite material with  $n-1$  inclusions in a  $\varepsilon_n = \varepsilon_h$  background as

$$\frac{\varepsilon_e}{\varepsilon_n} \equiv \zeta(v_1, v_2, \dots v_{n-1}) = \frac{1}{V} \int H_v (\nabla \psi)^2 dV \quad (23)$$

The normalized scalar potential field (22) in expression (23), is a solution of partial differential equation; got from (14) with normalizing and using the (20), (21) (22)

$$\nabla \bullet (H_v \nabla \psi) = 0 \quad (24)$$

The boundary conditions (normalized via (20), (21), (22) in figure-2) as

$$\psi(x=0) = 0; \quad \psi(x=L) = L; \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0; \quad \text{at the walls} \quad (25)$$

The last boundary condition on the capacitor wall is introduced in order to specify completely the boundary value problem. In practice this last condition of derivative of potential field with respect to normal component of side walls, is realized by making the capacitor walls from good conductor. Another approach to realize the last boundary condition is to have infinite parallel plates of the capacitor. The generation of (24) (25) is described below.

Take (13) that is  $\nabla \bullet (\varepsilon(\mathbf{r})E(\mathbf{r})) = 0$ , divide this by  $\varepsilon_n |E_0|$ , and use (21) and (22) so we have (24), as follows

$$\nabla \bullet \left( \frac{\varepsilon(\mathbf{r}) E(\mathbf{r})}{\varepsilon_n |E_0|} \right) = 0 \quad \text{i.e} \quad \nabla \bullet (H_v \nabla \psi) = 0$$

Same way one can have normalized boundary condition as (25).

There is another way of getting characteristic equation which is inverse of (23). Put  $D(\mathbf{r}) = \varepsilon(\mathbf{r})E(\mathbf{r})$ , and  $D_0 = \varepsilon_e E_0$ , in (18) to get

$$\frac{D_0^2}{\varepsilon_e} = \frac{1}{V} \int \frac{D^2(\mathbf{r})}{\varepsilon(\mathbf{r})} dV \quad \frac{1}{\varepsilon_e} = \frac{1}{V} \int \frac{1}{\varepsilon(\mathbf{r})} \frac{D^2(\mathbf{r})}{D_0^2} dV \quad \frac{\varepsilon_n}{\varepsilon_e} = \frac{1}{V} \int \frac{\varepsilon_n}{\varepsilon(\mathbf{r})} \frac{D^2(\mathbf{r})}{D_0^2} dV \quad (26)$$

Consider  $4\pi\varepsilon_e |E_0|$  and  $(4\pi/S) \int D_n dS$ ; they represent average charge per unit area on the plates of figure-2, with surface area  $S$ . Furthermore,  $\text{div}D = 0$  everywhere, and  $D_n = 0$  on the sidewalls. Thus,  $D_0$  is equal to component of  $D$  perpendicular to plates (figure-2), and  $D_0$  is  $D_n$  averaged over entire volume or over any of the plates, because  $E_0$  is perpendicular to plates. We can express this statement as

$$E_0 \bullet D_0 = E_0 \bullet \frac{1}{V} \int D(\mathbf{r}) dV \quad |D_0| = \frac{1}{S} \int D_n(\mathbf{r}) dS \quad (27)$$



Define scalar potential  $\phi$  as

$$D = -|D_0| \frac{\varepsilon(\mathbf{r})}{\varepsilon_n} \nabla \phi = -|D_0| H_v \nabla \phi \quad (28)$$

We manipulate (26) and write  $\tilde{\zeta}(\tilde{v}) = \varepsilon_n / \varepsilon_e$ , with  $\tilde{v}_i = \varepsilon_n / \varepsilon_i$  (reciprocal of (20)), using (28) as

$$\tilde{\zeta}(\tilde{v}) = \frac{1}{\zeta(v)} = \frac{1}{V} \int \left( \frac{\varepsilon_n}{\varepsilon(\mathbf{r})} \right) \left( \frac{\varepsilon(\mathbf{r})}{\varepsilon_n} \right)^2 (\nabla \phi)^2 dV = \frac{1}{V} \int \left( \frac{\varepsilon(\mathbf{r})}{\varepsilon_n} \right) (\nabla \phi)^2 dV = \frac{1}{V} \int H_v (\nabla \phi)^2 dV \quad (29)$$

The (29) is reciprocal characteristic function, satisfies the following differential equation like (24), that is

$$\nabla \bullet (H_v \nabla \phi) = 0 \quad (30)$$

The boundary conditions are

$$\phi(x=0) = 0; \quad \phi(x=L) = \text{constant}; \quad \frac{\partial \phi}{\partial \mathbf{n}} = 0 \quad \text{at sidewalls} \quad \frac{1}{S} \int H_v \frac{\partial \phi}{\partial \mathbf{n}} dS = 1 \quad (31)$$

We discuss the solubility of the boundary value problem (24) and (25); to do that we introduce the associated eigen-value problem.

$$\nabla \bullet (H_v \nabla \psi_k) = \lambda_{\text{eigen-}k} \psi_k; \quad \psi_k(x=0) = \psi_k(x=L) = 0; \quad \frac{\partial \psi_k}{\partial \mathbf{n}} = 0 \quad \text{at the walls} \quad (32)$$

Suppose that  $\psi$  is a solution of (24), then by multiplying the differential equation (24) by any eigen-function  $\psi_k$  and integrating over the volume of the capacitor we get;

$$\begin{aligned} \nabla \bullet (H_v \nabla \psi) &= 0 \\ 0 &= \int \psi_k \nabla \bullet (H_v \nabla \psi) dV = - \int H_v \nabla \psi_k \bullet \nabla \psi dV \\ &= - \oint \psi H_v \nabla \psi_k \bullet dS + \int \psi \nabla \bullet (H_v \nabla \psi_k) dV \\ &= -L \int_{x=L} H_v \frac{\partial \psi_k}{\partial x} dS_x + \lambda_{\text{eigen-}k} \int \psi \psi_k dV \end{aligned} \quad (33)$$

The (33) uses integration by part to get the first identity then, Green's theorem is applied twice in as well as used are boundary conditions on  $\psi$  and on  $\psi_k$  and the eigen value equation satisfied by  $\psi_k$ . Hence, the condition of existence of solution of (24) is that any eigen-function  $\psi_0$ , with eigen-value  $\lambda_{\text{eigen-}0} = 0$  must satisfy

$$\int H_v \frac{\partial \psi_0}{\partial \mathbf{n}} dS = 0 \quad (34)$$

In order to have (34) as sufficient condition, for solution to (24) we must assume that (32) has complete set of eigen-function solution  $\psi_k$  so that any function which satisfies boundary condition of (32) can be expanded into series of  $\psi_k$ . Thus,  $\psi$  a solution of (32) can be expanded as

$$\psi = \sum A_k \psi_k + x \quad (35)$$

Where  $x$  is included to satisfy boundary condition (25) of differential equation (24). Substituting (35) in the (32) we get the following steps (36)

$$\begin{aligned}
\psi &= \sum A_k \psi_k + x \\
\nabla \cdot (H_v \nabla \psi) &= 0 \\
\nabla \cdot (H_v \nabla [\sum A_k \psi_k + x]) &= 0 \\
\nabla \cdot (\sum A_k H_v \nabla \psi_k + H_v) &= 0 \\
\sum A_k \nabla \cdot (H_v \nabla \psi_k) + \nabla \cdot H_v &= 0 \\
\nabla \cdot (H_v \nabla \psi_k) &= \lambda_{eigen-k} \psi_k, \quad \nabla \cdot H_v = \frac{\partial H_v}{\partial x} \\
0 &= \sum A_k \lambda_{eigen-k} \psi_k + \frac{\partial H_v}{\partial x}
\end{aligned} \tag{36}$$

Therefore, (35) will solve the differential equation (24) if  $A_k \lambda_{eigen-k}$  are chosen to be expansion coefficients of  $-\partial H_v / \partial x$  which is zero everywhere except at the boundaries of inclusion structures, from definition of  $H_v$  as in (21). Note that such expansions will not contain any eigen-function  $\psi_0$  for  $\lambda_{eigen-0} = 0$ .

The characteristic function  $\zeta$  and  $\tilde{\zeta}$  embeds all the geometrical information about the system that is required to determine the effective permittivity  $\varepsilon_e$  of the composite with inclusions for any set of values  $\varepsilon_i$  of inclusions. However, that information was introduced also by  $H_i$ -functions appearing in the function  $H_v$ . The  $\zeta$ -functions depend in general on the exact internal geometry of the structural inclusion in composite. For a given composite system with a given internal geometry, the functions  $\zeta$  gives us not only effective dielectric constant (permittivity as described here), but also effective magnetic permeability, electrical or thermal conductivity, diffusivity, and several (mathematically) similar processes. Besides applicability to several processes, electrical, magnetic, thermal, transport etc. the usefulness of the characteristic function lies in their 'analytical' properties.

## 5. Analyticity of the characteristic function

It is useful to extend the variables  $v_i$  into complex numbers. Thus the  $\zeta$  is a complex function of several complex variables  $v_i$ 's. the notion of complex variables and complex functions are easy to picture the poles and zeros and analyticity of the characteristic function. First we examine its differentiability. The characteristic function was defined for the structural inclusions in (23) and (29), with its solvability issues in the associated eigen-value problem in the previous section. The differentiability of (23) is described as

$$\begin{aligned}
\frac{\partial \zeta}{\partial v_i} &= \frac{\partial}{\partial v_i} \frac{1}{V} \int H_v (\nabla \psi)^2 dV = \frac{1}{V} \int dV (\nabla \psi)^2 \frac{\partial}{\partial v_i} \sum_{i=1}^n H_i v_i \\
&= \frac{1}{V} \int H_i (\nabla \psi)^2 dV
\end{aligned} \tag{37}$$

Similarly we will get differentiability of (29) as

$$\frac{\partial \tilde{\zeta}}{\partial \tilde{v}_i} = \frac{1}{V} \int H_i (H_v \nabla \phi)^2 dV \quad (38)$$

Whenever the boundary value problem for  $\psi$  (or  $\phi$ ) has a finite solution, the function  $\zeta(v_i)$  ( $\tilde{\zeta}(\tilde{v})$ ) is analytic. In the usual case both the differential equations (24) and (30) have unique solutions and it is clear that  $\phi$  must be a constant multiplier of  $\psi$ . In order to find types of non-analytic (or singular) points, the,  $\zeta(v)$  and  $\tilde{\zeta}(\tilde{v})$  can have; we must determine how the boundary value problem (24) (30) fail to have solution. The discussion on solubility of (24) as taken for eigen-value counterpart via (32) to (36) says that (24) has a solution when either of the following conditions is true

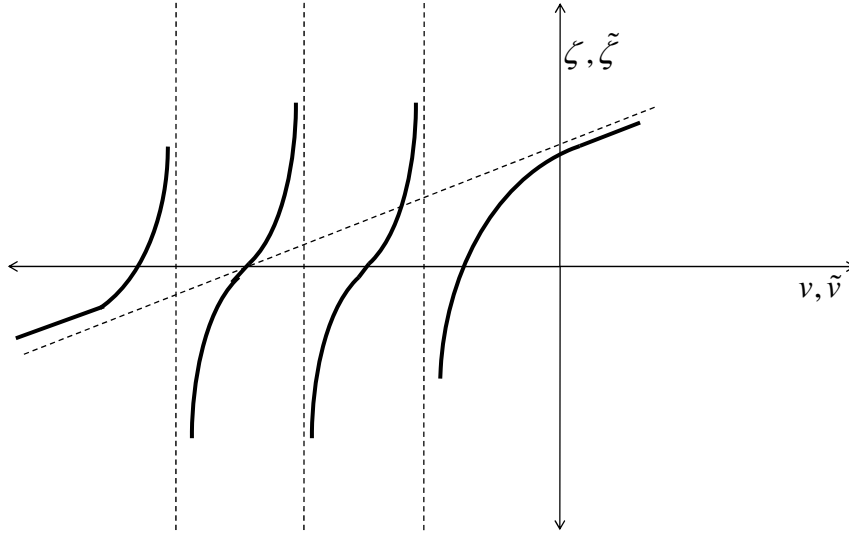
- (i) The homogeneous counterpart of (24) where  $\psi(0) = \psi(L) = 0$  has no solution. That is usual case, and then equation (24) has unique solution.
- (ii) The homogeneous counterpart of (24) does have a solution, but any such solution denoted by eigen-function  $\psi_0$  for  $\lambda_{eigen-0} = 0$  satisfies  $\int H_v (\partial \psi_0 / \partial \mathbf{n}) dS = 0$ , (34) then differential equation (24) will have more than one solution, but they all lead to same value for  $\zeta$ .

In both cases  $\zeta$  will have a regular point. The only other possibility is that homogeneous problem has a solution  $\psi_0$  where  $\int H_v (\partial \psi_0 / \partial \mathbf{n}) dS \neq 0$ . In that case equation (24) has no solution, but appropriate multiple of  $\psi_0$  will solve (30). Furthermore because  $\psi_0 = 0$  at  $x = 0$  and  $x = L$ , it means  $\tilde{\zeta} = \left(\frac{1}{V}\right) \int H_v (\nabla \phi)^2 dV = 0$  (using integration of parts). We thus conclude that the non-analytic points of  $\zeta(v_i)$  are the points where  $\tilde{\zeta} = 1/\zeta$  is analytic but zero, that is they are poles of  $\zeta$ . Obviously same conclusions hold for  $\tilde{\zeta}(\tilde{v}_i)$ . These conclusions include the cases when some of the  $v_i$  (or  $\tilde{v}_i$ ) are infinite. Therefore poles are the only type of singular points that  $\zeta$  and  $\tilde{\zeta}$  can have, including points at infinity. Consequently both  $\zeta$  and  $\tilde{\zeta}$  are rational functions in  $v_i$  and  $\tilde{v}_i$  variables.

For a two component composite case we have only  $v$  as one variable in one variable complex function  $\zeta(v)$ ; and  $H_v = vH_1 + H_2$ . If  $\tilde{v}$  is a zero of  $\tilde{\zeta}(\tilde{v})$ , then appropriate solution of (30), which is also a solution of homogeneous counterpart of (24) satisfies:

$$\begin{aligned} 0 &= \int \phi^* \nabla \cdot (H_v \nabla \phi) dV = - \int H_v \nabla \phi^* \cdot \nabla \phi dV \\ &= - \int H_v |\nabla \phi|^2 dV = - \int (vH_1 + H_2) |\nabla \phi|^2 dV \end{aligned} \quad (39)$$

The (39) vanishes for  $v$  negative and real. For such values of  $v$ , from (38) we get  $\partial \tilde{\zeta} / \partial \tilde{v}$  real positive, and hence  $\tilde{\zeta}(\tilde{v})$  has simple zero and  $\zeta(v)$  has simple pole, with negative residue. Qualitatively we may represent the pole-zero of characteristic function as in figure-3.



**Figure-3: Schematic of the function  $\zeta(v)$  or  $1/\zeta(v)$  for real values of  $v$  or  $(1/v)$  in case of two component composite system. This example show the characteristic function has three finite poles and one at infinity, interspersed with four zeros.**

Therefore we infer that zeros and poles of  $\zeta(v)$  and  $\tilde{\zeta}(\tilde{v})$  are real negative and simple with negative residues, and that for a finite system their total is finite. We have discussed and extended this inference from a two composite system.

The most striking feature of the characteristic function is the poles. Since as demonstrated these occur at only when  $v < 0$  one might wonder how. While no material exhibits negative value of d. c. dielectric constant (in fact  $\epsilon > 1$  always) such values are frequently found for the a. c. dielectric constants. For example metals in the range of frequencies between  $1/\tau$  and  $\omega_p$ , where  $\tau$  is the relaxation time for conductivity and  $\omega_p$  is the plasma frequency of the free electrons (electron gas in loose terms), have dielectric constant that is mostly real and negative, that is

$$\epsilon_M(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad (40)$$

Thus in metal insulator composites within that range of frequencies the ratio of dielectric constants can be made to vary along extended regions of negative  $v$ -axis (figure-3) by varying the frequency. In this way it should be possible to explore some of the poles of  $\zeta(v)$  for a given material experimentally.

The analytical properties of  $\zeta(v_i)$  we have seen how they are important in dealing with resonances. By mathematics we may construct a transformation as

$$p \equiv 1/(1-v) = \varepsilon_2 / (\varepsilon_2 - \varepsilon_1) = 1/u$$

$$F(p) \equiv f(u) = 1 - \zeta(v) = 1 - (\varepsilon_e / \varepsilon_2)$$
(41)

From the discussion on analyticity of  $\zeta(v)$  it is clear that the new transformed function  $F(p)$  is analytic everywhere except for finite number of simple poles on segment  $[0,1)$ , with residues of  $F(p)$  all positive. Thus transformed characteristic function may be represented as

$$1 - \frac{\varepsilon_e}{\varepsilon_n} = F(p) = \sum_n \frac{B_n}{(p - p_n)}$$
(42)

We conclude here having developed mathematical foundations in some way for getting into physics of composite what is essence in meta-material science. Several modifications and advances are yet to be covered in this direction.

## 6. Conclusion

This foundation mathematics described gives a lead, to generalize the theory of resonance, effective material properties, and many others by extending the mathematical analysis of characteristic functions and transformed characteristic functions. The further development of mathematics in regards to random shaped inclusions, randomly distributed inclusions with disorder, the fractal nature of inclusion in fractal geometry, extending the functional analysis to percolation theory and several other research frontiers, can be opened up in physical and mathematical science, in meta-material parlance.

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