Left Handed Maxwell Systems

PART-7

Magneto-Inductive Waves

SAMEER

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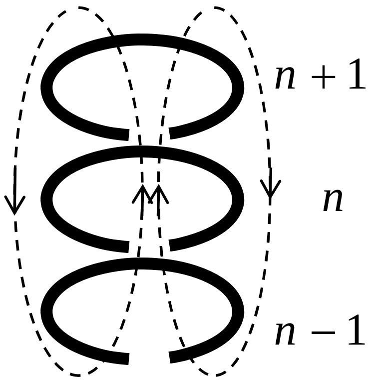
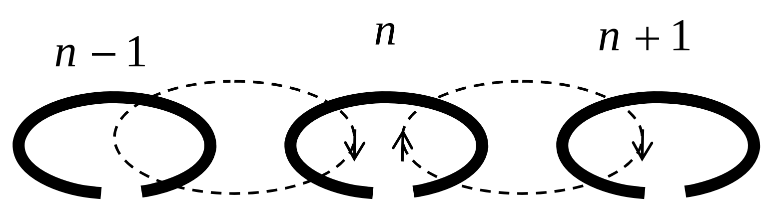
Magneto Inductive (MI) waves

Along with ‘backward waves’ the Magneto-Inductive (MI) waves are becoming popular in this particular field This happens or forms when two loops of SRR close to each other are ‘coupled’ to one another due to magnetic field of one loop ‘threading’ the other SRR. These coupling leads to waves are called MI waves

The dispersion equation is: $\omega = \dfrac{\omega_0}{\sqrt{1 + \dfrac{2M}{L}\cos(ka)}}$, ω_0 is resonant frequency of SRR, L is inductance, M the mutual inductance





Axial coupling $M > 0$

Planer coupling $M < 0$

For positive M the central ring’s voltage drop due to its own current $Z_0 I_n$ will try to increase due to the induced voltage due to adjacent rings $j\omega M I_{n-1}$ and $j\omega M I_{n+1}$

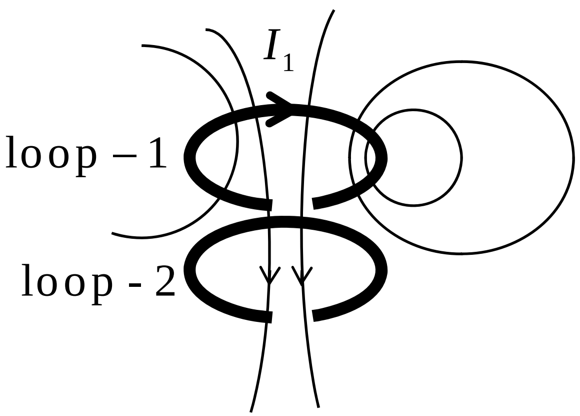
Thus we write KVL for SRR-n as: $Z_0 I_n + j\omega M I_{n-1} + j\omega M I_{n+1} = 0$

$$[R + j\omega L + (j\omega C)^{-1}] I_n = -j\omega M (I_{n-1} + I_{n+1})$$

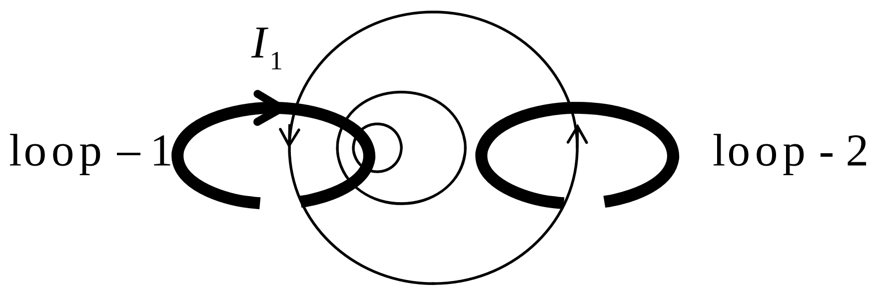
Magneto Inductive (MI) waves positive and negative mutual inductances

Mutual impedance between two elements is defined as ratio of the voltages in element-2 to the current in element-1, that introduced it. Corresponding vector potential is $A = (\mu_0 / 4\pi) \int (J / r) dv$ and the magnetic field is $H = \nabla \times A$ fitted over the area of loop-2. The flux threading the two loop-2 with mutual inductance M_{21} is $\phi_2 = M_{21} I_1$. Note the M_{21} is 'complex quantity' if the distance between elements becomes comparable to wave-length.

Mutual inductance is positive if magnetic lines cross the two loops in the same direction, and negative if the magnetic lines are in opposite direction



Axial coupling $M > 0$



Planer coupling $M < 0$

The MI waves considered here are threaded to SRR which does not form a very long line so that retardation effect and its losses due to radiation are not considered presently. So the MI lines are 'short'.

The Dispersion expression

From the KVL of the *n*-th SRR we have

$$Z_0 I_n + j\omega M I_{n-1} + j\omega M I_{n+1} = 0$$

$$[R + j\omega L + (j\omega C)^{-1}] I_n = -j\omega M (I_{n-1} + I_{n+1})$$

$$I_n = I_0 e^{-jkna} \qquad \text{wave-solution assumed and substitute}$$

$$\left(R + \frac{1}{j\omega C} + j\omega L \right) I_n = -j\omega M (I_{n-1} + I_{n+1})$$

$$\frac{j\omega RC + 1 + j^2\omega^2 LC}{j\omega C} I_0 e^{-jkna} = -j\omega M I_0 (e^{-jk(n-1)a} + e^{-jk(n+1)a})$$

$$j\omega RC + 1 - \omega^2 LC = -j^2\omega^2 MC (e^{jka} + e^{-jka}) = \omega^2 2MC \cos(ka)$$

Take R as zero lossless; and resonant frequency of SRR as ${\omega_0}^2 = (1/LC)$

$$1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2} \frac{2M}{L} \cos ka = 0$$

$$\frac{\omega^2}{{\omega_0}^2} \left(1 + \frac{2M}{L} \cos ka \right) = 1 \qquad \text{This is dispersion relation}$$

Dispersion *R* not equal to zero a lossy case with *k* as complex number (with attenuation)

From previous expansion of KVL for the *n*-th SRR with loss resistance *R* we have

$$j\omega RC + 1 - \omega^2 LC = -j^2\omega^2 MC(e^{jka} + e^{-jka}) = \omega^2 2MC\cos(ka)$$

$$\frac{j\omega^2 RLC}{\omega L} + 1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\kappa\cos ka = 0 \qquad \kappa = \frac{2M}{L} \qquad \text{put} \qquad Q = \frac{\omega L}{R}$$

$$j\frac{\omega^2}{\omega_0^2}\frac{1}{Q} + 1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\kappa\cos(\beta a - j\alpha a) = 0 \qquad \text{where} \qquad k = \beta - j\alpha$$

$$j\frac{\omega^2}{\omega_0^2}\frac{1}{Q} + 1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\kappa(\cos\beta a\cos j\alpha a + \sin\beta a\sin j\alpha a) = 0$$

$$\text{with} \qquad \cos jx = \cosh x \qquad \sin jx = j\sinh x$$

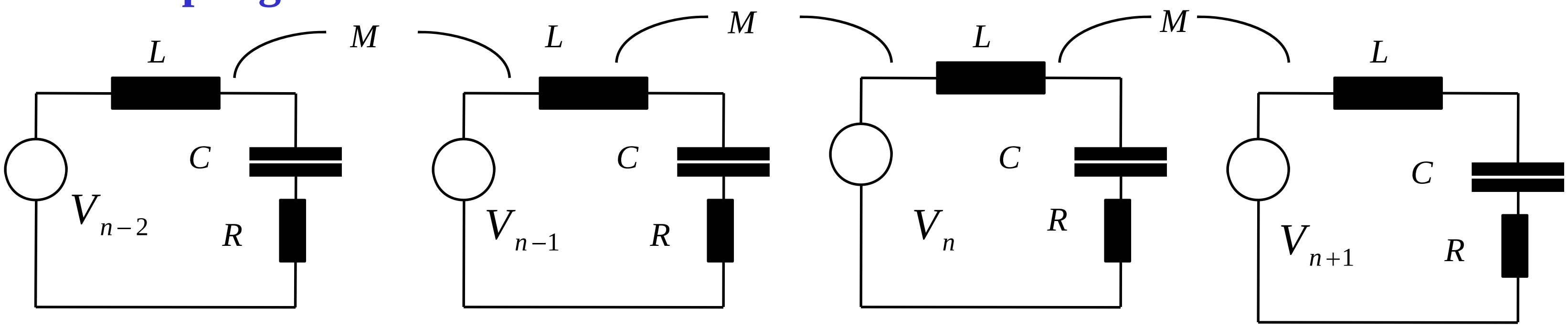
$$j\frac{\omega^2}{\omega_0^2}\frac{1}{Q} + 1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\kappa\cos\beta a\cosh\alpha a - j\frac{\omega^2}{\omega_0^2}\kappa\sin\beta a\sinh\alpha a = 0$$

segregating real & imaginary parts, and equating to zero

$$1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\kappa\cos\beta a\cosh\alpha a = 0$$

$$\frac{1}{Q} - \kappa\sin\beta a\sinh\alpha a = 0$$

Circuit coupling



$$Z_0 I_n + j\omega M (I_{n-1} + I_{n+1}) = 0 \qquad \text{KVL for } n^{th} \text{ loop}$$

$$Z_0 = j\omega L + \frac{1}{j\omega C} + R \qquad \text{self impedance}$$

Assume wave solution in form: $I_n = I_0 e^{-jkna}$ where $k = \beta - j\alpha$ complex quantity with β as propagation constant, α as attenuation. The dispersion relation

$$\omega = \omega_0 \left(1 + \frac{2M}{L}\cos ka\right)^{-1/2}$$

may be separated into real and imaginary parts yielding

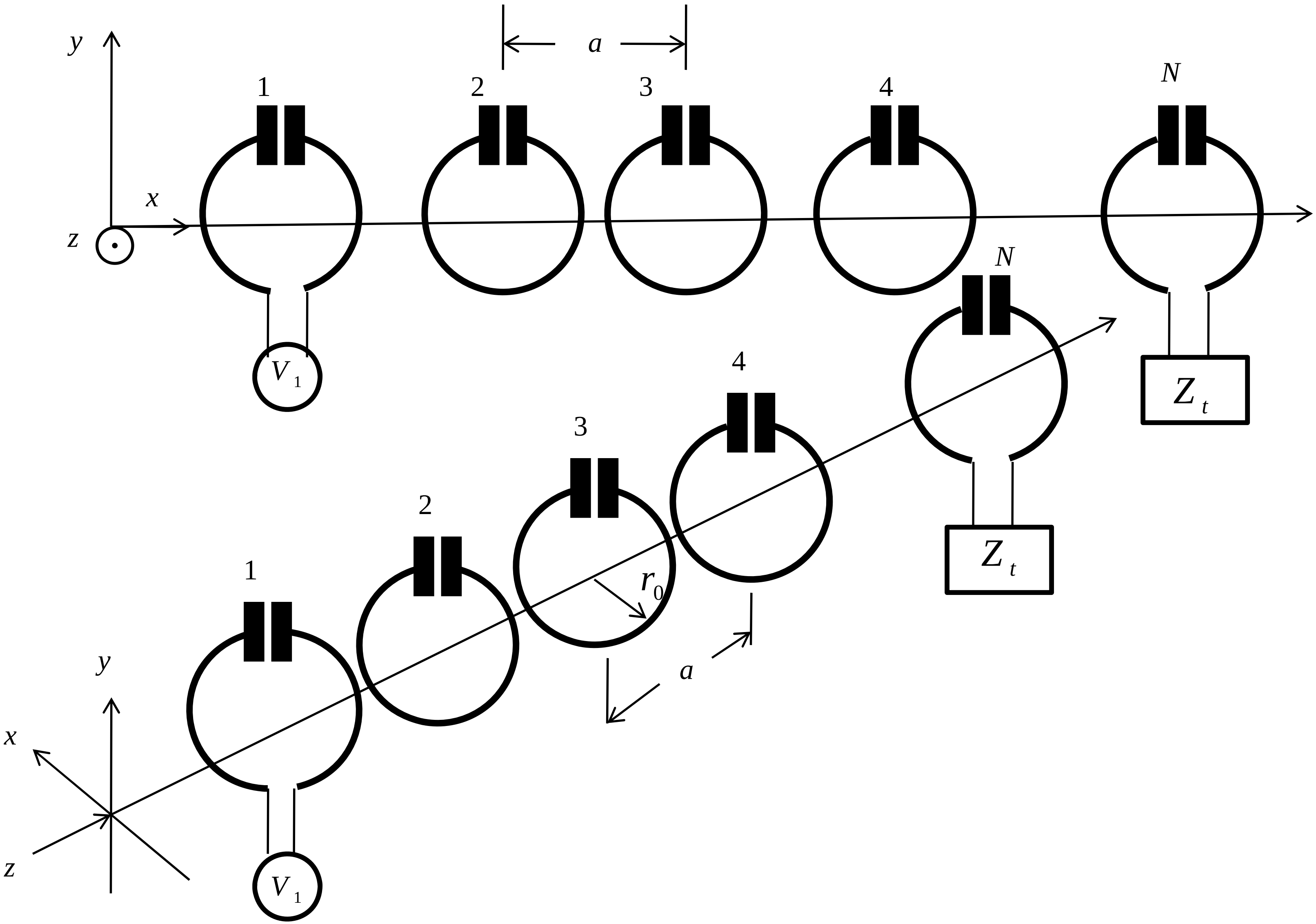
$$1 - \frac{\omega^2}{\omega_0^2}[1 + \kappa\cos(\beta a)\cosh(\alpha a)] = 0$$

$$\frac{1}{Q} - \kappa\sin(\beta a)\sinh(\alpha a) = 0$$

$$\kappa = 2M / L \qquad \text{coupling coefficient}$$

$$Q = \omega L / R \qquad \text{losses}$$

Planer and Axial coupling of SRR with excitation and termination to form ‘backward’ and ‘forward’ MI waves



Small losses dispersion expression

$$1-\frac{\omega^2}{\omega_0^{\,2}}[1+\kappa\cos(\beta a)\cosh(\alpha a)]=0$$

and

$$\frac{1}{Q}-\kappa\sin(\beta a)\sinh(\alpha a)=0$$

$\alpha \cong 0 \qquad \cosh\alpha a = 1 \qquad$ putting above

$$1-\frac{\omega^2}{\omega_0^{\,2}}[1+\kappa\cos(\beta a)]=0$$

$\omega = \omega_0 / \sqrt{1+\kappa\cos(\beta a)}$ is loss - less dispersion !

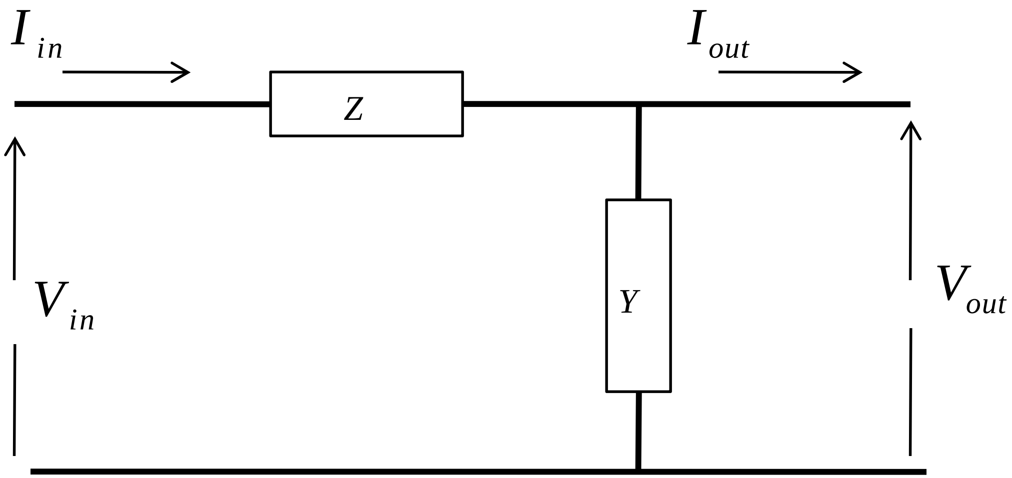
If the losses are small then and thus attenuation $\alpha \cong 0$ thus $\cosh(\alpha a) \cong 1$ and $\sinh(\alpha a) \cong \alpha a$ which means in the dispersion equation for phase change per element remains same, and losses per element given as:

$$\alpha a = \frac{1}{\kappa Q \sin(\beta \alpha)}$$

It may be expected that losses decline as the coupling coefficient κ and Q increases

Waves on four pole

Similar to part-2 where explanation of dispersion given through TL circuit approach



$$V_{in} = I_{in} Z + V_{out}$$

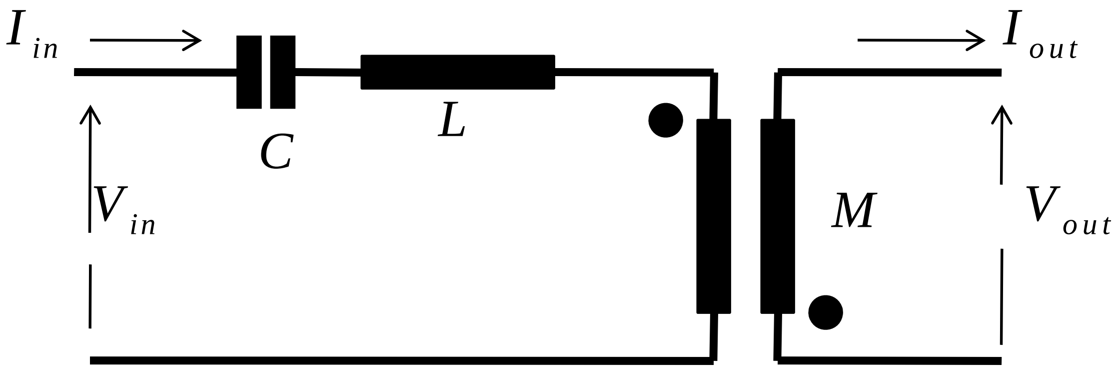
$$V_{out} = \frac{I_{in} - I_{out}}{Y}$$

$$\begin{bmatrix} V_{out} \\ I_{out} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} V_{in} \\ I_{in} \end{bmatrix}$$

$b_{11} = 1$ $\quad b_{12} = -Z$ $\quad b_{21} = -Y$ $\quad b_{22} = 1 + YZ$

General dispersion equation $\quad 2\cos ka = b_{11} + b_{22}$

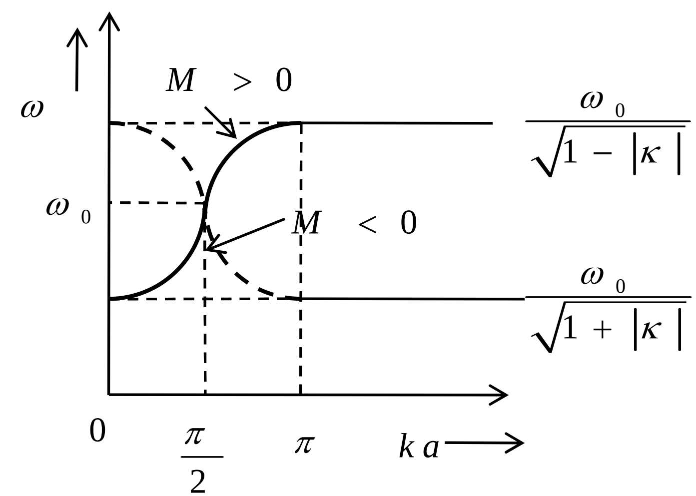
Coupled SRR circuit and dispersion



$$B = \begin{bmatrix} 0 & -j\omega M \\ -\dfrac{1}{j\omega M} & -\dfrac{L}{M}\left(1 - \dfrac{\omega_0^2}{\omega^2}\right) \end{bmatrix}$$

$$2\cos ka = b_{11} + b_{22} = -\frac{L}{M} + \frac{L}{M}\frac{\omega_0^2}{\omega^2}$$

$$\omega = \frac{\omega_0}{\sqrt{1 + \kappa\cos(ka)}} \qquad \kappa = 2M/L$$

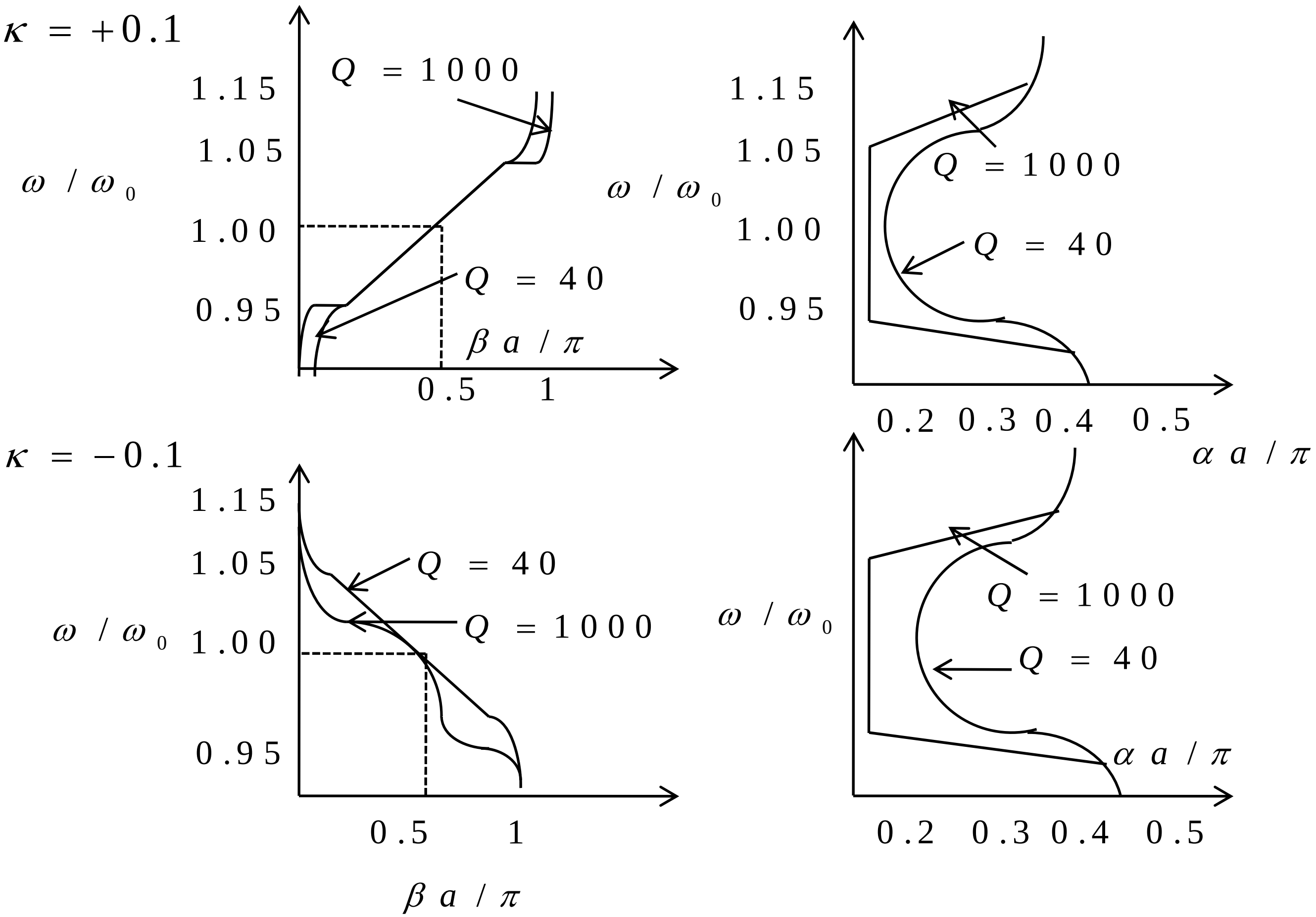


Dispersion lossless case

$\alpha = 0 \quad Q = \infty \quad \kappa = \pm 0.1$

Backward wave for $M < 0$

Forward and Backward MI wave dispersion in lossy case

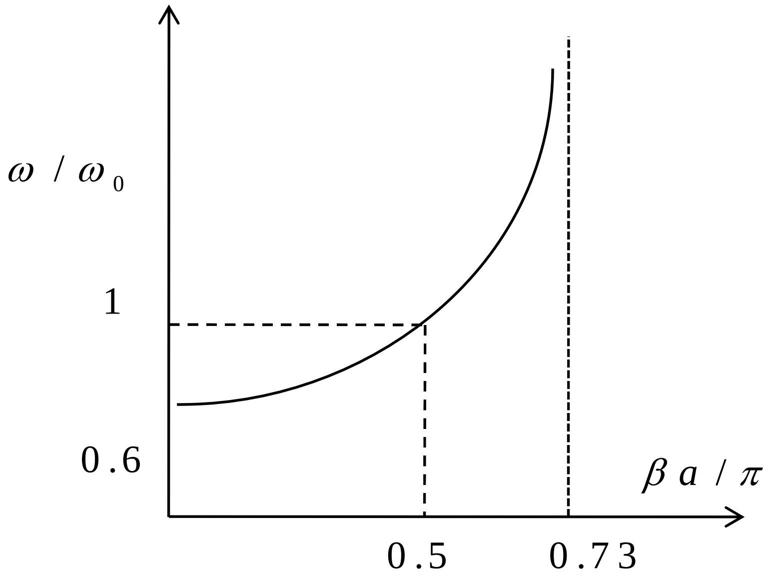


About value of coupling

The coupling κ need not be small as 0.1, but may be of large value say 2, in case of axial configuration when elements are very closely packed, and can be as high as -0.7 in planer configuration.

For $\kappa = +1.5$ and $Q = 100$ the dispersion curve is having a quite different look; but still there is lower frequency cut-off, but the bandwidth is extended to arbitrary high frequency with asymptote at

$$\beta a = \arccos(1/\kappa) = 0.73\pi$$



Matching of MI Transmission line (TL)

From what we have so far MI waves can propagate along the array whether planer or axial.

Such array can thus be regarded as TL having somewhat unusual dispersion.

For a TL a terminal impedance exists that can absorb all the incident power, this is called matching.

Waves travel from source to load but no reflection for matched case.

Relation between currents I_{n-1}, I_n and I_{n+1} is $Z_0 I_n + j\omega M (I_{n-1} + I_{n+1}) = 0$ true everywhere except the 1st and N-th element.

Can we substitute impedance for missing (N+1)-th element? KVL for the N-th element is

$$(Z_0 + Z_T) I_N + j\omega M I_{N-1} = 0$$

For a traveling wave (we do not want standing wave) the I_N and I_{N-1} are related by factor e^{-jka} whence Z_T may be obtained with aid of dispersion equation as $Z_T = j\omega M e^{-jka}$

$$Z_0 I_N + j\omega M I_{N-1} + Z_T I_N = 0$$

$$Z_0 I_n + j\omega M I_{n-1} + j\omega M I_{n+1} = 0 \qquad I_n = I_0 e^{-jkna}$$

It turns out that Z_T is not a constant as for co-axial line but a complex and frequency dependent.

MI waves with excitation

The dispersion equation obtained on assumption that there is no external excitation. As it happens it is quite easy to include possible excitation by ideal voltage source in any element. The excitation may come as incident plane wave or from separate voltage source. So we can write: $[V] = [Z][I]$
Where $[V] = [V_1, V_2, V_3, \ldots\ldots V_N]^T$, the $[I] = [I_1, I_2, I_3, \ldots I_N]^T$ and impedance is

$$[Z] = \begin{bmatrix} Z_0 & j\omega M & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ j\omega M & Z_0 & j\omega M & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & j\omega M & Z_0 & j\omega M & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j\omega M & Z_0 & j\omega M & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j\omega M & Z_0 & j\omega M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & j\omega M & Z_0 & j\omega M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & j\omega M & Z_0 & j\omega M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j\omega M & Z_0 & j\omega M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & j\omega M & Z_0 & j\omega M \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j\omega M & Z_0 \end{bmatrix}$$

A tri-diagonal matrix

Terminated Matrix

In the system $[Z]_{N \times N}$ if the last element is terminated then then *NN*, the last diagonal element is to be $Z_{NN} = Z_0 + Z_T$

For a known excitation $[V]$, we have to get: $[I] = [Z]^{-1}[V]$

Very often the first element is excited so $[V] = [V_1, 0, 0, 0, 0, \ldots\ldots..0]^T$

Well, this matrix representation enables us to get eigen solutions, the natural modes of the system!

Eigenvectors and Eigenvalues

Required to determine natural modes of the system (which resonates at n = 1, 2, ……., N).

For a loss less case get these for N elements.

Postulate that there are elements at n = 0 and N + 1 sites but the current is zero at that sites.

Also assume the current vary sinusoidal between these to (virtual) ends.

Hence the element of the l^{-th} eigenvector may be in form

$$I_n^{(l)} = I(N)\sin\left(\frac{nl\pi}{N+1}\right) \qquad n = 0,1,2,\ldots N, N+1$$

This expression gives current in n-th element

The I (N) is got from orthonormality , i.e. $I^{(l)}.I^{(m)} = \begin{cases} 0 & \text{if} \quad l \neq m \\ 1 & \text{if} \quad l = m \end{cases}$ Yielding $I(N) = \frac{2}{N+1} I_0$

The corresponding eigenvalues may be obtained from definition of eigenvalue $[Z][I]^{(l)} = \lambda_l [I]^{(l)}$ as $\lambda_l = Z_0 + 2j\omega M \cos\left(\frac{l\pi}{N+1}\right)$. Having the eigenvectors & eigenvalues then for voltage $[V]$ considering the matrix can be expanded in terms of eigenvectors like

$$V = \sum_{l=1}^{N} \mu_l [I]^{(l)} \text{ where } \mu_l = [V][I]^{(l)}$$

$$I = \sum_{l=1}^{N} (\mu_l / \lambda_l)[I]^{(l)}$$

Excitation of eigenvectors

From the previous discussions, for a general case all the eigenvectors are excited.

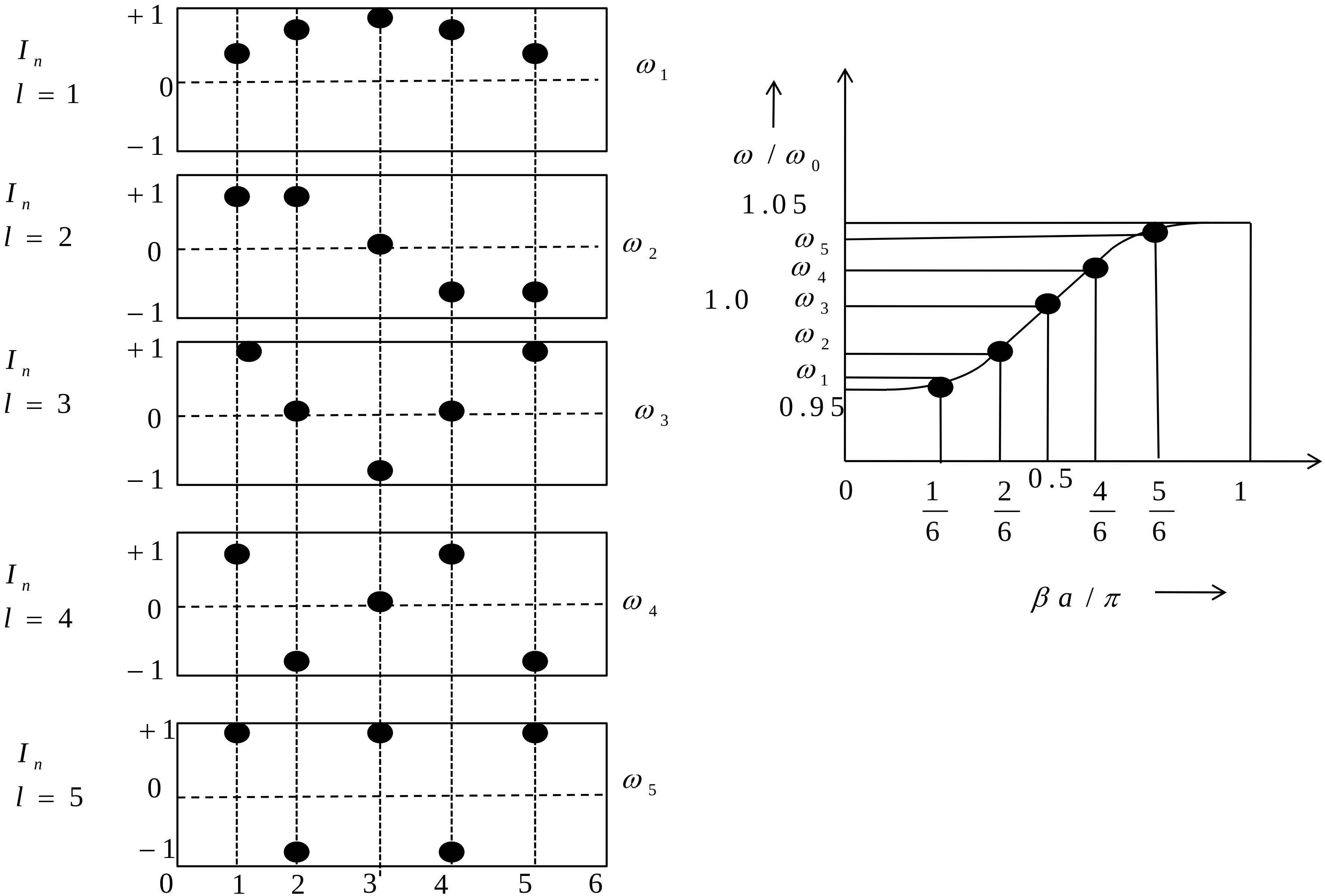
In order to excite a single mode, the corresponding eigenvalue should be close to zero

$$\lambda_l = Z_0 + 2j\omega M \cos\left(\frac{l\pi}{N+1}\right) = 0$$

This is the dispersion equation.

A single mode, i.e. a single value of ka can be excited for discrete set of frequencies.

For $N = 5$ elements and $\kappa = 0.1$ for axial configuration the eigenvectors example



Eigenvalues and eigenvectors for five element SRR

Eigenvectors for each element SRR

$$I_n^{(l)} = I(N)\sin\left(nl\pi/(N+1)\right) \qquad I(5) = (2/6)I_0 = (1/3)I_0$$

five eigenvectors for element – 1

$$I_1^{(1)} = \frac{I_0}{3}\sin\frac{\pi}{6} \quad I_1^{(2)} = \frac{I_0}{3}\sin\frac{2\pi}{6} \quad I_1^{(3)} = \frac{I_0}{3}\sin\frac{3\pi}{6} \quad I_1^{(4)} = \frac{I_0}{3}\sin\frac{4\pi}{6} \quad I_1^{(5)} = \frac{I_0}{3}\sin\frac{5\pi}{6}$$

five eigenvectors for element – 2

$$I_2^{(1)} = \frac{I_0}{3}\sin\frac{2\pi}{6} \quad I_2^{(2)} = \frac{I_0}{3}\sin\frac{4\pi}{6} \quad I_2^{(3)} = \frac{I_0}{3}\sin\frac{6\pi}{6} \quad I_2^{(4)} = \frac{I_0}{3}\sin\frac{8\pi}{6} \quad I_2^{(5)} = \frac{I_0}{3}\sin\frac{10\pi}{6}$$

five eigenvectors for element – 3

$$I_3^{(1)} = \frac{I_0}{3}\sin\frac{3\pi}{6} \quad I_3^{(2)} = \frac{I_0}{3}\sin\frac{6\pi}{6} \quad I_3^{(3)} = \frac{I_0}{3}\sin\frac{9\pi}{6} \quad I_3^{(4)} = \frac{I_0}{3}\sin\frac{12\pi}{6} \quad I_3^{(5)} = \frac{I_0}{3}\sin\frac{15\pi}{6}$$

and so on..............

The five Eigenvalues with corresponding points on the dispersion curve

$$\lambda_1 \qquad Z_0 + 2j\omega_1 M\cos(\pi/6) = 0 \qquad \beta a/\pi = 1/6 \qquad \omega_1$$

$$\lambda_2 \qquad Z_0 + 2j\omega_2 M\cos(2\pi/6) = 0 \qquad \beta a/\pi = 1/3 \qquad \omega_2$$

$$\lambda_3 \qquad Z_0 + 2j\omega_3 M\cos(3\pi/6) = 0 \qquad \beta a/\pi = 1/2 \qquad \omega_3 = \omega_0$$

$$\lambda_4 \qquad Z_0 + 2j\omega_4 M\cos(4\pi/6) = 0 \qquad \beta a/\pi = 2/3 \qquad \omega_4$$

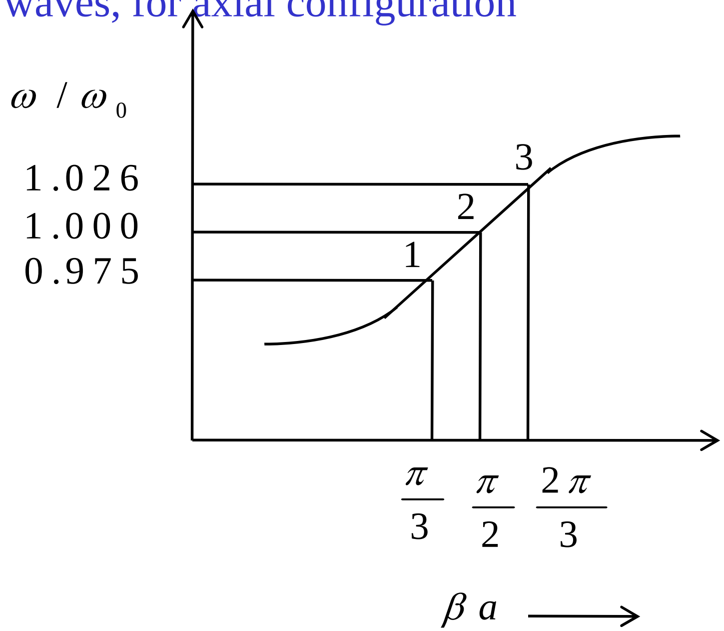
$$\lambda_5 \qquad Z_0 + 2j\omega_5 M\cos(5\pi/6) = 0 \qquad \beta a/\pi = 5/6 \qquad \omega_5$$

Travelling MI waves within pass band

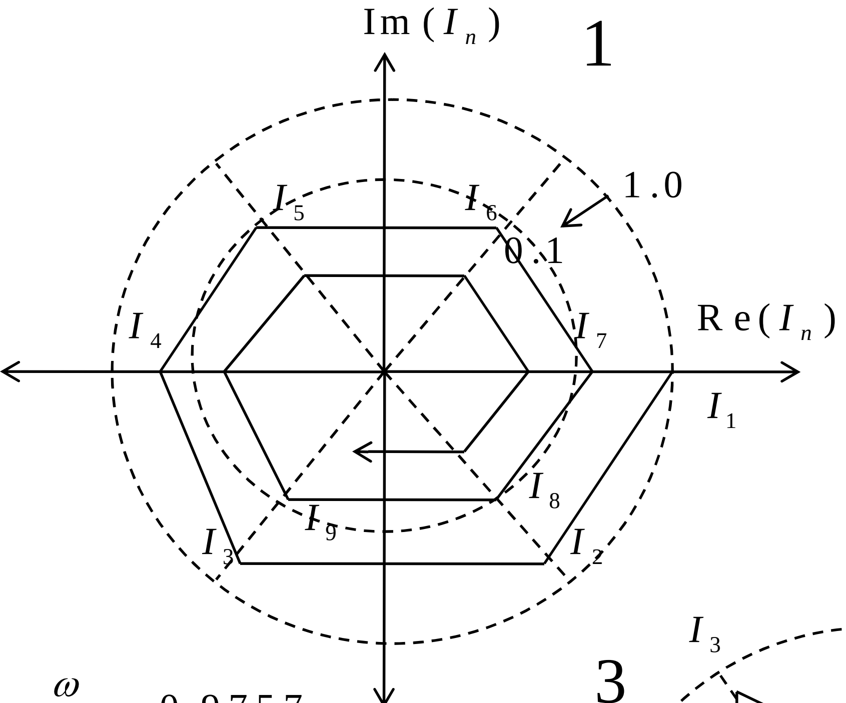
When the last element is terminated by Z_T , we get travelling MI waves, with first element excited $[V] = [V_1, 0, 0 \ldots\ldots..]^T$ calculate $[I] = [Z]^{-1}[V]$ for $N = 31$ $\kappa = 0.1$ $Q = 100$ for frequency points $(\omega / \omega_0) = 0.9757$ 1.00 1.0262 corresponding wave vector is thus,

$\beta a =$ $\pi / 3$ $\pi / 2$ $2\pi / 3$

Say for current in element one for frequency $\omega / \omega_0 = 0.9757$ the current in element-2 is $I_2 = I_1 e^{-\alpha a} e^{-\beta a} = I_1 e^{-\alpha a} e^{-(\pi/3)a}$ in element-3 will be $I_3 = I_1 e^{-\alpha 2a} e^{-2\beta a} = I_1 e^{-2\alpha a} e^{-2(\pi/3)a}$ and so on with attenuation $e^{-\alpha n}$and phase loss $-n(\pi/3)$ per element. This arrangement is for travelling forward MI waves, for axial configuration

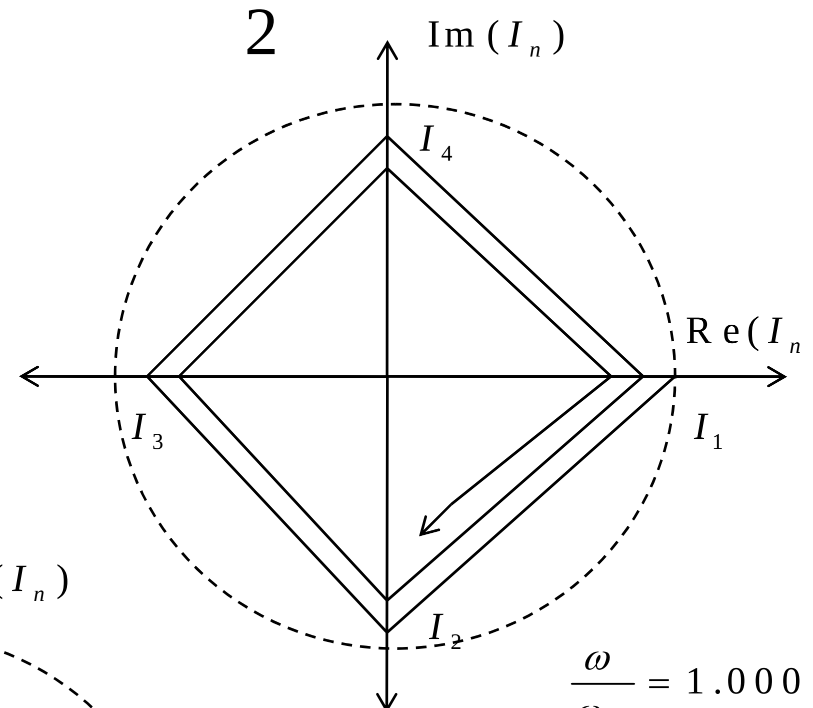


Travelling forward MI waves for axial configuration within pass-band



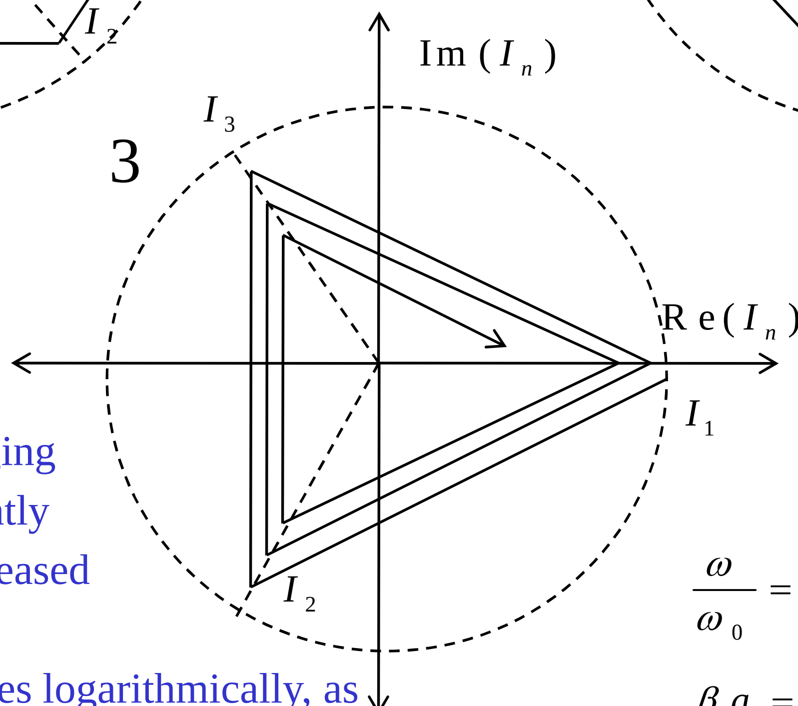
$$\frac{\omega}{\omega_0} = 0.9757$$

$$\beta a = \frac{\pi}{3}$$



$$\frac{\omega}{\omega_0} = 1.000$$

$$\beta a = \frac{\pi}{2}$$



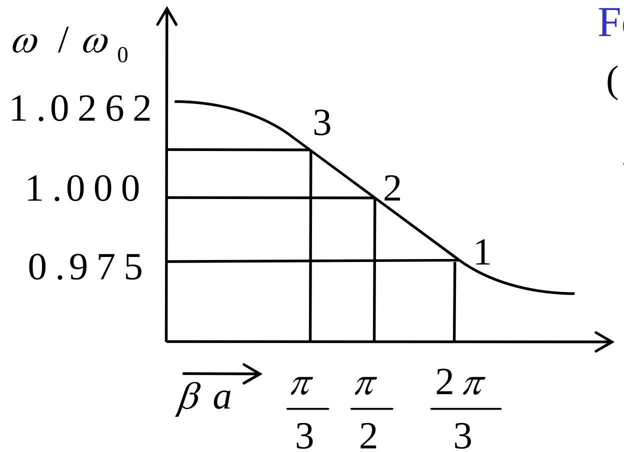
$$\frac{\omega}{\omega_0} = 1.0262$$

$$\beta a = \frac{2\pi}{3}$$

Annular spiral keeps changing the shape and becomes tightly wound as frequency is increased

The circular radius decreases logarithmically, as element number increases — well exponential attenuation.

Travelling MI backward waves in planer configuration within pass-band



For planer configuration, where the current in each element will gain phase

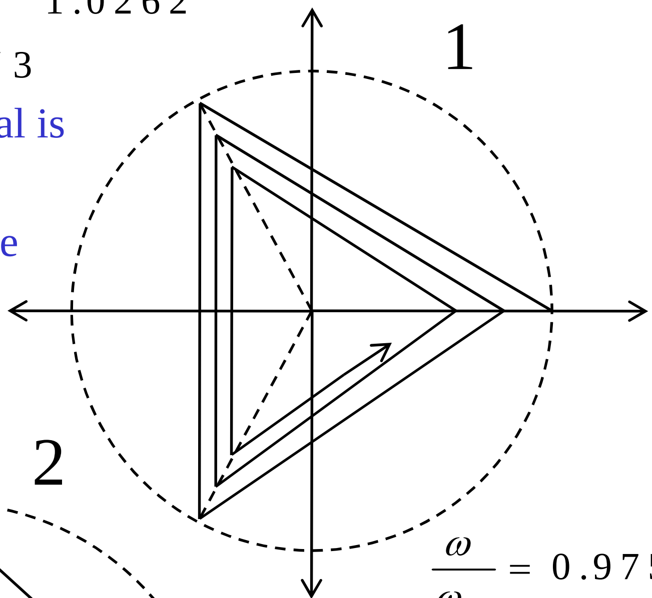
$(\omega / \omega_0) = 0.9757 \quad 1.00 \quad 1.0262$

$\beta a = \quad 2\pi/3 \quad \pi/2 \quad \pi/3$

1. The sense of rotation of spiral is changed.

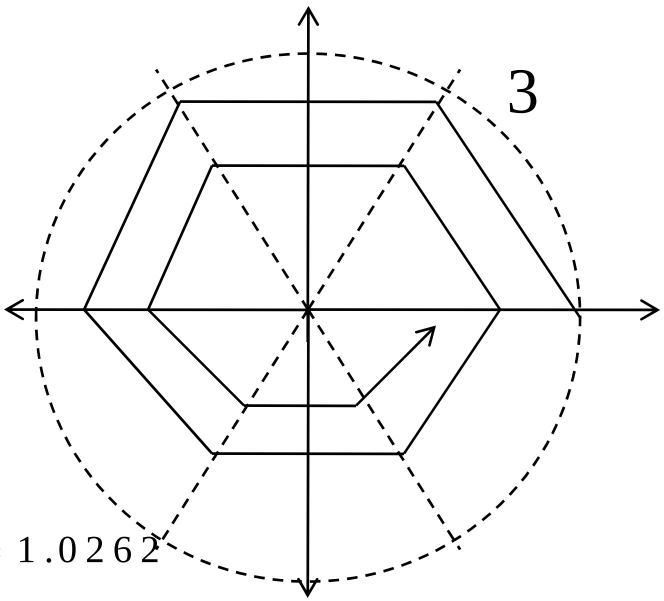
2. The higher the frequency the smaller the phase change from element to element.

3. This is due to backward nature of MI wave in planer array



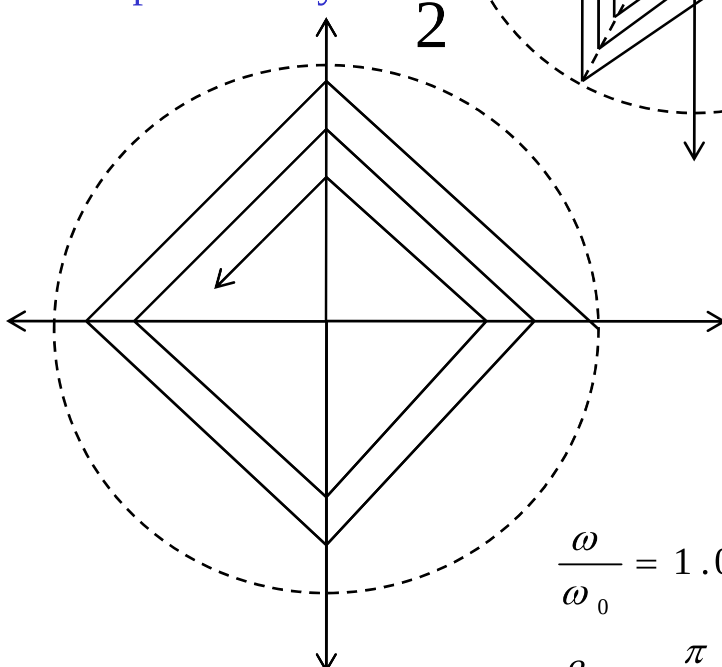
$\frac{\omega}{\omega_0} = 0.9757$

$\beta a = \frac{2\pi}{3}$



$\frac{\omega}{\omega_0} = 1.0262$

$\beta a = \frac{\pi}{3}$



$\frac{\omega}{\omega_0} = 1.000$

$\beta a = \frac{\pi}{2}$

Resonant MI waves within pass-band

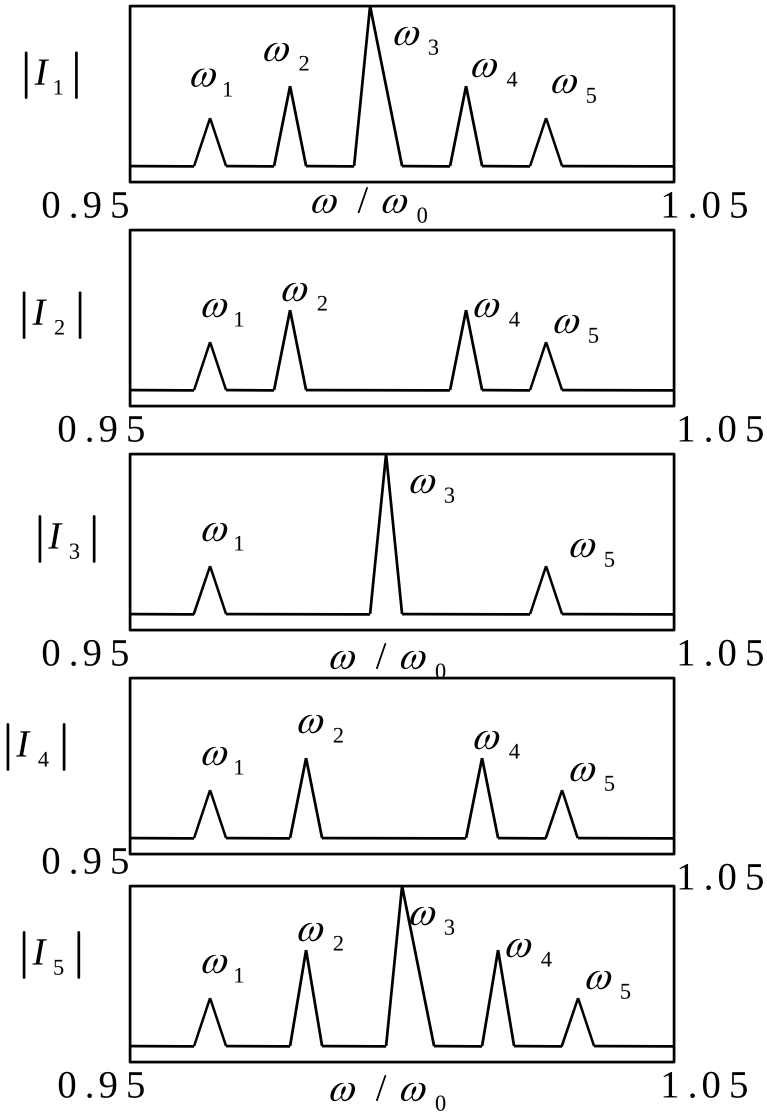
The line is not terminated by its matching impedance, reflection of MI waves from line's end takes place and standing wave pattern for the current is formed. Resonances are due to standing waves. The standing wave that have integral number of half periods corresponds to the eigenvectors of the system, which manifests themselves as resonances. In general line comprising of *N* discrete frequencies the eigenvalues are given by

$$Z_0 + 2j\omega M\cos[l\pi/(N+1)] = 0 \qquad l = 1,2,3,4\ldots N \qquad \text{corresponds to} \quad \omega_1, \omega_2, .., \omega_N$$

It is clear that all the resonances $\omega_1, \omega_2, \ldots\omega_N$ are in pass-band.

Natural resonances within MI pass band

First element is excited. The relative current amplitudes in all the elements $N = 5$, five resonances



This shows relative amplitudes of currents in elements 1 to 5; as a function of frequency. For this system the 1st element is excited. There are 5 resonant frequencies. Note that the 3rd resonance is missing in element 2 and element-4, and the 2nd and 4th resonances are missing in element-3

$$Z_0 + 2j\omega M \cos\left(\frac{l\pi}{N+1}\right) = 0$$

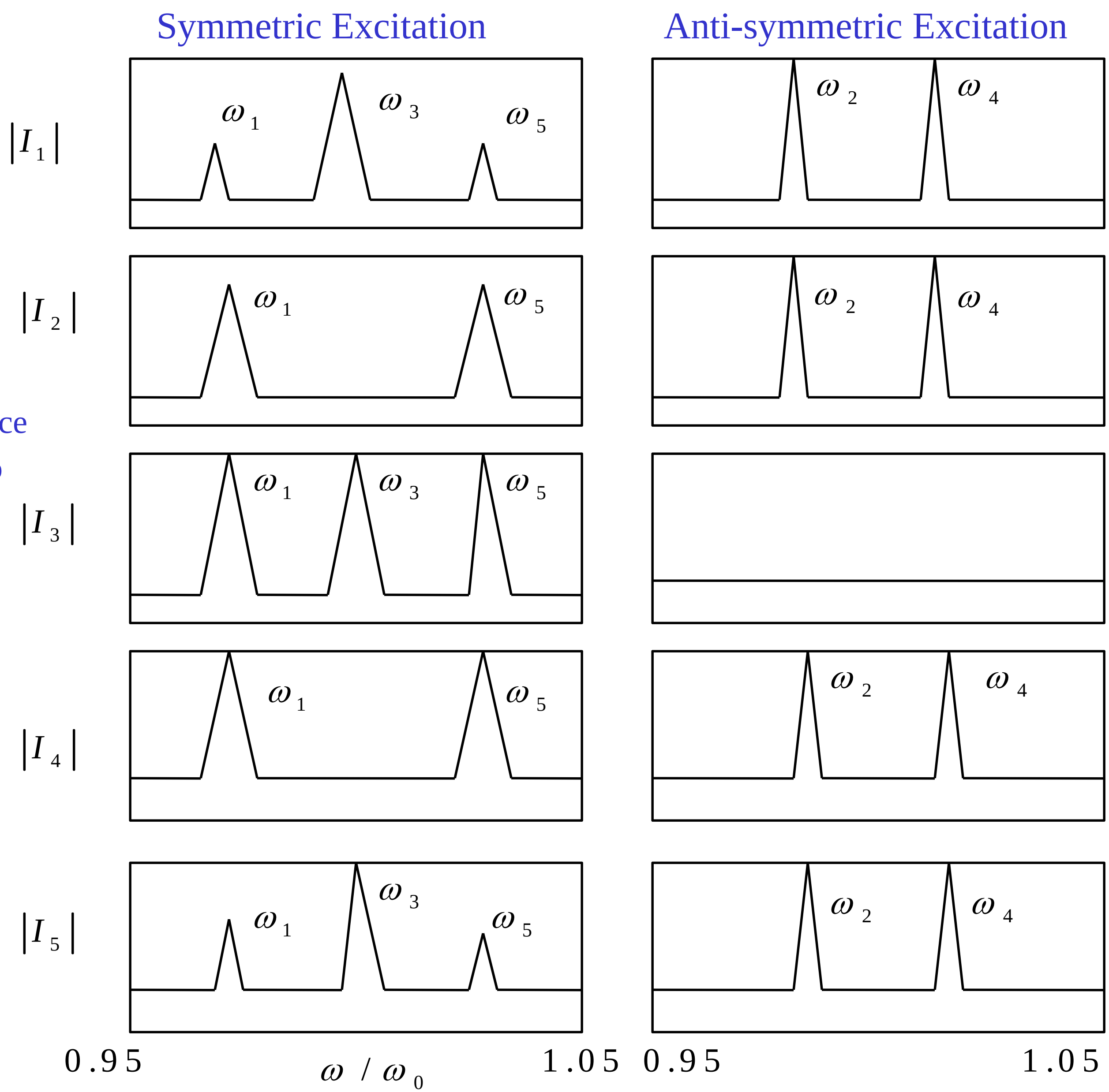
$$l = 1, 2, 3, 4, 5$$

$$\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$$

Selective excitation/Selection of resonance within pass-band

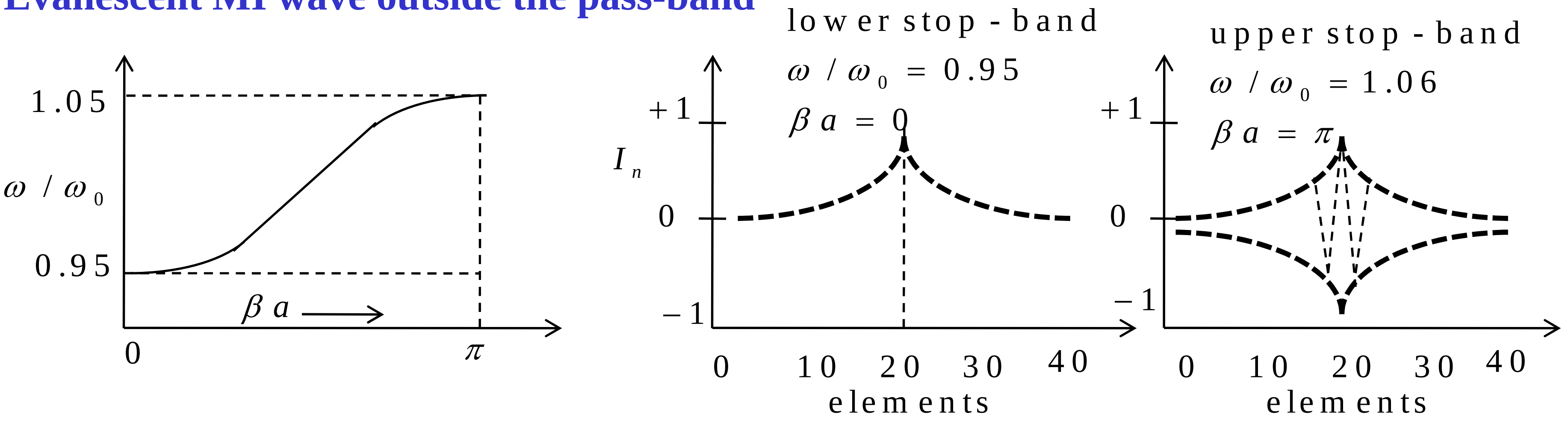
$\kappa = 0.1 \qquad Q = 100$

For symmetric excitation the central SRR is excited. Due to symmetry it is obvious that 2nd and 4th resonance that corresponds to anti-symmetric eigenmodes are missing



First and the last SRR is excited by external voltage source in anti-phase. In this case the central loop is not excited at all and only the 2nd and 4th resonances are present.

Evanescent MI wave outside the pass-band



In lower stop band $\beta = 0$ currents in all elements are in phase, in the upper stop band $\beta a = \pi$ the currents in neighboring elements are always in anti-phase. Centrally excited 20th element of the 41 SRR current amplitude decays strongly as we move from element that is driven by external source.

This is very important when operating in the stop band of MI, line is able to replicate pattern of excitation by pixel to pixel mechanism; well this is significant in design of MI near field lens!!

Poynting vector of MI waves

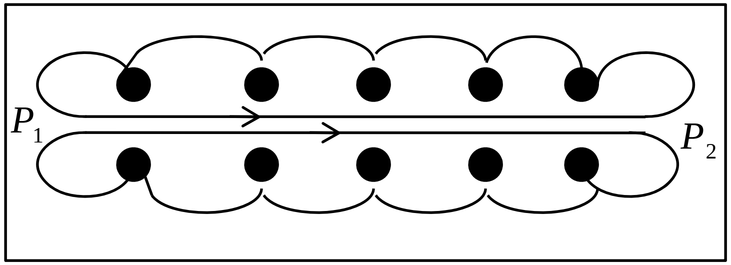
Current flowing in loop create magnetic field and magnetic vector potential $A = (\mu_0 / 4\pi) \int (\vec{J} / r) dv$
Here integration is over region where $\vec{J}$ is finite. Then via $H = \nabla \times A$, $E = -\nabla \phi - (\partial A / \partial t)$
and $I = \int H ds$. Having got H and E field Poynting vector is determined.

1 . Assume $[V] = [V_1, 0, 0, 0 \ldots\ldots]^T$

2. Use $[I] = [Z]^{-1} [V]$ to get $[I]$

3 . Find *H* and *E* field of each loop current

4 . Add vectorically to find total *E* and *H*



The stream lines originates on the first loop and reach by following variety of path, the last loop; terminated by matched impedance. Stream lines approaching the load from inside and outside are separated from each other by a so called P-point, where Poynting vector is zero.

Energy in MI wave

We can use Poynting vector and its integration between the two elements to determine the power. Also by analytical method by multiplying the stored energy per unit distance the group velocity, we can get power in MI wave. Stored energy is in *L*, *C*, and *M*

$$W = \frac{1}{2} L \left| I_n \right|^2 + \frac{1}{2} C \left| V_n \right|^2 + \frac{1}{2} M \left(I_n I_{n-1}^* + I_n I_{n+1}^* \right)$$

Here V_n is not the applied voltage, but the gap voltage across the capacitor of the *n*-th element related to current as: $I_n = j\omega C V_n$. Assuming travelling wave solution relating I_{n-1}, I_{n+1} to I_n, we write $I_{n+1} = I_n e^{+j\beta a}$ and $I_{n-1} = I_n e^{-j\beta a}$ also with aid of dispersion relation we get from *W* above

$$W = \frac{1}{2} L \left| I_n \right|^2 + \frac{1}{2} C \left| \frac{I_n^2}{(j\omega C)^2} \right| + \frac{1}{2} M \left(I_n I_n e^{-j\beta a} + I_n I_n e^{+j\beta a} \right)$$

$$= \frac{I_n^2}{2} \left[L + \frac{1}{\omega^2 C} + 2M \cos \beta a \right]$$

$$= \frac{I_n^2}{2\omega^2 C} \left[\omega^2 LC + 1 + 2\omega^2 MC \cos \beta a \right]$$

$$= \frac{I_n^2 L}{2\omega^2 CL} \left[\frac{\omega^2}{\omega_0^2} + 1 + \frac{2M}{L} \frac{\omega^2}{\omega_0^2} \cos \beta a \right] \qquad \omega_0^2 = (1/LC)$$

$$= \frac{I_n^2 L}{2} \frac{\omega_0^2}{\omega^2} \left[\frac{\omega^2}{\omega_0^2} \left(1 + \frac{2M}{L} \cos \beta a \right) + 1 \right] \qquad \frac{\omega^2}{\omega_0^2} = \frac{1}{1 + \kappa \cos \beta a}$$

$$= \frac{\omega_0^2}{\omega^2} L \left| I_n \right|^2$$

Group Velocity Energy and Power in MI wave

Energy is $W = \frac{\omega_0^2}{\omega^2} L |I_n|^2$

The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{\omega_0}{\sqrt{1+(2M/L)\cos ka}}\right)$$

$$= \frac{\omega_0 a M}{L} \frac{\sin ka}{\left(\sqrt{1+(2M/L)\cos ka}\right)^3}$$

$$= \frac{\omega_0 a M}{L}\left(\frac{\omega}{\omega_0}\right)^3 \sin ka = \frac{\omega_0 a}{2}\kappa\left(\frac{\omega}{\omega_0}\right)^3 \sin ka \qquad \kappa = 2M/L$$

Power in MI wave is

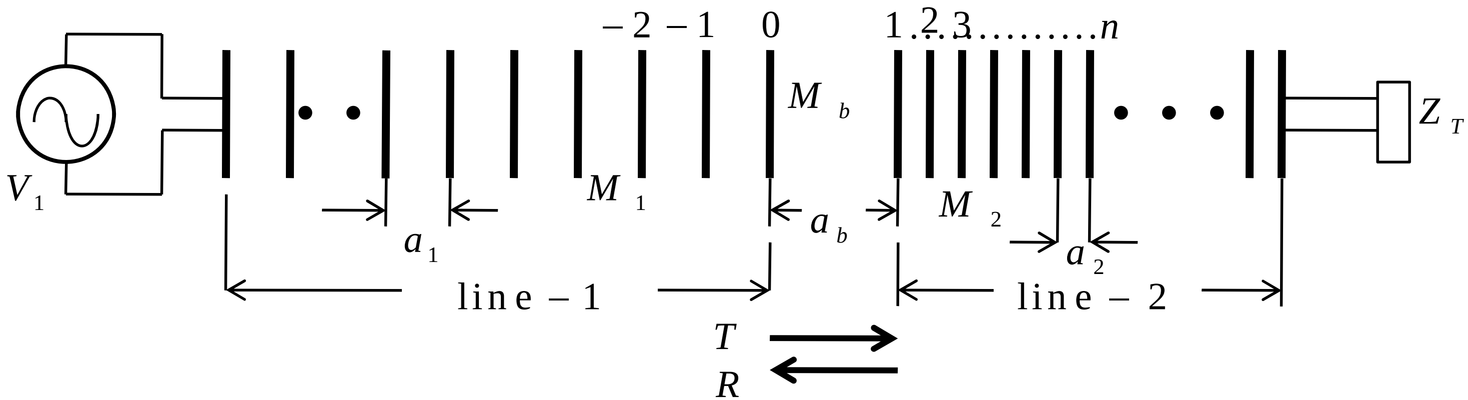
$$P = W v_g = \frac{\omega_0^2}{\omega^2} L |I_n|^2 \times \frac{\omega_0 a}{2}\kappa\left(\frac{\omega}{\omega_0}\right)^3 \sin ka$$

$$= \frac{1}{2}\omega M |I_0|^2 \sin ka \qquad \text{making} \quad I_n = I_0$$

Note that no power can be transferred at the band edges where group velocity is zero and optimum power is transferred at resonant frequency ω_0 where $ka = \pi/2$

Reflection and Transmission of MI waves at the boundary interface

Perpendicular incidence



$$\text{line}-1 \qquad I_n = I_{00}\left[e^{-jn(ka)_1} + R.e^{+j(ka)_1}\right]; \quad n \le 0 \text{........}(1)$$

$$\text{line}-2 \qquad I_n = I_{00}Te^{-j(ka)_2}; \quad n > 0 \text{............}(2)$$

KVL for element 0 and 1 on the opposite sides of the boundary as:

$$Z_{01}I_0 + j\omega M_1 I_{-1} + j\omega M_b I_1 = 0, \text{................}(3)$$

$$Z_{02}I_1 + j\omega M_2 I_2 + j\omega M_b I_0 = 0, \text{.................}(4)$$

Substituting (1) and (2) into (3) and (4) we get:

$$R = \frac{M_b^2 e^{-j(ka)_2} - M_1 M_2 e^{-j(ka)_1}}{M_1 M_2 e^{j(ka)_1} - M_b^2 e^{-j(ka)_2}} \text{.........}(5)$$

$$T = \frac{2jM_1 M_2 \sin(ka)_1}{M_1 M_2 e^{j(ka)_1} - M_b^2 e^{-j(ka)_2}} \text{............}(6)$$

Reflection and Transmission coefficients for MI waves

$$\text{For} \quad M_1 = M_b = M_2$$

$$R = \frac{e^{-j(ka)_2} - e^{-j(ka)_1}}{e^{j(ka)_1} - e^{-j(ka)_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$T = \frac{2j\sin(ka)_1}{e^{j(ka)_1} - e^{-j(ka)_2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

The above situation can arise when difference in the two media are in self impedances Z_{01} and Z_{02} hence the two media have different dispersion curves.

A further simplification of this for continuous limit when $(ka)_1 << 1$ and $(ka)_2 << 1$

Then

$$R = \frac{k_1 - k_2}{k_1 + k_2} \quad \text{and} \quad T = \frac{2k_1}{k_1 + k_2} \ldots\ldots\ldots\ldots\ldots(9)$$

There are no periodic media in the above, a case of simply a wave incident from media of propagation constant k_1 to k_2 .

$$P_1 = \frac{1}{2}\left(1 - |R|^2\right)\omega M_1 \sin(ka)_1 \ldots\ldots\ldots(10)$$

$$P_2 = \frac{1}{2}|T|^2 \omega M_2 \sin(ka)_2 \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Substituting (5) and (6) we get $P_1 = P_2$

Higher Order Interaction

Till now we have considered ‘nearest neighbor’ interaction only. This is usually a very good approximation when there is a ‘fast decay’ of the fields away from the element. Fast decay means ‘cubic’ decay as would be the case for elements that can be regarded a ‘static magnetic monopole’. The KVL for SRR array will thus be modified as: $Z_0 I_n + j\omega \sum_{m=1}^{\infty} M_m (I_{n+m} + I_{n-m}) = 0$. Where M_m is the mutual inductance between two elements a distance ma from each other.

Assume a wave solution $I_n = I_0 e^{-jkna}$, with this the dispersion can be obtained as earlier as:

$$1 - \frac{\omega_0^2}{\omega^2}[1 + \sum_{m=1}^{\infty} \kappa_m \cos(mka)] = 0$$

Where , $\kappa_m = 2M_m / L$ and above is loss less case

Well, if κ_m or decay of mutual inductances is cubic one, there is not significant difference but if the κ_m decay is quadratic then higher interactions are effective.

Can we say anything more in general terms? When κ_m can be expanded into a series in inverse power of distance as $\kappa_m = \sum_{i=2}^{\infty} c_i (na)^{-i}$; the dispersion equation reduces to

$$1 - \frac{\omega_0^2}{\omega^2}[1 + \sum_{i=2}^{\infty} c_i a^{-i} \sum_{n=1}^{\infty} n^{-i} \cos(nka)] = 0$$

Poly-logarithm function

$$\sum_{n=1}^{\infty} n^{-m} \cos(nka) = \frac{1}{2}\left[\mathrm{Li}_m(e^{jka}) + \mathrm{Li}_m(e^{-jka}) \right]$$

is a special function is monotonic. $\mathrm{Li}_s(z) \overset{\mathrm{def}}{=} \sum_{k=1}^{\infty} \frac{z^k}{k^s} \qquad |z| < 1$

$$\mathrm{Li}_1(z) = -\ln(1-z) \qquad \mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t} dt$$

Generalization of higher order interaction

The generalization to higher order interaction is also straight forward for the case of excitation

$$[V] = [Z][I]$$

But $[Z]$ is no longer a tri diagonal matrix

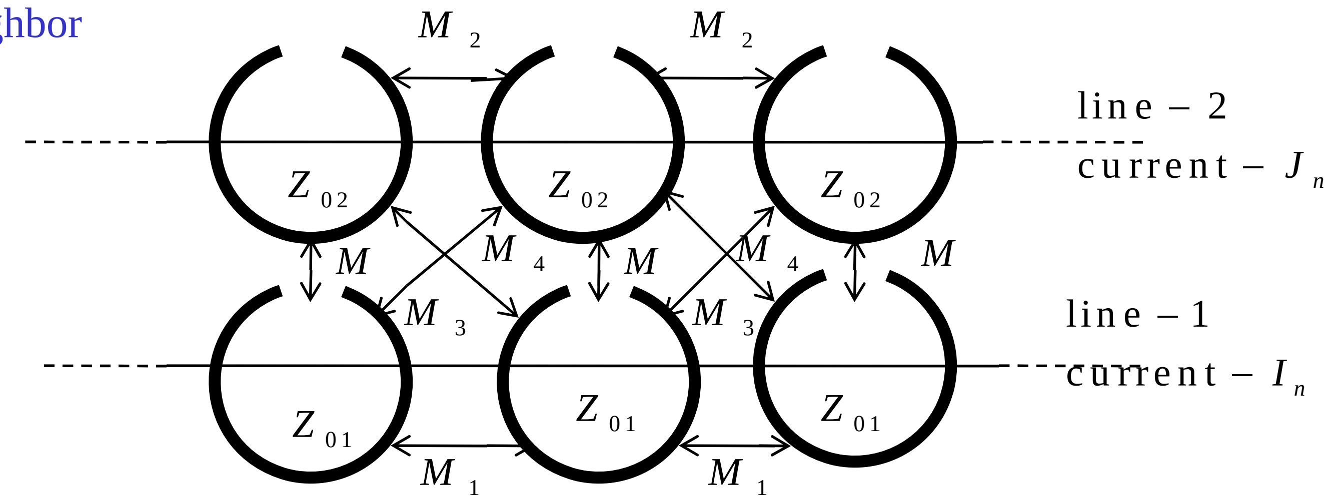
The main diagonal elements are still Z_0 but the off diagonal element are now equal to

$$Z_{ij} = j\omega M_{|i-j|} \qquad i \neq j$$

For a given set of $[V]$ the currents are $[I] = [Z]^{-1}[V]$

Coupled 1D lines

When two lines, both capable of carrying MI waves are coupled. Now there are not two but five nearest neighbor

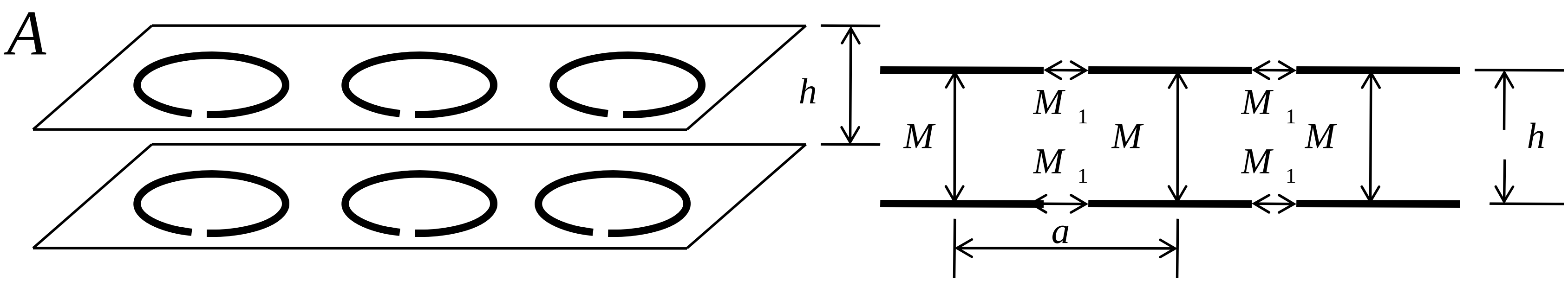


$$Z_{01} I_n + j\omega M_1 (I_{n-1} + I_{n+1}) + j\omega M J_n + j\omega M_4 J_{n-1} + j\omega M_3 J_{n+1} = 0$$

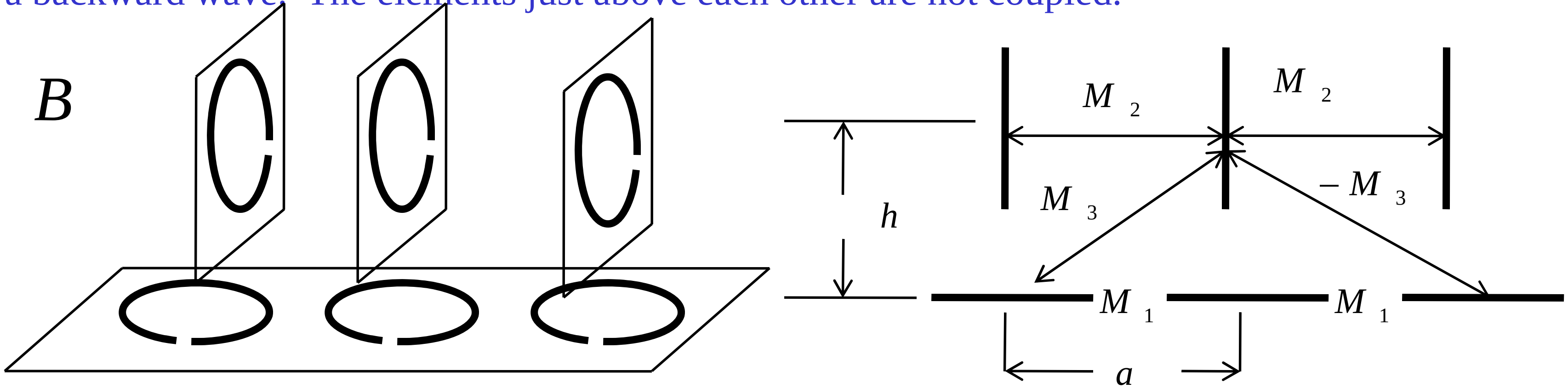
$$Z_{02} J_n + j\omega M_2 (J_{n-1} + J_{n+1}) + j\omega M I_n + j\omega M_3 I_{n-1} + j\omega M_4 J_{n+1} = 0$$

Bi-atomic SRR structure

Two planer lines one above another with intra line M_1 negative and inter line M positive as shown in *A*

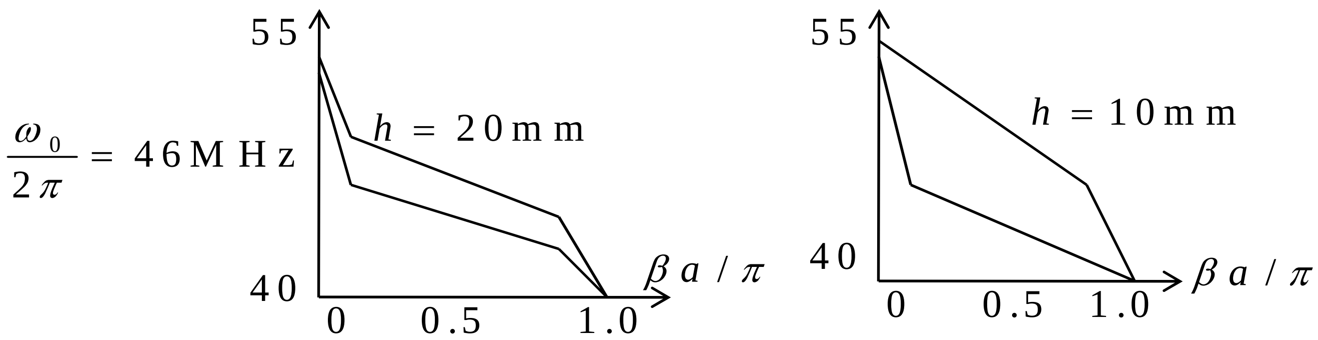


Planer line coupled to axial line, as shown in *B* , the upper one carries a forward MI wave and the lower one carries a backward wave. The elements just above each other are not coupled.

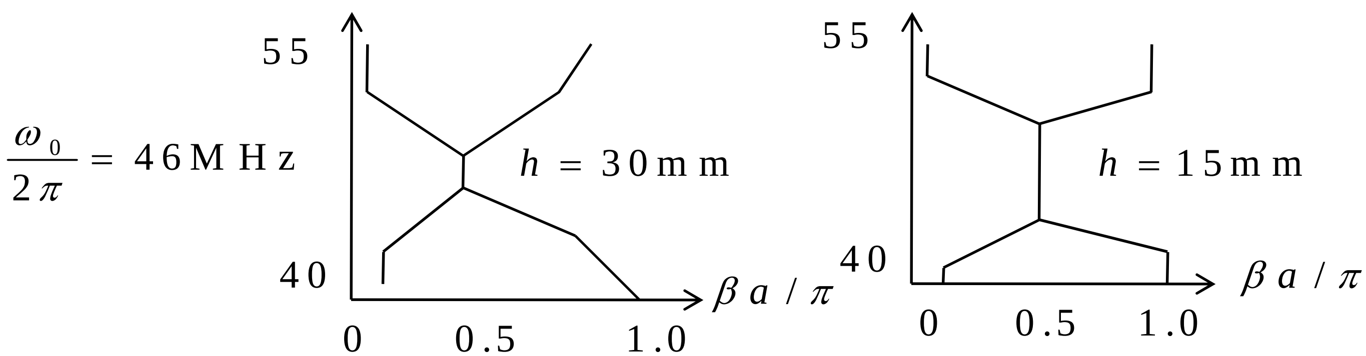


Dispersion of MI waves in in bi-atomic SRR structure

A For configuration both planar the explanation is like: a single planar line supports backward wave. If the coupling between the two is small say gap *h* is 20 mm, then we expect a small split in the dispersion curve, well if the coupling is strong say *h* is 10 mm, then we expect a large split. When the split is large we have effectively two pass bands with a stop band between them.



B For configuration with planar and axial the explanation is like: here the forward wave is coupled to a backward wave. We have combination of forward and backward wave. There is stop band around the resonant frequency (46 MHz) for small coupling *h* of 30 mm the stop band is small for larger coupling the say *h* of 15 mm the stop-band is large.



End of Part-7

The part-8 will have application of MI waves