Lecture Notes

Fractional Viscoelasticity Part-D

"Fractional diffusion-wave equation"

for
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Fractional diffusion-wave equation

This lecture deals with propagation of disturbance in the visco-elastic medium. We start with constitutive equation of visco-elastic material, and derive the wave equation and diffusion equation for one dimensional thin infinite rod. Thereafter we generalize the same by fractional time derivative and call that as Time Fractional diffusion-Wave Equation (TFDWE) and obtain Green's function for Cauchy type and signaling type problems. We see the reciprocity relation between these Green's function. As we have seen in earlier deliberation that Mittag-Leffler function is generalization of exponential function and has use in solution to fractional differential equation, similarly we find Mainardi-Wright function (M-Wright function) plays role in generalization of Gaussian probability distribution function (which is solution to Cauchy Diffusion problem in the integer order case). We see the plots of this M-Wright function for several fractional orders. We derive various properties (Laplace, Fourier transforms) and representations (integral and series) of this M-Wright function and show its usage in the solution to TFDWE. Also we take special manifestations of diffusion equation, integer order case, stretched time diffusion equation, fractional diffusion equation, and fractional diffusion equation with stretched time; and then show the use of M-Wright function used as Green's function of all these systems. We also find the second order moments of all these obtained Green's function, and show the variance as function of time; to classify slow and fast diffusion cases, and several classes of Brownian motion.

Contents

1. The constitutive equations for visco-elastic systems

We have done this before in the previous lectures. Let us reproduce the basic models of the 'visco-elastic' systems in figure-1, with changed notation for constants from previously used notations.

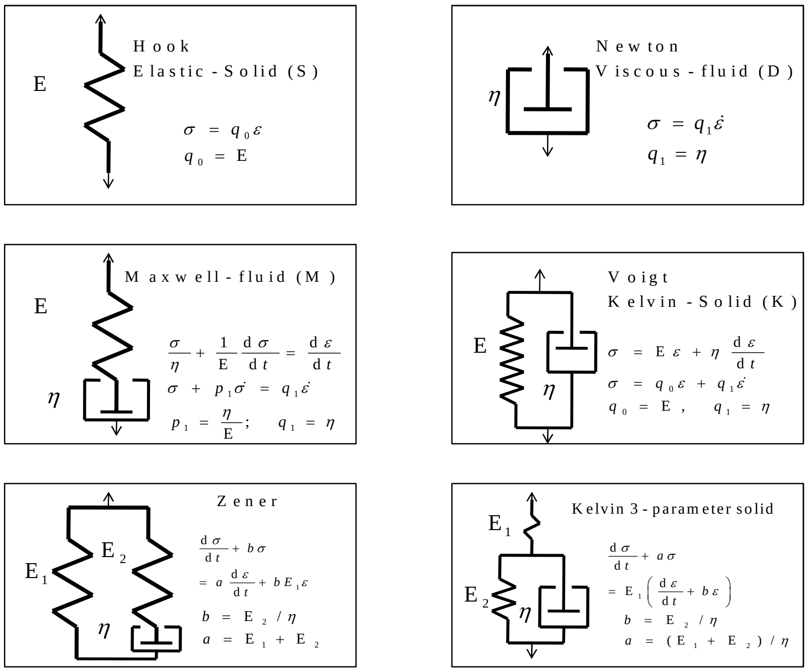


Figure-1: Basic building blocks for the visco-elastic systems

With these basic elements as in figure-1 we can generate complex models by connecting these elements in series or parallel, and we write their material properties, creep-compliance, relaxation modulus, complex-compliance (real & imaginary part); which we dealt in the previous lectures in detail.

Model S that is Elastic Solid (Hook's spring); its differential equation is $\sigma = q_0\varepsilon$, (differential with zero order) with creep compliance as $(1/q_0)$, relaxation modulus as q_0, real part of complex compliance is $(1/q_0)$, with imaginary part of complex compliance as 0.

Model D that is Newton fluid or viscous fluid or Dashpot having differential equation as $\sigma = q_1\dot{\varepsilon}$, creep-compliance as (t/q_1), relaxation modulus as $q_1\delta(t)$, real part of complex compliance as 0 with imaginary part of complex compliance as $-1/(q_1\omega)$; where ω is angular frequency (radian / s).

Model S series with D (S-D); is Maxwell fluid (M) has differential equation as $\sigma + p_1\dot{\sigma} = q_1\dot{\varepsilon}$; gives creep-compliance as $(p_1 + t)/q_1$, relaxation modulus as $(q_1 / p_1)e^{-(t/p_1)}$, with real part of complex compliance as p_1 / q_1, and imaginary part of complex compliance as $-1/(q_1\omega)$.

Model S parallel with D (S || D), is Kelvin solid (K) has differential equation as $\sigma = q_0\varepsilon + q_1\dot{\varepsilon}$, gives creep compliance as $(1/q_0)(1-e^{-\lambda t})$, $\lambda = q_0 / q_1$, relaxation modulus as $q_0 + q_1\delta(t)$, having real part of complex compliance as $q_0 /(q_0^2 + q_1^2\omega^2)$, and imaginary part of complex compliance as $-q_1\omega /(q_0^2 + q_1^2\omega^2)$.

Model S-K, (that is S series K) is a 3-element solid having differential equation as $\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}$, $q_1 > p_1q_0$, having creep-compliance as $(p_1 / q_1)e^{-\lambda t} + (1/q_0)(1-e^{-\lambda t})$, $\lambda = q_0 / q_1$, with relaxation modulus given by $(q_1 / p_1)e^{-t/p_1} + q_0(1-e^{-t/p_1})$, having real part of complex compliance as $(q_0 + p_1q_1\omega^2)/(q_0^2 + q_1^2\omega^2)$ and imaginary part of the complex compliance as $-(q_1 - q_0p_1)(\omega)/(q_0^2 + q_1^2\omega^2)$.

Model D-K (that is D series K) is 3-element fluid has differential equation $\sigma + p_1\dot{\sigma} = q_1\dot{\varepsilon} + q_2\ddot{\varepsilon}$, $p_1 q_1 > q_2$ has creep compliance as $(t/q_1) + [(p_1q_1 - q_2)/q_1^2](1-e^{-\lambda t})$, $\lambda = q_1 / q_2$, relaxation modulus is given by $(q_2 / p_1)\delta(t) + (1/p_1)[q_1 - (q_2 / p_1)]e^{-t/p_1}$ has real part of complex compliance equal to $(p_1q_1 - q_2)(q_1^2 + q_2^2\omega^2)$, and imaginary part of complex compliance as $-(q_1 + p_1q_2\omega^2)/[\omega(q_1^2 + q_2^2\omega^2)]$.

Let us take from the figure-1, the 3-parameter solid model which has the constitutive equations as following

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} + a\sigma = \mathrm{E}_1\left(\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + b\varepsilon\right) \quad \text{with} \quad b = \mathrm{E}_2/\eta; \quad a = (\mathrm{E}_1 + \mathrm{E}_2)/\eta$$

We can write this with new parameters $p_0 = 1, \quad p_1, \quad q_0, \quad q_1$, after algebraic manipulation as the following

$$\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}$$

Taking this as unit cell, and making chain out of this we may generally write as following

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \ldots = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} + \ldots$$

$$\sum_0^m p_k \frac{\mathrm{d}^k\sigma}{\mathrm{d}t^k} = \sum_0^n q_n \frac{\mathrm{d}^k\varepsilon}{\mathrm{d}t^k} \quad \text{with} \quad p_0 = 1$$

$$\mathbf{P}\sigma = \mathbf{Q}\varepsilon \quad \text{with} \quad \mathbf{P} = \sum_0^m p_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \qquad \mathbf{Q} = \sum_0^n q_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}$$

Looking at the solid model only in figure-1 we can say that either $m = n$, or $m = n - 1$. However the differential equation above, of its any form is the mathematical description of mechanical behavior of the visco-elastic material. It is called the constitutive equation. When the above differential equation is subjected to Laplace transformation, there results the following algebraic relation between Laplace transforms $\bar{\sigma}(s)$ and $\bar{\varepsilon}(s)$ of stress and strain

$$\sum_0^m p_k s^k \bar{\sigma}(s) = \sum_0^n q_k s^k \bar{\varepsilon}(s)$$

It may also be written as

$$\mathbf{P}(s).\bar{\sigma}(s) = \mathbf{Q}(s).\bar{\varepsilon}(s) \quad \text{with} \quad \mathbf{P}(s) = \sum_0^m p_k s^k \qquad \mathbf{Q}(s) = \sum_0^n q_k s^k$$

When we study visco-elastic systems simple models, we saw (in the previous lectures), that some of them display an instantaneous response and some do not (rather show retarded response). The solids which are under constant stress σ ultimately settle down at a finite strain ε ; and fluids which ultimately creep at constant strain rate $\dot{\varepsilon}$. The Table-1 shows how these typical behaviors are connected with patterns of non-zero coefficients ($p_k, \quad q_k$). If $q_0 = 0$, there are only derivatives on the RHS of constitutive equations, and at least the lowest of them must be non zero if there is stress σ . If $q_0 \neq 0$, stress and strain approach finite values for $t \to \infty$; and then all derivatives on both sides of constitutive differential equation vanish. In the limit we have $\sigma = q_0\varepsilon$, and the asymptotic modulus is $\mathrm{E}_\infty = q_0$. If $m = n$, an 'initial elastic response' with modulus E_0 occurs, and it is absent if $m = n - 1$.

In Table-1 first two columns show how the different viscoelastic materials may be represented by Kelvin or Maxwell models, consisting of at most one spring (S) and one dashpot (D)-and arbitrary number of Kelvin elements (K) in series or a spring, a dashpot and arbitrary number of Maxwell element (M) in parallel.

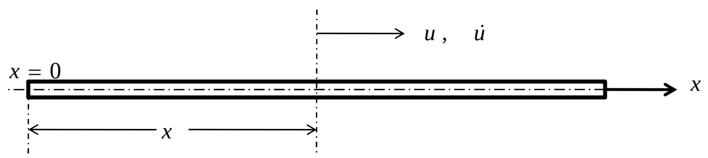
Kelvin Model	Maxwell Model	Nonzero coefficient of **P**	Nonzero coefficient of **Q**	Number of elements	E_0	Solid or Fluid
S	S	p_0	q_0	1	E_0	solid
D	D	p_0	q_1	1	-	fluid
K	S || D	p_0	$q_0 \quad q_1$	2	-	solid
S-D	M	$p_0 \quad p_1$	q_1	2	E_0	fluid
S-K	S || M	$p_0 \quad p_1$	$q_0 \quad q_1$	3	E_0	solid
D-K	D || M	$p_0 \quad p_1$	$q_1 \quad q_2$	3	-	fluid
K-K	S || D || M	$p_0 \quad p_1$	$q_0 \quad q_1 \quad q_2$	4	-	solid
S-D-K	M || M	$p_0 \quad p_1 \quad p_2$	$q_1 \quad q_2$	4	E_0	fluid
S-K-K	S || M || M	$p_0 \quad p_1 \quad p_2$	$q_0 \quad q_1 \quad q_2$	5	E_0	solid
D-K-K	D || M || M	$p_0 \quad p_1 \quad p_2$	$q_1 \quad q_2 \quad q_3$	5	-	fluid
K-K-K	S || D || M || M	$p_0 \quad p_1 \quad p_2$	$q_0 \quad q_1 \quad q_2 \quad q_3$	6	-	solid
S-D-K-K	M || M || M	$p_0 \quad p_1 \quad p_2 \quad p_3$	$q_1 \quad q_2 \quad q_3$	6	E_0	fluid
S-K-K-K	S || M || M || M	$p_0 \quad p_1 \quad p_2 \quad p_3$	$q_0 \quad q_1 \quad q_2 \quad q_3$	7	E_0	solid

Table-1: Several combinations of linear visco-elastic models

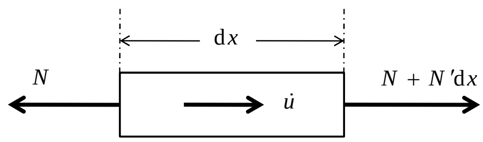
2. How do the constitutive differential equations for material generate wave propagation inside material?

Here we shall be describing a straight cylindrical (solid) bar which extends from $x = 0$ to infinity. At the end we apply a time-dependent tensile or compressive force or enforce a time dependent axial displacement. Changes in these excitation quantities are very rapid that 'inertia' of the material becomes important property in our analysis. We know that for such type of elastic bar any such excitation (i.e. changes in the prescribed force or displacement) produces a stress wave that propagates with a definite speed c, along the bar. We shall thus study the corresponding phenomena in a visco-elastic bar.

The bar is shown in the figure 2 a, its cross section area A and may have any compact shape. Here there is only axial force $\sigma_x = \sigma$ is the only non-vanishing stress component and deformation is described by component u in axial direction.



(a)



(b)

Figure-2: Semi-infinite visco-elastic bar (a) the bar, (b) the element

Refer figure 2 b, the bar element, where at its left end the force $N = \sigma A$ is acting, which is a function of x and t. Let N' (i.e. with prime) represent the derivative w.r.t space x, and $\dot{u}$ (i.e. with dot) represent derivative w.r.t time t. Then the force at the right end of element of figure- 2b, is $N + N'\mathrm{d}x$, and difference of these forces (left end and right end) produce acceleration that is $\ddot{u}$, for the mass in the element (say volume density is ρ kg / m^3); is $\rho(A\mathrm{d}x) = \mu \mathrm{d}x$; where $\mu = \rho A$ is linear mass density (kg / m); i.e. mass per unit length. The force balance for this element is therefore gives following dynamic equation

$$(\rho A \mathrm{d}x)\ddot{u} = (\mu \mathrm{d}x)\ddot{u} = (N + N'\mathrm{d}x) - N \qquad \mu\ddot{u} = N'$$

If at a certain time the displacement of the left end of element is u and that of the right end $u + u'\mathrm{d}x$, then the strain, that is, the difference between the two (right and left displacement), divided by the length of the element i.e. $\mathrm{d}x$ gives kinematic relation

$$\varepsilon = \frac{(u + u'\mathrm{d}x) - u}{\mathrm{d}x} \quad \text{i.e.} \quad \varepsilon = u'$$

We have constitutive differential equation for viscoelastic material as

$$\sum_0^m p_k \frac{\mathrm{d}^k \sigma}{\mathrm{d}t^k} = \sum_0^n q_k \frac{\mathrm{d}^k \varepsilon}{\mathrm{d}t^k} \qquad \sum_0^m p_k \frac{\mathrm{d}^k (\sigma A)}{\mathrm{d}t^k} = A\sum_0^n q_k \frac{\mathrm{d}^k \varepsilon}{\mathrm{d}t^k}; \quad \sigma A = N; \quad \varepsilon = u'$$

$$\sum_0^m p_k \frac{\mathrm{d}^k N}{\mathrm{d}t^k} = A\sum_0^n q_k \frac{\mathrm{d}^k u'}{\mathrm{d}t^k} \qquad \sum_0^m p_k \frac{\mathrm{d}^k N}{\mathrm{d}t^k} = A\sum_0^n q_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\sum_0^m p_k \frac{\mathrm{d}^k N}{\mathrm{d}t^k} = A\sum_0^n q_k \frac{\mathrm{d}^{k+1} u}{\mathrm{d}x\mathrm{d}t^k} \quad \text{differentiating w.r.t. } x$$

$$\frac{1}{A}\sum_0^m p_k \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^k N}{\mathrm{d}t^k} = \sum_0^n q_k \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}^{k+1} u}{\mathrm{d}x\mathrm{d}t^k} \qquad \frac{1}{A}\sum_0^m p_k \frac{\mathrm{d}^k N'}{\mathrm{d}t^k} = \sum_0^n q_k \frac{\mathrm{d}^{k+2} u}{\mathrm{d}x^2\mathrm{d}t^k}$$

$$\frac{1}{A}\sum_0^m p_k \frac{\mathrm{d}^k (\mu\ddot{u})}{\mathrm{d}t^k} = \sum_0^n q_k \frac{\mathrm{d}^{k+2} u}{\mathrm{d}x^2\mathrm{d}t^k}$$

$$\frac{\mu}{A}\sum_0^m p_k \frac{\mathrm{d}^{k+2} u}{\mathrm{d}t^{k+2}} = \sum_0^n q_k \frac{\mathrm{d}^{k+2} u}{\mathrm{d}x^2\mathrm{d}t^k}$$

Now change d to ∂ and obtain the partial differential equation for our problem

$$\frac{\mu}{A}\sum_0^m p_k \frac{\partial^{k+2} u}{\partial t^{k+2}} = \sum_0^n q_k \frac{\partial^{k+2} u}{\partial x^2\partial t^k}$$

This is the differential equation of our problem, a partial differential equation for the displacement u . We can prescribe for cases $N(0,t)$ or $u(0,t)$ either an impulse function, step function or as harmonic oscillator-as boundary condition for the fixed end. From what we might expect to find in the step input case, a ‘wave-front’ running from left to the right along the bar, separating the part before it, which does not yet ‘know’ that something has happened at $x = 0$, from the part behind it, which is under stress and in motion.

3. The Wave Front in visco-elastic system

We have derived the differential equation for a thin cylinder bar

$$\frac{\mu}{A}\sum_{0}^{m} p_k \frac{\partial^{k+2}u}{\partial t^{k+2}} - \sum_{0}^{n} q_k \frac{\partial^{k+2}u}{\partial x^2 \partial t^k} = 0$$

It may also be applied to simple wave propagation problems in two and three dimensions, if we replace the visco-elastic coefficients by proper tensor quantities. However, for all the materials as listed in table-1, there is either $m = n$ or $m+1 = n$. In both the cases the general differential equation is of order $(n+2)$. If $m = n$ there are two terms of order $(n+2)$, the $m = n$ cases are listed as follows

$$m = n \qquad \frac{\mu}{A}\sum_{k=0}^{n} p_k \frac{\partial^{k+2}u}{\partial t^{k+2}} = \sum_{k=0}^{n} \frac{\partial^{k+2}}{\partial x^2 \partial t^k}$$

$$m = n = 0 \qquad \frac{\mu}{A} p_0 \frac{\partial^2 u}{\partial t^2} = q_0 \frac{\partial^2 u}{\partial x^2}$$

$$m = n = 1 \qquad \frac{\mu}{A}\left[p_0 \frac{\partial^2 u}{\partial t^2} + p_1 \frac{\partial^3 u}{\partial t^3} \right] = q_0 \frac{\partial^2 u}{\partial x^2} + q_1 \frac{\partial^3 u}{\partial x^2 \partial t}$$

The above case with $m = n$ are equations with the second space derivative and second time derivative of $\partial^n u / \partial t^n$.

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^n u}{\partial t^n}\right) = C_n \frac{\partial^2}{\partial x^2}\left(\frac{\partial^n u}{\partial t^n}\right) \qquad C_n = \frac{q_n A}{p_n \mu}$$

In the system with $m = n$ we have all the coefficients as positives p_k, $q_k > 0$, and the differential equation is hyperbolic. As earlier noted these are the material having impact modulus $\mathrm{E}_0 = q_n / p_n$. Physically this type of equation of hyperbolic type represents that any discontinuity manifesting at the initial boundary will be running all along the bar-in form of a shock wave. Here we are not going to discuss the shock-waves.

For the material with $n = m+1$, there is only one term of order $(n+2)$

$$\frac{\mu}{A}\sum_{k=-1}^{n-1} p_k \frac{\partial^{k+2}}{\partial t^{k+2}} = \sum_{0}^{n} q_k \frac{\partial^{k+2}u}{\partial x^2 \partial t^k}$$

Writing the last terms on the RHS and LHS of the above dropping the summation sign we have the following

$$\frac{\mu}{A} p_{n-1} \frac{\partial^{n+1}u}{\partial t^{n+1}} = q_n \frac{\partial^{n+2}u}{\partial x^2 \partial t^n} \quad \text{for} \quad n = 0 \quad \frac{\mu}{A} p_{-1} \frac{\partial u}{\partial t} = q_0 \frac{\partial^2 u}{\partial x^2}$$

The above equation for $m = n-1$ is of parabolic type, like diffusion or heat conduction in this case the wave propagation is having no shock-waves. We will generalize these in subsequent sections, to get fractional diffusion-wave equation.

4. Time Fractional Diffusion-Wave Equation Cauchy and Signaling Problem

The standard Partial Differential equation (PDE) governing the known phenomena of diffusion and the wave propagation are governed through Fourier Diffusion Equation (popularly called Fick's law) and D' Alembert's wave equation, as follows.

$$\frac{\partial}{\partial t}u(x,t)=\mathbb{D}\frac{\partial^2}{\partial x^2}u(x,t) \qquad \frac{\partial^2}{\partial t^2}u(x,t)=c^2\frac{\partial^2}{\partial x^2}u(x,t)$$

Where $u(x,t)$ denoting response variable, $\mathbb{D}$ as diffusivity constant and c as characteristic velocity. We denote the Time Fractional Diffusion Wave (TFDWE) equation by

$$\frac{\partial^\beta}{\partial t^\beta}u(x,t)=a\frac{\partial^2}{\partial x^2}u(x,t) \qquad 0<\beta\le 2$$

The fractional order β is the order of Fractional Derivative of Caputo type ${}_0^C D_t^\beta$, i.e. fractional derivative w.r.t. time. We define these derivatives for $0<\beta\le 1$ and $1<\beta\le 2$ as follows:

$$\frac{\partial^\beta u}{\partial t^\beta}:=\begin{cases}\dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t\left[\frac{\partial u(x,\tau)}{\partial\tau}\right]\frac{\mathrm{d}\tau}{(t-\tau)^\beta} = {}_0I_t^{1-\beta}\,{}_0D_t^1u(x,t) & 0<\beta<1\\[2ex] \dfrac{\partial u}{\partial t} & \beta=1\end{cases}$$

$$\frac{\partial^\beta u}{\partial t^\beta}:=\begin{cases}\dfrac{1}{\Gamma(2-\beta)}\displaystyle\int_0^t\left[\frac{\partial^2 u(x,\tau)}{\partial\tau^2}\right]\frac{\mathrm{d}\tau}{(t-\tau)^{\beta-1}} = {}_0I_t^{2-\beta}\,{}_0D_t^2u(x,t) & 1<\beta<2\\[2ex] \dfrac{\partial^2 u}{\partial t^2} & \beta=2\end{cases}$$

For $1<\beta<2$, we integrate the Caputo Fractional Derivative, by order β and get the following

$${}_0I_t^\beta\circ{}_0^C D_t^\beta u(x,t)={}_0I_t^\beta\circ\left({}_0I_t^{2-\beta}D_t^2\right)u(x,t)$$

$$={}_0I_t^2D_t^2u(x,t)=u(x,t)-u(x,0^+)-tu_t(x,0^+)$$

$$u_t(x,0^+)=\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0^+}$$

For $0<\beta<1$, we integrate the Caputo Fractional Derivative, by order β and get the following

$${}_0I_t^\beta\circ{}_0^C D_t^\beta u(x,t)={}_0I_t^\beta\circ\left({}_0I_t^{1-\beta}D_t^1\right)u(x,t)$$

$$={}_0I_t^1D_t^1u(x,t)=u(x,t)-u(x,0^+)$$

Integrating by order β the LHS and RHS of TFDWE and by using the above expression of fractional integration of Caputo Fractional Derivative, for $0<\beta<1$, we have (by using definition of fractional integration to RHS), we obtain the following integral equation

$$ {}_0I_t^{\beta}\left[\frac{\partial^{\beta}}{\partial t^{\beta}}\right]u(x,t)={}_0I_t^{\beta}\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 0<\beta<1 $$

$$ u(x,t)-u(x,0^+)=\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}} $$

$$ u(x,t)=u(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}} $$

Integrating by order β the LHS and RHS of TFDWE and by using the above expression of fractional integration of Caputo Fractional Derivative, for $1<\beta<2$, we have (by applying definition of fractional integration to RHS), we obtain the following integral equation

$$ {}_0I_t^{\beta}\left[\frac{\partial^{\beta}}{\partial t^{\beta}}\right]u(x,t)={}_0I_t^{\beta}\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 1<\beta<2 $$

$$ u(x,t)-u(x,0^+)-tu_t(x,0^+)=\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}} $$

$$ u(x,t)=u(x,0^+)+tu_t(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}} $$

Now we set $\beta=2\upsilon$, to write FTDWE as the following with constant a replaced by $\mathbb{D}$-as diffusivity constant or proportional to wave propagation constant (velocity).

$$ \frac{\partial^{2\upsilon}u}{\partial t^{2\upsilon}}=\mathbb{D}\frac{\partial^2 u}{\partial x^2} \qquad 0<\upsilon\le 1 \qquad \mathbb{D}>0 $$

The dimension of $\mathbb{D}$ is $L^2/T^{2\upsilon}$; and $u(x,t)$ is ‘field’ variable which is assumed to be causal function of time, i.e. $u(x,t)=0$ for $t<0$. The above is one dimensional Time Fractional Diffusion Wave equation (TFDW).

For $\upsilon=1/2$, the integer order diffusion equation is recovered, that is

$$ \frac{\partial u}{\partial t}=\mathbb{D}\frac{\partial^2 u}{\partial x^2} $$

For $\upsilon=1$, the integer order classical wave equation gets recovered

$$ \frac{\partial^2 u}{\partial t^2}=\mathbb{D}\frac{\partial^2 u}{\partial x^2} $$

With the fractional order υ, as in above, the meaning of $\mathbb{D}$ is different.

The ‘Cauchy Problem’ is Initial Value Problem (IVP) where the initial values are given as:

$$u(x,0^+) = g(x) \quad \text{for} \quad -\infty < x < \infty$$

$$u(\mp\infty,t) = 0 \quad \text{for} \quad t > 0$$

The ‘Signaling Problem’ is Boundary Value Problem (BVP) where the boundary conditions are given as

$$u(x,0^+) = 0 \quad \text{for} \quad x > 0$$

$$u(0^+,t) = h(t) \quad \text{for} \quad t > 0$$

$$u(+\infty,t) = 0 \quad \text{for} \quad t > 0$$

However, to ensure the continuous dependence of our solution on the ‘fractional parameter’ υ, where $0 < \upsilon < 1$, also on the transition on $\upsilon = (1/2)^-$ to $\upsilon = (1/2)^+$, we assume

$$u_t(x,0^+) = \frac{\partial u}{\partial t} \Big|_{t=0^+} = 0$$

The reason is that for $\upsilon < 1/2$, $u_t(x,0^+)$ condition is not required, and that is zero; and that is continued for $\upsilon > 1/2$, in order to have continuity w.r.t υ at ½.

5. Green’s Function

We call the two Green’s functions (represented by symbol $G_*^{**}(x,t)$) for Cauchy and Signaling problem as follows

$G_c(x,t)$ for Cauchy Problem with $g(x) = \delta(x)$

$G_s(x,t)$ for Signaling Problem with $h(t) = \delta(t)$

The solutions to the above problems in terms of respective Green’s function is therefore

$$u(x,t) = \int_{-\infty}^{+\infty} G_c(\xi,t) g(x-\xi) \mathrm{d}\xi \qquad u(x,t) = \int_0^t G_s(x,\tau) h(t-\tau) \mathrm{d}\tau$$

The above are convolution of the Green’s function with forcing functions. We also point out that that $G_c(x,t) = G_c(|x|,t)$, since the Green’s function turns out to be even function of x.

6. Green’s Function for Diffusion Equation

For $\upsilon = 1/2$, we get integer order diffusion equation and its Green’s function is as follows

$$G_c^d(x,t) = \frac{1}{2\sqrt{\pi \mathbb{D} t}} e^{-x^2/4\mathbb{D}t} \overset{\Im}{\leftrightarrow} e^{-\mathbb{D}tk^2} \qquad G_s^d(x,t) = \frac{x}{2\sqrt{\pi \mathbb{D} t^3}} e^{-x^2/4\mathbb{D}t} \overset{\mathcal{L}}{\leftrightarrow} e^{-x\sqrt{\frac{s}{\mathbb{D}}}}$$

The above show respective Fourier Transform pair and Laplace Transform pair.

We define similarity variable as

$$z = \frac{|x|}{\sqrt{\mathbb{D}t}}$$

For $x > 0$, we get

$$xG_c^d(x,t) = tG_s^d(x,t) = F^d(z) = \frac{z}{2}M^d(z) \qquad \text{where} \qquad M^d(z) = \frac{1}{\sqrt{\pi}}e^{-z^2/4}$$

We call them $M^d(z)$ as to the auxiliary function for the diffusion equation, in that it provides the fundamental solution satisfying the condition as $\int_0^{+\infty} M^d(z)\mathrm{dz} = 1$.

It is easy to see that $\int_0^{\infty} M^d(z)\mathrm{dz} = 1$, by putting $(z/2) = y$, $\mathrm{dz} = (2\mathrm{d}y)$, we have following derivation

$$\int_0^{\infty} M^d(z)\mathrm{dz} = \int_0^{\infty} \frac{1}{\sqrt{\pi}}e^{-z^2/4}\mathrm{dz} = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z^2/4}\mathrm{dz}$$

$$= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}(2\mathrm{d}y) = 1 \qquad \mathrm{using} \qquad \int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y = \sqrt{\pi}$$

In above the superscript d denotes diffusion process.

7. Green's function for Wave Equation

For $\upsilon = 1$, we obtain ordinary wave equation as mentioned above, also put $c = \sqrt{\mathbb{D}}$

Cauchy Problem with $g(x) = \delta(x)$, we obtain

$$G_c^w(x,t) = \frac{1}{2}\delta(x - t\sqrt{\mathbb{D}}) + \frac{1}{2}\delta(x + t\sqrt{\mathbb{D}}) \overset{\Im}{\leftrightarrow} \frac{1}{2}e^{+itk\sqrt{\mathbb{D}}} + \frac{1}{2}e^{-itk\sqrt{\mathbb{D}}}$$

$$-\infty < x < \infty \qquad\qquad c = \sqrt{\mathbb{D}}$$

Signaling Problem with $h(t) = \delta(t)$, for $x > 0$ we obtain

$$G_s^w(x,t) = \delta(t - x/\sqrt{\mathbb{D}}) \overset{\mathcal{L}}{\leftrightarrow} e^{-(x/\sqrt{\mathbb{D}})s} \qquad\qquad x > 0 \qquad\qquad c = \sqrt{\mathbb{D}}$$

From the above explicit relations we recognize the reciprocity relation between the two Green functions (wave and signaling) for $x > 0$ and $t > 0$:

$$2xG_c^w(x,t) = tG_s^w(x,t) = F^w(z) = zM^w(z)$$

$$M^w(z) = \delta(1-z) \qquad\qquad z = \frac{x}{\sqrt{\mathbb{D}t}}$$

However, for the case $\upsilon = 1$, the similarity variable $z = |x|/\sqrt{\mathbb{D}t}$ is not defined, but for general case $0 < \upsilon < 1$, we can have; $z = |x|/\sqrt{\mathbb{D}t^{2\upsilon}} = |x|/t^{\upsilon}\sqrt{\mathbb{D}}$; $\quad 0 < \upsilon < 1$. Then for generalized case of TFDWE for $x > 0$ a certain auxiliary function $M_{\upsilon}(z)$ may exist such that

$$xG_{c,\upsilon}(x,t) = \frac{t}{2\upsilon}G_{s,\upsilon}(x,t) = \frac{z}{2}M_{\upsilon}(z); \quad 0 < \upsilon < 1 \qquad\qquad \int_0^{\infty} M_{\upsilon}(z)\mathrm{dz} = 1$$

8. Solution of TDFWE via Laplace Transform for Green's Function for Cauchy and Signaling Problems

We will get $\tilde{G}_c(x,s)$ and $\tilde{G}_s(x,s)$ for $0 < \upsilon < 1$ by solving ordinary differential equation of second order in x and then by inversion we will get $G_c(x,t)$ and $G_s(x,t)$

Cauchy Problem is with condition at initial time zero i.e. $g(x) = \delta(x)$ then $u(x,t) = G_c(x,t)$

$$\frac{\partial^{2\upsilon}u}{\partial t^{2\upsilon}} = \mathbb{D}\frac{\partial^2 u}{\partial x^2}$$

$$s^{2\upsilon}\tilde{u}(x,s) - s^{2\upsilon-1}\tilde{u}(x,0^+) = \mathbb{D}\frac{\mathrm{d}^2\tilde{u}(x,s)}{\mathrm{d}x^2}$$

$$s^{2\upsilon}\tilde{u}(x,s) - s^{2\upsilon-1}g(x) = \mathbb{D}\frac{\mathrm{d}^2\tilde{u}(x,s)}{\mathrm{d}x^2}$$

for $\quad g(x) = \delta(x) \quad$ we use $\quad \tilde{u}(x,s) = \tilde{G}_c(x,s)$

$$s^{2\upsilon}\tilde{G}_c(x,s) - s^{2\upsilon-1}\delta(x) = \mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}$$

There is a singular term $\delta(x)$ at $x = 0$, we thus need to consider solution of above for $x > 0$ and $x < 0$ imposing boundary condition at $x = \pm\infty$, and matching condition of continuity at $x = 0^{\pm}$. Look at the following; which is obtained via rearrangement of above expression

$$\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2} - \left(\frac{s^{2\upsilon}}{\mathbb{D}}\right)\tilde{G}_c(x,s) = -\frac{s^{2\upsilon-1}}{\mathbb{D}}\delta(x)$$

The indicial polynomial for above second order differential equation with roots is the following

$$p^2 - \frac{s^{2\upsilon}}{\mathbb{D}} = 0 \qquad p = \pm\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}$$

We expect the solution as

$$\tilde{G}_c(x,s) = \begin{cases} c_1(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x} + c_2(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x > 0 \\ c_3(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x} + c_4(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x < 0 \end{cases}$$

Since at $x = \pm\infty$, $\tilde{G}_c(x,s) = 0$ so we have $c_2(s) = c_1(s) = 0$, ensuring solution vanishes at $|x| \to 0$. Thus following is solution

$$\tilde{G}_c(x,s) = \begin{cases} c_1(s)e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x > 0 \\ c_4(s)e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x < 0 \end{cases}$$

Continuity at $x = 0^+$ and $x = 0^-$ states; $\tilde{G}_c(0^+,s) - \tilde{G}_c(0^-,s) = 0$, that is $\tilde{G}_c(0^+,s) = c_1(s) = \tilde{G}_c(0^-,s) = c_4(s)$, making $c_1(s) = c_4(s)$. The first derivative at $x = 0^{\pm}$, is as follows

$$\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} = -c_1(s)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)e^{-(s^{\upsilon}/\sqrt{\mathbb{D}})x} = -c_1(s)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)$$

$$\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} = +c_4(s)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)e^{+(s^{\upsilon}/\sqrt{\mathbb{D}})x} = +c_4(s)\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right)$$

$$\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} = -\left[c_1(s) + c_4(s)\right]\left(\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}\right) \quad \text{as} \quad c_1(s) = c_4(s)$$

$$= -2c_1(s)\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}} \neq 0$$

Integrating $s^{2\upsilon}\tilde{G}_c(x,s) - s^{2\upsilon-1}\delta(x) = \mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}$, from $x = 0^-$ to $x = 0^+$ we get

$$\int_{0^-}^{0^+}(\mathrm{d}x)s^{2\upsilon}\tilde{G}_c(x,s) - \int_{0^-}^{0^+}(\mathrm{d}x)s^{2\upsilon-1}\delta(x) = \int_{0^-}^{0^+}(\mathrm{d}x)\mathbb{D}\frac{\mathrm{d}^2\tilde{G}_c(x,s)}{\mathrm{d}x^2}$$

$$s^{2\upsilon}\left(\int_{0^-}^{0^+}c_1(s)e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}}\mathrm{d}x + \int_{0^-}^{0^+}c_4(s)e^{+(x/\sqrt{\mathbb{D}})s^{\upsilon}}\mathrm{d}x\right) - s^{2\upsilon-1} = \mathbb{D}\left(\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-}\right)$$

The first term of LHS of above is zero since the function $\tilde{G}_c(x,s)$ is continuous at $x = 0$, the infinitesimal area under this from 0^- to 0^+ is zero. The second term is the integral of delta function which is unity, thus we get the $s^{2\upsilon-1}$(constant function w.r.t x). The RHS is found out as discontinuity in the first derivative of $\tilde{G}_c(x,s)$ w.r.t x. Therefore we have

$$-\frac{s^{2\upsilon-1}}{\mathbb{D}} = \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-}$$

Substituting the value of RHS as obtained above we get

$$\left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^+} - \left.\frac{\mathrm{d}\tilde{G}_c(x,s)}{\mathrm{d}x}\right|_{x=0^-} = -\left[c_1(s)+c_4(s)\right]\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}} = -\frac{s^{2\upsilon-1}}{\mathbb{D}} \qquad \text{with} \qquad c_1(s)=c_4(s)$$

We have

$$c_1(s)=c_4(s)=\frac{s^{\upsilon-1}}{2\sqrt{\mathbb{D}}}=\frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}$$

We write therefore

$$\tilde{G}_c(x,s)=\begin{cases}\dfrac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x>0\\ \dfrac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{+\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}x}, & x<0\end{cases}$$

$$=\frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}}e^{-\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}|x|} \qquad -\infty<x<\infty$$

For signaling problem we apply the Laplace Transform to $\dfrac{\partial^{2\upsilon}u(x,t)}{\partial t^{2\upsilon}}=\mathbb{D}\dfrac{\partial^2 u(x,t)}{\partial x^2}$, with $0<\upsilon<1$, and $\mathbb{D}>0$, to get the following

$$s^{2\upsilon}\tilde{u}(x,s)-s^{2\upsilon-1}u(x,0^+)=\mathbb{D}\frac{\partial^2\tilde{u}(x,s)}{\partial x^2}$$

For signaling problem, $u(x,0^+)=0$, for $x>0$ and $u(0^+,t)=h(t)=\delta(t)$ with $u(+\infty,t)=0$ for $t>0$. We therefore write the solution of above problem as Green's function $\tilde{G}_s(x,s)$, and get

$$\mathbb{D}\frac{\mathrm{d}^2\tilde{G}_s(x,s)}{\mathrm{d}x^2}-s^{2\upsilon}\tilde{G}_s(x,s)=0$$

The corresponding indicial polynomial is

$$p^2-\frac{s^{2\upsilon}}{\mathbb{D}}=0 \qquad\qquad p=\pm\frac{s^{\upsilon}}{\sqrt{\mathbb{D}}}$$

The solution is

$$\tilde{G}_s(x,s) = c_1(s)e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}} + c_2(s)e^{+(x/\sqrt{\mathbb{D}})s^{\upsilon}}$$

Clearly, we must put $c_2(s) = 0$, to ensure that the solution vanishes at $x \to +\infty$. Therefore

$$\tilde{G}_s(x,s) = c_1(s)e^{-(x/\sqrt{D})s^{\upsilon}} \text{, for } x > 0 .$$

At initial position that is at the origin $x = 0^+$, we have

$$\tilde{G}_s(0^+,s) = c_1(s) \qquad u(0^+,t) = \delta(t) \quad \text{so} \quad \tilde{u}(0^+,s) = \mathcal{L}\{\delta(t)\} = 1 = \tilde{G}_s(0^+,s) = c_1(s)$$

Gives value of $c_1(s) = 1$. Therefore Green's function in Laplace domain for signaling problem is the following

$$\tilde{G}_s(x,s) = e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}} \text{, for } x > 0$$

Therefore the transformed solutions for Cauchy and Signaling are the following

$$\tilde{G}_c(x,s) = \frac{1}{2\sqrt{\mathbb{D}}s^{1-\upsilon}} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \qquad \tilde{G}_s(x,s) = e^{-\left(x/\sqrt{\mathbb{D}}\right)s^{\upsilon}}$$

The above transformed solution must be inverted to provide the Green's function in space-time domain. For this we require some transformed pair of Wright type function that we write as follows

$$\frac{1}{\upsilon}F_{\upsilon}(1/t^{\upsilon}) = \frac{1}{t^{\upsilon}}M_{\upsilon}(1/t^{\upsilon}) \overset{\mathcal{L}}{\leftrightarrow} \frac{e^{-s^{\upsilon}}}{s^{1-\upsilon}} \qquad \frac{1}{t}F_{\upsilon}(1/t^{\upsilon}) = \frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}(1/t^{\upsilon}) \overset{\mathcal{L}}{\leftrightarrow} e^{-s^{\upsilon}} \qquad 0 < \upsilon < 1$$

The above formulas can be used to invert the transformed Green's function. Then introducing the similarity variable for $x > 0$ and $t > 0$; $z := x/(\sqrt{\mathbb{D}}t^{\upsilon}) > 0$ and using the rules of scale change of Laplace pair i.e.

$$f(t) \leftrightarrow \tilde{f}(s); \quad f(at) \leftrightarrow \frac{1}{a}\tilde{f}(s/a); \quad \frac{1}{a}f(t/a) \leftrightarrow \tilde{f}(as) \quad \text{with} \quad a > 0$$

and after some manipulations we obtain Green's function in space time as

$$G_c(x,t) = \frac{1}{2\upsilon x}F_{\upsilon}(z) = \frac{1}{2\sqrt{\mathbb{D}}t^{\upsilon}}M_{\upsilon}(z) \qquad G_s(x,t) = \frac{1}{t}F_{\upsilon}(z) = \frac{\upsilon x}{\sqrt{\mathbb{D}}t^{1+\upsilon}}M_{\upsilon}(z)$$

We get the reciprocity relation of the Green's function as following

$$2\upsilon x G_c(x,t) = tG_s(x,t) = F_{\upsilon}(z) = \upsilon z M_{\upsilon}(z)$$

Where the $F_{\upsilon}(z)$ and $M_{\upsilon}(z)$ are auxiliary functions for general case $0 < \upsilon \le 1$, which generalize those for standard diffusion given for $\upsilon = 1/2$ by following

$$xG_c^d(x,t) = tG_s^d(x,t) = F^d(z) = \frac{1}{2}zM^d(z)$$

$$M^d(z) = \frac{1}{\sqrt{\pi}}e^{-z^2/4}; \quad z = \frac{x}{\sqrt{\mathbb{D}}t^{1/2}} > 0$$

and for standard wave equation given as

$$2xG_c^w(x,t) = tG_s^w(x,t) = F^w(z) = zM^w(z)$$

$$M^w(z) = \delta(1-z); \quad z = \frac{x}{\sqrt{\mathbb{D}}t} > 0$$

In fact for $\upsilon = 1/2$ and $\upsilon = 1$ we recover the expressions as

$$M^d(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}; \quad z = \frac{x}{\sqrt{\mathbb{D}}t^{1/2}} > 0 \qquad \text{for} \qquad \upsilon = \frac{1}{2}$$

$$M^w(z) = \delta(1-z); \quad z = \frac{x}{\sqrt{\mathbb{D}}t} > 0 \qquad \text{for} \qquad \upsilon = 1$$

9. Relation between Green's function of Cauchy and Signaling Problem-The Reciprocal relation

We got the transformed Green's function (in Laplace domain) for Cauchy problem as

$$\tilde{G}_c(x,s) = \frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-(|x|/\sqrt{\mathbb{D}})s^{\upsilon}} \qquad \text{for} \qquad -\infty < x < +\infty$$

For Signaling Problem the transformed Green's function is

$$\tilde{G}_s(x,s) = e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}} \qquad \text{for} \qquad x > 0$$

Taking the derivative w.r.t s, for above expression we get the following

$$\frac{\mathrm{d}\tilde{G}_s(x,s)}{\mathrm{d}s} = \upsilon s^{\upsilon-1}\left(-\frac{x}{\sqrt{\mathbb{D}}}\right)e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}}$$

$$= -(2\upsilon x)\left(\frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-(x/\sqrt{\mathbb{D}})s^{\upsilon}}\right) = -2\upsilon x\tilde{G}_c(x,s)$$

Therefore, the relation of the two in Laplace domain is following

$$x\tilde{G}_c(x,s) = -\frac{1}{2\upsilon}\frac{\mathrm{d}}{\mathrm{d}s}\tilde{G}_s(x,s)$$

With inverse Laplace applied to above we obtain

$$xG_c(x,t) = \frac{t}{2\upsilon}G_s(x,t)$$

We have used the Laplace pair

$$t^n f(t) \leftrightarrow (-1)^n \left\{\mathrm{d}^n \tilde{f}(s)/\mathrm{d}s^n\right\}; \qquad \mathrm{d}\tilde{f}(s)/\mathrm{d}s \leftrightarrow (-t)f(t) \qquad \text{where} \qquad \tilde{f}(s) \leftrightarrow f(t)$$

10.Application of Laplace inversion & origin of the Auxiliary Function

We have the transformed Green's function for Cauchy Problem and we then modify as following

$$\tilde{G}_c(x,s) = \frac{1}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \qquad |x|\tilde{G}_c(x,s) = \frac{|x|}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}}$$

Apply inverse Laplace to above and write the following steps

$$\begin{aligned}|x|G_c(x,t) &= \mathcal{L}^{-1}\left\{|x|\tilde{G}_c(x,s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{|x|}{2s^{1-\upsilon}\sqrt{\mathbb{D}}} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}}\right\} \\ &= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{st} e^{-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \frac{\mathrm{d}s}{s^{1-\upsilon}}\right] \\ &= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{st-\left(|x|/\sqrt{\mathbb{D}}\right)s^{\upsilon}} \frac{\mathrm{d}s}{s^{1-\upsilon}}\right]\end{aligned}$$

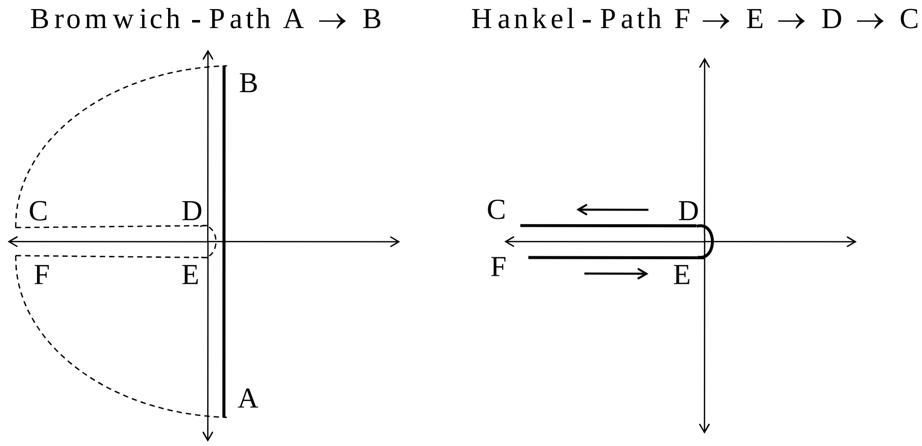
Substitute above, $\sigma = st$ and thus $\mathrm{d}s = (\mathrm{d}\sigma)/t$ to get the following

$$\begin{aligned}|x|G_c(x,t) &= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(|x|/\sqrt{\mathbb{D}}\right)\left(\sigma^{\upsilon}/t^{\upsilon}\right)} \frac{1}{\left(\sigma^{1-\upsilon}/t^{1-\upsilon}\right)}\left(\frac{\mathrm{d}\sigma}{t}\right)\right] \\ &= \frac{|x|}{2\sqrt{\mathbb{D}}}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(\frac{|x|}{(\sqrt{\mathbb{D}})t^{\upsilon}}\right)\sigma^{\upsilon}} \left(\frac{1}{t^{\upsilon}}\right)\left(\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\right] \\ &= \frac{|x|}{2(\sqrt{\mathbb{D}})t^{\upsilon}}\left[\frac{1}{2\pi i}\int_{Br} e^{\sigma-\left(\frac{|x|}{(\sqrt{\mathbb{D}})t^{\upsilon}}\right)\sigma^{\upsilon}} \left(\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right)\right]\end{aligned}$$

Using similarity variable $z = |x|/\left((\sqrt{\mathbb{D}})t^{\upsilon}\right)$ and substituting in above expression we get

$$|x|G_c(x,t) = \frac{z}{2}M_{\upsilon}(z) \quad \text{with} \quad M_{\upsilon}(z) = \frac{1}{2\pi i}\int_{Br} e^{\sigma - z\sigma^{\upsilon}} \frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}; \quad z>0; \quad 0<\upsilon<1$$

$M_{\upsilon}(z)$ is auxiliary function that is $M_{\upsilon}(z) := \frac{1}{2\pi i}\int_{Br} e^{\sigma - z\sigma^{\upsilon}} \frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}$, $z>0$, $0<\upsilon<1$. The Br Bromwich path on which Laplace inversion integration is done is a line on complex plane from $\sigma = \gamma - i\infty$ to $\sigma = \gamma + i\infty$. This above integral representation of $M_{\upsilon}(z)$ (Bromwich representation) can be analytically continued for all z in $\mathbb{C}$, adopting suitable integral and series representations valid in all of $\mathbb{C}$. For this purpose let us deform the Bromwich path Br into the Hankel path Ha, the Hankel contour that begins at $\sigma = -\infty - ia$, $(a>0)$, encircles the branch cut that lies along the negative real axis, and ends up at $\sigma = -\infty + ib$, $(b > 0)$ which is equivalent to original path of Bromwich. This is shown in figure-3.



$$\int_{\text{Bromwich}} F(\sigma)\mathrm{d}\sigma = \int_{\text{Hankel}} F(\sigma)\mathrm{d}\sigma$$

Figure-3: Showing Bromwich Path and Hankel Path

The Laplace inversion is carried out as integral on the Bromwich path which is A to B as indicated in figure-3. Notionally we represent as following ($F(\sigma)$ is inclusive of reciprocal of $2\pi i$ and exponential kernel in Laplace invert formula).

$$f(t) = \int_{A \to B} F(\sigma)\mathrm{d}\sigma$$

We make a closed contour across the branch cut line (i.e. the negative real axis) and call the closed contour as A, B, C, D, E, F, A in counterclockwise direction. Assuming the contour thus made does not include any poles, and then we have

$$\oint_{A \to B \to C \to D \to E \to F \to A} F(\sigma)\mathrm{d}\sigma = 0$$

$$\int_{A \to B} F(\sigma)\mathrm{d}\sigma + \int_{B \to C} F(\sigma)\mathrm{d}\sigma + \int_{C \to D} F(\sigma)\mathrm{d}\sigma + \int_{D \to E} F(\sigma)\mathrm{d}\sigma + \int_{E \to F} F(\sigma)\mathrm{d}\sigma + \int_{F \to A} F(\sigma)\mathrm{d}\sigma = 0$$

As the radius of the arcs BC and FA grows to infinity, the function $F(\sigma) \to 0$, (for a well behaved function to have Laplace inverse), therefore the integrals on these arcs vanishes. So we are left with the following paths on which we do the integration

$$\int_{A\to B} F(\sigma)\mathrm{d}\sigma + \int_{C\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to F} F(\sigma)\mathrm{d}\sigma = 0$$

$$\int_{Bromwich} F(\sigma)\mathrm{d}\sigma = \int_{A\to B} F(\sigma)\mathrm{d}\sigma = -\left(\int_{C\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to F} F(\sigma)\mathrm{d}\sigma \right)$$
$$= \int_{D\to C} F(\sigma)\mathrm{d}\sigma + \int_{E\to D} F(\sigma)\mathrm{d}\sigma + \int_{F\to E} F(\sigma)\mathrm{d}\sigma$$
$$= \int_{F\to E} F(\sigma)\mathrm{d}\sigma + \int_{E\to D} F(\sigma)\mathrm{d}\sigma + \int_{D\to C} F(\sigma)\mathrm{d}\sigma = \int_{Hankel} F(\sigma)\mathrm{d}\sigma$$

In addition if the contour $ABCDEFA$ encloses poles we write

$$\int_{Bromwich} F(\sigma)\mathrm{d}\sigma = \int_{Hankel} F(\sigma)\mathrm{d}\sigma + 2\pi i \sum \text{Residues of poles}$$

11. Integral and Series Representation of Auxiliary function and its Equivalence to the Wright's Function and its Moments

Therefore this equivalence of Bromwich path and Hankel path redefines the auxiliary function on Hankel path as

$$M_\upsilon(z) := \frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^\upsilon}\,\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}; \quad 0<\upsilon<1$$

This above representation is Integral representation of the auxiliary function. This auxiliary function as originated earlier is also called M-Wright's function, as it is very similar to Wright's function. (M-for Mainardi)

We study this auxiliary function $M_\upsilon(z)$, and get its series representation as described in the following steps

$$
\begin{aligned}
M_{\upsilon}(z) &= \frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^{\upsilon}} \frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left[e^{-z\sigma^{\upsilon}}\right]\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \qquad \text{use} \qquad e^{-z\sigma^{\upsilon}} = \sum_{n=0}^{\infty}\frac{(-1)^n (z\sigma^{\upsilon})^n}{n!} \\
&= \frac{1}{2\pi i}\int_{Ha} e^{\sigma}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\sigma^{\upsilon n}\right)\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma}\sigma^{\upsilon n+\upsilon-1} d\sigma\right] \qquad \text{use} \qquad \frac{1}{\Gamma(z)} := \frac{1}{2\pi i}\int_{Ha} e^{x} x^{-z}\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]}
\end{aligned}
$$

We used, in above expansion the Hankel representation of reciprocal Gamma function i.e.

$$
\frac{1}{\Gamma(z)} := \frac{1}{2\pi i}\int_{Ha} e^{x} x^{-z}\mathrm{d}x
$$

Therefore we have a series representation of this Auxiliary function as mentioned below

$$
M_{\upsilon}(z) := \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]} \qquad 0 < \upsilon < 1
$$

Using the reflection formula $\frac{1}{\Gamma(z)} = \frac{1}{\pi}\Gamma(1-z)\sin \pi z$, we write the auxiliary function in following form

$$
\begin{aligned}
M_{\upsilon}(z) &= \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]} = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\Gamma\left[\upsilon(n+1)\right]\sin[\pi\upsilon(n+1)]\frac{z^n}{n!} \\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\Gamma\left[\upsilon(n+1)\right]\sin[\pi\upsilon(n+1)] \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\upsilon n)\sin(\pi\upsilon n)
\end{aligned}
$$

The Wright's function is defined as

$$
W(z;\lambda,\mu) = W_{\lambda,\mu}(z) \triangleq \sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)} = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^{\mu}}; \quad \lambda > -1, \quad \mu > 0
$$

Therefore Auxiliary function in terms of the Wright's function is

$$
M_{\upsilon}(z) = W(-z;-\upsilon,1-\upsilon) = W_{-\upsilon,1-\upsilon}(-z) \qquad 0 < \upsilon < 1
$$

For $\upsilon = 1/2$, we get integer order diffusion problem, and earlier we wrote, for this case $M_{1/2}(z) = M^{d}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2/4}$, and wrote the reciprocity relation as $xG_c^d(x,t)=tG_s^d(x,t)=\dfrac{z}{2}M^d(z)$, also we showed that $\int_0^{+\infty}M^d(z)\mathrm{d}z=1$. Here we try and show that we get; $\int\limits_0^{\infty}M_\upsilon(z)\mathrm{d}z=1$; via following derivation.

$$\int\limits_0^{\infty}M_\upsilon(z)\mathrm{d}z=\int\limits_0^{\infty}\frac{1}{2\pi i}\int\limits_{Ha}\left[e^{\sigma-\sigma^\upsilon z}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right]\mathrm{d}z=\frac{1}{2\pi i}\int\limits_{Ha}e^{\sigma}\left[\int\limits_0^{\infty}e^{-\sigma^\upsilon z}\mathrm{d}z\right]\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}$$

$$=\frac{1}{2\pi i}\int\limits_{Ha}e^{\sigma}\left[\frac{e^{-\sigma^\upsilon z}}{-\sigma^\upsilon}\right]_{z=0}^{z=\infty}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}=\frac{1}{2\pi i}\int\limits_{Ha}\frac{e^{\sigma}}{\sigma}\mathrm{d}\sigma$$

Now we need evaluate integral on Hankel path for the function $F(\sigma)=e^{\sigma}/\sigma$. First leg is section FE (figure-3). Here we write $\sigma=re^{-i\pi}=-r$, $\mathrm{d}\sigma=-\mathrm{d}r$. The r varies from ∞ to 0^+; thus the integration on FE is

$$\int\limits_{FE}\frac{e^{\sigma}}{\sigma}\mathrm{d}\sigma=\int\limits_{\infty}^{0^+}\frac{e^{-r}}{r}\mathrm{d}r$$

Similarly second leg of Hankel path DC with $\sigma=re^{i\pi}=-r$; $\mathrm{d}\sigma=-\mathrm{d}r$ and with r varying from 0^+ to ∞, yields the integration on path DC (figure-3) as

$$\int\limits_{DC}\frac{e^{\sigma}}{\sigma}\mathrm{d}\sigma=\int\limits_{0^+}^{\infty}\frac{e^{-r}}{r}\mathrm{d}r$$

Summing these two above the integration on the legs of Hankel path FE and DC gives zero. Thus we are left with small circle on the Hankel path as ED, where we write $\sigma=\in e^{i\theta}$, with $\in$ a small constant such that $\in\to 0$; and θ varying from $-\pi$ to $+\pi$. With this we have $\mathrm{d}\sigma=\in ie^{i\theta}\mathrm{d}\theta$, and integration on this small circle is

$$\int\limits_{ED}\frac{e^{\sigma}}{\sigma}\mathrm{d}\sigma=\lim_{\in\to 0}\int\limits_{\theta=-\pi}^{\theta=\pi}\frac{e^{\in\exp(i\theta)}}{\in e^{i\theta}}\in ie^{i\theta}\mathrm{d}\theta=\lim_{\in\to 0}\int\limits_{-\pi}^{\pi}e^{\in\exp(i\theta)}i\mathrm{d}\theta$$

$$=i\int\limits_{-\pi}^{\pi}\mathrm{d}\theta=2\pi i$$

Therefore the total integration on Hankel path gives

$$\int\limits_0^{\infty}M_\upsilon(z)\mathrm{d}z=\frac{1}{2\pi i}\int\limits_{Ha}\frac{e^{\sigma}}{\sigma}\mathrm{d}\sigma=1$$

Very important deduction is as follows is the absolute moments of this M-Wright’s function derived for α order moment with z in $\mathbb{R}^+$, for $M_\upsilon(z)$ with $\alpha>-1$ and $0\le\upsilon<1$.

$$\int_0^\infty z^\alpha M_\upsilon(z)\mathrm{d}z = \int_0^\infty z^\alpha \left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right]\mathrm{d}z$$

$$= \frac{1}{2\pi i}\int_{Ha} e^\sigma \left[\int_0^\infty e^{-z\sigma^\upsilon} z^\alpha \mathrm{d}z\right]\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}} \qquad \text{use} \quad \int_0^\infty e^{-z\sigma^\upsilon} z^\alpha \mathrm{d}z = \frac{\Gamma(\alpha+1)}{(\sigma^\upsilon)^{\alpha+1}}$$

$$= \frac{\Gamma(\alpha+1)}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{\upsilon\alpha+1}}\mathrm{d}\sigma \qquad \text{use} \quad \frac{1}{\Gamma(\upsilon\alpha+1)} = \frac{1}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{\upsilon\alpha+1}}\mathrm{d}\sigma$$

$$= \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)} \qquad \alpha > -1; \quad 0 \le \upsilon < 1$$

So we have generalized moment expression for auxiliary function as

$$\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)} \qquad x \in \mathbb{R}^+; \quad \alpha > -1, \quad 0 \le \upsilon < 1$$

12.The General Wright Function

The Wright function we denote by $W_{\lambda,\mu}(z)$ is named after E. Maitland Wright (British mathematician), who introduced this function in 1933. The function is defined by the series representation convergent in the entire $z-$complex plane

$$W_{\lambda,\mu}(z) := \sum_{n=0}^\infty \frac{z^n}{n!\Gamma(\lambda n+\mu)}, \quad \lambda > -1, \quad \mu \in \mathbb{C}$$

The integral representation of the Wright's function reads as

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^\mu} \quad ; \quad \lambda > -1, \quad \mu \in \mathbb{C}$$

The origin of the above can be related similar to as obtained in previous section for $M_\upsilon(z)$. Using Hankel representation of reciprocal of Gamma function that is

$$\frac{1}{\Gamma(\varsigma)} = \int_{Ha} e^u u^{-\varsigma}\mathrm{d}u \quad ; \quad \varsigma \in \mathbb{C}$$

We obtain the following series representation

$$W_{\lambda,\mu}(z) = \frac{1}{2\pi i}\int_{Ha} e^{\sigma + z\sigma^{-\lambda}}\frac{\mathrm{d}\sigma}{\sigma^\mu} = \frac{1}{2\pi i}\int_{Ha} e^\sigma \left[\sum_{n=0}^\infty \frac{z^n}{n!}\sigma^{-\lambda n}\right]\frac{\mathrm{d}\sigma}{\sigma^\mu}$$

$$= \sum_{n=0}^\infty \frac{z^n}{n!}\left[\frac{1}{2\pi i}\int_{Ha} e^\sigma \sigma^{-\lambda n-\mu}\mathrm{d}\sigma\right] = \sum_{n-0}^\infty \frac{z^n}{n!\Gamma(\lambda n+\mu)}$$

For $\lambda = 0$, we get $W_{0,\mu}(z) = e^z/\Gamma(\mu)$ provided $\mu \ne 0,-1,-2,...$

13. The auxiliary Wright functions, the M_v (z) and F_v(z) functions as fractional generalization of Gaussian function.

The auxiliary functions we introduced as in previous sections is

$$M_\upsilon(z) := W_{-\upsilon,1-\upsilon}(-z); \quad 0<\upsilon<1 \qquad F_\upsilon(\upsilon) := W_{-\upsilon,0}(-z) \quad 0<\upsilon<1$$

These two auxiliary functions are related by

$$F_\upsilon(z) = \upsilon z M_\upsilon(z)$$

As a matter of fact $F_\upsilon(z)$ and $M_\upsilon(z)$ are particular cases of Wright function $W_{\lambda,\mu}(z)$ by setting $\lambda=-\upsilon$ for both $\mu=0$ and $\mu=1$ respectively.

The series representation for auxiliary function is

$$F_\upsilon(z) := \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\upsilon n)} \quad \text{use} \quad \Gamma(\varsigma)\Gamma(1-\varsigma) = \frac{\pi}{\sin(\pi\varsigma)}; \quad \frac{1}{\Gamma(-\upsilon n)} = \Gamma[1-(-\upsilon n)]\frac{\sin(-\pi\upsilon n)}{\pi}$$

$$= \sum_{n=1}^{\infty} \frac{(-z)^n}{n!}\left[\frac{1}{\pi}\Gamma(1+\upsilon n)\sin(-\pi\upsilon n)\right] = \sum_{n=1}^{\infty} -\frac{(-z)^n}{n!}\left[\frac{1}{\pi}\Gamma(1+\upsilon n)\sin(\pi\upsilon n)\right]$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}\Gamma(\upsilon n+1)\sin(\pi\upsilon n)$$

$$F_\upsilon(z) := \sum_{n=1}^{\infty} \frac{(-z)^n}{n!\Gamma(-\upsilon n)} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}\Gamma(\upsilon n+1)\sin(\pi\upsilon n)$$

$$M_\upsilon(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{n!\Gamma\left[-\upsilon n+(1-\upsilon)\right]} = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\Gamma\left[\upsilon(n+1)\right]\sin[\pi\upsilon(n+1)]\frac{z^n}{n!}$$

$$= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-z)^n}{n!}\Gamma\left[\upsilon(n+1)\right]\sin[\pi\upsilon(n+1)]$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\upsilon n)\sin(\pi\upsilon n)$$

$$M_\upsilon(z) := \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma\left[-\upsilon n+(1-\upsilon)\right]} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\upsilon n)\sin(\pi\upsilon n)$$

Furthermore $F_\upsilon(0)=0$ and $M_\upsilon(0)=1/\Gamma(1-\upsilon)$, thus we get relation between $F_\upsilon(z)$ and $M_\upsilon(z)$ expressed as follows

$$F_\upsilon(z)=\sum_{n=1}^{\infty}\frac{(-z)^n}{n!\Gamma(-\upsilon n)}=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}\Gamma(\upsilon n+1)\sin(\pi\upsilon n)\quad \text{use}\quad \Gamma(z+1)=z\Gamma(z)$$

$$=-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{n!}(\upsilon n)\Gamma(\upsilon n)\sin(\pi\upsilon n)$$

$$=-\upsilon\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)(-z)^{n-1}}{n!}n\Gamma(\upsilon n)\sin(\pi\upsilon n)$$

$$=\upsilon z\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^{n-1}}{(n-1)!}\Gamma(\upsilon n)\sin(\pi\upsilon n)=\upsilon zM_\upsilon(z)$$

Therefore $F_\upsilon(z)=\upsilon zM_\upsilon(z)$

The integral representation for the auxiliary functions is as follows

$$F_\upsilon(z):=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\mathrm{d}\sigma \qquad M_\upsilon(z):=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}$$

The interrelation between them can be derived from this integral representation as depicted below

$$M_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma-z\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}=\frac{1}{2\pi i}\int_{Ha}e^{\sigma}\left(\frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}}\right)\mathrm{d}\sigma=\frac{1}{2\pi i}\int_{Ha}e^{\sigma}\left(-\frac{1}{\upsilon z}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma$$

The above is due to following, which we are substituting to get above expression

$$\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}=-z(\upsilon\sigma^{\upsilon-1})e^{-z\sigma^\upsilon}=-z\upsilon\frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}}\quad \text{thus}\quad \frac{e^{-z\sigma^\upsilon}}{\sigma^{1-\upsilon}}=-\frac{1}{z\upsilon}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}$$

Now we do integration by parts $\int f(u)g(u)\mathrm{d}u=f(u)\int g(u)\mathrm{d}u-\int\left[f'(u)\left(\int g(u)\mathrm{d}u\right)\right]\mathrm{d}u$ for the integral expression

$$M_\upsilon(z)=\frac{1}{2\pi i}\int_{Ha}e^{\sigma}\left(-\frac{1}{\upsilon z}\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma=\frac{1}{2\pi i}\left(\frac{1}{\upsilon z}\right)\int_{Ha}e^{\sigma}\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma$$

$$=\frac{1}{2\pi i}\left(\frac{1}{\upsilon z}\right)\left[e^{\sigma}\int_{Ha}\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma-\int_{Ha}\left\{\int\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma\right\}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}e^{\sigma}\right)\mathrm{d}\sigma\right]$$

$$=\left(\frac{1}{\upsilon z}\right)\left[\frac{1}{2\pi i}\left(\int_{Ha}(e^{-z\sigma^\upsilon})(e^{\sigma})\mathrm{d}\sigma\right)\right]=\frac{1}{\upsilon z}\left[\frac{1}{2\pi i}\left(\int_{Ha}e^{\sigma-z\sigma^\upsilon}\mathrm{d}\sigma\right)\right]=\frac{1}{\upsilon z}F_\upsilon(z)$$

$$M_\upsilon(z)=\frac{1}{\upsilon z}F_\upsilon(z)\qquad F_\upsilon(z)=\upsilon zM_\upsilon(z)$$

In above derivation we used $\int_{Ha}\left(-\frac{\mathrm{d}e^{-z\sigma^\upsilon}}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma=0$; as this integral along the closed contour not enclosing any singularity as the Hankel path from $-\infty$ to origin encircling it and going again to $-\infty$. We may write this as $\left[-e^{-z\sigma^{\upsilon}}\right]_{\sigma=-\infty-i\in}^{\sigma=-\infty+i\in} \to 0$; with $\in \to 0$. The other steps are obvious.

For special values say for $\upsilon = 1/2$, we obtain

$$M_{1/2}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma\left(\frac{1-n}{2}\right)}$$

$$= \frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{z}{\Gamma(0)} + \frac{z^2}{2!\Gamma\left(-\frac{1}{2}\right)} - \frac{z^3}{3!\Gamma(-1)} + \frac{z^4}{4!\Gamma\left(-\frac{3}{2}\right)} + \ldots$$

$$= \frac{1}{\sqrt{\pi}} - 0 + \frac{z^2}{2!(-2)\sqrt{\pi}} - 0 + \frac{z^4}{4!\frac{4}{3}\sqrt{\pi}} + \ldots$$

$$= \frac{1}{\sqrt{\pi}}\left[1 + \left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)} + \left(\frac{3}{4}\right)\frac{(+1)z^4}{4!} + \ldots\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[1 + \left(\frac{1}{2}\right)\frac{(-1)^1(z)^{2\times1}}{(2\times1)!} + \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\frac{(-1)^2(z)^{2\times2}}{(2\times2)!} + \ldots\right]$$

$$= \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)_n\frac{(z)^{2n}}{(2n)!} \qquad (m)_n = m(m+1)(m+2)..(m+n-1); \quad (m)_0 = 1$$

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\left[1 + \left(\frac{1}{2}\right)\frac{(-1)z^2}{(2!)} + \left(\frac{3}{4}\right)\frac{(+1)z^4}{4!} + \ldots\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[1 + \frac{(-z^2)}{2\times2\times1} + \frac{3}{4}\frac{(-z^2)^2}{(4\times3\times2\times1)} + \ldots\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[1 + \frac{(-z^2/4)}{1!} + \frac{(-z^2/4)^2}{2!} + \ldots\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\sum_{n=0}^{\infty}\frac{(-z^2/4)^n}{n!}\right] = \frac{1}{\sqrt{\pi}}e^{(-z^2/4)}$$

Therefore we write

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)_n\frac{z^{2n}}{(2n)!} = \frac{1}{\sqrt{\pi}}e^{(-z^2/4)}$$

The notation $(m)_n$ is 'rising factorial' (also called Pochhammer Symbol) as defined above, which is also $(m)_n = \Gamma(m+n)/\Gamma(m)$.

For $\upsilon = 1/3$, we have the following formula

$$M_{1/3}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma\left(\frac{2-n}{3}\right)}=\frac{1}{\Gamma\left(\frac{2}{3}\right)}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)_n\frac{z^{3n}}{(3n)!}=\frac{1}{\Gamma\left(\frac{1}{3}\right)}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)_n\frac{z^{3n+1}}{(3n+1)!}$$

From the series representation of above for $\upsilon=0$, we will have

$$F_0(z)\equiv 0 \qquad M_0(z)=e^{-z}$$

We know that Mittag-Leffler function is (fractional) generalization of exponential function; similarly we may call this $M_\upsilon(z)$ as fractional generalization of the Gaussian function, which is the solution of integer order diffusion equation.

14. Laplace and Fourier transforms of the auxiliary Wright functions

We now state some relevant properties of the $M_\upsilon(z)$ and $F_\upsilon(z)$ in view of its role in TFDWE processes. The Laplace transform pair is

$$\frac{1}{t}F_\upsilon(1/t^\upsilon)=\frac{\upsilon}{t^{\upsilon+1}}M_\upsilon(1/t^\upsilon)\overset{\mathcal{L}}{\leftrightarrow}e^{-s^\upsilon} \qquad \frac{1}{\upsilon}F_\upsilon(1/t^\upsilon)=\frac{1}{t^\upsilon}M_\upsilon(1/t^\upsilon)\overset{\mathcal{L}}{\leftrightarrow}\frac{e^{-s^\upsilon}}{s^{1-\upsilon}}; \qquad 0<\upsilon<1$$

$$\mathcal{L}^{-1}\left\{e^{-s^\upsilon}\right\}=\frac{1}{2\pi i}\int_{Br}\mathrm{d}s e^{st}(e^{-s^\upsilon})=\frac{1}{2\pi i}\int_{Ha}e^{st-s^\upsilon}\mathrm{d}s \qquad \text{put} \qquad \sigma=st; \quad \frac{\mathrm{d}\sigma}{t}=\mathrm{d}s; \quad s=\frac{\sigma}{t}$$

$$=\left(\frac{1}{t}\right)\frac{1}{2\pi i}\int_{Ha}e^{\sigma-(\sigma/t)^\upsilon}\mathrm{d}\sigma$$

$$=\left(\frac{1}{t}\right)F_\upsilon(1/t^\upsilon)=\frac{\upsilon}{t^{\upsilon+1}}M_\upsilon(1/t^\upsilon)$$

We used $F_\upsilon(z)=\upsilon z M_\upsilon(z)$, with $z\equiv(1/t^\upsilon)$

The above result is also obtained via expanding in power-series the Laplace transform and inverting term by term as follows:

$$\mathcal{L}^{-1}\left\{e^{-s^\upsilon}\right\}=\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n s^{\upsilon n}}{n!}\right\}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathcal{L}^{-1}\left\{s^{\upsilon n}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{-\upsilon n-1}}{\Gamma(-\upsilon n)}=\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n t^{-\upsilon n}}{n!\Gamma(-\upsilon n)}$$

$$=\frac{1}{t}F_\upsilon(1/t^\upsilon)=\frac{\upsilon}{t^{\upsilon+1}}M_\upsilon(1/t^\upsilon)$$

Furthermore we have $\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(t^{-\upsilon})\right\}=\mathcal{L}\left\{\upsilon^{-1}F_\upsilon(t^{-\upsilon})\right\}=s^{\upsilon-1}e^{-s^\upsilon}$, as shown in following derivation

$$\mathcal{L}^{-1}\left\{e^{-s^{\upsilon}}\right\}=\frac{\upsilon}{t^{\upsilon+1}}M_{\upsilon}(1/t^{\upsilon})$$

$$\mathcal{L}\left\{\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon+1}}\right\}=\frac{e^{-s^{\upsilon}}}{\upsilon}$$

$$\mathcal{L}\left\{t\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon+1}}\right\}=(-1)\frac{\mathrm{d}}{\mathrm{d}s}(e^{-s^{\upsilon}}/\upsilon)=(-1)(-\upsilon s^{\upsilon-1})(e^{-s^{\upsilon}}/\upsilon)$$

$$\mathcal{L}\left\{\frac{M_{\upsilon}(t^{-\upsilon})}{t^{\upsilon}}\right\}=s^{\upsilon-1}e^{-s^{\upsilon}}$$

Mittag-Leffler function is defined for any z in complex $\mathbb{C}$ -plane, for any $\upsilon \geq 0$, by the following integral representation:

$$E_{\upsilon}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\upsilon n+1)}=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^{x}}{x^{\upsilon}-z}\mathrm{d}x \qquad \upsilon>0; \quad z\in\mathbb{C}$$

The above comes from the Laplace pair of Mittag-Leffler function which is

$$E_{\upsilon}(at^{\upsilon})\leftrightarrow\frac{s^{\upsilon-1}}{s^{\upsilon}-a}$$

From this pair we write inverse Laplace integral as

$$E_{\upsilon}(at^{\upsilon})=\frac{1}{2\pi i}\int_{Br}\frac{s^{\upsilon-1}}{s^{\upsilon}-a}e^{st}\mathrm{d}s \qquad \text{put} \quad st=x, \quad \mathrm{d}s=\frac{\mathrm{d}x}{t}, \quad s=\frac{x}{t}$$

$$=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}/t^{\upsilon-1}}{(x^{\upsilon}/t^{\upsilon})-a}e^{x}\frac{\mathrm{d}x}{t}$$

$$=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}}{t^{\upsilon-1}}\frac{t^{\upsilon}}{x^{\upsilon}-at^{\upsilon}}\frac{e^{x}}{t}\mathrm{d}x$$

$$=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^{x}}{x^{\upsilon}-at^{\upsilon}}\mathrm{d}x \qquad \text{put} \qquad at^{\upsilon}=z$$

$$E_{\upsilon}(z)=\frac{1}{2\pi i}\int_{Ha}\frac{x^{\upsilon-1}e^{x}}{x^{\upsilon}-z}\mathrm{d}x$$

This integral representation we wrote above for the Mittag-Leffler function. We know that this Mittag-Leffler function plays important role in the fractional order differential equation especially relaxation and oscillations. We now write a Laplace pair for the M-Wright function, that is

$$M_{\upsilon}(t)\overset{\mathcal{L}}{\leftrightarrow}E_{\upsilon}(-s), \qquad 0<\upsilon<1$$

$$\mathcal{L}\{M_\upsilon(t)\} = \int_0^\infty e^{-st} M_\upsilon(t)\mathrm{d}t = \int_0^\infty e^{-st}\left[\frac{1}{2\pi i}\int_{Ha} e^{\sigma - t\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right]\mathrm{d}t$$

$$= \frac{1}{2\pi i}\int_0^\infty e^{-st}\left[\int_{Ha} e^{\sigma - t\sigma^\upsilon}\frac{\mathrm{d}\sigma}{\sigma^{1-\upsilon}}\right]\mathrm{d}t$$

$$= \frac{1}{2\pi i}\int_{Ha}\frac{e^\sigma}{\sigma^{1-\upsilon}}\left[\int_0^\infty e^{-t(s+\sigma^\upsilon)}\mathrm{d}t\right]\mathrm{d}\sigma = \frac{1}{2\pi i}\int_{Ha} e^\sigma \sigma^{\upsilon-1}\left[-\frac{e^{-t(s+\sigma^\upsilon)}}{s+\sigma^\upsilon}\right]_{t=0}^{t=\infty}\mathrm{d}\sigma$$

$$= \frac{1}{2\pi i}\int_{Ha} e^\sigma \sigma^{\upsilon-1}\left[\frac{1}{s+\sigma^\upsilon}\right]\mathrm{d}\sigma$$

$$= \frac{1}{2\pi i}\int_{Ha}\frac{\sigma^{\upsilon-1}e^\sigma}{\sigma^\upsilon - (-s)}\mathrm{d}\sigma = E_\upsilon(-s)$$

In above we used the interchangeability of integrals in view of analyticity property of the involved functions; and we also used the derived integral representation of the one-parameter Mittag-Leffler function.

In the second approach we expand in series the exponential kernel e^{-st} and use the absolute moments of the $M_\upsilon(z)$ obtained earlier that is $\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x = \Gamma(\alpha+1)/\Gamma(\upsilon\alpha+1)$,and then the series definition of the one-parameter Mittag-Leffler function as follows

$$\mathcal{L}\{M_\upsilon(t)\} = \int_0^\infty e^{-st}M_\upsilon(t)\mathrm{d}t = \int_0^\infty\left(\sum_{n=0}^\infty \frac{(-st)^n}{n!}\right)M_\upsilon(t)\mathrm{d}t$$

$$= \sum_{n=0}^\infty \frac{(-s)^n}{n!}\int_0^\infty t^n M_\upsilon(t)\mathrm{d}t \qquad \text{use} \qquad \int_0^\infty t^n M_\upsilon(t)\mathrm{d}t = \frac{\Gamma(n+1)}{\Gamma(\upsilon n+1)}$$

$$= \sum_{n=0}^\infty \frac{(-s)^n}{n!}\frac{\Gamma(n+1)}{\Gamma(\upsilon n+1)} = \sum_{n=0}^\infty \frac{(-s)^n}{n!}\frac{n!}{\Gamma(\upsilon n+1)} \qquad \text{use} \qquad \Gamma(n+1) = n! \qquad \text{for} \quad n\in\mathbb{N}$$

$$= \sum_{n=0}^\infty \frac{(-s)^n}{\Gamma(\upsilon n+1)} = E_\upsilon(-s) \qquad \text{by using} \qquad E_\upsilon(z) \overset{\text{def}}{=} \sum_{n=0}^\infty \frac{z^n}{\Gamma(\upsilon n+1)}$$

The other way of getting above result is to use series definition of $M_\upsilon(z)$, and the Laplace expression, $\int_0^\infty e^{-st}(-t)^n\mathrm{d}t = \mathcal{L}\{(-t)^n\} = (-1)^n\mathcal{L}\{(t)^n\} = (-1)^n\{n!/s^{n+1}\}$ as indicated below

$$\mathcal{L}\{M_\upsilon(t)\} = \int_0^\infty e^{-st} M_\upsilon(t)\mathrm{d}t = \int_0^\infty e^{-st} \sum_{n=0}^\infty \frac{(-t)^n}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]}\mathrm{d}t$$

$$= \sum_{n=0}^\infty \frac{\int_0^\infty e^{-st}(-t)^n \mathrm{d}t}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]}$$

$$= \sum_{n=0}^\infty \frac{1}{n!\Gamma\left[-\upsilon n + (1-\upsilon)\right]}\left[\frac{(-1)^n n!}{s^{n+1}}\right]$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{\Gamma\left[-\upsilon n + (1-\upsilon)\right]}\left(\frac{1}{s^{n+1}}\right) \qquad \text{put} \qquad n+1 = m$$

$$= \sum_{m=1}^\infty \frac{(-1)^{m-1}}{\Gamma(1-\upsilon m)}\left(\frac{1}{s^m}\right) = \frac{s^{-1}}{\Gamma(1-\upsilon)} + \ldots \sim E_\upsilon(-s)$$

The expression $(s^{-1})/\Gamma(1-\upsilon)$ is asymptotic expansion for large s, i.e. for $s \to \infty$, we have $\left(s^{-1}/\Gamma(1-\upsilon)\right) \to E_\upsilon(-s)$

Now we write Fourier transform of the symmetric M-Wright function, where it should be extended to the negative real axis as an even function. The Fourier is then related to Mittag-Leffler function via following pair.

$$M_\upsilon\left(|x|\right) \overset{\mathfrak{I}}{\leftrightarrow} 2E_{2\upsilon}(-k^2) \qquad\qquad 0 < \upsilon < 1$$

We do the one-sided cosine Fourier integration as demonstrated below, where we used the formula for generalized absolute moments for $M_\upsilon(z)$ and made use of series definition of $E_\upsilon(z)$.

$$\int_0^\infty \cos(kx) M_\upsilon(x)\mathrm{d}x = \int_0^\infty \left[\sum_{n=0}^\infty (-1)^n \frac{(kx)^{2n}}{(2n)!}\right] M_\upsilon(x)\mathrm{d}x$$

$$= \sum_{n=0}^\infty (-1)^n \frac{k^{2n}}{(2n)!}\int_0^\infty x^{2n} M_\upsilon(x)\mathrm{d}x \qquad \text{use} \qquad \int_0^\infty x^{2n} M_\upsilon(x)\mathrm{d}x = \frac{\Gamma(2n+1)}{\Gamma(2\upsilon n+1)}$$

$$= \sum_{n=0}^\infty (-1)^n \frac{k^{2n}}{(2n)!}\left(\frac{(2n)!}{\Gamma(2n\upsilon+1)}\right) \qquad \text{writing} \qquad \Gamma(2n+1) = (2n)!$$

$$= \sum_{n=0}^\infty (-1)^n \frac{(k^2)^n}{\Gamma[(2\upsilon)n+1]} = E_\upsilon(-k^2)$$

From this we get the two-sided Fourier integral and thus Fourier pair for $M_\upsilon\left(|x|\right)$ and write

$$\mathfrak{I}\left\{M_\upsilon\left(|x|\right)\right\} = 2\int_0^\infty \cos(kx) M_\upsilon(x)\mathrm{d}x = 2E_{2\upsilon}(k^2)$$

15.Graphical representation of M-Wright function

The figures- 4 and 5 give graph of $M_{\upsilon}(x)$ for υ equaling zero to 0.5 and υ from 0.5 to 1.0, respectively for $|x| \leq 5$. For $\upsilon = 0$, the function $M_{0}\left(|x|\right)$ is $e^{-|x|}$ gets transformed to $(1/\sqrt{\pi})e^{-x^2}$ for $\upsilon = 1/2$, as depicted in the figure-4. In the figure-5 the $(1/\sqrt{\pi})e^{-x^2}$ gets transformed to almost $\delta(x \pm 1)$ for $\upsilon = 1$.

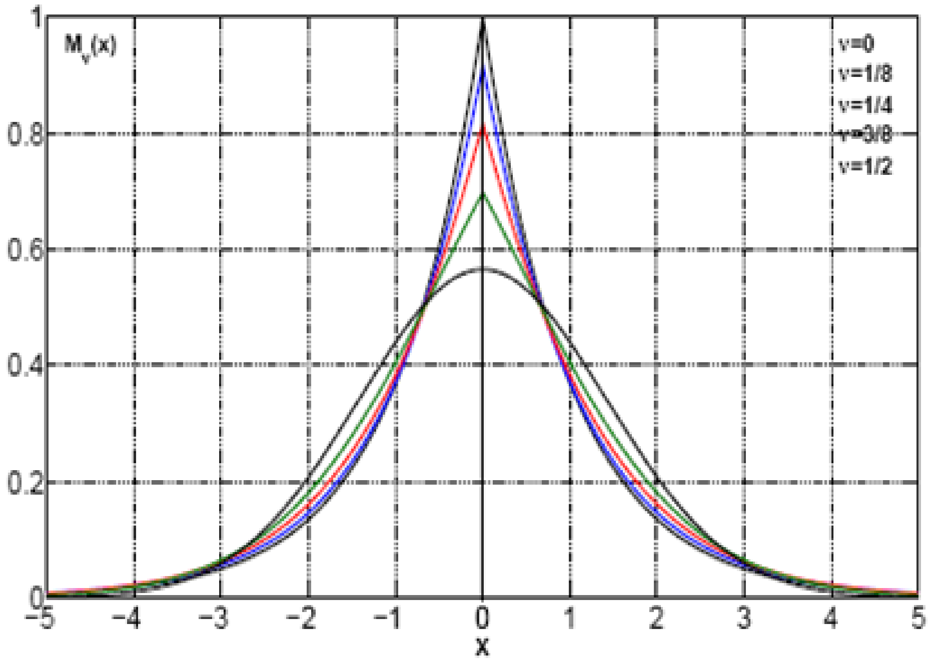


Figure-4: Plot of symmetric M-Wright function for υ zero to 0.5

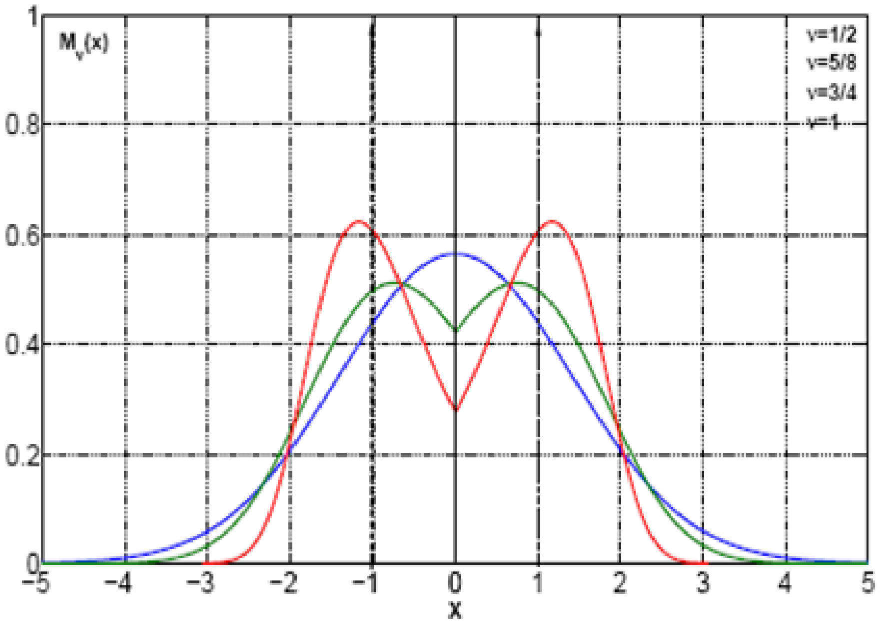


Figure-5: Plot of symmetric M-Wright function for υ 0.5 to 1.0

16.The auxiliary function in two variables

The two variables M-Wright function is introduced as

$$M_{\upsilon}(x,t) := t^{-\upsilon} M_{\upsilon}(xt^{-\upsilon}) \qquad 0 < \upsilon < 1 \qquad x,t \in \mathbb{R}^{+}$$

This above function defines probability density function (pdf) in space x, evolving with time t, with exponent υ. This exponent is also called self-similarity exponent or Hurst parameter, or Hurst index $H = \upsilon$. Obviously we should consider symmetric version, obtained from above definition and multiplying by ½ and replacing x by $|x|$ for $x \in \mathbb{R}$. From the Laplace pair $\upsilon^{-1} F_{\upsilon}(t^{-\upsilon}) = (t^{-\upsilon}) M_{\upsilon}(t^{-\upsilon}) \overset{\mathcal{L}}{\leftrightarrow} s^{\upsilon-1} e^{-s^{\upsilon}}$; $t \to s$, $t \in \mathbb{R}^{+}$, $s \in \mathbb{C}$ we derive the Laplace transform of $M_{\upsilon}(x,t)$ w.r.t $t \in \mathbb{R}^{+}$ as

$$\mathcal{L}\left\{M_{\upsilon}(x,t); t \to s\right\} = \mathcal{L}\left\{t^{-\upsilon} M_{\upsilon}(xt^{-\upsilon}); t \to s\right\} = s^{\upsilon-1} e^{-xs^{\upsilon}}$$

The derivation is as follows

$$\mathcal{L}^{-1}\left\{e^{-xs^{\upsilon}};s\to t\right\}=\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n x^n s^{\upsilon n}}{n!}\right\}=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\mathcal{L}^{-1}\left\{s^{\upsilon n}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\frac{t^{-\upsilon n-1}}{\Gamma(-\upsilon n)}=\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n x^n t^{-\upsilon n}}{n!\Gamma(-\upsilon n)}=\frac{1}{t}\sum_{n=0}^{\infty}\frac{(-1)^n (xt^{-\upsilon})^n}{n!\Gamma(-\upsilon n)}$$

$$=\frac{1}{t}F_{\upsilon}(xt^{-\upsilon})=\frac{\upsilon x}{t^{\upsilon+1}}M_{\upsilon}(1/t^{\upsilon})\qquad \text{use}\qquad F_{\upsilon}(z)=\upsilon z M_{\upsilon}(z);\quad z=xt^{-\upsilon}$$

$$\mathcal{L}^{-1}\left\{e^{-xs^{\upsilon}}\right\}=\frac{1}{t}F_{\upsilon}(xt^{-\upsilon})=\frac{\upsilon x}{t^{\upsilon+1}}M_{\upsilon}(xt^{-\upsilon})\qquad \text{or}\qquad \mathcal{L}\left\{\frac{1}{t^{\upsilon+1}}M_{\upsilon}(xt^{-\upsilon})\right\}=\frac{1}{\upsilon x}e^{-xs^{\upsilon}}$$

$$\mathcal{L}\left\{t^{-\upsilon}M_{\upsilon}(xt^{-\upsilon});t\to s\right\}=\mathcal{L}\left\{t\left(\frac{1}{t^{\upsilon+1}}M_{\upsilon}(xt^{-\upsilon})\right)\right\}=(-1)\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{\upsilon x}e^{-xs^{\upsilon}}\right]$$

$$=\frac{(-1)}{\upsilon x}(-x\upsilon s^{\upsilon-1})e^{-xs^{\upsilon}}=s^{\upsilon-1}e^{-xs^{\upsilon}}$$

Again from the Laplace pair $M_{\upsilon}(x,t)=t^{-\upsilon}M_{\upsilon}(xt^{-\upsilon})\overset{\mathcal{L}}{\leftrightarrow}E_{\upsilon}(-s);\quad x\to s$, we derive Laplace transform of $M_{\upsilon}(x,t)$ w.r.t $x\in\mathbb{R}^{+}$ as

$$\mathcal{L}\left\{M_{\upsilon}(x,t);x\to s\right\}=\mathcal{L}\left\{t^{-\upsilon}M_{\upsilon}(t^{-\upsilon}x);x\to s\right\}=\int_0^{\infty}e^{-sx}(t^{-\upsilon})M_{\upsilon}(t^{-\upsilon}x)\mathrm{d}x$$

$$=\int_0^{\infty}\left(\sum_{n=0}^{\infty}\frac{(-sx)^n}{n!}\right)(t^{-\upsilon})M_{\upsilon}(t^{-\upsilon}x)\mathrm{d}x$$

$$=\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}(t^{-\upsilon})\int_0^{\infty}x^n M_{\upsilon}(xt^{-\upsilon})\mathrm{d}x$$

Now put $xt^{-\upsilon}=y$, $\mathrm{d}x=t^{\upsilon}\mathrm{d}y$, $x=t^{\upsilon}y$

$$\mathcal{L}\left\{M_{\upsilon}(x,t);x\to s\right\}=\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}(t^{-\upsilon})\int_0^{\infty}(yt^{\upsilon})^n M_{\upsilon}(y)t^{\upsilon}\mathrm{d}y$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(s)^n(t^{\upsilon})^n}{n!}\int_0^{\infty}y^n M_{\upsilon}(y)\mathrm{d}y\qquad \text{use}\qquad \int_0^{\infty}y^n M_{\upsilon}(y)\mathrm{d}y=\frac{n!}{\Gamma(\upsilon n+1)}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(st^{\upsilon})^n}{n!}\frac{n!}{\Gamma(\upsilon n+1)}$$

$$=\sum_{n=0}^{\infty}\frac{(-st^{\upsilon})^n}{\Gamma(\upsilon n+1)}=E_{\upsilon}(-st^{\upsilon})\qquad \text{by using}\qquad E_{\upsilon}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\upsilon n+1)}$$

$$\mathcal{L}\left\{M_{\upsilon}(x,t);x\to s\right\}=\mathcal{L}\left\{t^{-\upsilon}M_{\upsilon}(xt^{-\upsilon});x\to s\right\}=E_{\upsilon}(-st^{\upsilon})$$

Note in above the variable x if representing quantity position, then the frequency s , be of spatial frequency generally represented as pure complex number ik (signifying the steady state response). In that case we may write

$$\mathcal{L}\left\{M_\upsilon(x,t);x\to ik\right\}=\mathcal{L}\left\{t^{-\upsilon}M_\upsilon(xt^{-\upsilon});x\to ik\right\}=E_\upsilon(-ikt^{\upsilon});\quad x\in\mathbb{R}^+,\quad k\in\mathbb{R}$$

From $M_\upsilon\left(|x|\right)\overset{\Im}{\leftrightarrow}2E_{2\upsilon}(-k^2)$, we write Fourier of $M_\upsilon\left(|x|,t\right)$ as

$$\Im\left\{M_\upsilon\left(|x|,t\right);x\to k\right\}=2E_{2\upsilon}(-k^2t^{2\upsilon})$$

$$\begin{aligned}\Im\left\{M_\upsilon(|x|,t);x\to k\right\}&=2\int_0^\infty\cos(kx)\left[t^{-\upsilon}M_\upsilon(xt^{-\upsilon})\right]\mathrm{d}x\\&=2\int_0^\infty\left(\sum_{n=0}^\infty\frac{(-1)^n(kx)^{2n}}{(2n)!}\right)\left[t^{-\upsilon}M_\upsilon(xt^{-\upsilon})\right]\mathrm{d}x\\&=2\sum_{n=0}^\infty\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(x^{2n})(t^{-\upsilon})M_\upsilon(xt^{-\upsilon})\mathrm{d}x\end{aligned}$$

Put $xt^{-\upsilon}=y,\quad \mathrm{d}x=t^{\upsilon}\,\mathrm{d}y$ to get following

$$\begin{aligned}\Im\left\{M_\upsilon(|x|,t);x\to k\right\}&=2\sum_{n=0}^\infty\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(x^{2n})(t^{-\upsilon})M_\upsilon(xt^{-\upsilon})\mathrm{d}x\\&=2\sum_{n=0}^\infty\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(yt^{\upsilon})^{2n}(t^{-\upsilon})M_\upsilon(y)(t^{\upsilon})\mathrm{d}y\\&=2\sum_{n=0}^\infty\frac{(-1)^n(k)^{2n}}{(2n)!}\int_0^\infty(yt^{\upsilon})^{2n}M_\upsilon(y)\mathrm{d}y\\&=2\sum_{n=0}^\infty\frac{(-1)^n(kt^{\upsilon})^{2n}}{(2n)!}\int_0^\infty(y)^{2n}M_\upsilon(y)\mathrm{d}y\\&=2\sum_{n=0}^\infty\frac{(-1)^n(k^2t^{2\upsilon})^n}{(2n)!}\times\frac{(2n)!}{\Gamma(2\upsilon n+1)}=2E_{2\upsilon}(-k^2t^{2\upsilon})\end{aligned}$$

As special case for $\upsilon=1/2$

$$\begin{aligned}\frac{1}{2}M_{\frac{1}{2}}\left(|x|,t\right)&=\frac{1}{2}t^{-1/2}M_{\frac{1}{2}}\left(xt^{-1/2}\right)=\frac{1}{2\sqrt{t}}\frac{1}{\sqrt{\pi}}e^{-(xt^{-1/2})^2/4}\qquad\text{use}\qquad M_{\frac{1}{2}}(z)=\frac{1}{\sqrt{\pi}}e^{-z^2/4}\\&=\frac{1}{2\sqrt{\pi t}}e^{-x^2/(4t)}\end{aligned}$$

For limiting case $\upsilon=1$ we obtain

$$\frac{1}{2}M_1(x,t)=\frac{1}{2}\delta(x-t)+\frac{1}{2}\delta(x+t)$$

17. The use of two variable auxiliary functions

Let us have Time Fractional Diffusion Equation (with Caputo fractional derivative of order β)

$$\left[\frac{\partial^\beta}{\partial t^\beta}\right]u(x,t)=\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 0<\beta<1$$

Integrating fractionally by order β both sides we obtain the following

$$ {}_0I_t^\beta\left[\frac{\partial^\beta}{\partial t^\beta}\right]u(x,t)= {}_0I_t^\beta\left[a\frac{\partial^2 u}{\partial x^2}\right] \qquad 0<\beta<1$$

$$u(x,t)-u(x,0^+)=\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}$$

$$u(x,t)=u(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}}$$

$$u(x,t)=u(x,0^+)+\frac{a}{\Gamma(\beta)}\int_0^t\left(\frac{\partial^2 u}{\partial x^2}\right)\frac{\mathrm{d}\tau}{(t-\tau)^{1-\beta}} \qquad \text{take} \qquad a=1$$

$$u(x,0^+)+{}_0I_t^\beta\frac{\partial^2}{\partial t^2}u(x,t)$$

$$u(x,0^+)=\delta(x)=G_\beta(x,0^+)$$

Therefore we obtain fundamental solution $u(x,t)=G_\beta(x,t)$ and we write

$$G_\beta(x,t)=\delta(x)+{}_0I_t^\beta\frac{\partial^2}{\partial x^2}G_\beta(x,t)$$

We take first Fourier transform of above equation, by using $ik\overset{\Im}{\leftrightarrow}\partial/\partial x$, $1\overset{\Im}{\leftrightarrow}\delta(x)$ and $x\overset{\Im}{\leftrightarrow}k$.

$$\hat{G}_\beta(k,t)=1+{}_0I_t^\beta(-k^2)\hat{G}_\beta(x,t)$$

Now we apply Laplace transform to above by using ${}_0I_t^\beta\overset{\mathcal{L}}{\leftrightarrow}s^{-\beta}$, $1\overset{\mathcal{L}}{\leftrightarrow}s^{-1}$, and $t\overset{\mathcal{L}}{\leftrightarrow}s$.

$$\tilde{\hat{G}}_\beta(k,s)=\frac{1}{s}-k^2s^{-\beta}\tilde{\hat{G}}_\beta(k,s)$$

We get

$$\tilde{\hat{G}}_\beta(k,s)=\frac{s^{\beta-1}}{s^\beta+k^2}$$

We can have two methods, for inverting the above

First we Fourier invert then Laplace invert. That is recalling Fourier pair as

$$\frac{2|b|}{b^2+\omega^2}\overset{\Im}{\leftrightarrow}e^{-|b|t} \qquad \text{gives} \qquad \frac{a}{b+k^2}\overset{\Im}{\leftrightarrow}\frac{a}{2\sqrt{b}}e^{-|x|\sqrt{b}} \qquad a,b>0$$

We set $a=s^{\beta-1}$ and $b=s^\beta$, and apply above transform pair to get

$$\tilde{G}_\beta(x,s)=\frac{s^{\beta-1}}{2\sqrt{s^\beta}}e^{-|x|\sqrt{s^\beta}}=\frac{1}{2}s^{(\beta/2)-1}e^{-|x|s^{\beta/2}}$$

Now we Laplace invert by using

$M_{\upsilon}(x,t)=t^{-\upsilon}M_{\upsilon}(x,t)\overset{\mathcal{L}}{\leftrightarrow}s^{\upsilon-1}e^{-xs^{\upsilon}}$, here we set $\upsilon=\beta/2$ to get

$$G_{\beta}(x,t)=\frac{1}{2}t^{-(\beta/2)}M_{\beta/2}(|x|t^{-(\beta/2)})$$

By second method first we Laplace invert the $\hat{\tilde{G}}_{\beta}(k,s)$ and then apply Fourier inversion. That is first recalling Laplace pair

$$\frac{s^{\beta-1}}{s^{\beta}+c}\overset{\mathcal{L}}{\leftrightarrow}E_{\beta}(-ct^{\beta}) \qquad c>0$$

and then by setting $c=k^2$, we get

$$\hat{G}_{\beta}(k,t)=E_{\beta}(-k^2t^{\beta})$$

Use $\Im\left\{M_{\upsilon}\left(|x|,t\right);x\to k\right\}=2E_{2\upsilon}(-k^2t^{2\upsilon})$ to Fourier invert the above with $2\upsilon=\beta;\quad \upsilon=\beta/2$, to get

$$G_{\beta}(x,t)=\frac{1}{2}M_{\beta/2}(x,t)=\frac{1}{2}t^{-\left(\frac{\beta}{2}\right)}M_{\beta/2}\left(\frac{|x|}{t^{\beta/2}}\right)$$

We shall now demonstrate the use of M-Wright function as solution to Time fractional drift equation that is the following

$$\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t)=-\frac{\partial}{\partial x}u(x,t);\qquad -\infty<x<\infty,\quad t\geq 0 \qquad \text{with} \qquad u(x,0^+)=\delta(x)$$

Applying Laplace and then to the above equation we get the following with $t\to s$

$$s^{\beta}\tilde{u}(x,s)-s^{\beta-1}u(x,0^+)=-\frac{\partial}{\partial x}\tilde{u}(x,s) \qquad u(x,0^+)=\delta(x) \qquad \text{then} \qquad u(x,t)=G_{\beta}(x,t)$$

$$s^{\beta}\tilde{G}_{\beta}(x,s)-s^{\beta-1}\delta(x)=-\frac{\partial}{\partial x}\tilde{G}_{\beta}(x,s)$$

Now we do Fourier transform, with $1\overset{\Im}{\leftrightarrow}\delta(x)$, $ik\overset{\Im}{\leftrightarrow}\partial/\partial x$ to have

$$s^{\beta}\hat{\tilde{G}}_{\beta}(k,s)-s^{\beta-1}(1)=-ik\hat{\tilde{G}}_{\beta}(k,s)$$

$$\hat{\tilde{G}}_{\beta}(k,s)=\frac{s^{\beta-1}}{s^{\beta}+ik}$$

We first Fourier invert the above by recalling

$$\frac{1}{|b|+i\omega}\overset{\Im}{\leftrightarrow}e^{-|b|t} \qquad \frac{a}{b+ik}\overset{\Im}{\leftrightarrow}ae^{-bx}; \qquad a,b>0;\quad x>0$$

We set $a = s^{\beta-1}$ and $b = s^{\beta}$, we obtain

$$\hat{G}_{\beta}(x,s) = s^{\beta-1}e^{-xs^{\beta}}$$

Now we inverse Laplace recalling $\mathcal{L}\left\{M_{\upsilon}(x,t);t \to s\right\} = s^{\upsilon-1}e^{-xs^{\upsilon}}$, to get

$$G_{\beta}(x,t) = M_{\beta}(x,t) = t^{-\beta}M_{\beta}(xt^{-\beta})$$

We apply the second method of first Laplace inverting $\tilde{\hat{G}}_{\beta}(k,s) = (s^{\beta-1})/(s^{\beta}+ik)$, by using the Laplace pair $(s^{\beta-1})/(s^{\beta}+c) \overset{\mathcal{L}}{\leftrightarrow} E_{\beta}(-ct^{\beta}); \quad c > 0$. Set $c = ik$, to get

$$\hat{G}_{\beta}(k,t) = E_{\beta}(-ikt^{\beta})$$

As indicated above we transform $ik \to x$ to get from obtained Laplace pair that is

$$\mathcal{L}\left\{M_{\upsilon}(x,t);x \to ik\right\} = \mathcal{L}\left\{t^{-\upsilon}M_{\upsilon}(xt^{-\upsilon});x \to ik\right\} = E_{\upsilon}(-ikt^{\upsilon}); \quad x \in \mathbb{R}^{+}, \quad k \in \mathbb{R}$$

The space x is transformed to complex spatial frequency ik is same as in Fourier case. So above Laplace pair we use as Fourier pair, and write the following.

$$G_{\beta}(x,t) = M_{\beta}(x,t) = t^{-\beta}M_{\beta}\left(\frac{x}{t^{\beta}}\right)$$

18.Reviewing Time Fractional diffusion equations

We consider a variety of diffusion equations including the standard integer order diffusion equation whose fundamental solution the Green's function are expressed in terms of the M-Wright function depending on space or time variables. The two variables however are related to through a self similarity condition (Hurst exponent $H \in (0,1)$)

The standard integer order diffusion equation for the field variable $u(x,t)$ with initial condition $u(x,0) = u_0(x)$ is

$$\frac{\partial u}{\partial t} = K_1\frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty; \quad t > 0$$

Where K_1 is suitable diffusion coefficient with dimensions of cm^2/s . This is initial Boundary value Problem, and we perform integration of both sides to write solution in terms of Volterra. Integral Equation

$$\int_0^t \frac{\partial u}{\partial \tau}\mathrm{d}\tau = K_1\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \qquad u(x,t) = u_0(x) + K_1\int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

It is well known that the fundamental solution also called Green's function corresponding to $u_0(x)=\delta(x)$ is the Gaussian probability density evolving in time with mean square displacement proportional to time (that is variance or second moment). We write Green's function as

$$G_1(x,t)=\frac{1}{2\sqrt{\pi K_1 t}}e^{-x^2/(4K_1 t)}$$

Having variance as $\sigma_1^2(t)=\int\limits_{-\infty}^{+\infty}x^2 G_1(x,t)\mathrm{d}x=2K_1 t$. The above Green's function in terms of M-Wright function is

$$G_1(x,t)=\frac{1}{2}\frac{1}{\sqrt{K_1 t^{1/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t}}\right)$$

Let us find the moment of the above Green's function

$$\sigma^2=\int\limits_{-\infty}^{+\infty}x^2\frac{1}{2\sqrt{K_1 t}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t}}\right)\mathrm{d}x$$

Put $\frac{|x|}{\sqrt{K_1 t}}=y;\quad x^2=y^2K_1 t;\quad \mathrm{d}x=\mathrm{d}y\sqrt{K_1 t}$, in the above expression to get

$$\begin{aligned}\sigma^2&=\int\limits_{-\infty}^{\infty}\frac{y^2K_1 t}{2\sqrt{K_1 t}}M_{1/2}(y)\mathrm{d}y\sqrt{K_1 t}\\&=\frac{K_1 t}{2}\int\limits_{-\infty}^{\infty}y^2M_{1/2}(y)\mathrm{d}y\\&=\frac{K_1 t}{2}\times 2\int\limits_{0}^{\infty}y^2M_{1/2}(y)\mathrm{d}y=K_1 t\int\limits_{0}^{\infty}y^2M_{1/2}(y)\mathrm{d}y\end{aligned}$$

We use already derived expression about α -moments of the M-Wright function that is $\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x=\frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)}$. In this general expression put $\alpha=2,\quad \upsilon=1/2$, so we have the following

$$\int_0^\infty x^2M_{1/2}(x)\mathrm{d}x=\frac{\Gamma(2+1)}{\Gamma\left(\frac{1}{2}\times 2+1\right)}=\frac{\Gamma(3)}{\Gamma(2)}=\frac{2!}{1!}=2$$

Use of the above derived expression gives the variance as proportional to time

$$\sigma^2=K_1 t\int\limits_0^\infty y^2M_{1/2}(y)\mathrm{d}y=2K_1 t$$

From the self-similarity of Green's function, as for following

$$G_1(x,t)=\frac{1}{2\sqrt{\pi K_1}t^{1/2}}e^{-x^2/(4K_1t)} \qquad G_1(x,t)=\frac{1}{2\sqrt{K_1}t^{1/2}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1}t^{1/2}}\right)$$

we can write

$$G_1(x,t)=\frac{1}{\sqrt{K_1}t^H}G_1(z) \qquad z=\frac{|x|}{\sqrt{K_1}t^H}$$

where $H=1/2$, is the self similarity (or Hurst) exponent and $z=|x|/\left(\sqrt{K_1}t^H\right)$ acts as the similarity variable, with $G_1(z)$ as one variable 'reduced' Green's function. Note that the above derived variance law $\propto t$ characterizes normal diffusion as it emerges from Brownian motion approach. With $H\neq 0.5$, $H\in(0,1)$ the case is of fractional Brownian motion with persistent motion $H\in(0.5,1)$ and anti-persistent with $H\in(0,0.5)$; giving case for anomalous diffusion.

Now we stretch the time variable in the integer order diffusion equation replacing t with t^α where $\alpha\in(0,2)$, and we write the modified equation as

$$\frac{\partial u}{\partial(t^\alpha)}=K_\alpha\frac{\partial^2 u}{\partial x^2} \qquad -\infty<x<+\infty; \quad t\geq 0$$

We call K_α as stretched diffusion coefficient having dimension $\mathrm{cm}^2/\mathrm{s}^\alpha$. We recognize easily that above diffusion equation is akin to the standard diffusion equation, but with diffusion coefficient depending on time $K_1(t)=\alpha t^{\alpha-1}K_\alpha$, which follows from following derivation

$$\frac{\partial u}{\partial t^\alpha}=\frac{\partial u}{\alpha t^{\alpha-1}\partial t}=K_\alpha\frac{\partial^2 u}{\partial x^2} \qquad \text{using} \qquad \mathrm{d}(t^\alpha)=\alpha t^{\alpha-1}\mathrm{d}t$$
$$\frac{\partial u}{\partial t}=\left(\alpha t^{\alpha-1}K_\alpha\right)\frac{\partial^2 u}{\partial x^2}=K_1(t)\frac{\partial^2 u}{\partial x^2}$$
$$K_1(t)=\alpha t^{\alpha-1}K_\alpha$$

The Volterra integral corresponding to the time stretched case is obtained by taking integration of both LHS and RHS of the stretched time diffusion equation $\frac{\partial u}{\partial t}=\left(\alpha t^{\alpha-1}K_\alpha\right)\frac{\partial^2 u}{\partial x^2}$; to get the following

$$u(x,t)=u_0(x)+\alpha K_\alpha\int_0^t \tau^{\alpha-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

We write the corresponding fundamental solution that is stretched-time Gaussian function

$$G_\alpha(x,t)=\frac{1}{2\sqrt{\pi K_\alpha t^{\alpha/2}}}e^{-x^2/\left(4K_\alpha t^\alpha\right)}=\frac{1}{2\sqrt{K_\alpha t^{\alpha/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_\alpha t^{\alpha/2}}}\right)$$

The above comes from $G_1(x,t)=\frac{1}{2\sqrt{K_1 t^{1/2}}}M_{1/2}\left(\frac{|x|}{\sqrt{K_1 t^{1/2}}}\right)$, by placing t^α instead t and K_α in place of K_1.

We can derive the corresponding variance (via the method described earlier) as

$$\sigma_\alpha^2=\int_{-\infty}^{\infty}x^2G_\alpha(x,t)\mathrm{d}x=2K_\alpha t^\alpha$$

This comes from $\sigma^2=2K_1t$ by placing t^α instead t and K_α in place of K_1.

This is characteristic of a general process of anomalous diffusion precisely slow diffusion for $0<\alpha<1$, and fast diffusion for $1<\alpha<2$.

Now we take time fractional diffusion equation and write that for $\beta\in(0,1)$ as following equivalent fractional differential equations (with fractional Caputo derivative)

$$\frac{\partial u}{\partial t}=K_\beta\left({}_0D_t^{1-\beta}\frac{\partial^2u}{\partial x^2}\right)\qquad \beta\in(0,1)\qquad \text{apply } {}_0D_t^{\beta-1}$$

$$\left({}_0D_t^{\beta-1}\circ\frac{\partial u}{\partial t}\right)=K_\beta\left({}_0D_t^{\beta-1}\circ{}_0D_t^{1-\beta}\frac{\partial^2u}{\partial x^2}\right)$$

$$\left({}_0D_t^{\beta}\circ{}_0D_t^{-1}\frac{\partial u}{\partial t}\right)=K_\beta\left(\frac{\partial^2u}{\partial x^2}\right)\qquad {}_0D_t^{-1}={}_0I_t^1$$

$${}_0D_t^\beta u=K_\beta\frac{\partial^2u}{\partial x^2}$$

So basically we have following equivalent time fractional diffusion equations

$$\frac{\partial u}{\partial t}=K_\beta\left({}_0D_t^{1-\beta}\frac{\partial^2u}{\partial x^2}\right)\qquad\qquad {}_0D_t^\beta u=K_\beta\frac{\partial^2u}{\partial x^2}$$

In this time fractional diffusion equation K_β is sort of fractional diffusion coefficient having dimension of $\mathrm{cm}^2/\mathrm{s}^\beta$.

Integrating LHS and RHS of $\frac{\partial u}{\partial t}=K_\beta\left({}_0D_t^{1-\beta}\frac{\partial^2u}{\partial x^2}\right)$ we obtain the following

$$u(x,t)-u_0(x)=K_\beta\left({}_0I_t^1\circ{}_0D_t^{1-\beta}\frac{\partial^2 u}{\partial x^2}\right)$$

$$=K_\beta\left({}_0I_t^\beta\frac{\partial^2 u}{\partial x^2}\right)\qquad {}_0D_t^{-\beta}={}_0I_t^\beta$$

$$=\frac{K_\beta}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

Like the diffusion equation of integer order we consider the equivalent Volterra integral equation corresponding to the fractional diffusion equation of above as by taking β order fractional integration for both the sides LHS and RHS of, ${}_0D_t^\beta u=K_\beta\frac{\partial^2 u}{\partial x^2}$ as demonstrated below

$$ {}_0D_t^\beta u=K_\beta\frac{\partial^2 u}{\partial x^2}$$

$$ {}_0I_t^\beta\circ{}_0D_t^\beta u=K_\beta\left({}_0I_t^\beta\frac{\partial^2 u}{\partial x^2}\right)$$

$$u(x,t)-u_0(x)=K_\beta\left({}_0I_t^\beta\frac{\partial^2 u}{\partial x^2}\right)$$

$$u(x,t)=u_0(x)+\frac{K_\beta}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

The Green's function, $G_\beta(x,t)$ for this time fractional diffusion equation can be expressed in terms of the M-Wright function

$$G_\beta(x,t)=\frac{1}{2\sqrt{K_\beta}t^{\beta/2}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta}t^{\beta/2}}\right)$$

The above we have derived in the previous section.

Let us find the moment of the above Green's function

$$\sigma_\beta^2=\int_{-\infty}^{+\infty}x^2\frac{1}{2\sqrt{K_\beta}t^{\beta/2}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta}t^{\beta/2}}\right)\mathrm{d}x$$

Put $\frac{|x|}{\sqrt{K_\beta t^\beta}}=y$; $x^2=y^2K_\beta t^\beta$; $\mathrm{d}x=\mathrm{d}y\sqrt{K_1t^\beta}$, in the above expression to get

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{y^2 K_\beta t^\beta}{2\sqrt{K_\beta t^\beta}} M_{\beta/2}(y)\mathrm{d}y\sqrt{K_\beta t^\beta}$$

$$= \frac{K_\beta t^\beta}{2} \int_{-\infty}^{\infty} y^2 M_{\beta/2}(y)\mathrm{d}y$$

$$= \frac{K_\beta t^\beta}{2} \times 2\int_0^{\infty} y^2 M_{\beta/2}(y)\mathrm{d}y = K_\beta t \int_0^{\infty} y^2 M_{\beta/2}(y)\mathrm{d}y$$

We use already derived expression about α -moments of the M-Wright function that is $\int_0^\infty x^\alpha M_\upsilon(x)\mathrm{d}x = \frac{\Gamma(\alpha+1)}{\Gamma(\upsilon\alpha+1)}$. In this general expression put $\alpha = 2, \quad \upsilon = \beta/2$, so we have the following

$$\int_0^\infty x^2 M_{\beta/2}(x)\mathrm{d}x = \frac{\Gamma(2+1)}{\Gamma\left(\frac{\beta}{2}\times 2+1\right)} = \frac{\Gamma(3)}{\Gamma(\beta+1)} = \frac{2}{\Gamma(\beta+1)}$$

Use of the above derived expression gives the variance as proportional to time

$$\sigma_\beta^2 = K_\beta t^\beta \int_0^\infty y^2 M_{\beta/2}(y)\mathrm{d}y = 2K_\beta t^\beta / \Gamma(\beta+1)$$

The corresponding variance we write as we obtained from absolute moments of M-Wright function as

$$\sigma_\beta^2(t) = \int_{-\infty}^{+\infty} x^2 G_\beta(x,t)\mathrm{d}x = \frac{2}{\Gamma(\beta+1)} K_\beta t^\beta$$

As a consequence for $0 < \beta < 1$the variance is consistent with a process of slow diffusion with similarity exponent $H = \beta/2$.

Now we make one more step as advancement, that is, stretched the time-variable with fractional diffusion equation; by stretching the time by replacing t by $t^{\alpha/\beta}$, where $0 < \alpha < 2$ and $0 < \beta \le 1$.; so we get the following (with Caputo fractional derivative)

$$\frac{\partial u}{\partial(t^{\alpha/\beta})} = K_{\alpha\beta}\left({}_0D_{t^{\alpha/\beta}}^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right)$$

Where, $K_{\alpha\beta}$ is a sort of stretched diffusion coefficient of dimension $\mathrm{cm}^2/\mathrm{s}^\alpha$ that reduces to K_α for $\beta = 1$ and to K_β for $\alpha = \beta$. From above we get the following equivalent fractional diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\alpha}{\beta}t^{\frac{\alpha}{\beta}-1}K_{\alpha\beta}\left({}_0D^{1-\beta}_{t^{\alpha/\beta}}\frac{\partial^2 u}{\partial x^2}\right)$$

The steps to get above expression are as follows.

$$\frac{\partial u}{\partial(t^{\alpha/\beta})}=\frac{\partial u}{\frac{\alpha}{\beta}t^{(\alpha/\beta)-1}\partial t}=K_{\alpha\beta}\left({}_0D^{1-\beta}_{t^{\alpha/\beta}}\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{\partial u}{\partial t}=\frac{\alpha}{\beta}t^{\frac{\alpha}{\beta}-1}K_{\alpha\beta}\left({}_0D^{1-\beta}_{t^{\alpha/\beta}}\frac{\partial^2 u}{\partial x^2}\right)$$

Now we integrate both sides LHS and RHS of the above obtained expression

$$u(x,t)-u_0(x)=\frac{\alpha}{\beta}K_{\alpha\beta}\int_{\tau=0}^{\tau=t}\tau^{\frac{\alpha}{\beta}-1}\left({}_0D^{1-\beta}_{\tau^{\alpha/\beta}}\frac{\partial^2 u(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau$$

Let $\tau^{\alpha/\beta}=\xi$, $\tau=\xi^{\beta/\alpha}$ for $\tau=0$; $\xi=0$ and for $\tau=t$ we have $\xi=t^{\alpha/\beta}$. Furthermore we have $(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau=\mathrm{d}\xi$. Substituting all these we have

$$\frac{\alpha}{\beta}K_{\alpha\beta}\int_{\tau=0}^{\tau=t}\tau^{\frac{\alpha}{\beta}-1}\left({}_0D^{1-\beta}_{\tau^{\alpha/\beta}}\frac{\partial^2 u(x,\tau)}{\partial x^2}\right)\mathrm{d}\tau$$

$$=\frac{\alpha}{\beta}K_{\alpha\beta}\int_{\xi=0}^{\xi=t^{\alpha/\beta}}\tau^{\frac{\alpha}{\beta}-1}\left({}_0D^{1-\beta}_{\xi}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial x^2}\right)\left(\frac{\mathrm{d}\xi}{\frac{\alpha}{\beta}\tau^{(\alpha/\beta)-1}}\right)$$

$$=K_{\alpha\beta}\int_{\xi=0}^{\xi=t^{\alpha/\beta}}\left({}_0D^{1-\beta}_{\xi}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial x^2}\right)\mathrm{d}\xi$$

$$=K_{\alpha\beta}\left({}_0I^1_{\xi}\,{}_0D^{1-\beta}_{\xi}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial x^2}\right)=K_{\alpha\beta}\left({}_0I^1_{\xi}\,{}_0D^1_{\xi}\,{}_0D^{-\beta}_{\xi}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial x^2}\right)$$

$$=K_{\alpha\beta}\left({}_0D^{-\beta}_{\xi}\frac{\partial^2 u(x,\xi^{\beta/\alpha})}{\partial x^2}\right)=K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\varsigma)^{\beta-1}\frac{\partial^2 u(x,\varsigma^{\beta/\alpha})}{\partial x^2}\mathrm{d}\varsigma \qquad \varsigma\equiv\xi$$

We have $\tau^{\alpha/\beta}=\xi$, $\mathrm{d}\xi=(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau$ .Take $\tau^{\alpha/\beta}=\varsigma$ and $(\alpha/\beta)\tau^{(\alpha/\beta)-1}\mathrm{d}\tau=\mathrm{d}\varsigma$, to have

$$K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\varsigma)^{\beta-1}\frac{\partial^2 u(x,\varsigma^{\beta/\alpha})}{\partial x^2}\mathrm{d}\varsigma=K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau\left(\frac{\alpha}{\beta}\tau^{(\alpha/\beta)-1}\right)$$

Put $\xi=t^{\alpha/\beta}$, so we have $\varsigma\equiv\xi=\tau^{\alpha/\beta}$, thus for $\varsigma=t^{\alpha/\beta}$, we have upper limit $\tau=t$

$$K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=\xi}(\xi-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau\left(\frac{\alpha}{\beta}\tau^{(\alpha/\beta)-1}\right)$$

$$=K_{\alpha\beta}\frac{\alpha}{\beta}\frac{1}{\Gamma(\beta)}\int_{\varsigma=0}^{\varsigma=t^{\alpha/\beta}}\tau^{\frac{\alpha}{\beta}-1}(t^{\alpha/\beta}-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau \qquad \varsigma\equiv\xi=\tau^{\alpha/\beta}$$

$$=K_{\alpha\beta}\frac{\alpha}{\beta}\frac{1}{\Gamma(\beta)}\int_{\tau=0}^{\tau=t}\tau^{\frac{\alpha}{\beta}-1}(t^{\alpha/\beta}-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

Therefore our integral equation becomes

$$u(x,t)=u_0(x)+K_{\alpha\beta}\frac{1}{\Gamma(\beta)}\frac{\alpha}{\beta}\int_0^t\tau^{\frac{\alpha}{\beta}-1}(t^{\alpha/\beta}-\tau^{\alpha/\beta})^{\beta-1}\frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau$$

This has Green's function as

$$G_{\alpha\beta}(x,t)=\frac{1}{2}\frac{1}{\sqrt{K_{\alpha\beta}}t^{\alpha/2}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_{\alpha\beta}}t^{\alpha/2}}\right)$$

Comes from $G_\beta(x,t)=\frac{1}{2\sqrt{K_\beta}t^{\beta/2}}M_{\beta/2}\left(\frac{|x|}{\sqrt{K_\beta}t^{\beta/2}}\right)$, by placing $t^{\alpha/\beta}$ in place of t and changing K_β with $K_{\alpha\beta}$.

The variance is

$$\sigma^2_{\alpha\beta}=\int_{-\infty}^{+\infty}x^2G_{\alpha\beta}(x,t)\mathrm{d}x=\frac{2}{\Gamma(\beta+1)}K_{\alpha\beta}t^{\alpha}$$

Comes from $\sigma^2_\beta(t)=\int_{-\infty}^{+\infty}x^2G_\beta(x,t)\mathrm{d}x=\frac{2}{\Gamma(\beta+1)}K_\beta t^\beta$, by placing $t^{\alpha/\beta}$ in place of t and $K_{\alpha\beta}$ in place of K_β.

As a consequence the resulting diffusion process turns out to be self similar with Hurst exponent $H=\alpha/2$ and the variance law consistent with both with slow diffusion $(0<\alpha<1)$ and fast diffusion $(1<\alpha<2)$. We note that the parameter β explicitly enter the variance expression only to modify the proportionality constant. It is straight forward to note that the evolution equations of this process reduce to those for fractional-time diffusion if $\alpha=\beta<1$, for stretched diffusion if $\alpha\neq1$ and $\beta=1$; and finally to the integer order diffusion equation if $\alpha=\beta=1$.

19. Fractional diffusion and stochastic process with self similar increments giving several classes of Brownian motion

We have seen that any Green's associated to the diffusion like equation can be interpreted as the time-evolving one-point pdf of certain self similar process. However it is not possible (generally) to define a unique self similar stochastic process because the determination of any multi-point probability distribution is required. Meaning, starting from a master equation which describes the dynamic evolution of pdf $f(x,t)$, it is always possible to define an equivalence class of stochastic process with the same marginal density function $f(x,t)$. All these processes provide suitable stochastic representations for the starting equation. It is clear that additional requirements may be stated in order to uniquely select the probability model.

For instant considering

$$\frac{\partial u}{\partial t} = \frac{\alpha}{\beta} t^{\frac{\alpha}{\beta}-1} K_{\alpha\beta} \left({}_0D_{t^{\alpha/\beta}}^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right)$$

the additional requirement of stationary increments can lead to a class of generalized grey Brownian motion (ggBm) which by construction is made up of self-similar processes with stationary increments and Hurst exponent $H = \alpha/2$. This ggBm is is a special class of H-self-similar-stationary-increments processes, which provide non-Markovian stochastic models for anomalous diffusion, both of slow type with $0 < \alpha < 1$ and fast type $1 < \alpha < 2$. The ggBm includes some well known processes, so that it defines an interesting general theoretical framework. The fractional Brownian motion (fBm) appears for $\beta = 1$ and is associated with equation

$$\frac{\partial u}{\partial (t^\alpha)} = K_\alpha \frac{\partial^2 u}{\partial x^2}$$

The grey Brownian motion (gBm) corresponds to choice of $\alpha = \beta$, with $0 < \beta < 1$, and is associated with

$$\frac{\partial u}{\partial t} = K_\beta \left({}_0D_t^{1-\beta} \frac{\partial^2 u}{\partial x^2} \right) \qquad {}_0D_t^\beta u = K_\beta \frac{\partial^2 u}{\partial x^2}$$

Finally the standard Brownian motion (Bm) is recovered by setting $\alpha = \beta = 1$ and is associated with following and is Markovian process

$$\frac{\partial u}{\partial t} = K_1 \frac{\partial^2 u}{\partial x^2}$$

End of Part-D