Lecture Notes

Fractional Viscoelasticity Part-C

"Fractional visco-elastic systems"

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Fractional visco-elastic systems

Scott-Blair writes, "I was working on the assessing of the firmness of various materials (e.g. cheeses and clay by experts handling them) these systems are of course both elastic and viscous but I felt sure that judgments were made not on an addition of elastic and viscous parts but on something in between the two so I introduced fractional differentials of strain with respect to time. I gave up the work eventually, mainly because I could not find a definition of fractional differential that would satisfy the mathematicians". The fractional differentials equations we saw in the earlier lecture series that describe the visco-elastic systems; in this series we will formalize the treatment with the theory of linear visco-elasticity what we described in the part-B of the lecture series and extend it to a continuous order differential equation system.

Contents

1. The Fractional Calculus and visco-elasticity-from Thermo-elastic Coupling consideration and Thermal relaxation with memory

The use of fractional calculus in linear visco-elastic systems leads us to generalize the classical mechanical models, in the basic Newton element (the dashpot/piston) is substituted by a more general intermediate material model (of Scott-Blair) of order υ. We can construct the several standard linear visco-elastic systems model Voigt, Maxwell, Zenner etc by using Hooke and this intermediate (Scott-Blair) element-as series and parallel combinations. What is the justification of using fractional order calculus?

Zener first postulated and then interpreted that an-elasticity in the solid (especially metals), is linked to 'spectrum of relaxation' phenomena-i.e. the time constant distributions of strain creep τ_ε and stress relaxation τ_σ. These are related to the 'thermal relaxation' phenomena. Particularly, the thermal relaxation due to diffusion in the thermo-elastic coupling is essential to derive the constitutive differential equations (stress-strain relationship), in linear as well as fractional order visco-elastic systems. We have discussed the various linear models in lecture series-B, here we write constitutive equation in the form as below for uniaxial stress $\sigma(t)$ and uniaxial strain $\varepsilon(t)$

$$\sigma + \tau_\varepsilon \frac{\mathrm{d}\sigma}{\mathrm{d}t} = M_r \left(\varepsilon + \tau_\sigma \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \right) \tag{1}$$

The above constitutive expression is like Zener system also called Standard Linear Solid (S.L.S), we discussed in lecture series-B. The three parameters are M_r called the 'relaxed-modulus', τ_σ and τ_ε denoting the relaxation times under the constant strain and under the constant stress respectively, appears in the corresponding material functions. An additional parameter is unrelaxed modulus M_u, given by

$$\frac{\tau_\sigma}{\tau_\varepsilon} = \frac{M_u}{M_r} > 1 \tag{2}$$

The above Zener system can be derived from the basic equations of the thermo-elastic coupling, provided that τ_σ and τ_ε also corresponds to the relaxation times for the temperature ΔT relaxation at constant stress and constant strain respectively, while M_r and M_u represents the isothermal and adiabatic modulii, respectively. Then the basic equations of thermo-elasticity are

$$\varepsilon = \frac{1}{M_r}\sigma + \lambda \Delta T \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Delta T = -\frac{1}{\tau_\varepsilon}\Delta T - \gamma \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{3}$$

Denoting ΔT as the deviation in temperature from its standard value, with λ as coefficient of linear thermal expansion; $\gamma = \left(\partial T / \partial \varepsilon\right)_{\text{adiabatic}}$, The above dynamic relaxation equation for ΔT; is manifestation of phenomena which induces temperature change, the first one is diffusion equation that relaxes the ΔT, i.e.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta T\right)_{\text{diffusion}} = -\frac{1}{\tau_\varepsilon}\Delta T \tag{4}$$

The other one is due to adiabatic strain change

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Delta T\right)_{\text{adiabatic}} = -\gamma \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{5}$$

Putting $1 + \lambda\gamma = \tau_\sigma / \tau_\varepsilon = M_u / M_r$ and eliminating ΔT, in the thermo-elasticity equations as in following steps we get

$$\varepsilon = \frac{1}{M_r}\sigma + \lambda\Delta T \qquad \text{or} \qquad \Delta T = \left(\frac{\varepsilon}{\lambda} - \frac{\sigma}{\lambda M_r}\right) \tag{6}$$

Substitute in $\Delta\dot{T} = -(1/\tau_\varepsilon)\Delta T - \gamma(\dot{\varepsilon})$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varepsilon}{\lambda} - \frac{\sigma}{\lambda M_r}\right) = -\frac{1}{\tau_\varepsilon}\left(\frac{\varepsilon}{\lambda} - \frac{\sigma}{\lambda M_r}\right) - \gamma\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

$$\frac{1}{\lambda\tau_\varepsilon M_r}\sigma + \frac{1}{\lambda M_r}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\varepsilon}{\lambda\tau_\varepsilon} + \left(\frac{1+\lambda\gamma}{\lambda}\right)\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \qquad \text{use} \qquad 1+\lambda\gamma = \frac{\tau_\sigma}{\tau_\varepsilon} \tag{7}$$

$$\sigma + \tau_\varepsilon\frac{\mathrm{d}\sigma}{\mathrm{d}t} = M_r\left(\varepsilon + \tau_\sigma\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right)$$

We have obtained this Zener type constitutive equation.

Now if we say the diffusion is of fractional order i.e. having a long memory associated with it, we write

$$\left(\frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\Delta T\right) = -\frac{1}{\bar{\tau}_\varepsilon^\upsilon}\Delta T \qquad 0<\upsilon<1 \tag{8}$$

The $\bar{\tau}_\varepsilon$ is a suitable relaxation time and write the adiabatic part as

$$\left(\frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\Delta T\right) = -\gamma\frac{\mathrm{d}^\upsilon\varepsilon}{\mathrm{d}t^\upsilon} \tag{9}$$

We get fractional order differential equation relaxation for ΔT as

$$\frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\Delta T = -\frac{1}{\bar{\tau}_\varepsilon^\upsilon}\Delta T - \gamma\frac{\mathrm{d}^\upsilon\varepsilon}{\mathrm{d}t^\upsilon} \tag{10}$$

By following similar elimination procedure for ΔT as carried out above, by using ε relation with ΔT and σ, and using $1+\lambda\gamma = \left(\bar{\tau}_\sigma/\bar{\tau}_\varepsilon\right)^\upsilon = M_u/M_r$, we get fractional counter part of Zener system as

$$\sigma + \bar{\tau}_\varepsilon^\upsilon\frac{\mathrm{d}^\upsilon\sigma}{\mathrm{d}t^\upsilon} = M_r\left(\varepsilon + \bar{\tau}_\sigma^\upsilon\frac{\mathrm{d}^\upsilon\varepsilon}{\mathrm{d}t^\upsilon}\right) \qquad 0<\upsilon<1 \tag{11}$$

Thus the fractional differential equations describing the visco-elastic systems come from the fact that thermo-elastic relaxation has memory while relaxing. Now what are the modes of memory, how many types of memory etc are further manifestations of the fractional order differential equations with several fractional orders or having a continuous order distributed over an interval.

2. Creep-Compliance and Relaxation-modulus as Power Law and their Rate Distribution functions

The first one to state this power law was Scott-Blair, we may call this as Scot-Blair system, where we define creep-compliance $J(t)$, as proportional to t^υ, and write with suitable constant of proportionality as

$$J(t) = \frac{a}{\Gamma(1+\upsilon)}t^\upsilon \qquad a>0; \quad 0<\upsilon<1 \tag{12}$$

The above power law is compatible with the mathematical theory and observation that this retardation function is non-decreasing function. This function therefore should be having a retardation spectrum $R_\varepsilon(\tau)$ in time and $S_\varepsilon(f)$ in frequency. (We have developed this in lecture series-B), we try to find out these rates. First we differentiate the creep-compliance w.r.t. time.

$$\dot{J}(t) = \frac{a\upsilon}{\Gamma(1+\upsilon)} t^{\upsilon-1} = \frac{a\upsilon t^{\upsilon-1}}{\upsilon\Gamma(\upsilon)} = a\frac{t^{\upsilon-1}}{\Gamma(\upsilon)} \tag{13}$$

Inverse Laplace of $s^{-(1-\upsilon)} = t^{1-\upsilon-1}/\Gamma(1-\upsilon+1) = t^{-\upsilon}/\Gamma\{1+(1-\upsilon)\}$; by using Laplace pair $\mathcal{L}\left\{t^n\right\} = n!/s^{n+1} = (n!)s^{-(n+1)}$, and $n! = \Gamma(n+1)$; $\quad n \in \mathbb{R}$. We have to invert $\dot{J}(t)$ so instead of Laplace variable s, we have Laplace variable t, gets inverted and we get function in variable f (instead of usual t). So we write

$$\mathcal{L}^{-1}\left\{\dot{J}(t)\right\} = \mathcal{L}^{-1}\left\{\frac{a}{\Gamma(\upsilon)}t^{-(1-\upsilon)}\right\} = \frac{a}{\Gamma(\upsilon)}\frac{f^{-\upsilon}}{\Gamma[1+(1-\upsilon)]} = \frac{a}{(1-\upsilon)\Gamma(\upsilon)\Gamma(1-\upsilon)} f^{-\upsilon} \tag{14}$$

We used $\Gamma(1+z) = z\Gamma(z)$ and use reflexion formula for Gamma function $\Gamma(\upsilon)\Gamma(1-\upsilon) = \pi/\sin\upsilon\pi$, to get

$$\mathcal{L}^{-1}\left\{\dot{J}(t)\right\} = \frac{a\sin\upsilon\pi}{(1-\upsilon)\pi} f^{-\upsilon} = -f\left[S_\varepsilon(f)\right] \tag{15}$$

Therefore, we have

$$S_\varepsilon(f) = -\frac{a\sin\upsilon\pi}{(1-\upsilon)\pi}\frac{1}{f^{\upsilon+1}} \tag{16}$$

Using the formula (as described in lecture series-B),

$$R_\varepsilon(\tau) = -\frac{f^2}{a}S_\varepsilon(f) \tag{17}$$

we obtain the following

$$R_\varepsilon(\tau) = -\frac{f^2}{a}\left[-\frac{a\sin(\upsilon\pi)}{(1-\upsilon)\pi f^{\upsilon+1}}\right] = \frac{\sin(\upsilon\pi)}{(1-\upsilon)\pi} f^{1-\upsilon} \qquad \text{using} \qquad f = \frac{1}{\tau} \tag{18}$$

We have the retardation rate for the power law creep-compliance as

$$R_\varepsilon(\tau) = \frac{\sin\pi\upsilon}{(1-\upsilon)\pi}\left(\frac{1}{\tau}\right)^{1-\upsilon} = \frac{\sin\pi\upsilon}{(1-\upsilon)\pi}\frac{1}{\tau^{1-\upsilon}} \qquad \text{and} \qquad S_\varepsilon(f) = -\frac{a\sin\pi\upsilon}{(1-\upsilon)f^{\upsilon+1}} \tag{19}$$

In virtue of the reciprocity relationship, in Laplace domain, as described in lecture series-B, that is the following relation

$$s\tilde{J}(s) = \frac{1}{s\tilde{Y}(s)} \qquad \text{or} \qquad \tilde{J}(s)\tilde{Y}(s) = \frac{1}{s^2} \tag{20}$$

We can obtain the relaxation modulus $Y(t)$ from the described power-law creep $J(t) = a\left[\Gamma(1+\upsilon)\right]^{-1} t^{\upsilon}$ here to write

$$Y(t) = \frac{b}{\Gamma(1-\upsilon)}t^{-\upsilon} \qquad\qquad b = \frac{1}{a} > 0 \tag{21}$$

The steps are $J(s) = \mathcal{L}\left\{J(t)\right\} = a\left[\Gamma(1+\upsilon)\right]^{-1}\mathcal{L}\left\{t^{\upsilon}\right\} = a\left[\Gamma(1+\upsilon)\right]^{-1}\Gamma(1+\upsilon)s^{-(\upsilon+1)}$; then $Y(s) = s^{-2}\left[J(s)\right]^{-1}$, gives $Y(s) = s^{-2}\left[as^{-\upsilon-1}\right]^{-1} = a^{-1}s^{-2}s^{\upsilon+1} = bs^{\upsilon-1}$; $\quad b = a^{-1}$. Use the Laplace pair $s^{-(n+1)} \leftrightarrow t^n/\Gamma(n+1)$; to write $\mathcal{L}^{-1}\{s^{\upsilon-1}\} = \mathcal{L}^{-1}\{s^{-(-\upsilon+1)}\} = t^{-\upsilon}/\Gamma(1-\upsilon)$. We have the result as $Y(t) = bt^{-\upsilon}/\Gamma(1-\upsilon)$; again this time power-law decay.

This relaxation-modulus will then have spectrums as

$$R_\sigma(\tau) = \frac{\sin\pi\upsilon}{\pi}\frac{1}{\tau^{1+\upsilon}} \qquad\qquad S_\sigma(f) = -b\frac{\sin\pi\upsilon}{\pi}\frac{1}{f^{1-\upsilon}} \tag{22}$$

This can be obtained by similar steps as done for $R_\varepsilon(\tau)$. We first differentiate the $Y(t) = bt^{-\upsilon} / \Gamma(1-\upsilon)$, and obtain: $\dot{Y}(t) = -b\upsilon t^{-\upsilon-1} / \Gamma(1-\upsilon)$

$$f[S_\sigma(f)] = \mathcal{L}^{-1}\{\dot{Y}(t)\} = \frac{-b\upsilon}{\Gamma(1-\upsilon)} \mathcal{L}^{-1}\left\{t^{-(\upsilon+1)}\right\} = \frac{-b\upsilon(f^{\upsilon})}{\Gamma(1-\upsilon)\Gamma(1+\upsilon)}$$

$$= \frac{-b\upsilon(f^{\upsilon})}{[\Gamma(1-\upsilon)](\upsilon)\Gamma(\upsilon)} = \frac{-bf^{\upsilon}}{\Gamma(\upsilon)\Gamma(1-\upsilon)} = -b\frac{\sin\pi\upsilon}{\pi} f^{\upsilon} \tag{23}$$

used is reflexion formula $\qquad \Gamma(\upsilon)\Gamma(1-\upsilon) = \dfrac{\pi}{\sin\pi\upsilon}$

Therefore

$$S_\sigma(f) = -b\frac{\sin\pi\upsilon}{\pi} f^{\upsilon-1} = -b\frac{\sin\pi\upsilon}{\pi}\frac{1}{f^{1-\upsilon}} \tag{24}$$

$$\begin{aligned} R_\sigma(\tau) &= -f^2\frac{S_\sigma(f)}{b} = -\left(\frac{f^2}{b}\right)\left(-b\frac{\sin\pi\upsilon}{\pi}\right)\left(\frac{1}{f^{1-\upsilon}}\right) \\ &= \frac{\sin\pi\upsilon}{\pi} f^{1+\upsilon} = \frac{\sin\pi\upsilon}{\pi}\frac{1}{\tau^{1+\upsilon}} \qquad \text{use} \qquad f = \frac{1}{\tau} \end{aligned} \tag{25}$$

3. Using Boltzmann's superposition for obtaining fractional derivative and integral expression for material with power law creep-compliance and power law relaxation modulus-the intermediate material.

We have this power law creep as $J(t) = a\left[\Gamma(1+\upsilon)\right]^{-1} t^{\upsilon}$, as mentioned in the previous section. For the creep we write Boltzmann super position, using this power law creep and assuming differentiability of the stress history as follows (defined in lecture series-B);

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\mathrm{d}\sigma(\tau) = \frac{a}{\Gamma(1+\upsilon)}\int_{-\infty}^{t}(t-\tau)^{\upsilon}\mathrm{d}\sigma(\tau) \qquad \text{use} \qquad \mathrm{d}\sigma(\tau) = \dot{\sigma}(\tau)\mathrm{d}\tau \tag{26}$$

$$\varepsilon(t) = \frac{a}{\Gamma(1+\upsilon)}\int_{-\infty}^{t}(t-\tau)^{\upsilon}\dot{\sigma}(\tau)\mathrm{d}\tau \tag{27}$$

We do now the integration by parts as follows

$$\varepsilon(t) = \frac{a}{\Gamma(1+\upsilon)} \int_{-\infty}^{t} (t-\tau)^{\upsilon} \dot{\sigma}(\tau) \mathrm{d}\tau$$
$$= \frac{a}{\Gamma(1+\upsilon)} \left[(t-\tau)^{\upsilon} \int_{-\infty}^{t} \dot{\sigma}(\tau)\mathrm{d}\tau - \int_{-\infty}^{t} \{[-\upsilon(t-\tau)^{\upsilon-1}] \int_{-\infty}^{t} \dot{\sigma}(\tau)\mathrm{d}\tau\}\mathrm{d}\tau \right]$$
$$= \frac{a}{\Gamma(1+\upsilon)} \left[(t-\tau)\sigma(\tau)\Big|_{\tau=-\infty}^{\tau=t} + \int_{-\infty}^{t} \upsilon(t-\tau)^{\upsilon-1}\sigma(\tau)\mathrm{d}\tau \right]$$
$$= \frac{a}{\Gamma(1+\upsilon)} \left[\{0 - [(t-\tau) \ \Big|_{\tau=-\infty} \sigma(-\infty)\} + \upsilon \int_{-\infty}^{t} (t-\tau)^{\upsilon-1}\sigma(\tau)\mathrm{d}\tau \right]$$
$$= \frac{a\upsilon}{\Gamma(1+\upsilon)} \int_{-\infty}^{t} (t-\tau)^{\upsilon-1}\sigma(\tau)\mathrm{d}\tau \qquad \text{recognizing} \qquad \sigma(-\infty) = 0$$
$$= \frac{a\upsilon}{\upsilon\Gamma(\upsilon)} \int_{-\infty}^{t} (t-\tau)^{\upsilon-1}\sigma(\tau)\mathrm{d}\tau \qquad \text{using} \qquad \Gamma(1+\upsilon) = \upsilon\Gamma(\upsilon)$$
$$= \frac{a}{\Gamma(\upsilon)} \int_{-\infty}^{t} (t-\tau)^{\upsilon-1}\sigma(\tau)\mathrm{d}\tau = a\left({}_{-\infty}I_t^{\upsilon}\left[\sigma(t)\right]\right) \tag{28}$$

We have used in the last step the definition of fractional integration as

$${}_{-\infty}I_t^{q}[f(t)] = \frac{1}{\Gamma(q)} \int_{-\infty}^{t} (t-\tau)^{q-1} f(\tau)\mathrm{d}\tau \qquad q \in \mathbb{R}^{+} \tag{29}$$

With lower limit as $-\infty$, giving Liouvelli-Weyl definition of fractional integration of order $q \in \mathbb{R}$. Similarly we have Boltzmann superposition for stress as

$$\sigma(t) = \int_{-\infty}^{t} Y(t-\tau)\mathrm{d}\varepsilon(\tau) = \frac{b}{\Gamma(1-\upsilon)} \int_{-\infty}^{t} (t-\tau)^{-\upsilon}\dot{\varepsilon}(\tau)\mathrm{d}\tau \qquad \dot{\varepsilon}(t) = \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t}$$
$$= b\frac{1}{\Gamma(1-\upsilon)} \int_{-\infty}^{t} (t-\tau)^{(1-\upsilon)+1} \frac{\mathrm{d}}{\mathrm{d}\tau}\varepsilon(\tau)\mathrm{d}\tau \tag{30}$$
$$= b\,{}_{-\infty}I_t^{1-\upsilon}D_t^{1}\varepsilon(t) = b\,{}_{-\infty}^{\;C}D_t^{\upsilon}\varepsilon(t)$$

We note here that in above derivation we used the Caputo fractional derivative definition which is one integer order derivative followed by fractional order integration of order $1-\upsilon$. However a point be noted as the lower limit of the fractional order derivative is at $-\infty$, as Weyl sense; and in that case the Riemann-Liouvelli and Caputo fractional derivative are same.

$${}_{-\infty}^{\;C}D_t^{\upsilon} \triangleq {}_{-\infty}I_t^{1-\upsilon} \circ D_t = D_t \circ {}_{-\infty}I_t^{1-\upsilon} \triangleq {}_{-\infty}^{RL}D_t^{\upsilon} \qquad \text{where} \qquad D_t \triangleq \frac{\mathrm{d}}{\mathrm{d}t} \tag{31}$$

Now we replace the lower terminal $-\infty$ to 0 considering the causal histories; and without loss of generality we get constitutive equations for the assumed power law as creep-compliance and relaxation modulus:

$$\varepsilon(t) = a\,{}_0I_t^{\upsilon}\left[\sigma(t)\right] \qquad \sigma(t) = b\,{}_0D_t^{\upsilon}\left[\varepsilon(t)\right] = b\,{}_0^CD_t^{\upsilon}\left[\varepsilon(t)\right] \qquad \text{with} \qquad ab = 1 \tag{32}$$

Thus we got the constitutive equation for the material with intermediate between pure viscous material and pure elastic material.

4. The material functions and correspondence principle between integer order calculus and fractional order calculus

The material functions are obtained by series parallel combination of Hooke element and the intermediate Scott-Blair element described in previous section. Their determination is made easy if we take into account the correspondence principle between integer order calculus and fractional order calculus. Taking the fractional order visco-elastic intermediate element, with order υ with $0<\upsilon<1$, the correspondence are in following table-1.

Integer Order Calculus based Material Functions in time domain and its Laplace	Fractional Order Calculus based Material Functions in time domain and its Laplace
$\delta(t)\leftrightarrow 1$	$\frac{t^{-\upsilon}}{\Gamma(1-\upsilon)}\leftrightarrow s^{1-\upsilon}$
$t\leftrightarrow \frac{1}{s^2}$	$\frac{t^{\upsilon}}{\Gamma(1+\upsilon)}\leftrightarrow \frac{1}{s^{\upsilon+1}}$
$e^{-t/\tau}\leftrightarrow \frac{1}{s+(1/\tau)}=\frac{\tau}{1+s\tau}$	$E_{\upsilon}\left((-t/\tau)^{\upsilon}\right)\leftrightarrow \frac{s^{\upsilon-1}}{s^{\upsilon}+(1/\tau)^{\upsilon}}$

Table: 1 Correspondence principle between integer order calculus and fractional calculus based material function components

Why these correspondences are shown is because recalling the material functions $J(t)$ and/or $Y(t)$, for various linear mechanical models described in the Lecture-B we find that the composition are due to these three basic functions $\delta(t)$, t and $e^{-t/\tau}$ plus a constant function; these components gets replaced in fractional order models with Scott-Blair element by the correspondence shown in the table. In the table $E_{\upsilon}[(-t/\tau)^{\upsilon}]$ is the one-parameter Mittag-Leffler function, we had discussed in earlier lectures. We again write the asymptote values and near origin approximates of Mittag-Leffler function as

$$E_{\upsilon}(-t^{\upsilon})\sim 1-\frac{t^{\upsilon}}{\Gamma(1+\upsilon)};\quad t\to 0^{+}\qquad\text{and}\qquad E_{\upsilon}(-t^{\upsilon})\sim\frac{t^{-\upsilon}}{\Gamma(1-\upsilon)};\quad t\to+\infty \tag{33}$$

For $\upsilon=1$, $E_{\upsilon}(-t^{\upsilon})=e^{-t}$ its time derivative is $-e^{-t}$, and at near origin, its rate of change is -1. Whereas, the rate of change of $E_{\upsilon}(-t^{\upsilon})$ is $\sim -\upsilon t^{\upsilon-1}/\Gamma(1+\upsilon)\to-\infty,\quad t\to 0^{+}$. Therefore the initial fall of Mittag-Leffler function is much steeper than the exponential function. While at late times $t\to\infty$, the rate of fall Mittag-Leffler function is much slower than that of exponential function, (can compare the two rates at say $t=1$). At late times the decay of Mittag-Leffler function is algebraic decay $\sim -\upsilon t^{-\upsilon-1}/\Gamma(1-\upsilon)$, whereas the exponential function at late time is exponential decay $-e^{-t}$, which is must faster. This comparison is shown in figure-1

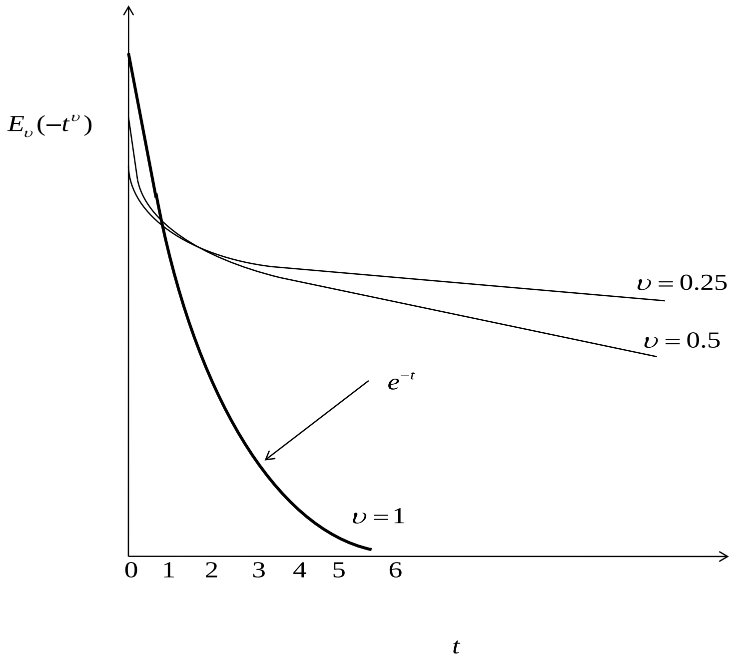

Figure-1: The Mittag-Leffler Function at early and late times compared with exponential function

5. The Fractional Visco-elastic Systems

In the previous lectures we have seen various models Newton, Voigt, Maxwell, Zener etc. They all had integer order derivative in their constitutive equations, and we derived various material functions. Here we note the fractional counterparts of those systems, where the viscous element the piston or the dash-pot is replaced by its intermediate Scott-Blair fractional system.

The fractional Newton system is

$$\sigma(t) = b_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon} \varepsilon(t) \tag{34}$$

The material functions are

$$J(t) = \frac{t^\upsilon}{b_1 \Gamma(1+\upsilon)} \qquad Y(t) = b_1 \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \tag{35}$$

The integer order model of Newton system is $\sigma(t) = b_1[\mathrm{d}\varepsilon(t)/\mathrm{d}t]$, which has material function as $J(t) = t/b_1$ and $Y(t) = b_1[\delta(t)]$. By use of Table-1 we replaced $\delta(t)$ by $t^{-\upsilon}/\Gamma(1-\upsilon)$ and t by $t^\upsilon/\Gamma(1+\upsilon)$; to obtain the corresponding material functions for the fractional Newton system.

The fractional Voigt system is

$$\sigma(t) = m\varepsilon(t) + b_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon} \varepsilon(t) \tag{36}$$

The material functions are

$$J(t) = \frac{1}{m}\left[1 - E_\upsilon\{-(t/\tau_\varepsilon)^\upsilon\}\right] \qquad Y(t) = m + b_1 \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \quad \text{with} \quad (\tau_\varepsilon)^\upsilon = \frac{b_1}{m} \tag{37}$$

The integer Voigt system is having $\sigma = m\varepsilon + b_1[\mathrm{d}\varepsilon / \mathrm{d}t]$, with material function as $J_1(1-e^{-t/\tau_\varepsilon})$, with $J_1 = 1/m$ and $\tau_\varepsilon = b_1/m$; and $Y(t) = m + b_1\delta(t)$. With the correspondence rule we come to the fractional order material functions with $\delta(t)$ replaced by $t^{-\upsilon}/\Gamma(1+\upsilon)$ and e^{-t/τ_ε} replaced by $E_\upsilon[-(t/\tau_\varepsilon)^\upsilon]$. Similarly we get the other cases.

Fractional Maxwell system is

$$\sigma(t) + a_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t}\sigma(t) = b_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\varepsilon(t) \tag{38}$$

The material functions are

$$J(t) = \frac{a_1}{b_1} + \frac{1}{b_1}\frac{t^\upsilon}{\Gamma(1+\upsilon)} \qquad Y(t) = \frac{b_1}{a_1}E_\upsilon[-(t/\tau_\sigma)^\upsilon] \quad \text{with} \quad (\tau_\sigma)^\upsilon = a_1 \tag{39}$$

In this case the integer order of Maxwell system had a time ramp function component in the creep-compliance, which is replaced correspondingly by $t^\upsilon/\Gamma(1+\upsilon)$, the rest is same as earlier done for above systems.

Fractional Zener system is

$$\left[1 + a_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\right]\sigma(t) = \left[m + b_1 \frac{\mathrm{d}^\upsilon}{\mathrm{d}t^\upsilon}\right]\varepsilon(t) \tag{40}$$

The material functions are

$$\begin{gathered} J(t) = J_g + J_1\left[1 - E_\upsilon[-(t/\tau_\varepsilon)^\upsilon\right] \qquad Y(t) = Y_e + Y_1E_\upsilon[-(t/\tau_\sigma)^\upsilon] \\ J_g = \frac{a_1}{b_1}, \quad J_1 = \frac{1}{m} - \frac{a_1}{b_1}, \quad \tau_\varepsilon = \frac{b_1}{m}, \quad Y_e = m, \quad Y_1 = \frac{b_1}{a_1} - m, \quad \tau_\sigma = a_1 \end{gathered} \tag{41}$$

We can extend the procedure as done in integer order classical visco-elastic systems, to get the generalized fractional order operator as

$$\left[1 + \sum_{k=1}^{p} a_k \frac{\mathrm{d}^{\upsilon_k}}{\mathrm{d}t^{\upsilon_k}}\right]\sigma(t) = \left[m + \sum_{k=1}^{q} b_k \frac{\mathrm{d}^{\upsilon_k}}{\mathrm{d}t^{\upsilon_k}}\right]\varepsilon(t) \tag{42}$$

With material functions as

$$\begin{gathered} J(t) = J_g + \sum_n J_n\left[1 - E_\upsilon[-(t/\tau_{\varepsilon,n})^\upsilon\right] + J_+ \frac{t^\upsilon}{\Gamma(1+\upsilon)} \qquad J_g, J_n, J_+ \ge 0 \\ Y(t) = Y_e + \sum_n Y_n E_\upsilon[-(t/\tau_{\sigma,n})^\upsilon] + Y_- \frac{t^{-\upsilon}}{\Gamma(1-\upsilon)} \qquad Y_e, Y_n, Y_- \ge 0 \end{gathered} \tag{43}$$

6. Finding the Rate Distribution functions for generalized Fractional Order visco-elastic system

We have the time varying retardation and relaxation functions as

$$\begin{gathered} J_\tau(t) = J_1\left[1 - E_\upsilon(t/\tau_\varepsilon)^\upsilon\right] \equiv J_1\int_0^\infty R_\varepsilon(\tau)\left(1 - e^{-t/\tau}\right)\mathrm{d}\tau \\ Y_\tau(t) = Y_1E_\upsilon(t/\tau)^\upsilon \equiv Y_1\int_0^\infty R_\sigma(\tau)e^{-t/\tau}\mathrm{d}\tau \end{gathered} \tag{44}$$

For evaluating, $R_\varepsilon(\tau)$ and $R_\sigma(\tau)$ we need to differentiate the $J_\tau(t)$ and $Y_\tau(t)$, with respect to time; and that process gives:

$$\dot{J}_{\tau}(t) = -\frac{J_1}{\tau_\varepsilon}\frac{\mathrm{d}E_\upsilon[(-t/\tau_\varepsilon)^\upsilon]}{\mathrm{d}t} \qquad \dot{Y}_\tau(t) = \frac{Y_1}{\tau_\sigma}\frac{\mathrm{d}E_\upsilon[(-t/\tau_\sigma)]}{\mathrm{d}t} \tag{45}$$

We write as general for both as

$$\dot{J}_\tau(t) \quad \text{or} \quad \dot{Y}_\tau(t) = C\frac{\mathrm{d}}{\mathrm{d}x}E_\upsilon(-x^\upsilon) \qquad x = t/\tau_*, \quad * = \varepsilon, \sigma, \quad C = (J_1 \text{ or } \mathrm{Y}_1)/\tau_* \tag{46}$$

Then next step is to evaluate $\mathcal{L}^{-1}\left\{-\dot{J}_\tau(t)\right\}$ or $\mathcal{L}^{-1}\left\{\dot{Y}_\tau(t)\right\}$ as described in lecture series-B, to get $f\{S_\varepsilon(f)\}$ and $f\{S_\sigma(f)\}$, the frequency spectra with relation

$$f\left\{S_*(f)\right\} = C\frac{R_*(\tau)}{f}; \quad \tau = \frac{1}{f} \tag{47}$$

Therefore we need to evaluate inverse Laplace of derivative of Mittag-Leffler function, to get the rates distribution function (both for relaxation function as well as retardation function); i.e. $\mathcal{L}^{-1}\left\{\mathrm{d}E_\upsilon(-x^\upsilon)/\mathrm{d}x\right\}$. First we shall evaluate $\mathcal{L}^{-1}\left\{E_\upsilon(-x^\upsilon)\right\}$, by use of Berberan-Santos method, which we detailed in lecture series-A.

We describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. First is change of variable i.e. from real time to complex time variable as $t = c + i\varpi$, $c = 0$. The symbol ϖ is different from imaginary part of frequency in the usual Laplace variable ω in complex frequency parameter s. With this change we have

$$\begin{aligned} H(k) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(t)e^{kt}\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(i\varpi)e^{ik\varpi}\mathrm{d}\varpi \qquad \text{put} \quad c = 0 \\ H(k) &= \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} f(i\varpi)\cos(k\varpi)\mathrm{d}\varpi + i\int_{-\infty}^{+\infty} f(i\varpi)\sin(k\varpi)\mathrm{d}\varpi\right] \end{aligned} \tag{48}$$

The function $H(k)$ is inverse Laplace of $f(t)$

Write $f(i\varpi) = \mathrm{Re}\left[f(i\varpi)\right] + i\,\mathrm{Im}\left[f(i\varpi)\right]$ and place above, we get

$$\begin{aligned} H(k) &= \frac{1}{2\pi}\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi) - \mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} \\ &+ \frac{1}{2\pi}i\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi) + \mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} \end{aligned} \tag{49}$$

Given that $H(k)$ is a real function, we get

$$\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi) + \mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} = 0 \tag{50}$$

And the above expression reduces to

$$H(k) = \frac{1}{2\pi}\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi) - \mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} \tag{51}$$

But we have $f(t) = \int_0^\infty H(k)e^{-kt}\mathrm{d}k$ putting $t = i\varpi$ we get the following

$$f(i\varpi) = \int_0^\infty H(k)e^{-k(i\varpi)}\mathrm{d}k = \int_0^\infty H(k)\cos(k\varpi)\mathrm{d}\varpi - i\int_0^\infty H(k)\sin(k\varpi)\mathrm{d}\varpi \tag{52}$$

$$\mathrm{Re}\left[f(i\varpi)\right]=\int_0^\infty H(k)\cos(k\varpi)\mathrm{d}k \qquad \mathrm{Im}\left[f(i\varpi)\right]=-\int_0^\infty \sin(k\varpi)\mathrm{d}k \tag{53}$$

Using this in obtained expression for $H(k)$, we observe that integrand is even function for $k>0$ therefore we re-write the formula as

$$H(k)=\frac{1}{\pi}\int_0^\infty\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi \tag{54}$$

Using $\pi H(k)=\int_0^\infty\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi$ and the relation

$0=\int_0^{+\infty}\left[\mathrm{Re}[f(i\varpi)]\cos(k\varpi)+\mathrm{Im}[f(i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi$; adding and subtracting these we get

$$H(k)=\frac{2}{\pi}\int_0^\infty\mathrm{Re}\left[f(i\varpi)\right]\cos(k\varpi)\mathrm{d}\varpi \qquad H(k)=-\frac{2}{\pi}\int_0^\infty\mathrm{Im}\left[f(i\varpi)\right]\sin(k\varpi)\mathrm{d}\varpi \tag{55}$$

In reality of decay functions, we can take $c=0$; as decay function will not expected to have singularity at time $t>0$. For a case of 'exponential-decay' i.e. $f(t)=e^{-t/\tau_0}$, obviously this function has only one decay rate i.e. $1/\tau_0$. As per procedure discussed above we do Laplace inversion by taking complex time with $c=0+i\varpi$; making it

$$f(i\varpi)=e^{-i\varpi/\tau_0}=\cos(\varpi/\tau_0)-i\sin(\varpi/\tau_0) \qquad \mathrm{Re}[f(i\varpi)]=\cos(\varpi/\tau_0) \tag{56}$$

$$\begin{aligned}H(k)&=\frac{1}{\pi}\int_0^\infty\{\mathrm{Re}[f(i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(i\varpi)]\sin(k\varpi)\}\mathrm{d}\varpi\\&=\frac{1}{\pi}\int_0^\infty[\cos(\varpi/\tau_0)\cos(k\varpi)+\sin(\varpi/\tau_0)\sin(k\varpi)]\mathrm{d}\varpi\\&=\frac{1}{\pi}\int_0^\infty\cos\left[\varpi(k-1/\tau_0)\right]\mathrm{d}\varpi=\delta\left(k-\frac{1}{\tau_0}\right)\end{aligned} \tag{57}$$

We use the above derivation to find the inverse Laplace of derivative of Mittag-Leffler function. First the integral representation of the Mittag-Leffler function we will obtain from the described method of Berberan-Santos as we described in lecture series part-A. The start point is the known Laplace transform of $E_\upsilon(-x^\upsilon)$ with $x=t/\tau$, to complex frequency $s=\mathrm{Re}[s]+i\omega$ i.e.

$$\mathcal{L}\left\{E_\upsilon(-x^\upsilon)\right\}=\int_0^\infty E_\upsilon(-x^\upsilon)e^{-sx}\mathrm{d}x=\frac{s^{\upsilon-1}}{1+s^\upsilon} \tag{58}$$

This can be also got via representing $E_\upsilon(-x^\upsilon)$ as series and taking term by term $s-$domain transform. Here now we apply the Berberan-Santo technique on

$$F(s)=\frac{s^{\upsilon-1}}{1+s^\upsilon} \qquad F(i\omega)=\frac{(i\omega)^{\upsilon-1}}{1+(i\omega)^\upsilon}=\frac{\omega^{\upsilon-1}\left[\cos[(\upsilon-1)\pi/2]+i\sin[(\upsilon-1)\pi/2]\right]}{1+\omega^\upsilon\cos(\upsilon\pi/2)+i\omega^\upsilon\sin(\upsilon\pi/2)} \tag{59}$$

$$\mathrm{Re}\left[F(i\omega)\right]=\frac{\omega^{\upsilon-1}\sin(\upsilon\pi/2)}{1+2\omega^\upsilon\cos(\upsilon\pi/2)+\omega^{2\upsilon}} \tag{60}$$

Use inverse Laplace technique in this case from $s=i\omega$ domain to x domain, by following

$$E_\upsilon(-x^\upsilon) = \mathcal{L}^{-1}\left\{F(i\omega)\right\} = \frac{2}{\pi}\int_0^\infty \mathrm{Re}\left[F(i\omega)\right]\cos(x\omega)\mathrm{d}\omega$$

$$= \int_0^\infty \frac{2}{\pi}\frac{\omega^{\upsilon-1}\sin(\upsilon\pi/2)\cos(x\omega)\mathrm{d}\omega}{1+2\omega^\upsilon\cos(\upsilon\pi/2)+\omega^{2\upsilon}} \tag{61}$$

$$= \int_0^\infty \frac{2}{\pi}\frac{\omega^{\upsilon-1}\sin(\upsilon\pi/2)}{1+2\omega^\upsilon\cos(\upsilon\pi/2)+\omega^{2\upsilon}}\mathrm{Re}\left[e^{-x\omega}\right]\mathrm{d}\omega$$

The above is integral representation of $E_\upsilon(-x^\upsilon)$. The classical integral-representation of the Mittag-Leffler function which comes from Bromwich inversion integral is:

$$E_{\alpha,1}(-t^\alpha) = \int_0^\infty \frac{1}{\pi\alpha}\frac{t^\alpha\sin(\pi\alpha)}{r^2+2rt^\alpha\cos(\pi\alpha)+t^{2\alpha}}e^{-r^{1/\alpha}}\mathrm{d}r \tag{62}$$

Put, $r = x^\alpha t^\alpha$ so $\mathrm{d}r = \alpha x^{\alpha-1}t^\alpha \mathrm{d}x$ and $e^{-r^{1/\alpha}} = e^{-xt}$, we get the integral representation of $E_\alpha(-t^\alpha)$, as following

$$E_{\alpha,1}(-t^\alpha) = E_\alpha(-t^\alpha) = \int_0^\infty \frac{1}{\pi}\frac{x^{\alpha-1}\sin\pi\alpha}{x^{2\alpha}+2x^\alpha\cos\pi\alpha+1}e^{-xt}\mathrm{d}x \tag{63}$$

Compare with definition of Laplace transform definition

$$F(s) \triangleq \int_0^\infty f(t)e^{-st}\mathrm{d}t \qquad \text{or} \qquad F(t) \triangleq \int_0^\infty f(x)e^{-tx}\mathrm{d}x \tag{64}$$

Where we changed for LHS variable $s \equiv t$, and for RHS $t \equiv x$. We can therefore say that $F(t)$ is Laplace transform of $f(x)$ and $f(x)$ is inverse Laplace of $F(t)$. Therefore above integral representation of Mittag-Leffler function states that

$$f(x) = \frac{1}{\pi}\frac{x^{\alpha-1}\sin\pi\alpha}{x^{2\alpha}+2x^\alpha\cos\pi\alpha+1} = \mathcal{L}^{-1}\left\{E_\alpha(-t^\alpha)\right\} = F(t) \tag{65}$$

Similarly from Berberan-Santos method we got integral representation as (we changed $\omega \equiv x$ in the RHS and $x \equiv t$ at LHS), to write the following

$$E_\upsilon(-t^\upsilon) = \int_0^\infty \frac{2}{\pi}\frac{x^{\upsilon-1}\sin(\upsilon\pi/2)}{1+2x^\upsilon\cos(\upsilon\pi/2)+x^{2\upsilon}}\mathrm{Re}\left[e^{-tx}\right]\mathrm{d}x \tag{66}$$

Says that

$$\left(\frac{2}{\pi}\right)\frac{x^{\upsilon-1}\sin(\upsilon\pi/2)}{1+2x^\upsilon\cos(\upsilon\pi/2)+x^{2\upsilon}} = \mathcal{L}^{-1}\left\{E_\upsilon(-t^\upsilon)\right\} \tag{67}$$

To get inverse Laplace of the derivative of Mittag-Leffler function is simple as we have following relations

$$\mathcal{L}\left\{E_\upsilon(-x^\upsilon)\right\} = \frac{s^{\upsilon-1}}{1+s^\upsilon} \qquad \mathcal{L}\left\{\frac{\mathrm{d}}{\mathrm{d}x}E_\upsilon(-x^\upsilon)\right\} = s\left(\mathcal{L}\left\{E_\upsilon(-x^\upsilon)\right\}\right) = s\frac{s^{\upsilon-1}}{1+s^\upsilon}$$

$$F'(s) = s\frac{s^{\upsilon-1}}{1+s^\upsilon} \qquad F'(i\omega) = (i\omega)\frac{(i\omega)^{\upsilon-1}}{1+(i\omega)^\upsilon} \qquad F(i\omega) = \frac{(i\omega)^{\upsilon-1}}{1+(i\omega)^\upsilon} \tag{68}$$

Therefore in the above steps we need to multiply by $s = i\omega$, to obtain integral transform obtained for $E_\upsilon(-x^\upsilon)$ to get Laplace inverse of the derivative of the Mittag-Leffler, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}E_{\upsilon}(-x^{\upsilon})=i\omega L\left\{E_{\upsilon}(-x^{\upsilon})\right\}=i\omega F(i\omega)=i\omega\frac{(i\omega)^{\upsilon-1}}{1+(i\omega)^{\upsilon}}=i\omega[F(i\omega)]$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}E_{\upsilon}(-x^{\upsilon})&=\frac{2}{\pi}(i\omega)\int_0^{\infty}\mathrm{Re}\left[F(i\omega)\right]\cos(x\omega)\mathrm{d}\omega\\&=\frac{2}{\pi}(i\omega)\int_0^{\infty}\frac{\omega^{\upsilon-1}\sin(\upsilon\pi/2)\cos(x\omega)\mathrm{d}\omega}{1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}}\\&=\frac{2}{\pi}\int_0^{\infty}\frac{\omega^{\upsilon}\left(i\sin(\upsilon\pi/2)\right)}{1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}}\cos(x\omega)\mathrm{d}\omega\qquad\text{put}\qquad i\sin\theta=\cos\theta\\&=\int_0^{\infty}\frac{2}{\pi}\frac{\omega^{\upsilon}\left(\cos(\upsilon\pi/2)\right)\cos(x\omega)}{1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}}\mathrm{d}\omega=\int_0^{\infty}\frac{2}{\pi}\frac{\omega^{\upsilon}\left(\cos(\upsilon\pi/2)\right)\mathrm{Re}\left[e^{-\omega x}\right]}{[1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}]}\mathrm{d}x\end{aligned}\tag{69}$$

Comparing the above with definition of Laplace transform integral we get Laplace inverse of the derivative of the Mittag-Leffler function as

$$F'(\omega)=\frac{2}{\pi}\frac{\omega^{\upsilon}\cos(\upsilon\pi/2)}{[1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}]}\qquad\text{is}\qquad\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}x}E_{\upsilon}(-x^{\upsilon})\right\},\quad x=\frac{t}{\tau_*}\tag{70}$$

The time varying functions and its derivatives are

$$J_{\tau}(t)=J_1\left[1-E_{\upsilon}[-(t/\tau_{\varepsilon})^{\upsilon}\right]\qquad\qquad\dot{J}_{\tau}(x)=-J_1\frac{\mathrm{d}}{\mathrm{d}x}E_{\upsilon}(-x^{\upsilon}),\quad x=\frac{t}{\tau_{\varepsilon}}\tag{71}$$

Now as per procedure with $f=\omega=\tau_*/t$, $\quad x=\omega^{-1}$ as invert Laplace variable we write

$$\omega\left[S_*(\omega)\right]=\mathcal{L}^{-1}\left[\dot{J}_{\tau}(x)\right]=-J_1\mathcal{L}^{-1}\left[\frac{\mathrm{d}}{\mathrm{d}x}E_{\upsilon}(-x^{\upsilon})\right]=-J_1\frac{2}{\pi}\frac{\omega^{\upsilon}\cos(\upsilon\pi/2)}{[1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}]}\tag{72}$$

With the formula $R_*(\tilde{\tau})=-\omega^2S_*(\omega)/J_1$, $\quad\tilde{\tau}=\tau/\tau_*$, we write

$$\begin{aligned}R_*(\tilde{\tau})&=\left(-\frac{\omega^2}{J_1}\right)\left(-J_1\frac{2}{\pi}\frac{\omega^{\upsilon}\cos(\upsilon\pi/2)}{[1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}]}\right)\\&=\frac{2}{\pi}\frac{\omega^{\upsilon+2}\cos(\upsilon\pi/2)}{[1+2\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}]}\qquad\text{put}\qquad\omega=\frac{1}{\tilde{\tau}}\\&=\frac{2}{\pi}\frac{\left(\frac{1}{\tilde{\tau}}\right)^{\upsilon+2}\cos(\upsilon\pi/2)}{\left[1+2\left(\frac{1}{\tilde{\tau}}\right)^{\upsilon}\cos(\upsilon\pi/2)+\left(\frac{1}{\tilde{\tau}}\right)^{2\upsilon}\right]}\end{aligned}\tag{73}$$

We have $\tilde{\tau}=\omega^{-1}=x=t/\tau_*=\tau/\tau_*$

$$R_*(\tau)=\frac{2}{\pi}\frac{\left(\frac{\tau_*}{\tau}\right)^{\upsilon+2}\cos(\upsilon\pi/2)}{\left[1+2\left(\frac{\tau_*}{\tau}\right)^{\upsilon}\cos(\upsilon\pi/2)+\left(\frac{\tau_*}{\tau}\right)^{2\upsilon}\right]}\tag{74}$$

By change of variable as $\ln(\tau/\tau_*)=u$, $\quad(\tau_*/\tau)=e^{-u}$ gives

$$R_*(u)=\frac{2}{\pi}\frac{\left(e^{-u}\right)^{\upsilon+2}\cos(\upsilon\pi/2)}{\left[1+2\left(e^{-u}\right)^{\upsilon}\cos(\upsilon\pi/2)+\left(e^{-u}\right)^{2\upsilon}\right]}=\frac{e^{-2u}}{\pi}\frac{\cos(\upsilon\pi/2)}{\left[\cosh(u\upsilon)+\cos(\upsilon\pi/2)\right]} \tag{75}$$

With similar procedure we can have the rate distribution function using the integral transform obtained via contour integration as follows:

$$f(x)=\frac{1}{\pi}\frac{x^{\alpha-1}\sin\pi\alpha}{x^{2\alpha}+2x^{\alpha}\cos\pi\alpha+1}=\mathcal{L}^{-1}\left\{E_{\alpha}(-t^{\alpha})\right\}=F(t) \tag{76}$$

7. Dissipation in Fractional Order Visco-Elastic Systems

We saw dissipation fundamentals in lecture part-B, we need to calculate loss tangent, from the complex relaxation modulus $Y^*(\omega)$. We introduced a Caputo's notation in earlier lecture and we use that. Let us define following new parameters for fractional Zener system as following

$$\alpha\triangleq\frac{1}{\tau_{\varepsilon}^{\upsilon}}=\frac{m}{b_1}\qquad\text{and}\qquad\beta\triangleq\frac{1}{\tau_{\sigma}^{\upsilon}}=\frac{1}{a_1}\qquad\text{where}\qquad 0<\alpha<\beta<\infty \tag{77}$$

The constitutive equation is therefore for a fractional Zener system is

$$\left[1+\frac{1}{\beta}\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}t^{\upsilon}}\right]\sigma(t)=m\left[1+\frac{1}{\alpha}\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}t^{\upsilon}}\right]\varepsilon(t)\qquad\qquad m=Y_e=Y_g\frac{\alpha}{\beta} \tag{78}$$

Take Laplace of the constitutive equation to get

$$\tilde{\sigma}(s)+\frac{1}{\beta}s^{\upsilon}\tilde{\sigma}(s)=m\left[\tilde{\varepsilon}(s)+\frac{1}{\alpha}s^{\upsilon}\tilde{\varepsilon}(s)\right]\qquad\qquad \varepsilon(0)=\dot{\varepsilon}(0)=\sigma(0)=\dot{\sigma}(0)=0 \tag{79}$$

This gives the transfer function as

$$\Phi(s)=\frac{\tilde{\sigma}(s)}{\tilde{\varepsilon}(s)}=m\left[\frac{1+(s^{\upsilon}/\alpha)}{1+(s^{\upsilon}/\beta)}\right]=Y_e\left[\frac{1+(s^{\upsilon}/\alpha)}{1+(s^{\upsilon}/\beta)}\right]=Y_g\left[\frac{\alpha+s^{\upsilon}}{\beta+s^{\upsilon}}\right] \tag{80}$$

The complex relaxation modulus $Y^*(\omega)=\Phi(s)\big|_{s=i\omega}$, $\tilde{\varepsilon}(s)=1$, that is steady state ($s=\mathrm{Re}(s)+i\omega,\quad \mathrm{Re}(s)=0$) stress response in frequency domain with unit impulse strain input i.e. $\tilde{\varepsilon}(s)=1$. Thus we write the complex relaxation modulus as follows

$$\begin{aligned}
Y^*(\omega)&=Y_e\frac{1+\left[(i\omega)^{\upsilon}/\alpha\right]}{1+\left[(i\omega)^{\upsilon}/\beta\right]}=Y_g\frac{\alpha+(i\omega)^{\upsilon}}{\beta+(i\omega)^{\upsilon}}\\
&=Y_g\frac{\alpha+\omega^{\upsilon}i^{\upsilon}}{\beta+\omega^{\upsilon}i^{\upsilon}}=Y_g\frac{\alpha+\omega^{\upsilon}e^{i\upsilon\pi/2}}{\beta+\omega^{\upsilon}e^{i\upsilon\pi/2}}\\
&=Y_g\frac{\alpha+\omega^{\upsilon}\left[\cos(\upsilon\pi/2)+i\sin(\upsilon\pi/2)\right]}{\beta+\omega^{\upsilon}\left[\cos(\upsilon\pi/2)+i\sin(\upsilon\pi/2)\right]}\\
&=Y_g\frac{\left(\alpha+\omega^{\upsilon}\cos(\upsilon\pi/2)\right)+i\omega^{\upsilon}\sin(\upsilon\pi/2)}{\left(\beta+\omega^{\upsilon}\cos(\upsilon\pi/2)\right)+i\omega^{\upsilon}\sin(\upsilon\pi/2)}\\
&=Y_g\frac{\alpha\beta+(\alpha+\beta)\omega^{\upsilon}\cos(\upsilon\pi/2)+\omega^{2\upsilon}+i\left[(\beta-\alpha)\omega^{\upsilon}\sin(\upsilon\pi/2)\right]}{\beta^2+2\omega^{\upsilon}\beta\cos(\upsilon\pi/2)+\omega^{2\upsilon}}
\end{aligned} \tag{81}$$

Therefore for this Zener system we have the complex relaxation modulus as following

$$Y^*(\omega) = Y'(\omega) + iY''(\omega) \qquad \begin{cases} Y'(\omega) = Y_g \dfrac{\alpha\beta + (\alpha+\beta)\omega^{\upsilon}\cos(\upsilon\pi/2) + \omega^{2\upsilon}}{\beta^2 + 2\beta\omega^{\upsilon}\cos(\upsilon\pi/2) + \omega^{2\upsilon}} \\ Y''(\omega) = Y_g \dfrac{(\beta-\alpha)\omega^{\upsilon}\sin(\upsilon\pi/2)}{\beta^2 + 2\beta\omega^{\upsilon}\cos(\upsilon\pi/2) + \omega^{2\upsilon}} \end{cases} \tag{82}$$

From above we write the loss tangent as follows

$$\tan[\phi(\omega)] = \frac{(\beta-\alpha)\omega^{\upsilon}\sin(\upsilon\pi/2)}{\alpha\beta + (\alpha+\beta)\omega^{\upsilon}\cos(\upsilon\pi/2) + \omega^{2\upsilon}} \tag{83}$$

We can have approximation to above obtained expression for the loss tangent for small values of loss tangent as

$$\tan[\phi(\omega)] \cong (\beta-\alpha)\frac{\omega^{\upsilon}\sin(\upsilon\pi/2)}{\omega^{2\upsilon} + \alpha^2 + 2\alpha\omega^{\upsilon}\sin(\upsilon\pi/2)} \qquad \Delta \triangleq \frac{\beta-\alpha}{\alpha} \ll 1 \tag{84}$$

The above approximation is 'nearly elastic' case, and in such approximation we set

$$\omega_0^{\upsilon} = \alpha \quad \text{and} \quad \Delta = \frac{\beta-\alpha}{\alpha} \simeq \frac{\beta-\alpha}{\sqrt{\alpha\beta}} \tag{85}$$

Therefore with above defined parameters we get the approximate loss tangent as

$$\tan[\phi(\omega)] \cong \Delta \frac{\left(\dfrac{\omega}{\omega_0}\right)^{\upsilon}\sin\left(\dfrac{\upsilon\pi}{2}\right)}{1 + \left(\dfrac{\omega}{\omega_0}\right)^{2\upsilon} + 2\left(\dfrac{\omega}{\omega_0}\right)^{\upsilon}\cos\left(\dfrac{\upsilon\pi}{2}\right)} \tag{86}$$

The ω_0 is the frequency where the loss tangent has maximum value that is

$$\tan[\phi(\omega)]_{\max} = \frac{\Delta}{2}\frac{\sin(\upsilon\pi/2)}{1+\cos(\upsilon\pi/2)} \tag{87}$$

Alternatively the ω_0 may be replaced by $1/\bar{\tau}$ with $\bar{\tau}$ indicating characteristic time constant intermediate between τ_ε and τ_σ. In our approximation when $\alpha \simeq \beta$, with this we get

$$\omega_0 \triangleq \frac{1}{\tau_\varepsilon} \simeq \frac{1}{\tau_\sigma} \simeq \frac{1}{\sqrt{\tau_\varepsilon\tau_\sigma}} = \frac{1}{\bar{\tau}} \qquad \tan[\phi(\omega)] \cong \Delta \frac{(\omega\bar{\tau})^{\upsilon}\sin(\upsilon\pi/2)}{1 + (\omega\bar{\tau})^{2\upsilon} + 2(\omega\bar{\tau})^{\upsilon}\cos(\upsilon\pi/2)} \tag{88}$$

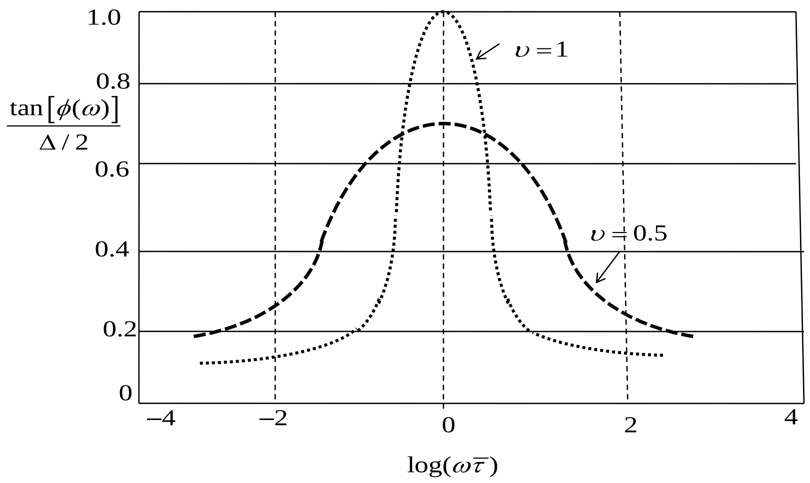

Figure-2: Loss tangent plot

The figure-2 shows the plot of loss tangent (normalized or scaled with $\Delta/2$, with fixed value for Δ) against log of $\omega\bar{\tau}=\omega/\omega_0$, a symmetrical function around the maximum value at $\omega\bar{\tau}=1$. For $\upsilon=1$, the plot is sharp, and as fractional order decreases, the plot broadens as the peak decreases. For lower fractional order system the loss tangent or dissipation factor or the specific dissipation factor i.e. Q^{-1} spreads for all frequency, and we tend to get dissipation Q^{-1} independent of frequency.

Now we write loss tangent for other systems (based on same derivation as done for the above). For Fractional Newton system we have constant loss tangent-also called constant phase element (CPE).

Fractional Newton system

$$\tan[\phi(\omega)]=\tan\frac{\upsilon\pi}{2} \qquad 0=\alpha<\beta=\infty \tag{88}$$

Fractional Voigt system

$$\tan[\phi(\omega)]=\frac{\omega^{\upsilon}\sin(\upsilon\pi/2)}{\alpha+\omega^{\upsilon}\cos(\upsilon\pi/2)} \qquad 0<\alpha<\beta=\infty \tag{89}$$

Fractional Maxwell system

$$\tan[\phi(\omega)]=\frac{\beta\omega^{\upsilon}\sin(\upsilon\pi/2)}{\omega^{2\upsilon}+\beta\omega^{\upsilon}\cos(\upsilon\pi/2)} \qquad 0=\alpha<\beta<\infty \tag{90}$$

8. Extension of fractional Calculus to continuous order differential equation systems

The constitutive equations relating stress strain we can write in various forms as follows

$$\sigma(t)=a_0\varepsilon(t) \tag{91}$$

The above is simple spring equation. The pure dashpot equation is following

$$\sigma(t) = a_1\dot{\varepsilon}(t) \tag{92}$$

The spring dashpot series equation is

$$\sigma(t) + b_1\dot{\sigma}(t) = a_1\dot{\varepsilon}(t) \tag{93}$$

The spring dashpot parallel equation is

$$\sigma(t) = a_0\varepsilon(t) + a_1\dot{\varepsilon}(t) \tag{94}$$

The spring connected to a parallel connection of spring dashpot will have constitutive equation as

$$\sigma(t) + b_1\dot{\sigma}(t) = a_0\varepsilon(t) + a_1\dot{\varepsilon}(t) \tag{95}$$

We can generalize our observation of stress strain from above expressions as

$$\sum_{k=0}^{m} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\sigma(t) = \sum_{k=0}^{n} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\varepsilon(t) \qquad k \in \mathbb{Z} \tag{96}$$

The above one is generalized integer order differential equation representation. Thus in general in the above integer order generalized system we can have k as real number giving fractional generalization of combination of spring dashpot system.

Let us generalize the system, which is

$$\varepsilon(t) + A\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}t^{\upsilon}}\varepsilon(t) = \sigma(t) \tag{97}$$

With several fractional orders z_i; $\quad i = 0,1,2,...(m-1)$, may be $\upsilon = \sum_{k=0}^{m-1} z_k$

$$\begin{gathered}\sum_{k=0}^{m-1} A_k(t)\,{}_0D_t^{z_k} f(t) = g(t) \\ \left[A_0(t) + A_1(t)\frac{\mathrm{d}^{z_1} f(t)}{\mathrm{d}t^{z_1}} + A_2(t)\frac{\mathrm{d}^{z_2} f(t)}{\mathrm{d}t^{z_2}} + ... + A_{m-1}(t)\frac{\mathrm{d}^{z_{m-1}} f(t)}{\mathrm{d}t^{z_{m-1}}}\right] = g(t)\end{gathered} \tag{98}$$

Where z_k are fractional numbers indicating fractional order derivatives, with $f(t)$ representing strain and $g(t)$ representing stress. We are extending the generalization of integer order system by 'integrating' the order in an interval of interest say $(a,b) \in (0,1)$; that is by changing the summation of above to integral and we get:

$$\int_a^b A(z)\left(\frac{\mathrm{d}^{m+z}}{\mathrm{d}t^{m+z}} f(t)\right)\mathrm{d}z = g(t); \quad 0 < a < b < 1; \quad m \in \mathbb{Z}^+ \tag{99}$$

This is generalization of equation several fractional orders differentials, substituting the summation with an integral where $A(z)$ limited in the interval $(a,b) \in (0,1)$, and m is positive integer. Why we did this? Well, the introduction of one parameter, rather interval a,b instead of z renders the fractional derivative operator d^{m+z} to become more flexible; because it includes a variety of memory mechanisms for relaxation! This is perhaps more apt to represent the dispersion acting with slightly different relaxation. The above generalized system is regarded as 'continuous order differential equation' where the order is continuous function in the designated interval. In other terms we can also say that the order has been weighted averaged!

9. Solving the continuous order differential equation

We apply the definition of fractional derivative to above continuous order system and obtain via Caputo's (1967) rule

$$\int_a^b \left[A(z)\frac{\mathrm{d}^{m+z} f(t)}{\mathrm{d}t^{m+z}}\right]\mathrm{d}z = \int_a^b \frac{[A(z)dz]}{\Gamma(1-z)}\int_0^t \left[\frac{f^{(m+1)}(u)}{(t-u)^z}\mathrm{d}u\right] = g(t); \quad m \in \mathbb{Z}^+ \tag{100}$$

Taking Laplace of above we write

$$\int_0^\infty e^{-st}\mathrm{d}t\left[\int_a^b \frac{A(z)}{\Gamma(1-z)}\mathrm{dz}\right]\left[\int_0^t \frac{f^{(m+1)}}{(t-u)^z}\mathrm{d}u\right]=G(s) \tag{101}$$

Interchanging the order of integration in above expression we get the following

$$\int_a^b \frac{A(z)}{\Gamma(1-z)}\left[\int_0^\infty e^{-st}\left[\int_0^t\left[\frac{f^{(m+1)}(u)}{(t-u)^z}\right]\mathrm{d}u\right]\mathrm{d}t\right]\mathrm{dz}=G(s) \tag{101}$$

$$\int_a^b A(z)\mathrm{dz}\left[s^{m+z}F(s)-s^z\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)\right]=G(s) \tag{102}$$

For above we write the LHS term by term as following

$$\int_a^b A(z)s^{m+z}F(s)\mathrm{dz}=F(s)s^m\int_a^b A(z)s^z\mathrm{dz} \tag{103}$$

$$\int_a^b A(z)s^z\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)\mathrm{dz}=\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)\times\left[\int_a^b A(z)s^z\mathrm{dz}\right] \tag{104}$$

With algebraic arrangement we get the expressions as follows

$$s^mF(s)\int_a^b s^zA(z)\mathrm{dz}-\left[\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)\right]\times\int_a^b A(z)s^z\mathrm{dz}=G(s) \tag{105}$$

$$s^mF(s)\int_a^b s^zA(z)\mathrm{dz}=G(s)+\left[\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)\right]\int_a^b s^zA(z)\mathrm{dz} \tag{106}$$

$$F(s)=\frac{G(s)}{s^m\int_a^b s^zA(z)\mathrm{dz}}+\frac{\sum_{n=0}^m s^{n-1}f^{(m-n)}(0)}{s^m} \tag{107}$$

In the expression above we do simplification of the second term of RHS. Dropping the summation sign we write the Laplace variables as $s^{n-1}\times\{s^{m-n}F(s)\}\times s^{-m}$. For the term s^{m-n}, we write for the Laplace of $(m-n)$ th derivative of function f. This we rearrange to get $s^{n-1}s^{-n}F(s)$, then $s^ns^{-1}s^{-n}F(s)$ and then $s^{-n-1}\{s^nF(s)\}$ to which we write, with n th derivative of function $f(t)$ at $t=0$ and obtain the expression $s^{-n-1}f^{(n)}(0)$. Here we apply the Laplace identity $\mathcal{L}\left\{t^n/n!\right\}=s^{-(n+1)}$ to get $s^{-n-1}f^{(n)}(0)\leftrightarrow(t^n/n!)f^{(n)}(0)$. We use this long simplification to write above in compact way as follows; the solution in Laplace and then solution in time domain, well in terms of convolution $(-)*(--)$ operation.

$$F(s)=\frac{G(s)}{s^m\int_a^b A(z)s^z\mathrm{dz}}+\sum_{n=0}^m s^{-n-1}f^{(n)}(0) \tag{108}$$

$$f(t)=g(t)*\mathcal{L}^{-1}\left\{\frac{1}{s^m\int_a^b A(z)s^z\mathrm{dz}}\right\}+\sum_{n=0}^m \frac{t^n}{n!}f^{(n)}(0) \tag{109}$$

If we consider the derivative order distribution function, $A(z)$ to be analytic then, $A(z)=\sum_{j=0}^{j=\infty}(A_jz^j/j!)$. We can substitute this analytic expansion in above and get

$$F(s)=\frac{G(s)}{s^m\sum_{j=0}^\infty\int_a^b(A_jz^js^z/j!)\mathrm{dz}}+\sum_{n=0}^m s^{-n-1}f^{(n)}(0)$$

$$f(t)=g(t)*\mathcal{L}^{-1}\left\{\frac{1}{s^m\sum_{j=0}^\infty\int_a^b(A_jz^js^z/j!)\mathrm{dz}}\right\}+\sum_{n=0}^m \frac{t^n}{n!}f^{(n)}(0) \tag{110}$$

10.Response function analysis for continuous order differential equation

The derivatives of order z_1 and z_2 with initial condition $f(0)=0$ imply has 'filtering effect', i.e. filtering the function $f(t)$ with high pass characteristics where response functions are s^{z_1} and s^{z_2}. Since $\left\{|s|^{z_2}-|s|^{z_1}\right\}/|s|^{z_1}=\left|s^{z_2-z_1}\right|-1$ is an increasing function of $|s|$ then the response function s^{z_2} is increasing more severely than s^{z_1}. The high pass filters what we stated has response function as $\Phi(s)=s^m\int_a^b s^z A(z)\mathrm{d}z$ which acts on $f(t)$. While we can have $A(z)=\delta(z-z_0)$, then

$$f(t)=g(t)\otimes\mathcal{L}^{-1}\left\{\frac{1}{s^m\int_a^b A(z)s^z\mathrm{d}z}\right\}=g(t)\otimes\mathcal{L}^{-1}\left\{\frac{1}{s^m\int_a^b \delta(z-z_0)s^z\mathrm{d}z}\right\}$$
$$=g(t)\otimes\mathcal{L}^{-1}\left\{\frac{1}{s^m s^{z_0}}\right\}=g(t)\otimes\mathcal{L}^{-1}\left\{\frac{1}{s^{m+z_0}}\right\} \qquad (111)$$

Figure-3 depicts the high pass filter characteristics for various discrete fractional orders z_0. The plot is between modulus of $|\Phi(i\omega)|$ and frequency ω. The figure-4 depicts the high pass characteristics of the response function when $A(z)=h$, for an interval (mentioned near each curve).

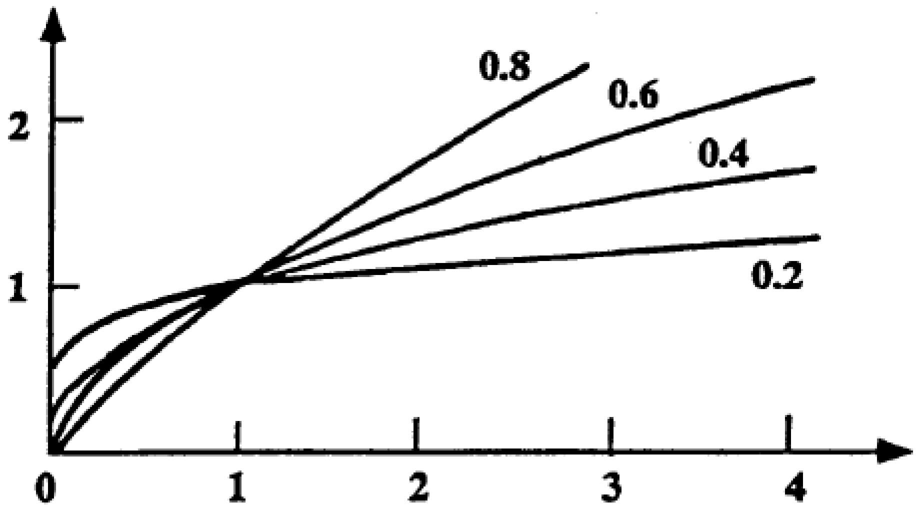

Figure-3: Plot show modulus of response function high passes characteristics when the order distribution function is $A(z)=\delta(z-z_0)$ for z_0 as fractional order of 0.2, 0.4, 0.6, and 0.8.

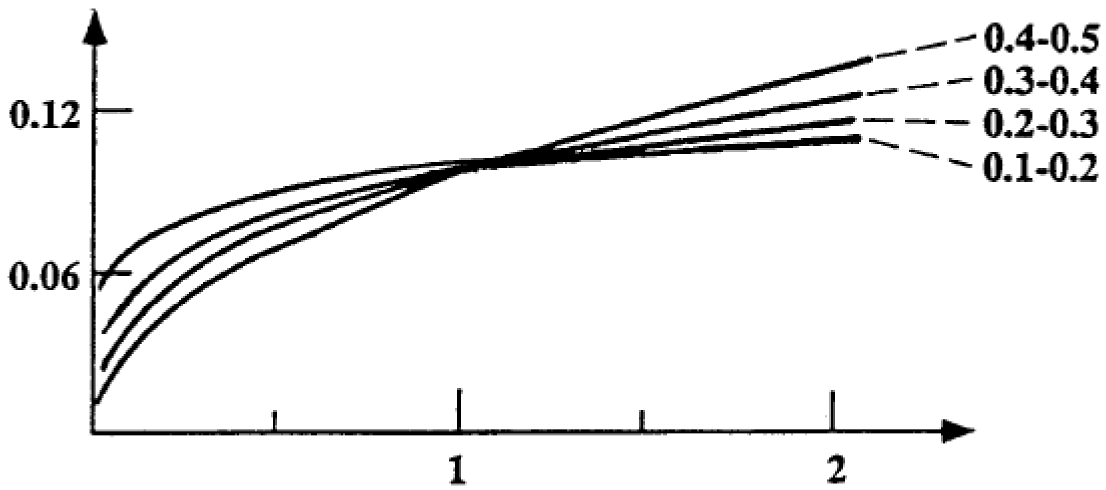

Figure-4: Plot show modulus of response function high passes characteristics when the order distribution function is $A(z)=h$ and with lower and upper limits of integration on the z

The term high pass means the transfer function plot says that as the frequency increases, the system allows those frequency signals to have larger gain, and as frequency is lesser we have more attenuation.

Now we consider $A(z)=kz+h$; the response function is

$$\begin{aligned}\Phi(s) &= s^m\int_a^b s^z A(z)\mathrm{d}z = \int_a^b s^z(kz+h)\mathrm{d}z \\ &= s^m\Big[\Big[(kb+h)s^b-(ka+h)s^a\Big]-k(s^b-s^a)/\ln s\Big]/\ln s\end{aligned} \tag{112}$$

With $k=0$ the continuous order is a simple case; meaning that the order distribution function of derivative order continuously placing same weights to all the derivatives (rather infinite numbers) of fractional orders in the interval (a,b). With $k=0$ and $h=1$; we obtain the response function as

$$\Phi(s)=s^m\int_a^b s^z A(z)\mathrm{d}z=\frac{(s^b-s^a)s^m}{\ln s} \tag{113}$$

What is of interest to us in above is the 'modulus' function of Φ; to get that we put $s=i\omega$; a standard procedure. Now what we see is $s^b=(i\omega)^b=(\omega e^{(\pi/2)i})^b=(\omega)^b e^{(\pi b/2)i}=\left|\omega^b\right|\angle(\pi b/2)$, is a vector with modulus as ω^b an argument (angle) of $\pi b/2$. Therefore s^b-s^a should be vector subtraction of two vectors shall give us resultant modulus as $\left|\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\pi(b-a)/2\right|$; using $|A-B|=\left||A|^2+|B|^2-2|A||B|\cos(\angle B-\angle A)\right|$. For taking modulus of $\ln s$, we just place $s=i\omega$ to have $\ln(i\omega)=\ln\left(|\omega|e^{i(\pi/2)}\right)=\ln|\omega|+i(\pi/2)$. From this we calculate $|\ln s|=\sqrt{(\ln\omega)^2+(\pi/2)^2}$. The modulus of s^m is nothing but ω^m. Using all these we get useful expression for modulus for above function which is following

$$|\Phi(i\omega)|=\frac{\left[\omega^m\right]\sqrt{\left|\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\pi(b-a)/2\right|}}{\sqrt{(\ln\omega)^2+(\pi/2)^2}} \tag{114}$$

For $m = 0$, we get the properties of response function $\Phi(s)$ at $s = \infty$ are governed by s^b while at $s = 0$ are governed by s^a . This property represents the difference between $\Phi(s)$ as expressed above type vis-à-vis response to single fractional derivative of order say z_0 .

This new type of response function arising out of continuous order differential equation allows us to study different behaviors for high frequency and low frequency; since it allows a filter to have filtering with independent properties at high frequency (early time) and low frequency (late times). This gives us extra freedom to study various complex relaxation processes and dynamic systems of nature.

We have used the term high pass filtering effect; nonetheless the inverse response function $(1/\Phi(s))$ acting on function $g(t)$ produces low pass filtering action. These are depicted in figure 5 for discrete and figure-6 for continuous order.

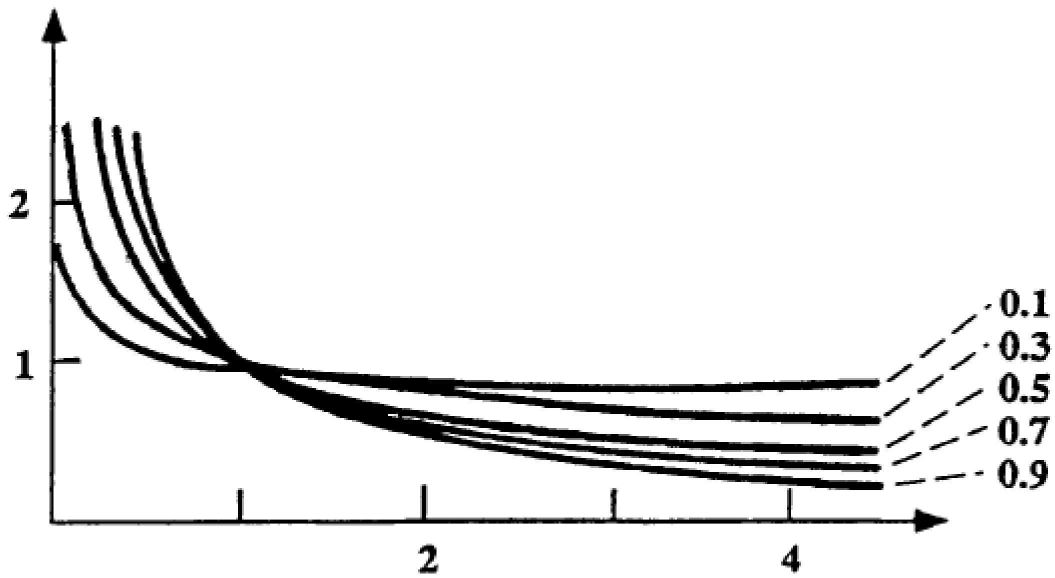

Figure-5 Plot show modulus of response function low passes characteristics when the order distribution function is $A(z) = \delta(z - z_0)$ for z_0 as fractional order of 0.1, 0.3, 0.5, 0.7 and 0.9.

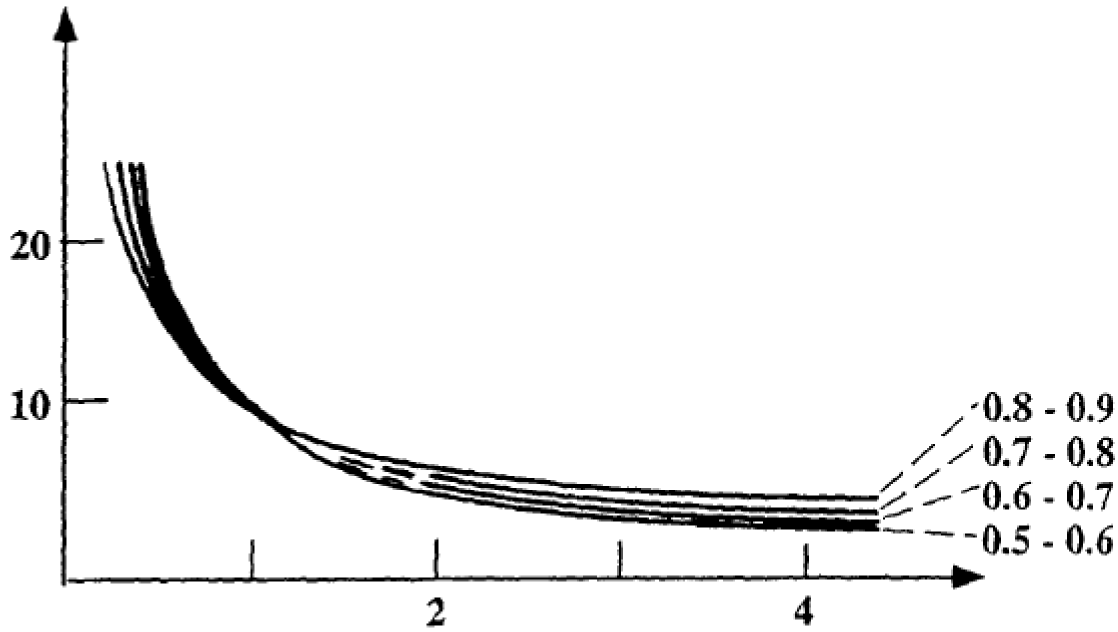

Figure-6: Plot show modulus of response function low passes characteristics when the order distribution function is $A(z)=h$ and with lower and upper limits of integration on the z

In order to see filtering effect of $1/\Phi(s)$ we again assume in $A(z)=kz+h$, with $k=0$ and with initial conditions $f^{(j)}(0)=0$ and write the following

$$F(s)=\frac{G(s)\ln s}{(s^b-s^a)h\omega^m}$$

$$|F(i\omega)|=\frac{|G(i\omega)||\ln(i\omega)|}{\left|(i\omega)^b-(i\omega)^a\right|h\omega^m}=\frac{|G(i\omega)|\left[(\ln\omega)^2+(\pi/2)^2\right]^{1/2}}{\omega^m h\left[\omega^{2b}+\omega^{2a}-2\omega^{a+b}\cos\pi(b-a)/2\right]^{1/2}} \quad (115)$$

The above implies low pass filtering action. Taking $h=1$; we can invert Laplace and write time response

$$f(t)=g(t)*\mathcal{L}^{-1}\left\{\frac{\ln s}{s^a}\right\}* \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^{b-a}-1)}\right\}*\frac{t^{m-1}}{\Gamma(m)} \quad (116)$$

In above the second expression $\ln s/s^\alpha$ is Laplace response as low pass filter, we do few algebraic manipulations on this expression and take invert Laplace in convolution form which gives us time response. While doing so we have used identity $\mathcal{L}^{-1}\left\{s^{-m}\right\}=t^{m-1}/(m-1)!=t^{m-1}/\Gamma(m)$. In above time domain expression the third term in RHS is Laplace invert of higher transcendental function called Robotnov-Hartley function $F_q(a,t)$ we can thus use the expression for the same as follows

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{b-a}-1}\right\}=F_{(b-a)}(1,t)=\sum_{n=0}^{\infty}\frac{t^{(n+1)(b-a)-1}}{\Gamma\left[(n+1)(b-a)\right]} \quad (117)$$

The Laplace invert for second term of above time domain expression that is $\mathcal{L}^{-1}\left\{\ln s/s^a\right\}$ requires numerical evaluation.

We have considered a uniform distribution of fractional order in the interval, with equal weights however it can have any functional form. Say if we have the form $A(z) = Az^n$, then we have solution (for initial conditions zero) as:

$$f(t) = g(t) * \mathcal{L}^{-1}\left\{ \frac{1}{As^m \left[\sum_{n=0}^{m} (-1)^n m(m-1).....(m-n+1)(b^{m-n}s^b - s^a a^{m-n}) / (\ln s)^{n+1} \right]} \right\} \quad (118)$$

We notice here that all previously obtained expressions after Laplace Transforms and then integration with respect to z are appearing as sums of powers of frequency s or ω, (meaning $A(z)$ having dimension of s^{-z}) which have different dimensions and which could be physically unacceptable. Thus we do the transformation of abscissa (normally frequency as in figure- 3 to a dimensionless scale. We assume that $A(z) = \tau^z B(z)$, where τ has dimensions of time. With this change of scale we will get the expressions obtained earlier as

$$F(s) = \frac{G(s)}{s^m} \int_a^b B(z)(s\tau)^z \mathrm{d}z + \sum_{n=0}^{m} s^{-n-1} f^{(n)}(0) \quad (119)$$

For constant $B(z) = h$, we thus obtain

$$F(s) = \frac{G(s)\ln(s\tau)}{hs^m \left((s\tau)^b - (s\tau)^a \right)} + \sum_{n=0}^{m} s^{-n-1} f^{(n)}(0) \quad (120)$$

With $m = 0$, $h = 1$, we obtain relaxation in Laplace Domain with a variety of slight different relaxations as $F(s) = (G(s)\ln(s\tau))/((s\tau)^b - (s\tau)^a)$, with abscissa as dimensionless τs, for figure 4 and 6.

11. A further generalization of natural system behavior with several memory and several relaxation modes

Let us write the following general constitutive one dimensional equation as

$$x(t) = Ky(t) \quad (121)$$

Generally if system is an-elastic media then $x(t)$ is the stress $\sigma(t)$ and $y(t)$ is the strain $\varepsilon(t)$ with K as the elastic parameter, if the system is dielectric media then the parameters are dielectric displacement($\vec{\mathrm{D}}$) electric field ($\vec{\mathrm{E}}$)and electric permittivity ($\in$) if the above equation represents relaxation of time-domain electromagnetic system, then the parameters are voltage, excitation current and resistivity. Even above expression can have generalized diffusion equation with memory acting on LHS and RHS say $x(t)$ representing flow or flux say $q(x,t)$ and $y(t)$ representing gradient of pressure or say concentration $\mathrm{grad}\{u(x,t)\}$. Here we assume that the above equation in fact be continuous order type mentioned below where memories acting on both the sides, as following generalization

$$\left(\frac{1}{b-a}\right)\int_a^b \alpha(z)\frac{\mathrm{d}^z y(t)}{\mathrm{d}t^z}\mathrm{d}z + \beta y(t) = \left(\frac{1}{f-c}\right)\int_c^f \gamma(z)\frac{\mathrm{d}^z x(t)}{\mathrm{d}t^z}\mathrm{d}z + \eta x(t) \quad (122)$$

Where $0 \le c < f < 1;\quad 0 \le a < b < 1$. The fractional derivatives inside the integrals are Caputo type

$$f^{(z)}(t) = \frac{\partial^z f(t)}{\partial t^z} = \frac{1}{\Gamma(1-z)}\int_0^t \frac{f'(u)\mathrm{d}u}{(t-u)^z} \quad (123)$$

The functions call it order distribution functions, $\alpha(z)$ and $\gamma(z)$ have dimensions of s^{-z}, β and η are dimensionless and the fractional order variable z is such that $0 < z < 1$. The division by length of the interval of integration $(b-a)$ for LHS and $(f-c)$ we do in above continuous order differential equation in order to normalize the operation and ensure that when $b \to a$ or $f \to c$ the integration will lead to the simple Fractional Order Derivative of order $z = a$ or $z = c$ for LHS of above continuous order system. We now do Laplace Transform of both the sides which gives

$$Y(s)\left[\left(\frac{1}{b-a}\right)\int_a^b s^z \alpha(z)\mathrm{d}z + \beta\right] = X(s)\left[\left(\frac{1}{f-c}\right)\int_c^f s^z \gamma(z)\mathrm{d}z + \eta\right] \tag{124}$$

Where s is Laplace variable (complex-frequency) $Y(s)$ and $X(s)$ are Laplace transforms of $y(t)$ and $x(t)$; it is assumed that initial conditions of $x(0) = y(0) = 0$. We have used Laplace Transform property of Fractional Order Derivative; as $\mathcal{L}\left\{\partial^z f / \partial t^z\right\} = s^z F(s) - s^{z-1} f(0)$; for $0 < z < 1$. We shall discuss here the case of interest when $\alpha(z) = \alpha$ and $\gamma(z) = \gamma$ are constants. This implies that all the fractional order derivatives are weighted equally in their respective intervals. Then the above Laplace transformed case becomes.

$$X(s) = Y(s)\frac{\left[\left(\frac{\alpha}{b-a}\right)\frac{(s^b - s^a)}{\ln s} + \beta\right]}{\left[\left(\frac{\gamma}{f-c}\right)\frac{(s^f - s^c)}{\ln s} + \eta\right]} = Y(s)\left(\frac{\beta}{\eta}\right)\frac{\left[\left(\frac{\alpha}{(b-a)\beta}\right)\frac{(s^b - s^a)}{\ln s} + 1\right]}{\left[\left(\frac{\gamma}{(f-c)\eta}\right)\frac{(s^f - s^c)}{\ln s} + 1\right]} \tag{125}$$

Let us write the above expression by partitioning the same with several functions as

$$X(s) = Y(s)\beta[\Omega(s,a,b)][\Psi(s,c,f)]/\eta = Y(s)\Lambda(s) \tag{126}$$

Where

$$\begin{aligned} &\Psi(s,c,f) = 1/\left\{(\gamma/(f-c)\eta)(s^f - s^c)/\ln s + 1\right\} \\ &\Omega(s,a,b) = (\alpha/(b-a)\beta)(s^b - s^a)/\ln s + 1 \\ &x(t) = (\eta/\beta)\left[\mathcal{L}^{-1}\left\{\Psi(s,c,f)\right\}\right] * \left[\mathcal{L}^{-1}\left\{\Omega(s,a,b)/s\right\}\right] * [y'(t)] \end{aligned} \tag{127}$$

The third expression above is the time domain expression in convolution $(*)$ terms. We have purposely divided the original Laplace equation by Laplace variable s so that we get expression for $x(t)$ if derivative of $y(t)$ that is $y'(t)$ being the excitation.

12. Time domain solution for continuous order differential equation by Laplace inversion with contour integration of the response function in frequency domain

Now we apply some technique to obtain the Laplace inverse of $\Psi(s,f,c)$. It may be seen that function $\Psi(s,f,c)$ has pole in $s = 1$, however apply L'Hopital's rule and we can say that $\lim_{s\to 1}\psi(s,f,c) = 1/\left\{\gamma/\eta(f-c) + 1\right\}$. Also we shall verify that $\lim_{s\to 0}\Psi(s,f,c) = 1$. Thus we can assume that $\Psi(s,f,c)$ is analytic in s; which also enables us to easily say the contour integration $\oint e^{st}\Psi(s)ds$, around the contour of figure-12 is zero. We shall be finding the $\mathcal{L}^{-1}\left\{\Psi(s,f,c)\right\}$ by integrating $\exp(st)\Psi(s,c,f)$ along the path shown in figure-12; extending the outer radius to infinity and inner one to zero. The choice of branch-cut as shown in figure-12 is standard logarithmic branch cut. The figure-12 has a vertical line AB having a constant real part and extended imaginary part. Let the vertical line be $s = u + i\omega$, actually standard is $s = \sigma + i\omega$, where Real part has same symbol as 'stress' thus we have used the said symbol. The vertical line in this figure-12 is the integration path for obtaining Laplace inverse. The

Laplace inverse is thus defined as integration on the said vertical line extending from minus infinity to plus infinity in the complex s plane.

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{u-i\omega}^{u+i\omega} e^{st}F(s)\mathrm{d}s \tag{128}$$

The above integration is done along this vertical line $\mathrm{Re}(s) = u$ in the complex plane such that u is greater than the 'real part of all singularities' of $F(s)$. Well if there are poles in $F(s)$ all of them in Left Half Plane then vertical line $\mathrm{Re}(s) = u = 0$, this is also true for the function $F(s)$ being real valued. Integrating along the designated closed path and taking the radius of inner circle $r \to 0$ and outer circle $R \to \infty$, we will get our Laplace inverse of the function $\Psi(s)$. Path begins at A and follows the contour. Inside the integration path there are no poles of the function. $\exp(st)\Psi(s)$. Because this has no poles in the negative complex plane s and then the integral in this closed contour is 'zero'; because 'residues' are nil. The integrals along BC, CD, HK, and KA are nil, when $R \to \infty$, the integral on EF is nil when inner radius $r \to 0$. We can see that when $r \to \infty$; the contributions from all the parts of the circuit are zero except those on the negative real axis and on the straight line in the positive real plane and obtain the following. Therefore, the integration of a complex valued function $\Psi(s)$, along the path of figure-12, means:

$$\frac{1}{2\pi i}\oint e^{st}\Psi(s)\mathrm{d}s = \frac{1}{2\pi i}\left\{\int_{u-i\infty}^{u+i\infty} e^{st}\Psi(s)\mathrm{d}s + \int_{D}^{E} e^{st}\Psi(s)\mathrm{d}s + \int_{F}^{H} e^{st}\Psi(s)\mathrm{d}s\right\} \tag{129}$$

Well we have purposely divided both sides by $(2\pi i)$. Call the circle in s plane of figure-12 as $s = r\exp(i\theta) = r(\cos\theta + i\sin\theta)$, with r and θ as polar coordinates, in that complex (frequency) plane; on $\theta = \pm\pi$, we have $s = -r$ and $ds = -dr$; where r is modulus of s. We observe the integration on DE means putting $\theta = \pi$ and integration on FH means $\theta = -\pi$. The closed integral above is zero by above arguments of residues 'path does not include any poles', thus we get

$$\begin{aligned} 0 = \frac{1}{2\pi i}\left\{\int_{u-i\infty}^{u+i\infty} e^{st}\Psi(s)\mathrm{d}s + \right\} + \frac{1}{2\pi i}\left[\int_{\infty}^{0} e^{-rt}\Psi(r,\theta=\pi)(-\mathrm{d}r)\right] \\ + \frac{1}{2\pi i}\left[\int_{F}^{H} e^{-rt}\Psi(r,\theta=-\pi)(-\mathrm{d}r)\right] \end{aligned} \tag{130}$$

The first term in the above simplified expression is nothing but $\mathcal{L}^{-1}\left\{\Psi(s)\right\} = \psi(t)$; we got from basic definition of the Laplace Inverse. Therefore we write

$$\begin{aligned} \psi(t) = \mathcal{L}^{-1}\left\{\Psi(s)\right\} &= -\frac{1}{2\pi i}\left[-\int_{\infty}^{0} e^{-rt}\Psi(r,\theta=\pi)\mathrm{d}r - \int_{0}^{\infty} e^{-rt}\Psi(r,\theta=-\pi)\mathrm{d}r\right] \\ &= \frac{1}{2\pi i}\left[\int_{0}^{\infty} e^{-rt}\left(\Psi(r,\theta=\pi) - \Psi(r,\theta=-\pi)\right)\mathrm{d}r\right] \end{aligned} \tag{131}$$

The above is an integral whose integrand is difference of two ratios of complex conjugate numbers which leads imaginary number and then leads to following computations. While in some calculations we may get integral of difference of two exponentials whose exponents are ratios of complex conjugate numbers, which in turn too then gives an imaginary expression-which cancels the i of denominator above yielding 'real' time domain response. We mean

$$\frac{a+ib}{c+id} - \frac{a-ib}{c-id} = \frac{2i(bc-ad)}{c^2+d^2} \tag{132}$$

Using the above logic the time response from Laplace Inverse of $\Psi(s,c,f)$, we write as follows.

$$\psi(t)=(1/2\pi i)\int_{\infty}^{0}(e^{-rt}\mathrm{d}r)/\left[1+A\left\{r^{f}e^{if\pi}-r^{c}e^{ic\pi}\right\}/(\ln r+i\pi)\right]$$
$$-(1/2\pi i)\int_{0}^{\infty}(e^{-rt}\mathrm{d}r)/\left[1+A\left\{r^{f}e^{-if\pi}-r^{c}e^{-ic\pi}\right\}/(\ln r-i\pi)\right]$$
$$\psi(t) = (1/2\pi i)\int_{0}^{\infty}(e^{-rt}\mathrm{d}r)/\left[1/[1+A\left\{r^{f}e^{if\pi}-r^{c}e^{ic\pi}\right\}/(\ln r+i\pi)\right]$$
$$-1/\left[1+A\left\{r^{f}e^{-if\pi}-r^{c}e^{-ic\pi}\right\}/(\ln r-i\pi)\right] \quad (133)$$

Where, $A=\gamma/\eta(f-c)$. The above is an integral whose integrand is difference of two ratios of complex conjugate numbers which leads imaginary number and then leads to following computations

$$\psi(t)=\frac{1}{\pi}\int_{0}^{\infty}e^{-rt}\left\{\frac{\left[Ar^{f}\sin(f\pi)-Ar^{c}\sin(c\pi)\right]\ln r-\pi\left[Ar^{f}\cos(f\pi)-Ar^{c}\cos(c\pi)\right]}{\left\{\left[Ar^{f}\cos(f\pi)-Ar^{c}\cos(c\pi)+\ln r\right]^{2}+\left[Ar^{f}\sin(f\pi)-Ar^{c}\sin(c\pi)+\pi\right]^{2}\right\}}\right\}\mathrm{d}r \quad (134)$$

We can use the above Laplace inverse $\psi(t)$ and write time evolution of $y(t)$ with $\gamma=0$, which is $y(t)=(\eta/\beta)x(t)*\psi(t,a,b)$. In this case $\Psi(s,a,b)=(1/\Omega(s,a,b))$.

$$y(t)=(\eta/\beta)x(t)\otimes\psi(t,a,b)$$
$$=\left(\frac{\eta}{\beta\pi}\right)x(t)*\int_{0}^{\infty}e^{rt}\left\{\frac{\begin{bmatrix}Ar^{b}\sin(b\pi)-\\ Ar^{a}\sin(a\pi)+\pi\end{bmatrix}\ln r-\pi\begin{bmatrix}Ar^{b}\cos(b\pi)-\\ Ar^{a}\cos(a\pi)+\ln r\end{bmatrix}}{\begin{bmatrix}Ar^{b}\cos(b\pi)-\\ Ar^{a}\cos(a\pi)+\ln r\end{bmatrix}^{2}+\begin{bmatrix}Ar^{b}\sin(b\pi)-\\ Ar^{a}\sin(a\pi)+\pi\end{bmatrix}^{2}}\right\}\mathrm{d}r \quad (135)$$

Here $A=(\alpha/\beta)(b-a)$. Well we should be resorting to numerical integration for above time domain solution put as convolution.

For full solution with $\gamma=1$ and with manipulation we write the following:

$$X(s)=Y(s)\beta\Omega(s,a,b)\Psi(s,c,f)/\eta$$
$$=\left(\frac{\beta}{\eta}\right)sY(s)\left[\alpha K(s,a,b)/\beta(b-a)+1/s\right]$$
$$\gamma=0;\quad \Psi(s,c,f)=1;\quad K(s,a,b)=\Omega(s,a,b)-1$$
$$K(s,a,b)=\left(s^{b}-s^{a}\right)/s\ln s \quad (136)$$

In above we have re-arranged by multiplying and dividing by s. We have to get Laplace inverse of the $K(s,a,b)$, that is $k(t,a,b)$ by the same method, and write convolution solution as:

$$x(t) = (\beta/\eta)\{y(t) + y'(t) \otimes (\alpha/\beta(b-a))k(t,a,b)\}$$

$$= (\beta/\eta)\left\{ y(t) + y'(t) \otimes (\alpha/\beta(b-a)\pi)\int_0^\infty e^{-rt} \left[\frac{\begin{array}{c} r^{b-1}(\ln r(r\sin(b\pi)) + \pi\cos(b\pi)) \\ -r^{a-1}(\ln(r\sin(a\pi)) - \pi\cos(a\pi)) \end{array}}{[(\ln r)^2 + \pi^2]} \right] \mathrm{d}r \right\}$$

(137)

$$k(t,a,b) = \mathcal{L}^{-1}K(s,a,b) = \mathcal{L}^{-1}\left\{ \left(s^b - s^a\right)/s\ln s \right\}$$

$$= \left(\frac{1}{\pi}\right)\int_0^\infty e^{-rt}\left[\frac{r^{b-1}(\ln(r\sin(b\pi)) + \pi\cos(b\pi)) - r^{a-1}(\ln(r\sin(a\pi)) - \pi\cos(a\pi))}{[(\ln r)^2 + \pi^2]}\right]\mathrm{d}r \tag{138}$$

The explicit time domain solution when $\gamma \neq 0$ is

$$x(t) = \left(\frac{\beta}{\eta}\right)\left\{ y(t) + y'(t) * \left(\frac{\alpha}{\beta(b-a)}\right) k(t,a,b) * \psi(t,c,f) \right\}$$

$$= \left(\frac{\beta}{\eta}\right)\{ y(t) + y'(t) *$$

$$\left(\frac{\alpha}{\beta(b-a)\pi}\right)\int_0^\infty e^{-rt}\left[\frac{\begin{array}{c} r^{b-1}(\ln(r\sin(b\pi)) + \pi\cos(b\pi)) \\ -r^{a-1}(\ln(r\sin(a\pi)) - \pi\cos(a\pi)) \end{array}}{[(\ln r)^2 + \pi^2]}\right]\mathrm{d}r$$

$$*\int_0^\infty e^{-rt}\left[\frac{\begin{array}{c} [Br^f\sin(f\pi) - Br^c\sin(c\pi) + \pi]\ln r \\ -\pi[Br^f\cos(f\pi) - Br^c\cos(c\pi) + \ln r \end{array}}{\begin{array}{c} [Br^f\cos(f\pi) - Br^c\cos(c\pi) + \ln r]^2 \\ +[Br^f\sin(f\pi) - Br^c\sin(c\pi) + \pi]^2 \end{array}}\right]\mathrm{d}r\} \tag{139}$$

Here $B = \gamma/\eta(f-c)$

$$\psi(t) = (1/2\pi i)\int_\infty^0 (e^{-rt}\mathrm{d}r)/\left[1 + A\left\{r^f e^{if\pi} - r^c e^{ic\pi}\right\}/(\ln r + i\pi)\right]$$

$$-(1/2\pi i)\int_0^\infty (e^{-rt}\mathrm{d}r)/\left[1 + A\left\{r^f e^{-if\pi} - r^c e^{-ic\pi}\right\}/(\ln r - i\pi)\right]$$

$$\psi(t) \quad = \quad (1/2\pi i)\int_0^\infty (e^{-rt}\mathrm{d}r)/\left[1/[1 + A\left\{r^f e^{if\pi} - r^c e^{ic\pi}\right\}/(\ln r + i\pi)\right]$$

$$-1/\left[1 + A\left\{r^f e^{-if\pi} - r^c e^{-ic\pi}\right\}/(\ln r - i\pi)\right] \tag{140}$$

Where, $A = \gamma/\eta(f-c)$. The above is an integral whose integrand is difference of two ratios of complex conjugate numbers which leads to following computations

$$\psi(t) = (1/\pi)\int_0^{\infty} e^{-rt} \left\{ \frac{\begin{matrix} \left[Ar^f \sin(f\pi) - Ar^c \sin(c\pi)\right]\ln r \\ -\pi\left[Ar^f \cos(f\pi) - Ar^c \cos(c\pi)\right] \end{matrix}}{\left\{\begin{matrix} \left[Ar^f \cos(f\pi) - Ar^c \cos(c\pi) + \ln r\right]^2 \\ +\left[Ar^f \sin(f\pi) - Ar^c \sin(c\pi) + \pi\right]^2 \end{matrix}\right\}} \right\} dr \qquad (141)$$

Now we write the transfer function's modulus as

$$|\Lambda(\omega)| = \frac{f-c}{b-a} \frac{\left| \begin{matrix} \alpha^2\left(\omega^{2b} - \omega^{2a}\right) + (b-a)^2\beta^2 \\ -2\alpha^2\omega^{a+b}\cos(b-a)\pi/2 - 2\alpha\beta(b-a)\left(\omega^b \cos b\pi/2 - \omega^a \cos a\pi/2\right) \end{matrix} \right|^{1/2}}{\left| \begin{matrix} \gamma^2\left(\omega^{2f} - \omega^{2c}\right) + (f-c)^2\eta^2 \\ -2\gamma^2\omega^{f+c}\cos(f-c)\pi/2 - 2\gamma\eta(f-c)\left(\omega^f \cos f\pi/2 - \omega^c \cos c\pi/2\right) \end{matrix} \right|^{1/2}} \qquad (142)$$

The above has asymptotes, when $b < f$ then $\lim_{\omega\to\infty}|\Lambda(\omega)| = 0$, for $b > f$ the limit is $\lim_{\omega\to\infty}|\Lambda(\omega)| = \infty$. While $b = f$ then we get $\lim_{\omega\to\infty}|\Lambda(\omega)| = \alpha(f-c)/\gamma(f-a)$, and $\lim_{\omega\to 0}|\Lambda(\omega)| = \beta/\eta$. Which imply the possibility of the low pass and high pass filtering; concerning the filtering properties of the general case $\alpha,\beta,\gamma,\eta \neq 0$. We consider the particular case when $b = f$ and see here that when $\beta/\eta < (\alpha/\gamma)(f-c)/(f-a)$ the filtering is high pass. When, $\beta/\eta > (\alpha/\gamma)(f-c)/(f-a)$ the filtering is low pass. We also extend the hypothesis here that when percent of $y(t)$ subject to memory is larger than $x(t)$ (that is we mean that $(\alpha/\beta)(f-a) > (\gamma/\eta)(f-c)$) then the memory is like high pass action, when instead the percent memory in $x(t)$ is larger than percent $y(t)$, meaning $(\alpha/\beta)(f-a) < (\gamma/\eta)(f-c)$ then the memory is a low pass filter. Refer figure-7, plot of generalized transfer function, depicting high pass and low pass depending on percent memory. Figure-8 to 11 give the frequency plot for $\Lambda(i\omega)$ its real and imaginary parts for various conditions.

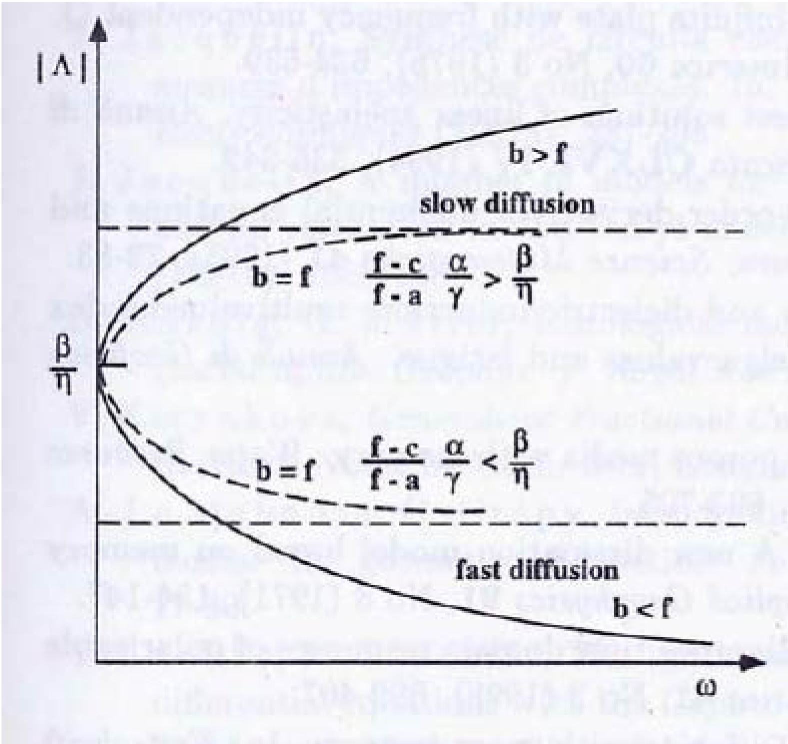

Figure-7: Filtering properties with generalized memory the plot of the generalized transfer function

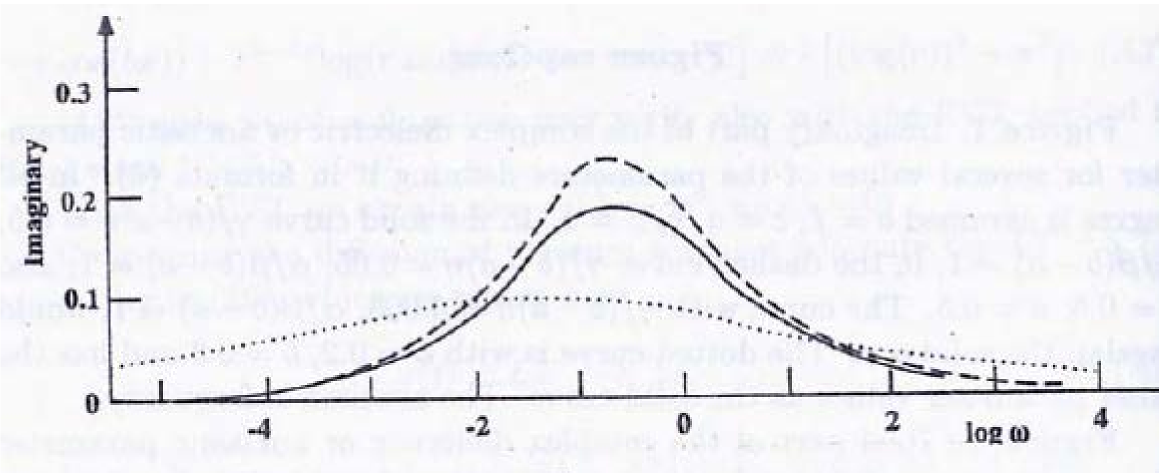

Figure-8: Imaginary part of $\Lambda(i\omega)$ for several values of parameters. In all curves it is assumed $b = f$ $c = a$ $\eta/\beta = 1$. The solid one has $\gamma(b-a)\eta = 0.5$, $\alpha/\beta(b-a) = 1$, in the dashed curve $\gamma(b-a)\eta = 0.05$, $\alpha/\beta(b-a) = 1$, with $a = 0.5$ and $b = 0.6$. The curve with $\gamma(b-a)\eta = 0.005$, $\alpha/\beta(b-a) = 1$ would overlap the solid one. The dotted curve is with $a = 0.2$, $b = 0.3$ and has the other parameter same as solid curve.

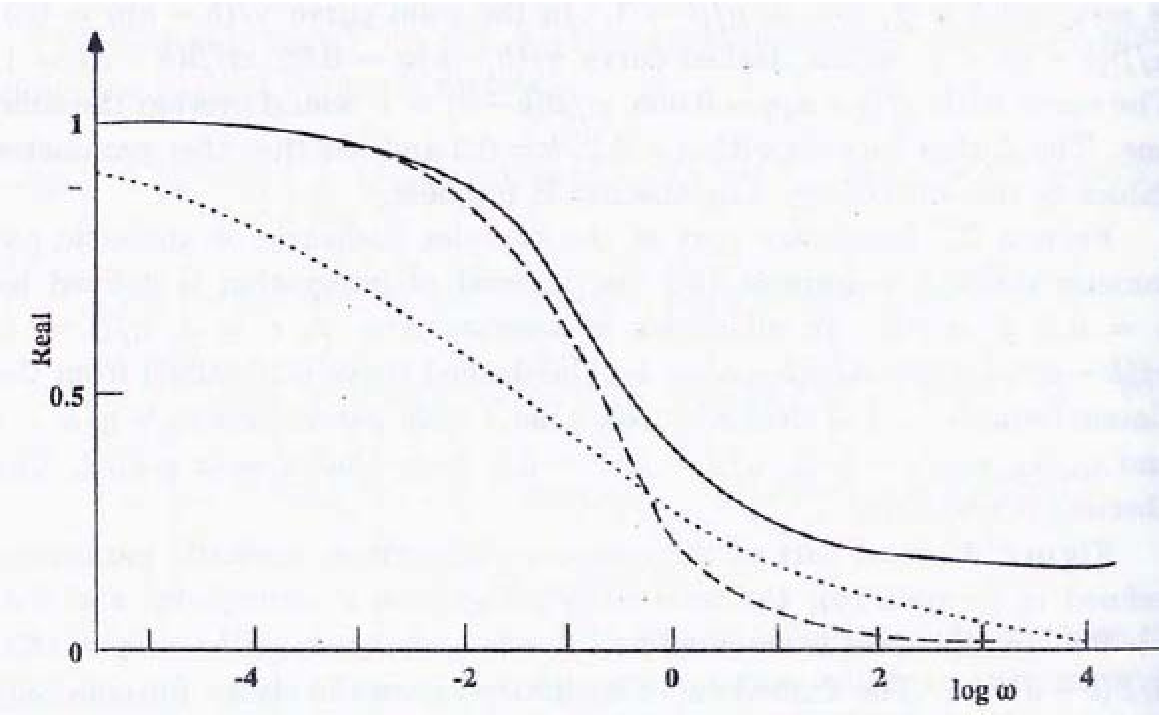

Figure-9: Real part of $\Lambda(i\omega)$ for several values of parameters. In all curves it is assumed $b = f \quad c = a \quad \eta/\beta = 1$. The solid one has $\gamma(b-a)\eta = 0.5$, $\alpha/\beta(b-a) = 1$ in the dashed curve $\gamma(b-a)\eta = 0.05$, $\alpha/\beta(b-a) = 1$ with $a = 0.5$ and $b = 0.6$. The curve with $\gamma(b-a)\eta = 0.005$, $\alpha/\beta(b-a) = 1$ would overlap the solid one. The dotted curve is with $a = 0.2$, $b = 0.3$ and has the other parameter same as solid curve.

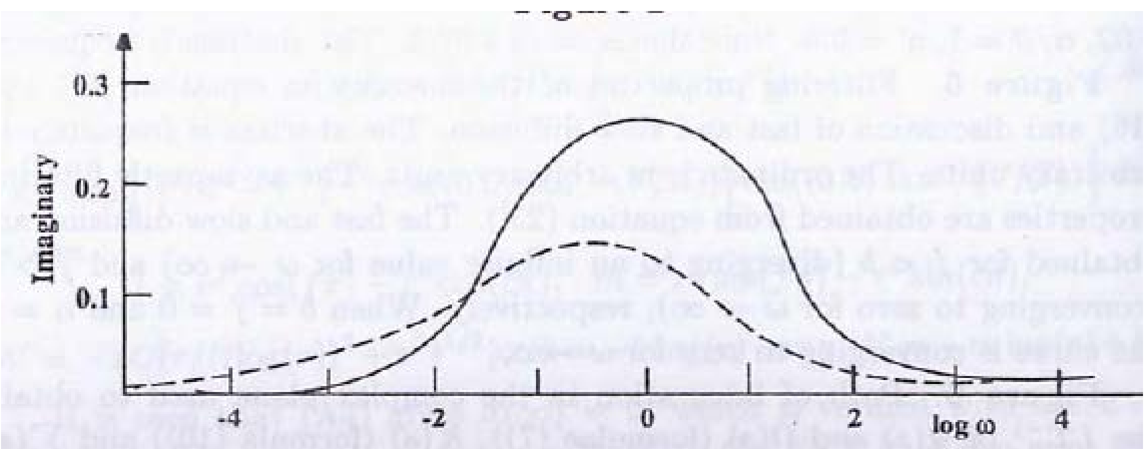

Figure-10: Imaginary part of $\Lambda(i\omega)$ for several values of parameters. The interval of integration is defined $a = 0.2 \quad b = 0.8$. In all curves it is assumed $b = f \quad c = a \quad \eta/\beta = 1$, $\gamma(b-a)\eta = 0.02$, $\alpha/\beta(b-a) = 1$.

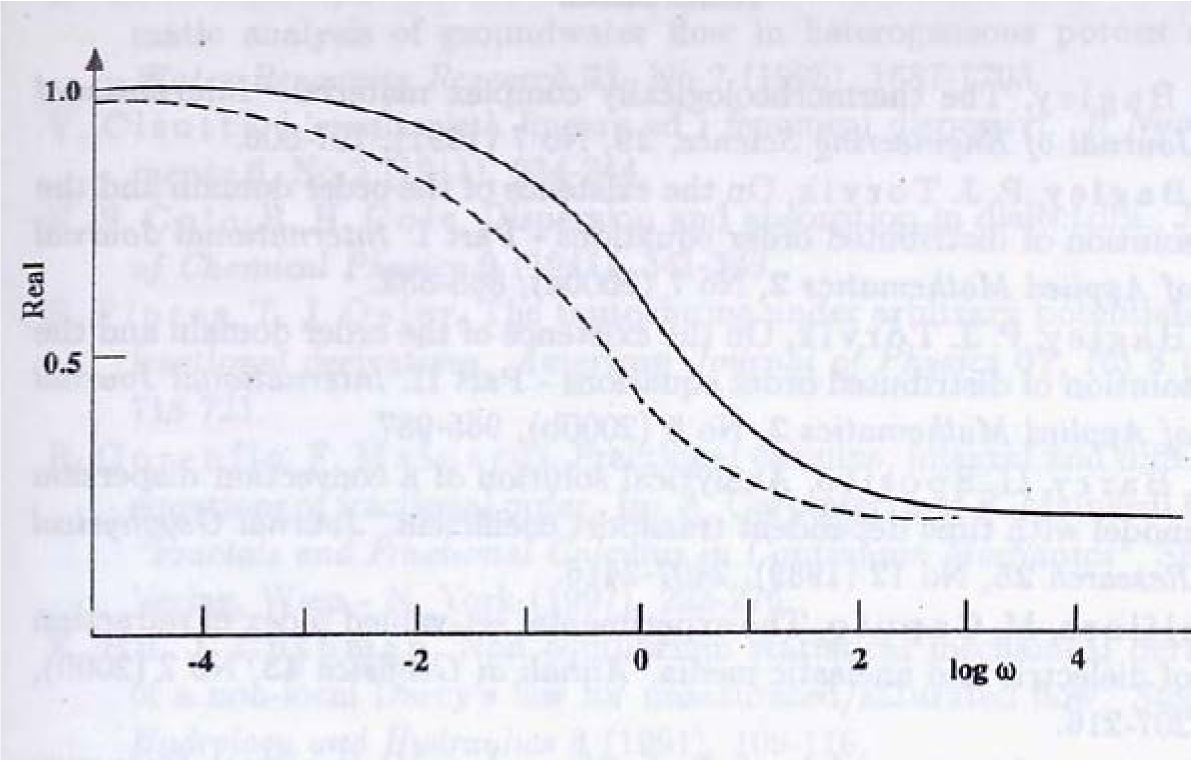

Figure-11: Real part of $\Lambda(i\omega)$ for several values of parameters. The interval of integration is defined $a=0.2$ $b=0.8$. In all curves it is assumed $b=f$ $c=a$ $\eta/\beta=1$, $\gamma(b-a)\eta=0.02$, $\alpha/\beta(b-a)=1$.

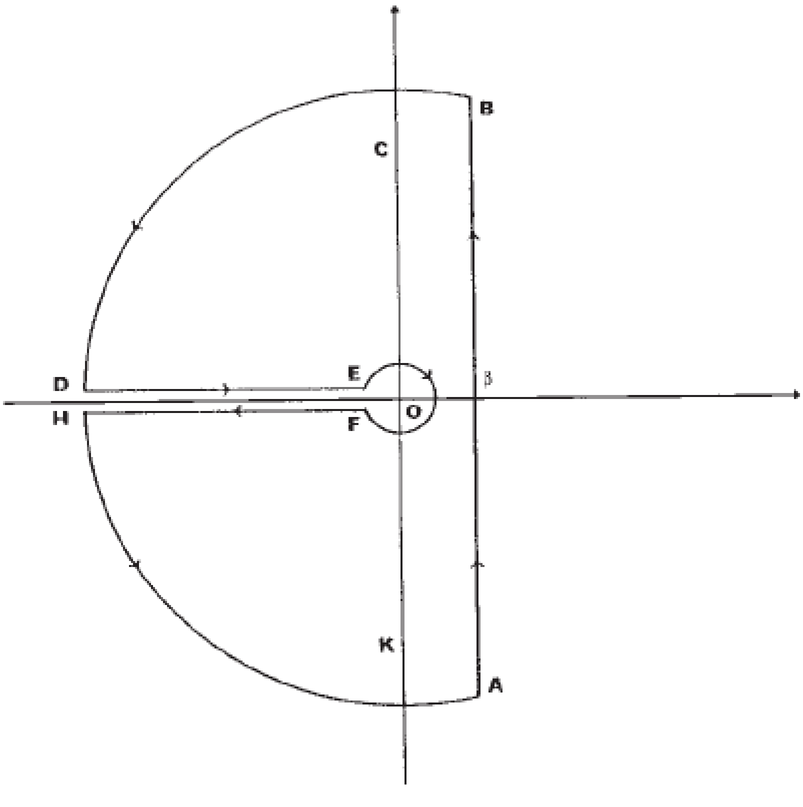

Figure-12: Integration path for obtaining Laplace invert for $\Psi(s,f,c)$

End of Part-C