Lecture Notes

Fractional Viscoelasticity Part-B

"Material Functions & Absorption Dissipation of Energy in Visco-elastic systems"

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Material Functions & Absorption Dissipation of Energy in Visco-elastic systems

We have gone through several facets of fractional visco-elastic systems, fractional derivatives, fractional integrals, and fractional differential equations; with several other interesting topics in Lecture series A. Here we are trimming down only to develop methodically the concept of visco-elastic systems. We will take and derive standard material functions describing the constitutive equations of integer order linear visco-elastic systems; thereafter we shall be developing the concepts of complex transfer functions; and see the energy status in the visco-elastic systems, i.e. how the deformation energy gets absorbed and dissipated in the visco-elastic systems. These basic ‘standard’ concepts we will utilize in further Lectures-dealing with several fractional visco-elastic systems; therefore concepts of integer order visco-elastic systems are essential, to further standardize the fractional order visco-elastic systems.

Contents

1. The Material Functions

Here we formalize the concepts of material functions which are important in description of the visco-elastic properties. First we deal with classical mechanical models and describe these functions, and after that we take the cases with fractional order visco-elastic models. We write the stress as $\sigma(t)$ and strain as $\varepsilon(t)$ and suitably normalize them with some reference values say σ_0 and ε_0. For all the linear systems (even for fractional order visco-elastic systems) we form constitutive equations, with two hypotheses.

The first one we say that the system is invariance for time translations, implying that a time shift in the input (strain or stress) results in an equal time shift in the output (stress or strain). The second one is to 'causality' implying that outputs at any instant say time t_1, depends on the values of the inputs only for past i.e. for times $t \le t_1$.

There are response functions 'creep-compliance' $J(t)$ and 'relaxation-modulus' $Y(t)$; they are the basic material functions. These are the functions or rather solution to the constitutive equations for a Heaviside step excitation $H(t)$ of stress and strain respectively.

$$H(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases} \tag{1}$$

The material functions $J(t)$ and $Y(t)$ give time evolution of strain for unit step stress input, and stress relaxation for unit step strain output.

However, the frequency domain analysis for the visco-elastic material has two functions namely 'complex-compliance' and 'complex-modulus' as $J^*(\omega)$ and $Y^*(\omega)$, respectively. These are just not Laplace transforms of $J(t)$ and $Y(t)$, a caution may be noted, rather are frequency-domain transforms of say $J^*(t)$ and $Y^*(t)$ which are responses to 'impulse-input' excitation of stress and stress. These frequency domain functions the complex-compliance and the complex modulus are also called dynamic functions of material, and important in extracting the information regarding energy absorbed by the material.

2. The Creep-Compliance and the Relaxation-Modulus

As we described above, we denote $J(t)$, the 'creep-compliance' as the strain-response $\varepsilon(t)$ to the unit-step of stress $\sigma(t) = H(t)$. We denote $Y(t)$ the stress-response $\sigma(t)$ to a unit step of strain $\varepsilon(t) = H(t)$. These $J(t)$ and $Y(t)$ are material functions. In terms of causality both are causal that is vanishing for $t < 0$. Implicitly $J(t)$ and $Y(t)$ gets multiplied by $H(t)$.

The limiting values of the material functions for $t \to 0^+$ and $t \to +\infty$ are related to instantaneous (or glass) and equilibrium behavior of the visco-elastic body, respectively. So we have 'glass-compliance' and 'equilibrium-compliance' as described below:

$$J_g \triangleq J(0^+) \qquad J_e \triangleq J(+\infty) \tag{2}$$

Similarly we have 'glass-modulus' and 'equilibrium-modulus' as described below:

$$Y_g \triangleq Y(0^+) \qquad Y_e \triangleq Y(+\infty) \tag{3}$$

From the experimental evidence we obtain that both the material function $J(t)$ and $Y(t)$ are non-negative functions.

For $0 < t < +\infty$, $J(t)$ is non-decreasing and $Y(t)$ is non-increasing. Assuming that $J(t)$ is differentiable increasing function of time, we write that

$$t \in \mathbb{R}^+ \qquad \frac{\mathrm{d}J}{\mathrm{d}t} > 0 \quad \text{i.e} \quad 0 \le J(0^+) < J(t) < J(+\infty) \le +\infty \tag{4}$$

Similarly, assuming that $Y(t)$ is a differentiable decreasing function of time

$$t \in \mathbb{R}^+ \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} < 0 \quad \text{i.e} \quad +\infty \ge Y(0^+) > Y(t) > Y(+\infty) \ge 0 \tag{5}$$

3. Boltzmann superposition principle

The statement of this Boltzmann superposition principle states that in a linear visco-elastic system the total response to a stress (or strain) history is equivalent (in some way) to the sum of responses to a sequence of incremental stress (or strain) histories.

Here the stress-strain is expressed in terms of one material functions [$J(t)$ or $Y(t)$], through linear hereditary integrals of Steieltjes type, indicated as

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\mathrm{d}\sigma(\tau) \qquad \sigma(t) = \int_{-\infty}^{t} Y(t-\tau)\mathrm{d}\varepsilon(\tau) \tag{6}$$

As the responses are to be time invariant for 'time-translation', we note that in $J(t)$ and $Y(t)$, the variable t represents time lag, since the application of stress or strain. The input, in other words are like:

$$\sigma(t) = \sigma_1 H(t-\tau_1) \quad \text{or} \quad \varepsilon(t) = \varepsilon_1 H(t-\tau_1) \tag{7}$$

Here σ_1 and ε_1 are constants. To these shifted inputs in time τ_1 therefore we will get output shifted as described below:

$$\varepsilon(t) = \sigma_1 J(t-\tau_1) \quad \text{or} \quad \sigma(t) = \varepsilon_1 Y(t-\tau_1) \tag{8}$$

Here we wrote response to a single step input shifted by time τ_1. If we apply series of N steps of stress as $\Delta\sigma_n = \sigma_{n+1} - \sigma_n$ for $n = 1,2,3,....N$ added consequently at times $\tau_N > \tau_{N-1} > ... > \tau_1 > -\infty$, we will introduce

$$\sigma(t) = \Delta\sigma_1 H(t-\tau_1) + \Delta\sigma_2 H(t-\tau_2) + ... + \Delta\sigma_N H(t-\tau_N) = \sum_{n=1}^{N} \Delta\sigma_n H(t-\tau_n) \tag{9}$$

For a unit step stress $H(t-\tau_n)$, we have strain response as $J(t-\tau_n)$, therefore we have the expression as following to give strain response for series of small steps of stress input

$$\varepsilon(t) = \Delta\sigma_1 J(t-\tau_1) + \Delta\sigma_2 J(t-\tau_2) + ... + \Delta\sigma_N J(t-\tau_N) = \sum_{n=1}^{N} \Delta\sigma_n J(t-\tau_n) \tag{10}$$

We can however, approximate any arbitrary stress excitation (nevertheless physically realizable) by the above mentioned small steps of stress input of large numbers and write the following (remembering $H(t) = 1$)

$$\sigma(t) = \int_{-\infty}^{t} H(t-\tau)\mathrm{d}\sigma(\tau) = \int_{-\infty}^{t} \mathrm{d}\sigma(\tau) \tag{11}$$

Gives the strain as

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\mathrm{d}\sigma(\tau) \tag{12}$$

As we wrote in above case the stress excitation, similarly we can write strain excitation by small steps $\Delta\varepsilon_n$.

$$\varepsilon(t)=\int_{-\infty}^{t}H(t-\tau)\mathrm{d}\varepsilon(\tau)=\int_{-\infty}^{t}\mathrm{d}\varepsilon(\tau) \qquad \sigma(t)=\int_{-\infty}^{t}Y(t-\tau)\mathrm{d}\varepsilon(\tau) \tag{13}$$

Whenever the stress 'history' $\sigma(t)$ or strain 'history' $\varepsilon(t)$ is differentiable, by $\mathrm{d}\sigma(\tau)$ or $\mathrm{d}\varepsilon(\tau)$, we can write as

$$\dot{\sigma}(\tau)=\frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau} \qquad \dot{\varepsilon}(\tau)=\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau} \tag{14}$$

Therefore, we have from above

$$\mathrm{d}\sigma(\tau)=\dot{\sigma}(\tau)\mathrm{d}\tau \qquad \mathrm{d}\varepsilon(\tau)=\dot{\varepsilon}(\tau)\mathrm{d}\varepsilon \tag{15}$$

For a discontinuity at say time τ_0 of a jump magnitude of $\Delta\sigma_0$ and $\Delta\varepsilon_0$ for the stress excitation and strain excitation respectively, those contributions are noted respectively as follows

$$\Delta\sigma_0 J(t-\tau_0) \quad \text{and} \quad \Delta\varepsilon_0 Y(t-\tau_0) \tag{16}$$

4. The Laplace Transform approach to stress and strain histories

We assume without loss of generality, the visco-elastic system is stationary and quiescent for all times prior to some starting point of application of input stress (or strain). We assume that the start point is at $t=0$. We assume the causal histories of stress and strain (meaning $\sigma(0^-)=0$ and $\varepsilon(0^-)=0$), excitation as differentiable, for $t\in\mathbb{R}^+$, the creep and relaxation are as follows

$$\begin{aligned}
\varepsilon(t)&=\int_{0^-}^{t}J(t-\tau)\mathrm{d}\sigma(\tau)=\int_{0^-}^{t}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau\\
&=\int_{0^-}^{0^+}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau+\int_{0^+}^{t}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau\\
&=J(t)\int_{0^-}^{0^+}\dot{\sigma}(\tau)\mathrm{d}\tau+\int_{0^+}^{t}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau\\
&=J(t)\left[\sigma(0^+)-\sigma(0^-)\right]+\int_{0^+}^{t}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau \qquad \sigma(0^-)=0\\
&=J(t)\sigma(0^+)+\int_{0^+}^{t}J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau
\end{aligned} \tag{17}$$

While doing above steps we assumed that in the short interval of time from $t=0^-$ to $t=0^+$, the function $J(t-\tau)=J(t)$ a constant. Following the above steps we write

$$\sigma(t)=\int_{0^-}^{t}Y(t-\tau)\mathrm{d}\varepsilon(\tau)=Y(t)\varepsilon(0^+)+\int_{0^+}^{t}Y(t-\tau)\dot{\varepsilon}(\tau)\mathrm{d}\tau \tag{18}$$

We take from above expression for $\varepsilon(t)$, and do integration by parts for the second term RHS, as following steps:

$$
\begin{aligned}
\varepsilon(t) &= J(t)\sigma(0^+) + \int_{0^+}^{t} J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau \\
&= J(t)\sigma(0^+) + \left[J(t-\tau)\sigma(\tau)\Big|_{0^+}^{t} - \int_{0^+}^{t} \left(-\dot{J}(t-\tau)\right)\sigma(\tau)\mathrm{d}\tau \right] \\
&= J(t)\sigma(0^+) + \left[\{J(0)\sigma(t) - J(t)\sigma(0^+)\} + \int_{0^+}^{t} \dot{J}(t-\tau)\sigma(\tau)\mathrm{d}\tau \right] \\
&= J(0)\sigma(t) + \int_{0}^{t} \dot{J}(t-\tau)\sigma(\tau)\mathrm{d}\tau \\
&= J_g\sigma(t) + \int_{0}^{t} \dot{J}(t-\tau)\sigma(\tau)\mathrm{d}\tau
\end{aligned} \tag{19}
$$

In the above derivation we have made 0^+ to 0, and wrote $J(0) = J_g$, the glass compliance. Also we note that in the integration by parts formula the differentiation is in respect to variable τ , thus the minus sign appears before $\dot{J}(t-\tau)$. Similarly from the expression of the stress $\sigma(t)$, we write (assuming $Y_g < 0$)

$$
\sigma(t) = Y_g\varepsilon(t) + \int_{0}^{t} Y(t-\tau)\varepsilon(\tau)\mathrm{d}\tau \tag{20}
$$

If we have $\mathcal{L}\{f(t)\} = \tilde{f}(s)$ and $\mathcal{L}\{g(t)\} = \tilde{g}(s)$ the convolution is

$$
f(t) * g(t) = \tilde{f}(s)\tilde{g}(s) \qquad f(t) * g(t) = \int_{-\infty}^{t} f(t-\tau)g(\tau)\mathrm{d}\tau \tag{21}
$$

Therefore

$$
\begin{aligned}
\varepsilon(t) &= \sigma(t)J_g + \int_{0}^{t} \dot{J}(t-\tau)\sigma(\tau)\mathrm{d}\tau = \sigma(t)J_g + \dot{J}(t) * \sigma(t) \\
\sigma(t) &= \varepsilon(t)Y_g + \int_{0}^{t} \dot{Y}(t-\tau)\varepsilon(\tau)\mathrm{d}\tau = \varepsilon(t)Y_g + \dot{Y}(t) * \varepsilon(t)
\end{aligned} \tag{22}
$$

Taking Laplace of above with property; $\mathcal{L}\{f(t)\} = s\tilde{f}(s) - f(0^+)$, we get

$$
\begin{aligned}
\tilde{\varepsilon}(s) &= J_g\tilde{\sigma}(s) + \left[s\tilde{J}(s) - J_g \right]\tilde{\sigma}(s) = s\tilde{J}(s)\tilde{\sigma}(s) \\
\tilde{\sigma}(s) &= Y_g\tilde{\varepsilon}(s) + \left[s\tilde{Y}(s) - Y_g \right]\tilde{\varepsilon}(s) = s\tilde{Y}(s)\tilde{\varepsilon}(s)
\end{aligned} \tag{23}
$$

The validity is for even unbounded glass-relaxation i.e. $G_g = \infty$.

5. Reciprocity

We got from above Laplace relations for stress history and strain history the expressions as $\tilde{\varepsilon}(s) = s\tilde{J}(s)\tilde{\sigma}(s)$ and $\tilde{\sigma}(s) = s\tilde{Y}(s)\tilde{\varepsilon}(s)$. From here we write

$$
\tilde{\varepsilon}(s) = s\tilde{J}(s)\tilde{\sigma}(s) = s\tilde{J}(s)\left[s\tilde{Y}(s)\tilde{\varepsilon}(s) \right] \qquad \tilde{J}(s)\tilde{Y}(s) = \frac{1}{s^2} \tag{24}
$$

Inverting the above we get

$$
\mathcal{L}^{-1}\left\{ \tilde{J}(s)\tilde{Y}(s) \right\} = \mathcal{L}^{-1}\left\{ s^{-2} \right\} \quad \text{i.e} \quad J(t) * Y(t) = \int_{0}^{t} J(t-\tau)Y(\tau)\mathrm{d}\tau = t \tag{25}
$$

These are reciprocity property. If the stress is $H(t)$ then the strain is $J(t)$. We have the relation

$$\sigma(t)=\int_{-\infty}^{t} Y(t-\tau)\mathrm{d}\varepsilon(\tau) \qquad \sigma(t)=H(t) \qquad \varepsilon(t)=J(t) \quad \text{then} \quad \mathrm{d}\varepsilon=\mathrm{d}J \tag{26}$$

The above becomes

$$\begin{aligned} H(t)&=\int_{0^-}^{t} Y(t-\tau)\mathrm{d}J(\tau)=\int_{0^-}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau \\ &=\int_{0^-}^{0^+} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau+\int_{0^+}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau=Y(t)J_g+\int_{0}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau \end{aligned} \tag{27}$$

Thus we get

$$H(t)=J_gY(t)+\int_{0}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau \tag{28}$$

Taking Laplace we get

$$\frac{1}{s}=J_g\tilde{Y}(s)+\tilde{Y}(s)\left[s\tilde{J}(s)-J_g\right] \qquad \frac{1}{s}=s\tilde{Y}(s)\tilde{J}(s) \qquad \frac{1}{s^2}=\tilde{Y}(s)\tilde{J}(s) \tag{29}$$

The reciprocity derived again this way. Further manipulation of the same we arrive at Voltera's integral equation

$$\begin{aligned} H(t)&=Y(t)J_g+\int_{0}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau \\ H(t)J_g^{-1}&=Y(t)+J_g^{-1}\int_{0}^{t} Y(t-\tau)\dot{J}(\tau)\mathrm{d}\tau=Y(t)+J_g^{-1}\int_{0}^{t}\dot{J}(t-\tau)Y(\tau)\mathrm{d}\tau \\ Y(t)&=J_g^{-1}-J_g^{-1}\int_{0}^{t}\dot{J}(t-\tau)Y(\tau)\mathrm{d}\tau \end{aligned} \tag{30}$$

We have used commutation in convolution integral in above derivation i.e. $Y(t)*\dot{J}(t)=\dot{J}(t)*Y(t)$. Similarly we have Voltera's integral equation for $J(t)$ is

$$J(t)=Y_g^{-1}-Y_g^{-1}\int_{0}^{t}\dot{Y}(t-\tau)J(\tau)\mathrm{d}\tau \tag{31}$$

6. Limiting Values of creep-compliance and stress relaxation function and material function for standard linear visco-elastic systems

We use final value theorem and initial value theorem to state the limiting values in time domain of material functions by use of their Laplace transforms.

$$f(0^+)=\lim_{s\to\infty} s\tilde{f}(s) \qquad f(+\infty)=\lim_{s\to 0} s\tilde{f}(s) \tag{32}$$

This is obvious as time variable t and Laplace variable s i.e. frequency are inversely related. Using the above two theorems and the reciprocity property of material function as discussed above we arrive at interesting expressions for material functions in time domain as

$$J_g=\frac{1}{Y_g} \qquad J_e=\frac{1}{Y_e} \tag{33}$$

We can classify the visco-elastic materials based on the limiting values of the material functions

TYPE	J_g	J_e	Y_g	Y_e
I	>0	$<\infty$	$<\infty$	>0
II	>0	$=\infty$	$<\infty$	$=0$
III	$=0$	$<\infty$	$=\infty$	>0
IV	$=0$	$=\infty$	$=\infty$	$=0$

Table-1: Showing classes of visco-elastic systems based on glass & equilibrium values of material functions

For Hooke's system of pure spring we have

$$\sigma(t) = m\varepsilon(t) \qquad J(t) = 1/m \qquad Y(t) = m \tag{34}$$

For a Hooke's spring we thus have no creep and no relaxation as both these functions are constant.

$$J(t) \equiv J_g = J_e = 1/m \qquad Y(t) \equiv Y_g = Y_e = m \tag{35}$$

For Newton system of pure piston we have

$$\sigma(t) = b_1 \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \qquad J(t) = \frac{t}{b_1} \qquad Y(t) = b_1 \delta(t) \tag{36}$$

Here we get linearly rising creep $J(t) = J_+ t$; $J_+ = 1/b_1$ and an instantaneous relaxation $Y(t) = Y_- \delta(t)$; $Y_- = (1/J_+) = b_1$.

For a Voigt's system we have the constitutive equation as

$$\sigma(t) = m\varepsilon(t) + b_1 \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

$$J(t) = J_1\left(1 - e^{-t/\tau_\varepsilon}\right), \quad J_1 = \frac{1}{m}, \quad \tau_\varepsilon = \frac{b_1}{m} \tag{37}$$

$$Y(t) = Y_e + Y_- \delta(t), \quad Y_e = m, \quad Y_- = b_1$$

For a Maxwell system we have the constitutive equation and the material function as

$$\sigma(t) + a_1 \frac{\mathrm{d}\sigma}{\mathrm{d}t} = b_1 \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$$

$$J(t) = J_g + J_+ t \quad ; \quad J_g = \frac{a_1}{b_1} \quad ; \quad J_+ = \frac{1}{b_1} \tag{38}$$

$$Y(t) = Y_1 e^{-t/\tau_\sigma} \quad ; \quad Y_1(t) = \frac{b_1}{a_1} \quad ; \quad \tau_\sigma = a_1$$

For a Zenner system the constitutive equation and material functions are as following

$$\left[1 + a_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\sigma(t) = \left[m + b_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\varepsilon(t)$$

$$J(t) = J_g + J_1(1 - e^{-t/\tau_\varepsilon}) \quad ; \quad J_g = \frac{a_1}{b_1} \quad ; \quad J_1 = \frac{1}{m} - \frac{a_1}{b_1} \quad ; \quad \tau_\varepsilon = \frac{b_1}{m} \tag{39}$$

$$Y(t) = Y_e + Y_1 e^{-t/\tau_\sigma} \quad ; \quad Y_e = m \quad ; \quad Y_1 = \frac{b_1}{a_1} - m \quad ; \quad \tau_\sigma = a_1$$

With condition that $0 < m < b_1 / a_1$ in order J_1, Y_1 be positive and hence $0 < Y_e < Y_g < \infty$. This makes the retardation time τ_ε must be greater than the relaxation time τ_σ i.e. $0 < \tau_\sigma < \tau_\varepsilon < \infty$.

7. The generalization of material functions

In the previous section we noted several types of the material functions depending on the constituent equations. We combine them to write the following generalized functions as;

$$J(t) = J_g + \sum_n J_n (1 - e^{-t/\tau_{\varepsilon,n}}) + J_+ t \qquad Y(t) = Y_e + \sum_n Y_n e^{-t/\tau_{\sigma,n}} + Y_- \delta(t)$$

The material functions are composed of basically three type of functions depicted in figure-1. From the combined function as written above, we can say that

$$\begin{array}{llll} J_e < \infty & \text{iff} & J_+ = 0 & \text{and} \quad \text{vice-versa} \\ J_e = \infty & \text{iff} & J_+ \neq 0 & \text{and} \quad \text{vice-versa} \\ G_g < \infty & \text{iff} & G_- = 0 & \text{and} \quad \text{vice-versa} \\ G_g = \infty & \text{iff} & G_- \neq 0 & \text{and} \quad \text{vice-versa} \end{array} \qquad (40)$$

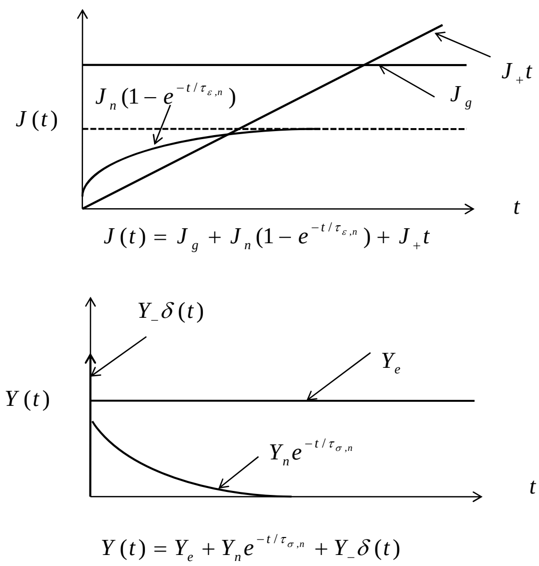

Figure-1 Showing component functions of consolidated material functions

The creep-compliance has a glass-compliance part which is constant J_g added to it several monotonically exponentially rising part denoted by retardation times $\tau_{\varepsilon,n}$, plus a linearly rising creep-compliance part added i.e. J_+. In material function $J(t)$ all these components or some of them may be present, depending on the constitutive equation. In the stress-relaxation function the components are constant Y_e added to this is exponentially decaying functions $Y_n e^{-t/\tau_{\sigma,n}}$ plus an instantaneous modulus as delta function i.e. $Y_-\delta(t)$. In the material function $Y(t)$ all these components or some of these component functions may be present depending on the constitutive equation.

Take the generalized expression for creep-compliance $J(t) = J_g + \sum_n J_n \left(1 - e^{-t/\tau_{\varepsilon,n}}\right) + J_+ t$; and apply Laplace transform to obtain the following equation in complex frequency domain

$$\tilde{J}(s) = \frac{J_g}{s} + \sum_n \frac{J_n}{s(1+s\tau_{\varepsilon,n})} + \frac{J_+}{s^2} \qquad \text{or} \qquad s\tilde{J}(s) = J_g + \sum_n \frac{J_n}{(1+s\tau_{\varepsilon,n})} + \frac{J_+}{s} \tag{41}$$

Similarly for the stress-relaxation function $Y(t) = Y_e + \sum_n Y_n e^{-t/\tau_{\sigma,n}} + Y_- \delta(t)$, we have Laplace domain equation as:

$$\tilde{Y}(s) = \frac{Y_e}{s} + \sum_n \frac{\tau_{\sigma,n} Y_n}{(1+s\tau_{\sigma,n})} + Y_- \qquad \text{or} \qquad s\tilde{Y}(s) = Y_e + \sum_n \frac{s\tau_{\sigma,n} Y_n}{(1+s\tau_{\sigma,n})} + Y_- s \tag{42}$$

Can expressed as

$$\begin{aligned} s\tilde{Y}(s) &= Y_e + \left(\sum_n Y_n - \sum_n \frac{Y_n}{1+s\tau_{\sigma,n}} \right) + Y_- s \\ s\tilde{Y}(s) &= (s+B) - \sum_n \frac{Y_n}{1+s\tau_{\sigma,n}} + Y_- s; \quad B \triangleq \sum_n Y_n \end{aligned} \tag{43}$$

The Laplace expressions $s\tilde{J}(s)$ or $s\tilde{Y}(s)$ are ratio of polynomials in s; a rational function with simple poles and zeros on the left half of the complex plane ($\mathrm{Re}(s) < 0$) and a simple pole at the origin $s = 0$; can be described as:

$$s\tilde{J}(s) = \frac{1}{s\tilde{Y}(s)} = \frac{N(s)}{D(s)}; \qquad \begin{cases} N(s) = 1 + \sum_{k=1}^{p} a_k s^k \\ D(s) = m + \sum_{k=1}^{q} b_k s^k \end{cases}; \quad q = p \quad \text{or} \quad q = p+1 \tag{44}$$

8. The operator equation

From above polynomial representation we thus have

$$s\tilde{J}(s) = \frac{1 + \sum_{k=1}^{p} a_k s^k}{m + \sum_{k=1}^{q} b_k s^k} = \frac{\tilde{\varepsilon}(s)}{\tilde{\sigma}(s)} = \tilde{J}^*(s) \qquad \left[1 + \sum_{k=1}^{p} a_k s^k\right]\tilde{\sigma}(s) = \left[m + \sum_{k=1}^{q} b_k s^k\right]\tilde{\varepsilon}(s) \tag{45}$$

Therefore we get the operator equation, by inverting the above Laplace equation

$$\left[1 + \sum_{k=1}^{p} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\right]\sigma(t) = \left[m + \sum_{k=1}^{q} b_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\right]\varepsilon(t) \tag{46}$$

Note that $\tilde{J}^*(s)$ is complex compliance is response (in frequency s-domain) to a unit-impulse stress $\delta(t)$, and the strain what is measured thus when transformed to Laplace domain gives complex-compliance. While $\tilde{J}(s)$ is Laplace transformed of creep-compliance $J(t)$ i.e. the strain measured for unit step $H(t)$ input. The both are related as $s\tilde{J}(s) = \tilde{J}^*(s)$. The same is true for complex modulus $\tilde{Y}^*(s)$, where excitation is strain instead of stress. Here we have $s\tilde{Y}(s) = \tilde{Y}^*(s)$.

Thus we have got this operator equation a differential equation encompassing all the linear models of visco-elasticity, relating $\sigma(t)$ and $\varepsilon(t)$. For solving the differential equation the initial conditions required are

$\sigma^{(h)}(0)$; $h = 0,1,2,\ldots p-1$ and $\varepsilon^{(k)}(0)$; $k = 0,1,2,\ldots q-1$

In the operator equation as above, the parameters m, a_k, b_k are such that the solution is physically realizable.

TYPE	ORDER	m	J_g	G_e	J_+	G_-
I	$q = p$	> 0	a_p / b_p	m	0	0
II	$q = p$	$= 0$	a_p / b_p	0	$1/b_1$	0
III	$q = p+1$	> 0	0	m	0	b_q / a_p
IV	$q = p+1$	$= 0$	0	0	$1/b_1$	b_q / a_p

Table-2 Type of visco-elastic system vis-à-vis order & parameters of operator equation

From this operator equation we get

Burger's system of I-kind

A four element visco-elastic system, the elements are a_1, a_2, b_1, b_2 where we take $m = 0$, $p = q = 2$, and the constitutive equation is

$$\left[1 + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right]\sigma(t) = \left[b_1 \frac{\mathrm{d}}{\mathrm{d}t} + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right]\varepsilon(t) \tag{47}$$

This gives the material functions as

$$J(t) = J_g + J_+ t + J_1\left(1 - e^{-t/\tau_\varepsilon}\right) \qquad G(t) = G_1 e^{-t/\tau_{\sigma,1}} + G_2 e^{-t/\tau_{\sigma,2}} \tag{48}$$

The values of J_g, J_+, J_1, τ_ε, $\tau_{\sigma,1}$ and $\tau_{\sigma,2}$ can be expressed in terms of a_1, a_2, b_1 and b_2

Burger's system of II-kind

Again we get a four element model, the elements are m, a_1, b_1 and b_2, with setting $p = 1$ and $q = 2$; to have constitutive equation as

$$\left[1 + a_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\sigma(t) = \left[m + b_1 \frac{\mathrm{d}}{\mathrm{d}t} + b_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right]\varepsilon(t) \tag{49}$$

Voigt system

Take $p = 0$, $q = 1$ and $m > 0$, the constitutive expression is

$$\sigma(t) = \left[m + b_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\varepsilon(t) \tag{50}$$

Maxwell system

Set $m = 0$, $p = q = 1$ to have the constitutive equation as

$$\left[1 + a_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\sigma(t) = \left[b_1 \frac{\mathrm{d}}{\mathrm{d}t}\right]\varepsilon(t) \tag{51}$$

Let us take the Voigt system as above, and we take Laplace transform of this to have

$\tilde{\sigma}(s)=m\tilde{\varepsilon}(s)+b_1\left[s\tilde{\varepsilon}(s)-\varepsilon(0^+)\right]$. With setting $\varepsilon(0^+)=0$, we get $\tilde{\sigma}(s)=\left[m+b_1 s\right]\tilde{\varepsilon}(s)$. For unit step stress excitation $\sigma(t)=H(t)$, implying $\tilde{\sigma}(s)=(1/s)$, we obtain $\tilde{\varepsilon}(s)=\tilde{J}(s)$ and also the complex-compliance $\tilde{J}^*(s)$.

$$\tilde{J}(s)=\frac{1}{s\left[m+b_1 s\right]} \qquad \text{or} \qquad \tilde{J}^*(s)=s\tilde{J}(s)=\frac{1}{m+b_1 s} \tag{52}$$

We have taken $\varepsilon(0^+)=0$, this initial condition is valid for a stress history $\sigma(t)=H(t)$, since $J_g=0$. This will not be valid for a reasonable strain history, in fact, if we consider the stress-relaxation tests for which the excitation is $\varepsilon(t)=H(t)$, we have in this case $\varepsilon(0^+)=1$. For Voigt system we have $J_g=0$, and $Y_g=+\infty$ (because of delta function's contribution in relaxation expression).

For a Maxwell's constitutive equation if we take Laplace, we get

$$\tilde{\sigma}(s)+a_1\left[s\tilde{\sigma}(s)-\sigma(0^+)\right]=b_1\left[s\tilde{\varepsilon}(s)-\varepsilon(0^+)\right] \tag{53}$$

For condition $a_1\sigma(0^+)=b_1\varepsilon(0^+)$, from above we get $\tilde{\sigma}(s)+a_1 s\tilde{\sigma}(s)=b_1 s\tilde{\varepsilon}(s)$, and write complex modulus $[\tilde{\varepsilon}(s)/\tilde{\sigma}(s)]_{\tilde{\sigma}(s)=1}=\tilde{J}^*(s)=s\tilde{J}(s)$ as

$$s\tilde{J}(s)=\tilde{J}^*(s)=\left.\frac{\tilde{\varepsilon}(s)}{\tilde{\sigma}(s)}\right|_{\sigma(t)=\delta(t)}=\frac{1+a_1 s}{b_1 s}=\frac{a_1}{b_1}+\frac{1}{b_1 s} \tag{54}$$

The condition $a_1\sigma(0^+)=b_1\varepsilon(0^+)$ is satisfied for any causal history, both in stress and in strain. This due to the fact that for Maxwell's system we have $J_g>0$ and $Y_g=1/J_g>0$.

9. Spectral Distribution Function in time

This is to study the distribution of characteristic times τ's, appearing in the exponential rising or falling functions contributions in the material functions $J(t)$ and $Y(t)$.

$$J(t)=J_g+\sum_n J_n\left(1-e^{-t/\tau_{\varepsilon,n}}\right)+J_+ t \qquad Y(t)=Y_e+\sum_n Y_n e^{-t/\tau_{\sigma,n}}+Y_-\delta(t) \tag{55}$$

We have discrete distribution of $\tau_{\varepsilon,1}$, $\tau_{\varepsilon,2}$, $\tau_{\varepsilon,3}$...., call it $R_\varepsilon(\tau)$ retardation spectrum a function of τ-time; and discrete distribution $\tau_{\sigma,1}$, $\tau_{\sigma,2}$...call it $R_\sigma(\tau)$ relaxation spectrum a function of τ-time. These discrete distributions we can write as follows depicted in figure-2.

$$\begin{aligned} R_\varepsilon(\tau)&=J_1\delta(\tau-\tau_{\varepsilon,1})+J_2\delta(\tau-\tau_{\varepsilon,2})+...=\sum_n J_n\delta(\tau-\tau_{\varepsilon,n}) \\ R_\sigma(\tau)&=Y_1\delta(\tau-\tau_{\sigma,1})+Y_2\delta(\tau-\tau_{\sigma,2})+...=\sum_n Y_n\delta(\tau-\tau_{\sigma,n}) \end{aligned} \tag{56}$$

From this discrete retardation and relaxation functions we can have a continuous function as we re-write the creep-compliance and stress-relaxation functions in the following way

$$J(t)=J_g+a\int_0^\infty R_\varepsilon(\tau)\left(1-e^{-t/\tau}\right)\mathrm{d}\tau+J_+ t \qquad Y(t)=Y_e+b\int_0^\infty R_\sigma(\tau)e^{-t/\tau}\mathrm{d}\tau+G_-\delta(t) \tag{57}$$

The time-dependent contributions are

$$J_\tau(t)\triangleq a\int_0^\infty R_\varepsilon(\tau)\left(1-e^{-t/\tau}\right)\mathrm{d}\tau \qquad Y_\tau(t)\triangleq b\int_0^\infty R_\sigma(\tau)e^{-t/\tau}\mathrm{d}\tau \tag{58}$$

$J_\tau(t)$ is the creep function is non-decreasing non-negative with limiting values as $J_\tau(0^+)=0$ and $J_\tau(+\infty)=a$ or $+\infty$. Whereas $Y_\tau(t)$ is relaxation function which is non-increasing and non-negative with limiting values as $Y_\tau(0^+) = b$ or $+\infty$ and $Y_\tau(+\infty) = 0$. Also with following conditions

$$\begin{aligned} &J_\tau(t) \geq 0 \qquad (-1)^n \frac{\mathrm{d}^n J_\tau(t)}{\mathrm{d}t^n} \leq 0 \qquad t \geq 0, \quad n = 1, 2, \ldots \\ &Y_\tau(t) \geq 0 \qquad (-1)^n \frac{\mathrm{d}^n Y_\tau(t)}{\mathrm{d}t^n} \geq 0 \qquad t \geq 0, \quad n = 1, 2, \ldots \end{aligned} \tag{59}$$

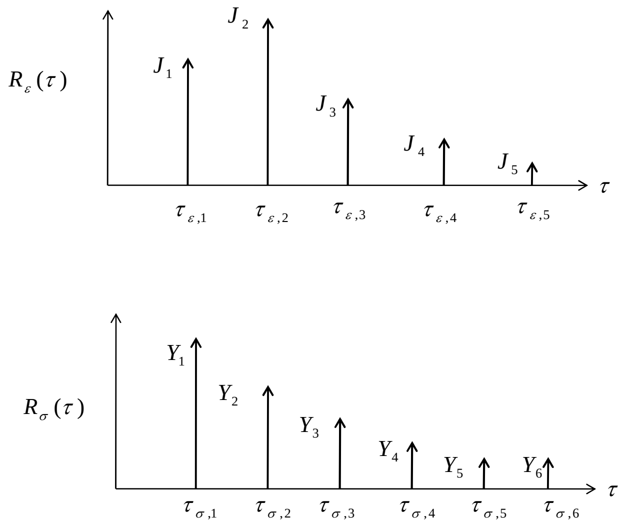

Figure-2 Time spectral (discrete) distribution functions

10. Determination of Time and Frequency Spectral Functions from Material Functions

We should have the knowledge of the creep-compliance and stress-relaxation functions, for determining $R_\varepsilon(\tau)$ and $R_\sigma(\tau)$. Define the frequency spectral functions as follows:

$$S_\varepsilon(f) \triangleq a\frac{R_\varepsilon(\tau)}{f^2} \quad \text{and} \quad S_\sigma(f) \triangleq b\frac{R_\sigma(\tau)}{f^2} \quad \text{where} \quad f = \frac{1}{\tau} \quad \text{and} \quad \frac{\mathrm{d}\tau}{\mathrm{d}f} = -\frac{1}{f^2} \tag{60}$$

From the above we re-write the definitions as

$$S_\varepsilon(f) = a\frac{R_\varepsilon(\tau)}{f^2} = -aR_\varepsilon(\tau)\frac{\mathrm{d}\tau}{\mathrm{d}f} \quad \text{or} \quad S_\varepsilon(f)\mathrm{d}f = -aR_\varepsilon(\tau)\mathrm{d}\tau \tag{61}$$

Similarly we have

$$S_\sigma(f)\mathrm{d}f = -bR_\sigma(\tau)\mathrm{d}\tau \tag{62}$$

With these relations we take the creep function $J_\tau(t)$ and differentiate that with respect to t

$$J_\tau(t) = a\int_0^\infty R_\varepsilon(\tau)\left(1 - e^{-t/\tau}\right)\mathrm{d}\tau \qquad \dot{J}_\tau(t) = a\int_0^\infty \frac{R_\varepsilon(\tau)e^{-t/\tau}}{\tau}\mathrm{d}\tau \tag{63}$$

Using $\tau^{-1} = f$ and the above obtained expression i.e. $aR_\varepsilon(\tau)\mathrm{d}\tau = -S_\varepsilon(f)\mathrm{d}f$, we write

$$\dot{J}_\tau(t) = a\int_0^\infty \frac{R_\varepsilon(\tau)}{\tau} e^{-t/\tau}\mathrm{d}\tau = \int_0^\infty f e^{-tf} a R_\varepsilon(\tau)\mathrm{d}\tau = -\int_0^\infty f\{S_\varepsilon(f)\}e^{-tf}\mathrm{d}f \tag{64}$$

Similarly for stress relaxation function $G_\tau(t) = b\int_0^\infty R_\sigma(\tau)e^{-t/\tau}\mathrm{d}\tau$, following similar steps we get

$$\dot{G}_\tau(t) = -b\int_0^\infty \frac{R_\sigma(\tau)e^{-t/\tau}}{\tau}\mathrm{d}\tau = \int_0^\infty f\{S_\sigma(f)\}e^{-tf}\mathrm{d}f \tag{65}$$

Therefore, we have the integral transformed pairs as

$$-\dot{J}_\tau(t) = \int_0^\infty [f\{S_\varepsilon(f)\}]e^{-tf}\mathrm{d}f \quad \text{and} \quad \dot{G}_\tau(t) = \int_0^\infty [f\{S_\sigma(f)\}].e^{-tf}\mathrm{d}f \tag{66}$$

The above pairs are nothing but Laplace transformed pairs, with Laplace variable t instead of usual parameter s.

$$f[S_\varepsilon(f)] = -a\frac{R_\varepsilon(1/f)}{f} = \mathcal{L}^{-1}\left\{-\dot{J}_\tau(t)\right\} \quad \text{and} \quad f\left[S_\sigma(f)\right] = -b\frac{R_\sigma(1/f)}{f} = \mathcal{L}^{-1}\left\{\dot{G}_\tau(t)\right\} \tag{67}$$

We take an example to elucidate the above n a system having creep-compliance function as $J_\tau(t) = a\int_0^{t/\tau_0}\frac{1-e^{-\xi}}{\xi}\mathrm{d}\xi$, here we put $\xi = \frac{t'}{\tau_0}$, $\mathrm{d}\xi = \frac{\mathrm{d}t'}{\tau_0}$ to get $J_\tau(t) = a\int_0^{\xi}\frac{1-e^{-t'/\tau_0}}{t'}\mathrm{d}t'$. Differentiating this we get

$$\dot{J}_\tau(t) = a\frac{1-e^{-t/\tau_0}}{t} = a\left(\frac{1}{t} - e^{-tf_0}\frac{1}{t}\right) \qquad -\dot{J}_\tau(t) = a\left(e^{-tf_0}\frac{1}{t} - \frac{1}{t}\right) \tag{68}$$

We have to take Laplace inverse of the above where Laplace variable is t. The known Laplace pairs are $\mathcal{L}^{-1}\{1/s\} = H(t)$; $\mathcal{L}^{-1}\left\{e^{-sT}/s\right\} = H(t-T)$. From here we write

$$f\left\{S_\varepsilon(f)\right\} = \mathcal{L}^{-1}\left\{-\dot{J}(t)\right\} = \mathcal{L}^{-1}\left\{ae^{-tf_0}t^{-1}\right\} - \mathcal{L}^{-1}\left\{at^{-1}\right\} = aH(f-f_0) - aH(f) \tag{69}$$

$$f\left\{S_\varepsilon(f)\right\} = \begin{cases} -a & 0 < f < f_0 \\ 0 & f_0 < f < \infty \end{cases} \quad \text{or} \quad S_\varepsilon(f) = \begin{cases} -\dfrac{a}{f} & 0 < f < f_0 \\ 0 & f_0 < f < \infty \end{cases} \tag{70}$$

With the relation $f\left\{S_\varepsilon(f)\right\} = -a\frac{R_\varepsilon(\tau)}{f}$; $R_\varepsilon(\tau) = -\frac{f^2\left\{S_\varepsilon(f)\right\}}{a}$, we write distribution in time or time spectrum as

$$R_\varepsilon(\tau) = \begin{cases} -\dfrac{f^2}{a}\left(-\dfrac{a}{f}\right) & 0 < f < f_0 \\ 0 & f_0 < f < \infty \end{cases} = \begin{cases} af & 0 < f < f_0 \\ 0 & f_0 < f < \infty \end{cases} = \begin{cases} \dfrac{a}{\tau} & \infty > \tau > \tau_0 \\ 0 & \tau_0 > \tau > 0 \end{cases} \tag{71}$$

We extracted the retardation-rate distribution function as

$$R_\varepsilon(\tau) = \begin{cases} 0 & \text{for} \quad 0 < \tau < \tau_0 \\ \dfrac{a}{\tau} & \text{for} \quad \tau_0 < \tau < \infty \end{cases} \tag{72}$$

We plot the $R_\varepsilon(\tau)$ and $S_\varepsilon(f)$, the time and frequency spectral distributions in figure-3

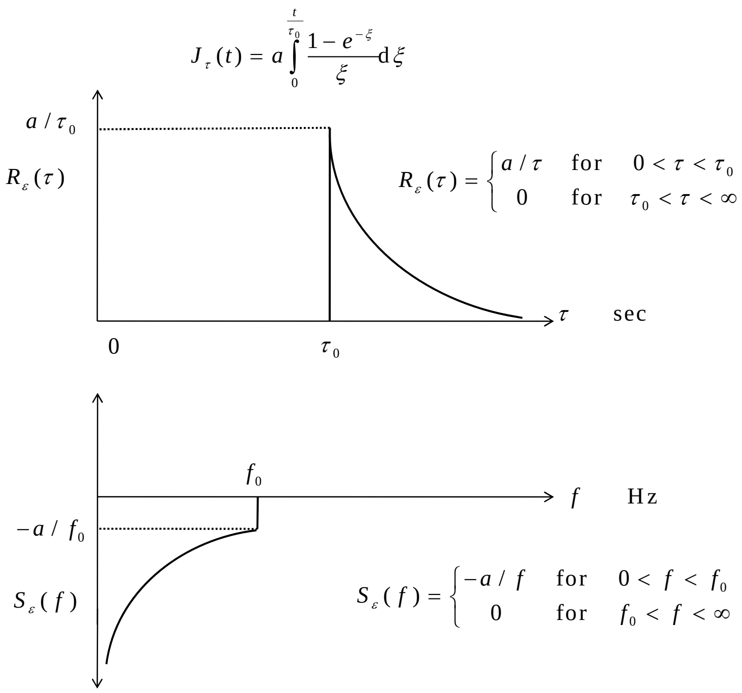

Figure-3; Time and frequency spectral density for creep-compliance

11. The complex-compliance and complex-relaxation functions the dynamic functions in frequency domain-Fourier Transform Approach

We had indicated earlier that we have complex-compliance $\tilde{J}^*(s)$ and complex-relaxation functions $\tilde{Y}^*(s)$ where the input excitation in time domain is delta function $\delta(t)$. Now we resort to excitation of sinusoidal nature $\sigma(\omega,t) = e^{i\omega t}$ and $\varepsilon(\omega,t) = e^{i\omega t}$, with time of application of input excitation as $t = -\infty$; the corresponding responses are called the 'dynamic-function'. This dynamic-function in other words gives steady-state response of the visco-elastic system and importantly provides information about the 'storage' and 'dissipation' of the mechanical energy.

For a stress excitation input of $\sigma(\omega,t) = e^{i\omega t}$, the strain response $\varepsilon(\omega,t) = J^*(\omega)e^{i\omega t}$ obtained where $J^*(\omega) = i\omega\hat{J}(\omega)$. Similarly for strain input excitation $\varepsilon(\omega,t) = e^{i\omega t}$ the stress response $\sigma(\omega,t) = Y^*(\omega)e^{i\omega t}$ where $Y^*(\omega) = i\omega\hat{Y}(\omega)$. The $\hat{J}(\omega)$ and $\hat{Y}(\omega)$ are Fourier transform of material functions (in time), causal functions $J(t)$ and $Y(t)$ i.e.

$$\hat{J}(\omega) = \int_0^\infty J(t)e^{-i\omega t}\mathrm{d}t \qquad\qquad \hat{Y}(\omega) = \int_0^\infty Y(t)e^{-i\omega t}\mathrm{d}t \tag{73}$$

Also as we indicated the following holds true

$$J^*(\omega) = \tilde{J}^*(s)\Big|_{s=i\omega} = s\tilde{J}(s)\Big|_{s=i\omega} \qquad\qquad Y^*(\omega) = \tilde{Y}^*(s)\Big|_{s=i\omega} = s\tilde{Y}(s)\Big|_{s=i\omega} \tag{74}$$

With $J^*(\omega)G^*(\omega) = 1$

12. Energy Dissipation and Storage in Visco-elastic System

While deformation of the visco-elastic body, part of the total work of deformation is dissipated as heat through viscous losses but the remainder of the deformation energy remains stored in the elasticity (spring), i.e. the energy is stored elastically. Elastically the stored energy is the potential energy. Energy can be also stored inertially as kinetic energy-such energy storage may be encountered in fast loading experiments, e.g. in impulsive excitation or in wave propagation at high-frequency. In this particular section however, inertial energy storage plays no role. Therefore, how we should determine these components of energy for a given sample of material of visco-elastic system. Thus it of interest to find out for a given system, in a given mode the amount of energy stored and the amount of energy dissipated. Similarly one may wish to find out the rate at which the energy of deformation is absorbed by the material or rate at which the energy gets stored or dissipated. The complex-compliance and the complex relaxation modulus give those figures, as described above, for a visco-elastic system.

We write the complex-compliance we write as

$$J^*(\omega) = J'(\omega) - iJ''(\omega) = \left|J^*(\omega)\right| e^{-i\phi(\omega)} \qquad \phi(\omega) > 0 \qquad 0 < \phi(\omega) < \pi/2 \tag{75}$$

The $\phi(\omega)$ is the phase lag angle as function of applied frequency of excitation ω ; it is lag as this is retarded response to stress excitation. The phase angle is the angle between the stress applied and its output strain response. Similarly we write for complex-relaxation function as

$$Y^*(\omega) = Y'(\omega) + iY''(\omega) = \left|Y^*(\omega)\right| e^{i\phi(\omega)} \qquad \phi(\omega) > 0 \qquad 0 < \phi(\omega) < \pi/2 \tag{76}$$

Here the phase angle $\phi(\omega)$ is the lead angle, for stress-relaxation response to applied strain. Also the real and imaginary parts are positive i.e. $J'(\omega), J''(\omega), Y'(\omega), Y''(\omega) > 0$.

The 'storage-compliance' is the real part of Y^* and J^* i.e. Y' and J', and 'loss-compliance' are the imaginary part of Y^* and J^* i.e. Y'' and J'' . The 'loss-tangent' or 'tan-delta' is tangent of the phase angle, i.e. $\tan\phi(\omega) = J''(\omega)/J'(\omega) = Y''(\omega)/Y'(\omega)$, this parameter gives the damping ability of the 'visco-elastic' system.

The energy absorption rate per volume of visco-elastic material during deformation is equal to the stress power i.e.

$$\dot{W}(t) = \sigma(t)\dot{\varepsilon}(t) \tag{77}$$

It is similar to electrical power $P(t) = v(t)i(t) = v(t)\dot{q}(t)$, i.e. product of instantaneous voltage and instantaneous current. Therefore, total work of deformation (or the mechanical-energy absorbed per unit volume of the material in deformation from time t_0 up to t, results in

$$W(t) = \int_{t_0}^{t} \dot{W}(\tau)\mathrm{d}\tau = \int_{t_0}^{t} \sigma(\tau)\dot{\varepsilon}(\tau)\mathrm{d}\tau \tag{78}$$

Also we have two parts, i.e. energy stored and energy dissipated component

$$W(t) = W_S(t) + W_d(t) \qquad \dot{W}(t) = \dot{W}_S(t) + \dot{W}_d(t) \tag{79}$$

The energy is dissipated in piston (dashpot) and stored in the spring. For energy dissipated in piston (dashpot) we write

$$\dot{W}_d(t) = \sum_n \sigma_{d,n}(t)\dot{\varepsilon}_{d,n}(t) = \sum_n \eta_n(t)\left[\dot{\varepsilon}_{d,n}(t)\right]^2 \qquad \sigma_d(t) = \eta\dot{\varepsilon}_d(t) \tag{80}$$

The energy storage in spring is

$$W_s(t)=\sum_n\int_{t_0}^{t}\sigma_{s,n}(\tau)\dot{\varepsilon}_{s,n}(\tau)\mathrm{d}\tau \qquad \dot{\varepsilon}_{s,n}(\tau)\mathrm{d}\tau=\mathrm{d}\varepsilon_{s,n}(\tau) \quad \text{and} \quad \sigma_{s,n}(t)=Y_n\varepsilon_{s,n}(t)$$
$$=\sum_n Y_n\int_{\varepsilon_{s,n}(0)}^{\varepsilon_{s,n}(t)}\varepsilon_{s,n}(\tau)\mathrm{d}\varepsilon_{s,n}(\tau)=\frac{1}{2}\sum_n Y_n\left[\varepsilon_{s,n}(t)\right]^2 \tag{81}$$

13. Computation of energy and rate of energy

We had stated that here we deal with sinusoidal excitation; say we are giving strain excitation then $\varepsilon(t)=\sin\omega t=\mathrm{Im}\left[e^{i\omega t}\right]$.

$$\sigma(t)=Y^*(\omega)\varepsilon(t)=\left[Y'(\omega)+iY''(\omega)\right]\sin\omega t=Y'(\omega)\sin\omega t+iY''(\omega)\sin\omega t$$
$$=Y'(\omega)\sin\omega t+Y''(\omega)\cos\omega t \tag{82}$$

We have used

$$i\sin\omega t=e^{i\pi/2}\,\mathrm{Im}\left\{e^{i\omega t}\right\}=\mathrm{Im}\left\{e^{i\pi/2}e^{i\omega t}\right\}=\mathrm{Im}\left\{e^{i(\omega t+\pi/2)}\right\}$$
$$=\mathrm{Im}\left\{\cos\left(\omega t+\frac{\pi}{2}\right)+i\sin\left(\omega t+\frac{\pi}{2}\right)\right\}=\sin\left(\omega t+\frac{\pi}{2}\right)=\cos\omega t \tag{83}$$

Looking at the obtained stress response to a strain excitation input as $\sin\omega t$, we identify that $Y'(\omega)\sin\omega t$ is in phase response with the input, while $Y''(\omega)\cos\omega t$ is out of phase response. Also with $\dot{\varepsilon}(t)=\omega\cos\omega t$, we write rate of total deformation energy as

$$\dot{W}(t)=\sigma(t)\dot{\varepsilon}(t)$$
$$=\left[Y'(\omega)\sin\omega t+Y''\cos\omega t\right]\left(\omega\cos\omega t\right)$$
$$=\omega Y'(\omega)\sin\omega t\cos\omega t+\omega Y''(\omega)\cos^2\omega t \tag{84}$$
$$=\frac{\omega}{2}\left[Y'(\omega)\sin 2\omega t+Y''(\omega)\left(1+\cos 2\omega t\right)\right]$$

Take $W(0)=0$, integrate the above to get total deformation energy as

$$W(t)=\int_0^t\dot{W}(\tau)d\tau=\frac{\omega}{4}\frac{Y'(\omega)}{\omega}(-\cos 2\omega t)\Big|_0^t+\frac{\omega}{2}Y''(\omega)t\Big|_0^t+\frac{\omega}{4}\frac{Y''(\omega)}{\omega}\sin 2\omega t\Big|_0^t$$
$$=\frac{1}{4}\left[Y'(\omega)\left(1-\cos 2\omega t\right)+Y''(\omega)\left(2\omega t+\sin 2\omega t\right)\right] \tag{85}$$

From above expression of $W(t)$ we cannot recognize the partial contributions to the storage and dissipation of energy; as in general all storage mechanisms are not in phase as well as the dissipations. If we assume 'phase-coherence' among the energy storing mechanisms on the one hand and 'phase-coherence' amongst the dissipating mechanisms; then we separate the components as following.

$$W_S^C(t)=\frac{Y'(\omega)}{4}(1-\cos 2\omega t) \qquad \dot{W}_S^C=\frac{\omega}{2}Y'(\omega)\sin 2\omega t$$
$$W_d^C(t)=\frac{Y''(\omega)}{4}(2\omega t+\sin 2\omega t) \qquad \dot{W}_d^C=\frac{\omega}{2}Y''(\omega)\left[1+\cos 2\omega t\right] \tag{86}$$

From here we write average stored energy as, average of coherently stored energy over a full cycle, derived below,

$$\begin{aligned}\langle W_s(\omega)\rangle &\triangleq \frac{1}{T}\int_t^{t+T} W_s^C(\tau)\mathrm{d}\tau \\ &= \frac{1}{T}\int_t^{t+T}\frac{Y'(\omega)}{4}(1-\cos 2\omega\tau)\mathrm{d}\tau \qquad T=\frac{2\pi}{\omega} \\ &= \frac{\omega Y'(\omega)}{8\pi}\int_t^{t+\frac{2\pi}{\omega}}(1-\cos 2\omega\tau)\mathrm{d}\tau = \frac{\omega Y'(\omega)}{8\pi}\int_0^{\frac{2\pi}{\omega}}(1-\cos 2\omega\tau)\mathrm{d}\tau \\ &= \frac{Y'(\omega)}{4}\end{aligned} \tag{87}$$

The dissipated energy we write as amount of energy that would be dissipated coherently over the full cycle of the excitation

$$\begin{aligned}\Delta W_d(\omega) &\triangleq \int_t^{t+T}\dot{W}_d^C(\tau)\mathrm{d}\tau \\ &= \frac{\omega Y''(\omega)}{2}\int_0^{2\pi/\omega}(1+\cos 2\omega\tau)\mathrm{d}\tau = \pi Y''(\omega)\end{aligned} \tag{88}$$

14.Specific Dissipation Function

This also known as ‘internal-function’ and defined as

$$Q^{-1}(\omega) \triangleq \frac{1}{2\pi}\frac{\Delta W_d}{W_s^*} \tag{89}$$

The ΔW_d is the amount of energy dissipated in one cycle, and W_s^* is the peak energy stored coherently during the cycle. This Q^{-1} is the reciprocal of the ‘quality factor-Q’ in resonating system as we all know.

$$\begin{aligned}Q^{-1}(\omega) &= \frac{1}{2\pi}\frac{\Delta W_d}{W_s^*} = \frac{1}{2\pi}\frac{\pi Y''(\omega)}{[Y'(\omega)/2]} \\ &= \frac{Y''(\omega)}{Y'(\omega)} = \tan\{\phi(\omega)\}\end{aligned} \tag{90}$$

The meaning of $Q^{-1}(\omega)$ physically is stating the damping ability of a linear visco-elastic system, depends only on the tangent of the phase angle of the relaxation-complex-modulus (and/or complex-compliance); i.e. the loss tangent; that is the function of frequency but independent of stress or strain amounts.

15.Example of Derivation of Dynamic Functions for visco-elastic systems and loss tangent

For a Zenner system the constitutive equation and material functions are

$$\begin{aligned}&\left[1+a_1\frac{\mathrm{d}}{\mathrm{d}t}\right]\sigma(t) = \left[m+b_1\frac{\mathrm{d}}{\mathrm{d}t}\right]\varepsilon(t) \\ &J(t) = J_g + J_1\left(1-e^{-t/\tau_\varepsilon}\right) \qquad J_g = \frac{a_1}{b_1},\quad J_1 = \frac{1}{m}-\frac{a_1}{b_1},\quad \tau_\varepsilon = \frac{b_1}{m} \\ &Y(t) = Y_e + Y_1 e^{-t/\tau_\sigma} \qquad Y_e = m,\quad Y_1 = \frac{b_1}{a_1}-m,\quad \tau_\sigma = \frac{b_1}{m}\end{aligned} \tag{91}$$

First we take usual Laplace transform of the constitutive equation

$$\tilde{\sigma}(s)+a_1 s\tilde{\sigma}(s)=m\tilde{\varepsilon}(s)+b_1 s\tilde{\varepsilon}(s) \qquad (1+a_1 s)\tilde{\sigma}(s)=(m+sb_1)\tilde{\varepsilon}(s)$$

$$\frac{\tilde{\varepsilon}(s)}{\tilde{\sigma}(s)}=\frac{1+a_1 s}{m+b_1 s} \tag{92}$$

The $J^*(s)$ is the strain-response for stress input as unit impulse $\sigma(t)=\delta(t)$ or $\tilde{\sigma}(s)=1$. Therefore we have the following

$$\begin{aligned} J^*(s) &= \frac{1+a_1 s}{m+b_1 s}=\frac{1}{m+b_1 s}+\frac{a_1 s}{m+b_1 s}=\frac{1}{m\left(1+\frac{b_1}{m}s\right)}+\frac{\frac{a_1}{b_1}s}{\left(\frac{m}{b_1}\right)\left(1+\frac{b_1}{m}s\right)} \\ &= \frac{1}{m\left(1+\frac{b_1}{m}s\right)}+\left(\frac{a_1}{b_1}\right)\left[\frac{\frac{b_1}{m}s}{1+\frac{b_1}{m}s}\right]=\frac{1}{m\left(1+\frac{b_1}{m}s\right)}+\frac{a_1}{b_1}\left(1-\frac{1}{1+\frac{b_1}{m}s}\right) \\ &= \frac{a_1}{b_1}+\left(\frac{1}{m}-\frac{a_1}{b_1}\right)\left(\frac{1}{1+\frac{b_1}{m}s}\right)=J_g+J_1\frac{1}{1+s\tau_\varepsilon} \end{aligned} \tag{93}$$

$$J_g=\frac{a_1}{b_1}, \quad J_1=\left(\frac{1}{m}-\frac{a_1}{b_1}\right), \quad \tau_\varepsilon=\frac{b_1}{m}$$

Therefore;

$$\begin{aligned} J^*(\omega) &= J_g+J_1\frac{1}{1+s\tau_\varepsilon}\bigg|_{s=i\omega} \\ &= J_g+J_1\frac{1}{1+i\omega\tau_\varepsilon}=\left(J_g+\frac{J_1}{1+\omega^2\tau_\varepsilon^2}\right)-i\frac{J_1\omega\tau_\varepsilon}{1+\omega^2\tau_\varepsilon^2}=J'(\omega)-iJ''(\omega) \end{aligned} \tag{94}$$

$$J'(\omega)=J_g+J_1\frac{1}{1+\omega^2\tau_\varepsilon^2} \quad \text{and} \quad J''(\omega)=J_1\frac{\omega\tau_\varepsilon}{1+\omega^2\tau_\varepsilon^2} \tag{95}$$

Similarly we can have the expression for $Y^*(\omega)$

$$\begin{aligned} & Y^*(\omega)=Y^*(s)\big|_{s=i\omega}=s\tilde{Y}(s)\big|_{s=i\omega}=Y_e+Y_1\frac{s\tau_\sigma}{1+s\tau_\sigma}\bigg|_{s=i\omega}=Y'(\omega)+iY''(\omega) \\ & Y'(\omega)=Y_e+Y_1\frac{\omega^2\tau_\sigma^2}{1+\omega^2\tau_\sigma^2} \quad \text{and} \quad Y''(\omega)=Y_1\frac{\omega\tau_\sigma}{1+\omega^2\tau_\sigma^2} \end{aligned} \tag{96}$$

We now have change of variables as

$$\tau \triangleq \sqrt{\tau_\varepsilon\tau_\sigma}$$

$$\Delta \triangleq \frac{\tau_\varepsilon-\tau_\sigma}{\sqrt{\tau_\varepsilon\tau_\sigma}}=\frac{\tau_\varepsilon-\tau_\sigma}{\tau}=\begin{cases}\dfrac{J_e-J_g}{\sqrt{J_eJ_g}} \\ \dfrac{Y_g-Y_e}{\sqrt{Y_gY_e}}\end{cases} \tag{97}$$

And write tan delta or loss tangent as

$$\tan[\phi(\omega)] = \frac{J''(\omega)}{J'(\omega)} = \frac{Y''(\omega)}{Y'(\omega)} = \frac{\Delta\omega\tau}{1+(\omega\tau)^2} \tag{98}$$

This loss tangent gets maximum value at $\omega = \tau^{-1}$ with value $\Delta/2$.

16. Caputo's notation and derivation of dynamic functions and loss tangent

We have Caputo's notation in order that when we deal with fractional models of visco-elastic systems, this will be helpful

$$\alpha \triangleq \frac{1}{\tau_\varepsilon} = \frac{m}{b_1} \quad \text{and} \quad \beta \triangleq \frac{1}{\tau_\sigma} = \frac{1}{a_1} \qquad 0 < \alpha < \beta < \infty \tag{99}$$

Consequently the Zenner system becomes

$$\left[1 + \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}t}\right]\sigma(t) = m\left[1 + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\right]\varepsilon(t) \qquad m = Y_e = Y_g\frac{\alpha}{\beta} \tag{100}$$

With same procedures we find that

$$Y^*(\omega) = Y_e\frac{1+i\dfrac{\omega}{\alpha}}{1+i\dfrac{\omega}{\beta}} = Y_g\frac{\alpha+i\omega}{\beta+i\omega} \qquad Y'(\omega) = Y_g\frac{\omega^2+\alpha\beta}{\omega^2+\beta^2}; \quad Y''(\omega) = Y_g\frac{\omega(\beta-\alpha)}{\omega^2+\beta^2} \tag{101}$$

Finally

$$\tan[\phi(\omega)] = \frac{Y''(\omega)}{Y'(\omega)} = (\beta-\alpha)\frac{\omega}{\omega^2+\alpha\beta} \tag{102}$$

The figure-4 depicts above quantities

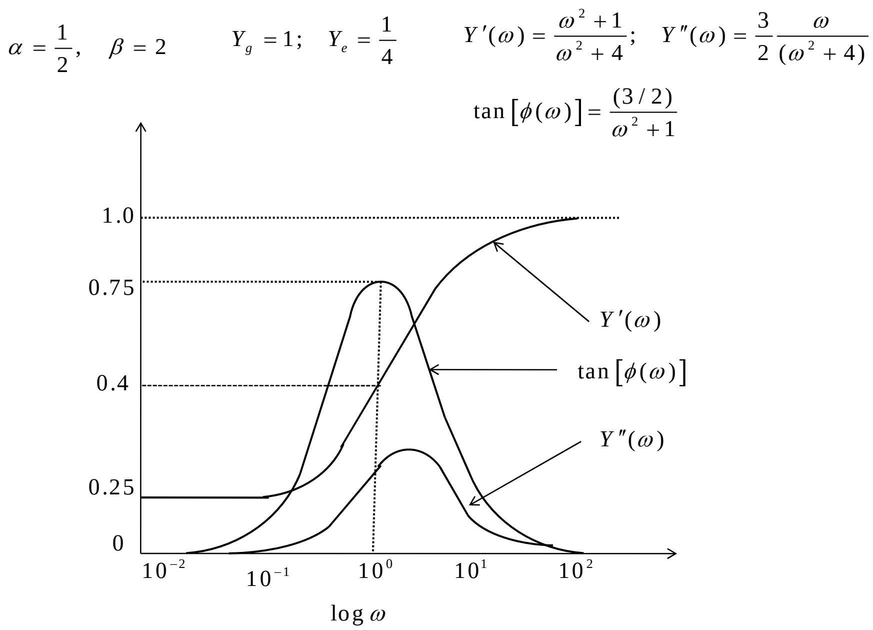

Figure-4 Plot for tan delta, storage and loss modulus

With respect to the Caputo's notation we write the following observation for various systems

Hooke	$\alpha = \beta = 0$	$\tan\left[\phi(\omega)\right] = 0$	Only energy storage
Newton	$0 = \alpha < \beta = \infty$	$\tan\left[\phi(\omega)\right] = \infty$	Only dissipation
Voigt	$0 < \alpha < \beta = \infty$	$\tan\left[\phi(\omega)\right] = \frac{\omega}{\alpha} = \omega\tau_{\varepsilon}$	Storage plus dissipation
Maxwell	$0 = \alpha < \beta < \infty$	$\tan\left[\phi(\omega)\right] = \frac{\beta}{\omega} = \frac{1}{\omega\tau_{\sigma}}$	Storage plus dissipation

Table-3: Various types of visco-elastic system with loss tangent

There were no fractional viscoelastic systems in this lecture-B series, from this background we go to make theories in a formal way for fractional visco-elasticity.

End of Part-B