Lecture Notes

Fractional Viscoelasticity Part-A

"Generalized Theory of Viscoelasticity with Fractional Differential Equation"

for
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Generalized Theory of Viscoelasticity with Fractional Differential Equation

We have linear visco-elastic models defined by Linear Integer Order Differential equation relating force per area called stress to deformation called strain. How the material dissipate mechanical energy the excitation of stress by undergoing deformation or by relaxing the stress when it is subjected to externally deformed, is no doubt the property of material, subjected to certain thermodynamic consideration. This mechanism may be very complex and may call for explanations through some other means other than combination of Hookian, Newtonian, Maxwellian, Zener, Burger, Kelvin, Voigt, models, based on integer order differential equations. Therefore generalization is done here to explain the anomalous behavior of the anomalous material, via fractional differential equation governed by Riemann-Liouvelli and/or Caputo fractional derivatives. In the middle of the 19th century Becquerel discovered radioactive decay, he did not use the pure exponential function to explain decay law-instead used a power law function in time-a compressed hyperbola! Therefore Becquerel's function of decay was anomalous having various decay rates distributed in some way as probability density function. In the visco-elastic experiments we do observe this type of non-exponential decay having several decay rates thus distributed in some form. What is the distribution function for decay rates is calculated here by means of Berberan-Santo method of Laplace inversion 'without going in for contour integration'. Well to take inverse Laplace of time function or decay function we get the distribution of decay rates. In the experiments we fit our results with Mittag-Leffler function, which arises due to fractional differential equation as constituent law. The 20-30 iterations blow this function out, though this function monotonically converges; that is due to Gamma function overflows. With this method of Berberan-Santo we arrive at integral representation of Mittag-Leffler function by inverting its Laplace transform, without contour integration; and this integral representation can be solved via numerical integration, without involving the Gamma function. With this method one gets distribution function for the decay rates for the anomalous relaxation-with power law; when the constitutive equation is of fractional order. This lecture takes one to different aspects of fractional calculus and generalized system of viscoelasticity, and several generalized ways to get solution of fractional differential equation.

Contents

1. The anomalous materials

The discovery of "new-materials" beginning in the middle of the 19^{th} century, such as plastics, rubbers, fibers, resins and polymers, have provided a set of materials whose mechanical behavior cannot be explained via the classical rheological means, such as those contained in the theories of Kelvin or Maxwell. The field of rheology studies among the other things, the flow and deformation of materials with unusual behavior, particularly the flow of non-Newtonian fluids, the behavior of solids which can flow, or the flow of substances across porous media with fractal geometry. Visco-elasticty is an intrinsic property of many of these above mentioned "new-material", and is also constitutes an element of study within rheology. This property describes varying degree of behavior between elasticity and viscosity, depending on the material. The figure-1 describes the various classical models.

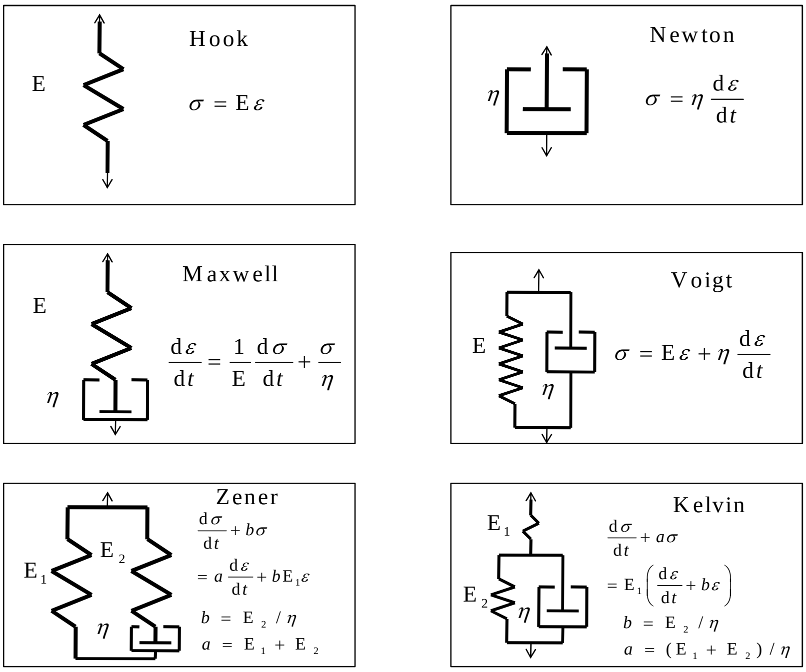


Figure-1: Various standard models of visco-elastic material

Depending on the temperature of the materials, these "new-materials" will achieve in general, four different states:

a. Vitreous state (in the vitreous region): this region contains organic crystals which allow small deformations.
b. Transition states (in the transition region): this region contains, among others, linear thermoplastic and reticular polymers (cellulosic material, polyamides, polyesters etc.)
c. Elastic state (in the elastic region): this region contains elastomeric materials (rubber, plastics etc.)
d. Fluid state (in the influence region).

At the ambient temperature a given polymer may be found in one of these four regions. Here we will consider the models which attempt to explain the visco-elastic deformation of certain materials.

2. Differential equation model to generalize the visco-elastic properties of anomalous material

Let us take the Kelvin model of figure-1. The total strain (deformation) ε is divided by a spring E_1 with deformation ε_1, and strain (deformation) ε_2 by parallel combination of spring

E_2 and piston η. So we have $\varepsilon = \varepsilon_1 + \varepsilon_2$. The force/surface acting on the system i.e. σ is same for both the deforming elements. We write the equation for spring E_1 as $\sigma(t) = E_1\varepsilon_1(t)$, and for the parallel connected spring E_2 and piston η as $\sigma(t) = E_2\varepsilon_2(t) + \eta\dot{\varepsilon}_2(t)$. We assume E_1, E_2, η are constants. Taking Laplace transforms of the above two equations assuming zero initial condition, we get; $\hat{\sigma}(s) = E_1\hat{\varepsilon}_1(s)$ and $\hat{\sigma}(s) = [E_2 + s\eta]\hat{\varepsilon}_2(s)$. Multiplying the first equation by $(E_2 + s\eta)$ and second equation by E_1; and then adding up the two equations, we have (1)

$$\begin{aligned} &(E_2 + s\eta)\hat{\sigma}(s) + E_1\hat{\sigma}(s) = (E_2 + s\eta)E_1\hat{\varepsilon}_1(s) + E_1(E_2 + s\eta)\hat{\varepsilon}_2(s) = E_1(E_2 + s\eta)(\hat{\varepsilon}_1(s) + \hat{\varepsilon}_2(s)) \\ &(E_2 + s\eta)\hat{\sigma}(s) + E_1\hat{\sigma}(s) = E_1(E_2 + s\eta)\hat{\varepsilon}(s) \end{aligned} \tag{1}$$

Taking Laplace inverse of the above (1), we obtain (2)

$$\begin{aligned} &(E_1 + E_2)\sigma(t) + \eta\frac{d\sigma(t)}{dt} = E_1E_2\varepsilon(t) + E_1\eta\frac{d\varepsilon(t)}{dt} \\ &a\sigma(t) + \frac{d\sigma(t)}{dt} = E_1\left[b\varepsilon(t) + \frac{d\varepsilon(t)}{dt}\right] \qquad a = \frac{E_1 + E_2}{\eta}; \quad b = \frac{E_2}{\eta} \end{aligned} \tag{2}$$

For the Zener model in figure-1, the deformation is ε common to both the elements i.e. the spring E_1 and parallel connected spring E_2 and dashpot η. The force/surface σ gets shared as σ_1, with equation $\sigma_1 = E_1\varepsilon$ and σ_2, with equation $(d\varepsilon/dt) = (1/E_2)(d\sigma_2/dt) + (\sigma_2/\eta)$; where $\sigma = \sigma_1 + \sigma_2$. The Laplace of these two equations are $\hat{\sigma}_1 = E_1\hat{\varepsilon}$ and $\hat{\sigma}_2 = (s\eta\hat{\varepsilon})/(s\eta + E_2)$. From here we add and get the following (3)

$$\begin{aligned} &\hat{\sigma} = \hat{\sigma}_1 + \hat{\sigma}_2 \\ &(s\eta + E_2)\hat{\sigma} = (s\eta E_1 + s\eta E_2 + E_1E_2)\hat{\varepsilon} \end{aligned} \tag{3}$$

Now we take Laplace inverse of above (3) to get (4)

$$\frac{d\sigma}{dt} + b\sigma = a\frac{d\varepsilon}{dt} + bE_1\varepsilon \qquad a = E_1 + E_2, \quad b = E_2/\eta \tag{4}$$

Many rheological models have been developed to study the behavior. In general we have the following model (5)

$$\sum_{k=0}^{n} a_k \frac{d^k\sigma}{dt^k} = \sum_{k=0}^{m} b_k \frac{d^k\varepsilon}{dt^k} \tag{5}$$

Next, we describe Burger's model (figure-2)

(i) If a sample of an amorphous polymer is subjected to a constant tension, an instantaneous deformation will be observed when the force is applied. If the force is immediately relaxed, the deformation disappears. This state is called the "elastic domain". Mechanically, this behavior is represented with a spring of modulus E_1, which describes the instantaneous elastic response.

$$\sigma(t) = E_1\varepsilon(t) \tag{6}$$

This is Hooke's law, where $\sigma(t)$ represents the tension (force/surface) and $\varepsilon(t)$ represents unit deformation ($\Delta L / L$).

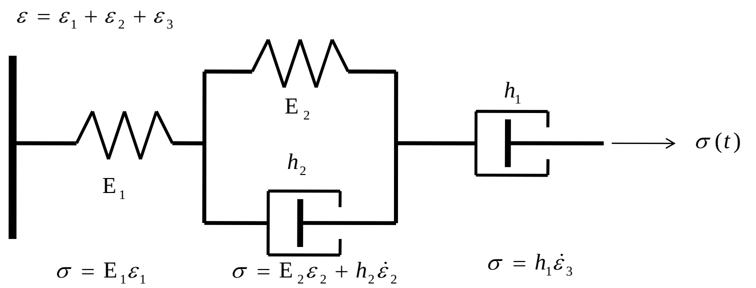
(ii) As the tension is prolonged we witness a transition to a "deformation as a function of time". The deformation slows until it progresses at a constant speed. This mechanism is represented as being made up of a "Voigt type" element, which corresponds to a "delayed elasticity", (represented via spring E_2 and piston h_2). When the tension is removed, the material begins to return to its original shape but the recovery slows and finally disappears before it is completed. Mechanically this behavior is represented by parallel grouping of a spring and dashpot (piston), i.e.

$$\sigma(t) = \mathrm{E}_2\varepsilon(t) + h_2 \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} \tag{7}$$

(iii) Lastly a piston (dashpot) represented as h_1 represents viscous fluid, and expressed as laws' of Newtonian fluid

$$\sigma(t) = h_1 \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} \tag{8}$$

B u r g e r



$$\mathcal{L}\{\varepsilon\} = \hat{\varepsilon}, \quad \mathcal{L}\{\sigma\} = \hat{\sigma}$$

$$\hat{\varepsilon} = \hat{\varepsilon}_1 + \hat{\varepsilon}_2 + \hat{\varepsilon}_3 = \frac{\hat{\sigma}}{\mathrm{E}_1} + \frac{\hat{\sigma}}{sh_2 + \mathrm{E}_2} + \frac{\hat{\sigma}}{sh_1} = \frac{s^2 h_1 h_2 + s(h_1\mathrm{E}_2 + h_1\mathrm{E}_1 + h_2\mathrm{E}_1) + \mathrm{E}_1\mathrm{E}_2}{s^2 h_1 h_2 \mathrm{E}_1 + sh_1\mathrm{E}_1\mathrm{E}_2}\hat{\sigma}$$

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}\sigma}{\mathrm{d}t} + a_0\sigma = b_2\frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2} + b_1\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + b_0\varepsilon$$

$$a_1 = \frac{\mathrm{E}_1 + \mathrm{E}_2}{h_2} + \frac{\mathrm{E}_1}{h_1}; \quad a_0 = \frac{\mathrm{E}_1\mathrm{E}_2}{h_1 h_2}; \quad b_2 = \mathrm{E}_1; \quad b_1 = \frac{\mathrm{E}_1\mathrm{E}_2}{h_2}; \quad b_0 = 0$$

Figure-2 Burger's model

Therefore, this Burger's model of figure-2 has the following (9) mathematical representation

$$a_2\sigma''(t) + a_1\sigma'(t) + a_0\sigma(t) = b_2\varepsilon''(t) + b_1\varepsilon'(t) + b_0\varepsilon(t) \tag{9}$$

where $a_2 = 1;\ a_1 = \frac{\mathrm{E}_1 + \mathrm{E}_2}{h_2} + \frac{\mathrm{E}_1}{h_1};\ a_0 = \frac{\mathrm{E}_1\mathrm{E}_2}{h_1 h_2};\ b_2 = \mathrm{E}_1;\ b_1 = \frac{\mathrm{E}_1\mathrm{E}_2}{h_2};\ b_0 = 0$

Therefore ordinary linear model was widely studied; e.g.

$$\left[1 + \sum_{k=1}^{p} a_k D^k\right]\sigma(t) = \left[b_0 + \sum_{k=1}^{q} b_k D^k\right]\varepsilon(t) \tag{10}$$

where $p = q$ or $p = q + 1$. This model represents a series or parallel combination of basic Zener or Kelvin type units.

3. Fractional Derivatives Riemann-Liouville (RL) Left Hand Definition (LHD)

Fractional Integration Riemann-Liouville (RL) is generalization of *n*-fold integer order repeated integration. The repeated *n*-fold integration is generalized by Gamma function for the factorial expression, when the integer n is real number α, with property of Gamma function as $\Gamma(n) = (n-1)!$ as follows

$$
\begin{aligned}
{}_0D_t^{-n} f(t) &= {}_0I_t^{\,n} f(t) = f_n(t) = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\mathrm{d}\tau \qquad n \in \mathbb{N} \\
{}_0D_t^{-\alpha} f(t) &= {}_0I_t^{\,\alpha} f(t) = f_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau \qquad \alpha \in \mathbb{R}^+
\end{aligned} \tag{11}
$$

The formulation of this definition of RL fractional derivative is:
Select an integer *m* greater than fractional number α

- (i) Integrate the function $(m-\alpha)$ folds by RL fractional integration method.
- (ii) Differentiate the above result by *m*.

Expression is given as:

$$
{}_0D_t^{\alpha} f(t) = ({}_0D_t^{\,m}\, {}_0I_t^{\,m-\alpha} f)(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau\right] \qquad m \in \mathbb{N}, \quad \alpha \in \mathbb{R}^+ \tag{12}
$$

In this LHD the limit of integration is from 0 to t. We thus denote the derivative by notation ${}_0D_t^{\alpha} f(t)$. In fractional calculus we find limit of derivative-i.e. derivatives are taken in interval. We call this as 'forward derivative' ${}_0D_{t,+}^{\alpha} f(t)$. Now if the limits of integration are changed to (t to 0) the derivative is denoted as ${}_tD_0^{\alpha} f(t)$; ${}_tD_{0,-}^{\alpha} f(t)$ the 'backward derivative'. The backward derivative is related to forward derivative by

$$
{}_tD_0^{\alpha} f(t) = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m}\, {}_tI_0^{m-\alpha} f(t) \tag{13}
$$

Therefore in order to obtain fractional derivative of a function at a point (say 0) we should have the values of these two derivatives same: forward derivative should equal the backward derivative. This implies not only one should know the function from past to the point of interest (say 0) but also the function should be known into the future-in order to have point fractional derivative at a point!

4. Fractional Derivatives Caputo Right Hand Definition (RHD)

The formulation is exactly opposite to LHD.
Select an integer *m* greater than fractional number

- (i) Differentiate the function *m* times.
- (ii) Integrate the above result $(m-\alpha)$ fold by RL fractional integration method.

In LHD and RHD the integer selection is made such that $(m-1) < \alpha < m$. For example differentiation of the function by order π will select $m = 4$. The formulation of RHD Caputo is as follows:

$$
{}_0^CD_t^{\alpha} f(t) = ({}_0I_t^{\,m-\alpha}\, {}_0D_t^{\,m} f)(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{\frac{\mathrm{d}^m f(\tau)}{\mathrm{d}t^m}}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau \tag{14}
$$

The definitions of Riemann-Liouville of fractional differentiation played an important role in development of fractional calculus. However the demands of modern science and engineering require a certain revision of the well established pure mathematical approaches. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable "initial conditions" which contain $f(a), f^{(1)}(a), f^{(2)}(a)$ and not fractional quantities (presently unthinkable!). The RL definitions require

$$\lim_{t\to a} {}_aD_t^{\alpha-1} f(t) = b_1$$
$$\lim_{t\to a} {}_aD_t^{\alpha-2} f(t) = b_2 \tag{15}$$

In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically, their solutions are practically useless, because there is no known physical interpretation for such initial conditions, presently. It is hard to interpret.

RHD is more restrictive than LHD. For RL $f(t)$ is causal. For LHD as long as initial function of t satisfies $f(0)=0$. For RHD because $f(t)$ is first made to m-th derivative i.e. $f^{(m)}(t)$, the condition $f(0)=0$ & $f^1 = f^2 = ...f^m = 0$ is required. In mathematical world this is vulnerable for RHD may be deliberating. For LHD $D^\alpha C \neq 0 = \dfrac{Ct^{-\alpha}}{\Gamma(1-\alpha)}$, the derivative of constant C is not zero. This fact led to using the RL or LHD approach with lower limit of differentiation $a \to -\infty$ in physical world this posses problem. The physical meaning of this lower limit extending towards minus infinity is starting of physical process at time immemorial! In such cases transient effects cannot be then studied. However making $a \to -\infty$ is necessary abstraction for consideration of steady state process, for example for study of sinusoidal analysis for steady state fractional order system.

5. Laplace transforms of RL and Caputo derivative and generalized stationary conditions or initial conditions

The meaning of type of fractional derivative is Caputo type and Riemann-Liouvelli (RL) type of fractional derivative definition. We have generalized the two definitions with type parameter β . The generalized Laplace transform in integer order theory is:

$$\mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0^+) \tag{16}$$

where $n \in \mathbb{N}$ natural number. Following this we have generalized the above for fractional order α , for Riemann-Liouvelli derivative as:

$$\begin{aligned}
\mathcal{L}\left\{{}_0D_x^\alpha f(x)\right\} &= \mathcal{L}\left\{\frac{\mathrm{d}^n}{\mathrm{d}x^n}\, {}_0I_x^{n-\alpha} f(x)\right\} \\
&= s^n \mathcal{L}\left\{{}_0I_x^{n-\alpha} f(x)\right\} - \sum_{k=0}^{n-1} s^k \frac{\mathrm{d}^{n-1-k}}{\mathrm{d}x^{n-1-k}}\left[{}_0I_x^{n-\alpha} f(0^+)\right] \\
&= s^n s^{-n+\alpha} F(s) - \sum_{k=0}^{n-1} s^k \mathrm{d}_x^{\;n-1-k} \mathrm{d}_x^{\;-n+\alpha} f(x) \;\Big|_{x=0^+} \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^k\, {}_0\mathrm{d}_x^{\alpha-1-k} f(x) \;\Big|_{x=0+}
\end{aligned} \tag{17}$$

Here $n-1 < \alpha < n$.

For $\alpha = 0.7$, we will have the following Laplace transformation of RL derivative operator

$$\mathcal{L}\left\{{}_0D_x^{0.7} f(x)\right\} = s^{0.7}F(s) - \left[{}_0\mathrm{d}_x^{-0.3} f(x)\right]_{x=0^+} = s^{0.7}F(s) - \left[{}_0I_x^{0.3} f(x)\right]_{x=0^+}$$

In above if we have initial condition defined as $\lim_{x\to 0^+} I^{1-\alpha} f(x) = C$ where C is constant; that means fractional derivative of order $1-\alpha = 0.3$ for the constant at limit of initial point is what is required i.e.; $\lim_{x\to 0+} f(x) = {}_0D_x^{1-\alpha} C = Cx^{-(1-\alpha)} / \Gamma(\alpha)$ or we have to state that at initial point in the limit we should have $\lim_{x\to 0+} x^{1-\alpha} f(x) = C/\Gamma(\alpha)$ as initial condition. Therefore if the start point is $x=a$; we have the following $\lim_{x\to a+} (x-a)^{1-\alpha} f(x) = C/\Gamma(\alpha) = k \quad 0<\alpha\le 1$ as initial condition requirement, with k a constant, for a RL derivative. The initial condition is limiting value of fractional integration of function to be a constant at the initial point, for α-order differential equation with $0<\alpha\le 1$.

For $\alpha = 1.7$, we have the following Laplace expression with two initial conditions required as

$$\begin{aligned}\mathcal{L}\left\{{}_0D_x^{1.7} f(x)\right\} &= s^{1.7}F(s) - {}_0\mathrm{d}_x^{0.7} f(x)\ \Big|_{x=0+} - s\left[{}_0\mathrm{d}_x^{-0.3} f(x)\ \Big|_{x=0+}\right] \\ &= s^{1.7}F(s) - {}_0\mathrm{d}_x^{0.7} f(x)\ \Big|_{x=0+} - s\left[{}_0I_x^{0.3} f(x)\right]_{x=0+}\end{aligned} \tag{18}$$

The initial conditions are fractional differ-integrals in nature, and those are $\lim_{x\to 0+}\left[{}_0\mathrm{d}_x^{0.7} f(x)\right] = C_1$; $\lim_{x\to 0+}\left[{}_0I_x^{0.3} f(x)\right] = C_2$. The above (18) is similar to $\mathcal{L}\left\{{}_0D_x^2 f(x)\right\} = s^2F(s) - f'(0+) - s[f(0+)]$, requiring conditions for $f(x)$ and $f'(x)$ at start point and here it is $x=0$.

For Caputo derivative the Laplace can be derived using the above method as:

$$\begin{aligned}\mathcal{L}\left\{{}_0^CD_x^{\alpha} f(x)\right\} &= \mathcal{L}\left\{{}_0I_x^{n-\alpha}\ {}_0D_x^n f(x)\right\} = s^{-(n-\alpha)}\mathcal{L}\left\{{}_0D_x^n f(x)\right\} \\ &= s^{-(n-\alpha)}\left[s^nF(s) - \sum_{k=0}^{n-1} s^k f^{(n-1-k)}(0+)\right] \\ &= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{k-n+\alpha} f^{(n-1-k)}(0+)\end{aligned} \tag{19}$$

Say we have $n=2$, and then we get from above, the Laplace of Caputo derivative operator as $\mathcal{L}\left\{{}_0^CD_x^{\alpha} f(x)\right\} = s^{\alpha}F(s) - s^{\alpha-2} f^{(1)}(0+) - s^{\alpha-1} f(0+)$; which we can rewrite as following

$$\mathcal{L}\left\{{}_0^CD_x^{\alpha} f(x)\right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+) \tag{20}$$

The essential difference between the two is RL requires ${}_0D_x^{\alpha-1-k} f(0^+)$, $k = 0,1,2...(n-1)$ number fractional initial states, the Caputo requires $f^{(k)}(0^+)$, $k=0,1,2,...(n-1)$ integer order initial states. One may generalize the above two as the fractional derivative as solution $g(x)$ of integral equation

$${}_0I_x^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1} g(u)\mathrm{d}u = f(x) \tag{21}$$

In case it exists. Let us denote the solution by ${}_0^{*}D_x^{\alpha} f(x)$ then following assertions are equivalent for $n-1\le\alpha\le n$ where $n\in\mathbb{N}$ and $\alpha\in\mathbb{R}^+$

a) $s^{\alpha}F(s) = G(s)$

b) $g(x) = {}_0^{*}D_x^{\alpha} f(x)$

c) $\dfrac{d^k\left[{}_0I_x^{n-\alpha} f(x)\right]}{dx^k}$ are equal to ZERO at $x=0$ for $k=0,1,2,...(n-1)$ together with ${}_0D_x^{\alpha} f(x) = g(x)$, the Riemann-Liouvelli derivative.

d) $f^{(k)}(x)$ are equal to ZERO at $x=0$ for $k=0,1,2,3,...(n-1)$, together with ${}_0^CD_x^{\alpha} f(x) = g(x)$, the Caputo derivative.

We have seen two types of fractional derivative and discussed them in above sections; those are Riemann-Liouvelli (RL) and Caputo. Following formula gives fractional derivative definition with in between RL-Caputo type, with β as type defining parameter as $0 \le \beta \le 1$. When $\beta = 1$ the definition is Caputo definition with symbol, ${}^{C}D_t^{\alpha} f(t)$ while $\beta = 0$ gives RL fractional derivative with symbol $D_t^{\alpha} f(t)$. The generalized definition is as following:

$$ {}_0^{\beta}D_t^{\alpha} f(t) = {}_0I_t^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t}\left[{}_0I_t^{(1-\beta)(1-\alpha)} f(t) \right] \tag{22} $$

The α is the order of derivative, here in this definition is $0 < \alpha < 1$ and β is the type of derivative with $0 \le \beta \le 1$. Therefore the nearest integer is $m = 1$ in the above generalization.

With type parameter $0 \le \beta \le 1$ we generalized the definition in Laplace transform, for fractional order derivative of order $0 < \alpha < 1$ as:

$$ \mathcal{L}\left\{ {}_a^{\beta}D_t^{\alpha} f(t) \right\} = s^{\alpha} F(s) - s^{\beta(\alpha-1)} \left[{}_0D_t^{(1-\beta)(\alpha-1)} f(t) \right]_{t=0^+} \tag{23} $$

Where ${}_0D_t^{(1-\beta)(\alpha-1)} f(t)$ is Riemann-Liouvelli fractional derivative type $\beta = 0$.

For, fractional order $0 < \alpha < 1$, and for any type, β we define fractional derivative as:

$$ {}_a^{\beta}D_x^{\alpha} f(x) = {}_aI_x^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} {}_aI_x^{(1-\beta)(1-\alpha)} f(x) \tag{24} $$

As an example consider the differential equation of order $0 < \alpha < 1$, with type $0 \le \beta \le 1$, and with C a constant. The given initial condition is ${}_0I_t^{(1-\beta)(1-\alpha)} f(0^+) = f_0$, a constant.

$$ {}_0^{\beta}D_t^{\alpha} f(t) = C \tag{25} $$

Using generalized Laplace identity as obtained above we write the Laplace transformed equation, recognizing that ${}_0I_t^{(1-\beta)(1-\alpha)} \equiv {}_0D_t^{(1-\beta)(\alpha-1)}$; as following.

$$ s^{\alpha} F(s) - s^{\beta(\alpha-1)} f_0 = \frac{C}{s} \tag{26} $$

Giving

$$ F(s) = \frac{C}{s^{\alpha+1}} + \frac{f_0}{s^{\alpha+\beta(1-\alpha)}} \tag{27} $$

Inverting this (26) we get

$$ f(t) = \frac{Ct^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_0 t^{(1-\beta)(\alpha-1)}}{\Gamma\left[(1-\beta)(1-\alpha)+1\right]} \tag{28} $$

For RL type with $\beta = 0$, we have solution

$$ f(t) = \frac{Ct^{\alpha}}{\Gamma(\alpha+1)} + \frac{f_0 t^{(\alpha-1)}}{\Gamma(2-\alpha)} \tag{29} $$

For Caputo type with $\beta = 1$, we have solution

$$ f(t) = \frac{Ct^{\alpha}}{\Gamma(\alpha+1)} + f_0 \tag{30} $$

Consider another example as generalized fractional relaxation equation as:

$$ {}_0^{\beta}D_t^{\alpha} f(t) = -Cf(t) \tag{31} $$

With given, initial condition as $[{}_0I_t^{(1-\beta)(1-\alpha)} f(t)]_{t=0^+} = f_0$ with C as 'relaxation-constant'. With $0 < \alpha < 1$ and $0 \le \beta \le 1$. Using the generalized Laplace identity as in above example we get:

$$F(s)=\frac{s^{\beta(\alpha-1)}f_0}{s^\alpha+C} \tag{32}$$

To get Laplace inverse re-write the above (31) expression as:

$$F(s)=\frac{s^{\alpha-\gamma}}{C+s^\alpha}f_0=f_0 s^{-\gamma}\frac{1}{Cs^{-\alpha}+1}=f_0\sum_{k=0}^{\infty}(-C)^k s^{-\alpha k-\gamma} \tag{33}$$

where $\gamma=\alpha+\beta(1-\alpha)$.

Using $\mathcal{L}^{-1}\left\{x^{\alpha-1}/\Gamma(\alpha)\right\}=s^{-\alpha}$, and applying this term by term we obtain the response as:

$$f(t)=f_0 t^{\gamma-1}\sum_{k=0}^{\infty}\frac{(-Ct^\alpha)^k}{\Gamma(\alpha k+\gamma)}=f_0 t^{(1-\beta)(\alpha-1)}E_{\alpha,\alpha+\beta(1-\alpha)}(-Ct^\alpha) \tag{34}$$

Here we have used the definition of 'two parameter Mittag-Leffler function' $E_{a,b}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(ak+b)}$, for all $a>0$ and $b\in\mathbb{C}$. This is an entire function of order $1/a$, and completely monotone for $0<a\le 1$ and $b\ge a$.

1) For $C=0$ the result is $f(t)=\frac{f_0 t^{(1-\beta)(\alpha-1)}}{\Gamma\left[(1-\beta)(\alpha-1)+1\right]}$, because $E_{a,b}(0)=\frac{1}{\Gamma(b)}$. In this case if; $\beta=0$,that is RL case, the solution is $f(t)=f_0 t^{\alpha-1}/\Gamma(\alpha)$.

   Take the ${}_0D_t^\alpha\left[t^{\alpha-1}\right]$, which is $\Gamma(\alpha)t^{-1}/\Gamma(0)$ is zero. For $\beta=1$ the Caputo case, the solution is $f(t)=f_0$, revealing Caputo fractional derivative of constant is zero.
2) For $\beta=1$ Caputo derivative type, $f(t)=f_0 E_\alpha(-Ct^\alpha)$, where $E_\alpha(x)=E_{\alpha,1}(x)$, denotes the ordinary 'one parameter Mittag-Leffler function'. For RL case with $\beta=0$, the solution is $f(t)=f_0 t^{\alpha-1}E_{\alpha,\alpha}(-Ct^\alpha)$.

These examples make us to think about the concept of 'stationarity' in fresh way, in fractional calculus domain.

Consider a fractional 'stationary' differential equation with order $0<\alpha<1$ and type $0\le\beta\le 1$ as ${}_0^\beta D_t^\alpha f(t)=0$ with initial condition as ${}_0D^{(1-\beta)(\alpha-1)}f(0^+)=f_0$, fractional Riemann-Liouvelli derivative initial condition as stationary. We can re-write as and state fractional integral initial value given as constant that is ${}_0I_t^{(1-\beta)(1-\alpha)}f(0^+)=f_0$. For $\alpha=1$, the conventional definition of stationary condition is recovered as ${}_0^\beta D_t^1 f(t)=0$, for any type β; with initial condition as $f(0^+)=f_0$. The solution of this fractional stationary differential equation is:

$$f(t)=\frac{f_0 t^{(1-\beta)(\alpha-1)}}{\Gamma\left[(1-\beta)(\alpha-1)+1\right]} \tag{35}$$

This may be seen by inserting the above $f(t)$ expression into, the generalized definition of fractional derivative as;

$${}_a^\beta D_x^\alpha f(x)={}_aI_x^{\beta(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\,{}_aI_x^{(1-\beta)(1-\alpha)}f(x) \tag{36}$$

for $0<\alpha<1$ and using the fractional integral of power function expression ${}_aI_x^\alpha(x-a)^\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}(x-a)^{\alpha+\lambda}$.

For RL case with $\beta = 0$ the function $f(t) = f_0 t^{\alpha-1} / \Gamma(\alpha)$ and for Caputo case with $\beta = 1$, the function is $f(t) = f_0$ a constant. Note that the integral ${}_0 I_t^{(1-\beta)(1-\alpha)} f(t) = f_0$ remains conserved and constant for all t ; while the function $f(t)$ itself varies. In particular $\lim_{t\to 0} f(t) = \infty$ and $\lim_{t\to\infty} f(t) = 0$. For $\beta = 1$ and $\alpha = 1$ one recovers $f(t) = f_0$ as usual. Fractional derivative of constant for Caputo type is zero, and integer order derivative of constant of any type is zero.

The new type of stationary states for which a fractional integral (rather than function itself) is constant is arrived at. It seems rather lack of knowledge about fractional stationarity is particularly responsible for this difficulty of deciding which type of fractional derivative should be used when generalizing traditional equation of motion thermodynamics electrodynamics control system and others. Nevertheless let us carry on with this dichotomy.

6. Fractional generalization for modeling anomalous visco-elasticity and definitions of creep-compliance, (retarded response) relaxation modulus (relaxation function), complex-compliance and transfer function

These integer order differential equation models do not adjust themselves well to the behavior demonstrated by many visco-elastic materials, as they are governed by anomalous dynamics. The first way is to generalize the classical model, where the tension (stress) could be proportional to the Riemann-Liouvelli (RL) fractional derivative of the deformation strain, that is

$$\sigma(t) = \mathrm{E}_\alpha \left(D_+^\alpha \varepsilon \right)(t) \qquad 0 < \alpha < 1 \tag{37}$$

This generalization is based on RL derivative D_{0+}^α, but could be a Caputo's derivative ${}^C D_{0+}^\alpha$. Which of these derivatives nature follows is an interesting question? The RL derivative requires to be operated on a function which is only a continuous everywhere, but Caputo's requirement is that the function need be differentiable too in the entire interval. Well why these generalizations are called for, is based on the experimental observations verified that 'complex-compliance' of a visco-elastic material in relation to its frequency that is

$$G(i\omega) = \frac{\hat{\varepsilon}(i\omega)}{\hat{\sigma}(i\omega)} = \mathrm{Re}\left\{G(i\omega)\right\} + i\left\{\mathrm{Im}(G(i\omega)\right\} \tag{38}$$

Where $\hat{\sigma}(i\omega)$ and $\hat{\varepsilon}(i\omega)$ represents Fourier-transform (or the Laplace transform with real part of complex frequency taken as zero i.e. $\mathrm{Re}\{s\} = 0;\quad s = i\omega$ as the case may be); of $\sigma(t)$ and $\varepsilon(t)$ respectively, adjust better to non-integer power of the differential equation of order α ; $0 < \alpha < 1$, of the frequency ω than to natural (integer) orders as classically thought. The frequency domain will have Laplace variable $s = \mathrm{Re}\left\{s\right\} + i\,\mathrm{Im}\left\{s\right\} \equiv i\omega$; for steady state analysis we take $\mathrm{Re}\left\{s\right\} = 0$; and draw the frequency response as $\left|G(i\omega)\right| = \sqrt{\left(\mathrm{Re}\{G(i\omega)\right)^2 + (\mathrm{Im}\{G(i\omega)\})^2}$.

Let us consider the above fractional model. With $\alpha = 0$, we obtain Hook's material as $\sigma(t) = \mathrm{E}_0 \varepsilon(t)$, and with $\alpha = 1$, we obtain Newton's material with $\sigma(t) = \mathrm{E}_1 \dot{\varepsilon}(t)$. The 'creep-compliance' is time response of strain for a unit step load (stress), this is also termed as 'retarded response'. Thus when $\sigma(t) = 1$ for $t > 0$; we get its Laplace as $\hat{\sigma}(s) = 1/s$. For

Hook's material the strain (in frequency domain) is thus $\hat{\varepsilon}(s) = 1/\mathrm{E}_0 s$; inverting this we get 'creep-compliance' for Hook's material as $J(t) = 1/\mathrm{E}_0$. The relaxation modulus is the stress response in time when we have unit step strain to the system. For Hook's case we have 'relaxation-modulus' as $Y(t) = \mathrm{E}_0$. For Hook's case; 'complex-compliance' is $G(i\omega) = \hat{\varepsilon}(i\omega)/\hat{\sigma}(i\omega) = 1/\mathrm{E}_0$. Thus for Hook's case $\mathrm{Re}\{G(i\omega)\} = 1/\mathrm{E}_0$; $\mathrm{Im}\{G(i\omega)\} = 0$.

The same way we see that for Newton's material (in fractional model have $\alpha = 1$) with $\sigma(t) = \mathrm{E}_1\dot{\varepsilon}(t)$, $\hat{\sigma}(s) = s\mathrm{E}_1\hat{\varepsilon}(s)$; will have 'creep-compliance' as $J(t) = \mathcal{L}^{-1}\{(1/\mathrm{E}_1 s)(1/s)\} = \mathcal{L}^{-1}\{1/s^2\mathrm{E}_1\} = t/\mathrm{E}_1$. The 'relaxation-modulus' in the Newton's material is $Y(t) = \mathcal{L}^{-1}\{(s\mathrm{E}_1)(1/s)\} = \mathcal{L}^{-1}\{\mathrm{E}_1\} = \mathrm{E}_1\delta(t)$. The complex compliance of the Newton's material is $G(i\omega) = \hat{\varepsilon}(i\omega)/\hat{\sigma}(i\omega) = (i\omega\mathrm{E}_1)^{-1} = 0 - i(\omega\mathrm{E}_1)^{-1}$. Therefore, the $\mathrm{Re}\{G(i\omega)\} = 0$, $\mathrm{Im}\{G(i\omega)\} = -(\omega\mathrm{E}_1)^{-1}$.

For the fractional model as described by $\sigma(t) = \mathrm{E}_\alpha\left(D_+^\alpha\varepsilon\right)(t)$; we have $\hat{\sigma}(s) = s^\alpha\mathrm{E}_\alpha\hat{\varepsilon}(s)$; will have creep compliance $J(t) = \mathcal{L}^{-1}\{(1/\mathrm{E}_\alpha s^\alpha)(1/s)\} = \mathcal{L}^{-1}\{1/s^{\alpha+1}\mathrm{E}_\alpha\} = t^\alpha/[\Gamma(1+\alpha)\mathrm{E}_\alpha]$. This expression is also fractional integration of a constant i.e. ${}_0D_t^{-\alpha}[1/\mathrm{E}_\alpha] = (1/\mathrm{E}_\alpha)[t^\alpha/\Gamma(1+\alpha)]$; obtained by fractionally integrating by order $1 > \alpha > 0$ the constant stress of unity multiplied by a constant factor i.e. $1/\mathrm{E}_\alpha$.

The relaxation-modulus is $Y(t) = \mathcal{L}^{-1}\{(s^\alpha\mathrm{E}_\alpha)(1/s)\} = \mathcal{L}^{-1}\{s^{-(1-\alpha)}\mathrm{E}_\alpha\} = \mathrm{E}_\alpha\left[t^{-\alpha}/\Gamma(1-\alpha)\right]$. This is also termed as 'relaxation function'. In the above cases we used Laplace pair $t^n \leftrightarrow n!/s^{n+1}$; and the identity $n! = \Gamma(n+1)$ for $n \notin \mathbb{Z}$. Therefore we wrote $\mathcal{L}^{-1}\{1/s^{1-\alpha}\} = t^{-\alpha}/(-\alpha)! = t^{-\alpha}/\Gamma(-\alpha+1)$. We also mention that fractional derivative of order α for a constant function of unity value is ${}_0D_t^\alpha[1] = t^{-\alpha}/\Gamma(1-\alpha)$. Therefore the relaxation modulus can be also obtained by fractionally differentiating a constant strain.

For this fractional order material, the 'complex-compliance' is $G(i\omega) = \hat{\varepsilon}(i\omega)/\hat{\sigma}(i\omega) = 1/(i\omega)^\alpha\mathrm{E}_\alpha$. Writing $i = e^{i\pi/2}$; we have $i^{-\alpha} = e^{-i\alpha\pi/2}$, we get the following relations. Therefore, we have complex compliance as,

$G(i\omega) = (e^{-i\alpha\pi/2})(\omega^{-\alpha}\mathrm{E}_\alpha{}^{-1}) = \{[\cos(\alpha\pi/2)]/\omega^\alpha\mathrm{E}_\alpha\} - i\{[\sin(\alpha\pi/2)]/\omega^\alpha\mathrm{E}_\alpha\}$.

Having the real and imaginary parts as

$\mathrm{Re}\{G(i\omega)\} = [\cos(\alpha\pi/2)]/\omega^\alpha\mathrm{E}_\alpha$; $\mathrm{Im}\{G(i\omega)\} = -[\sin(\alpha\pi/2)]/\omega^\alpha\mathrm{E}_\alpha$.

The transfer function $X(s)$ is defined as ratio of frequency domain $\hat{\sigma}(s)/\hat{\varepsilon}(s)$ as Laplace variable's ratio. Technically it is stress-response of a system with impulse deformation where $\hat{\varepsilon}(s) = 1$. For example the fractional model the transfer-function is $X(s) = s^\alpha\mathrm{E}_\alpha$. The Burger's model has transfer function as

$$X(s) = \frac{s^2 h_1 h_2 \mathrm{E}_1 + s h_1 \mathrm{E}_1 \mathrm{E}_2}{s^2 h_1 h_2 + s(h_1\mathrm{E}_2 + h_1\mathrm{E}_1 + h_2\mathrm{E}_1) + \mathrm{E}_1\mathrm{E}_2} \qquad X(i\omega) = \frac{-\omega^2 h_1 h_2 \mathrm{E}_1 + i\omega h_1 \mathrm{E}_1 \mathrm{E}_2}{(\mathrm{E}_1\mathrm{E}_2 - \omega^2 h_1 h_2) + i\omega(h_1\mathrm{E}_2 + h_1\mathrm{E}_1 + h_2\mathrm{E}_1)}$$

We note here that complex-compliance is just reciprocal of the defined transfer function. Thus complex-compliance is the strain response for a unit impulse stress to the system. Therefore, the transfer function and complex-compliance are in frequency domain representations of response function for unit-impulse input, whereas creep-compliance and relaxation modulus are the time domain responses for Heaviside's step input (stress and strain respectively).

7. Origination of fractional derivative in stress-strain model

We wrote in previous section $\sigma(t) = \mathrm{E}_\alpha\left(\mathrm{d}_t^\alpha \varepsilon\right)(t) \qquad 0<\alpha<1$, as generalization of stress-strain relation. In figure-3 (Bouwens model or tree model) the most elementary model is visualized, the visco-elastic response is determined via the transfer function $X(i\omega) = \hat{\sigma}(i\omega)/\hat{\varepsilon}(i\omega)$ is represented as elastic element of transfer-function E (Hook's material) deforming (straining) in a viscous medium (Newtonian material), having transfer function as $i\omega\eta$, and both these material connected to visco-elastic material with transfer function $X(i\omega)$. This is self similar fractal nature of material composite (as any part of this chain is similar to other part!).

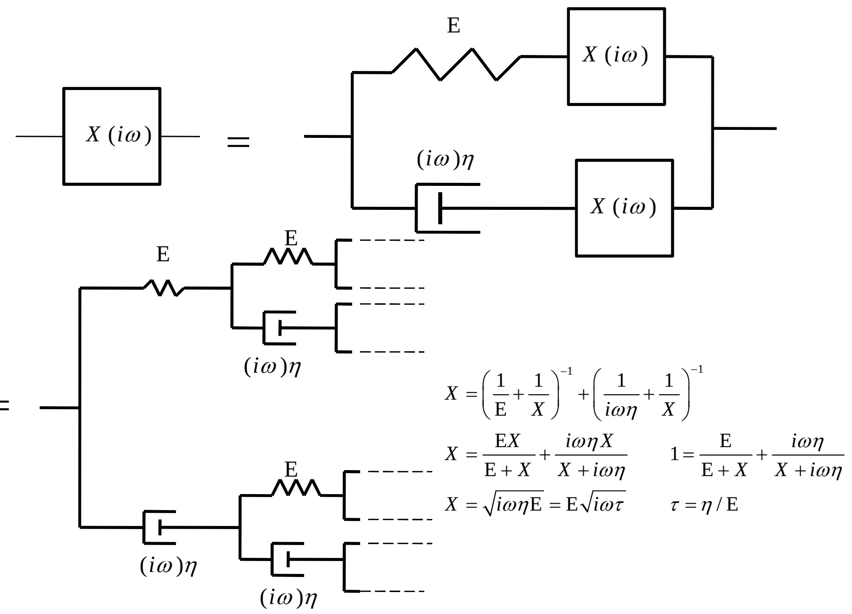


Figure-3: Self similar fractal tree model for visco-elastic material with fractional derivative

The derivation is simple, giving frequency domain transfer function as $X(i\omega) = \mathrm{E}\sqrt{\tau}\sqrt{i\omega}$, or $X(s) = \hat{\sigma}(s)/\hat{\varepsilon}(s) = s^{1/2}\tau^{1/2}\mathrm{E}$ where $\tau = \eta/\mathrm{E}$ is the time-constant of the basic element which is also the shortest time-constant of the model of figure-3. The corresponding differential equation is

$$\sigma(t) = \mathrm{E}_{1/2}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}\varepsilon(t); \quad \mathrm{E}_{1/2} = \tau^{1/2}\mathrm{E}, \quad \tau = \eta/\mathrm{E} \tag{39}$$

We can have order α as generalization with $\mathrm{E}_\alpha = \tau^\alpha \mathrm{E}$ in $\sigma(t) = E_\alpha\left(\mathrm{d}_t^\alpha \varepsilon\right)(t); \quad 0 \le \alpha \le 1$.

A further generalization of figure-3 is by replacing the Newton material (pure-viscous element) by a fractal element with transfer function $Z(i\omega)$ as depicted in figure-4

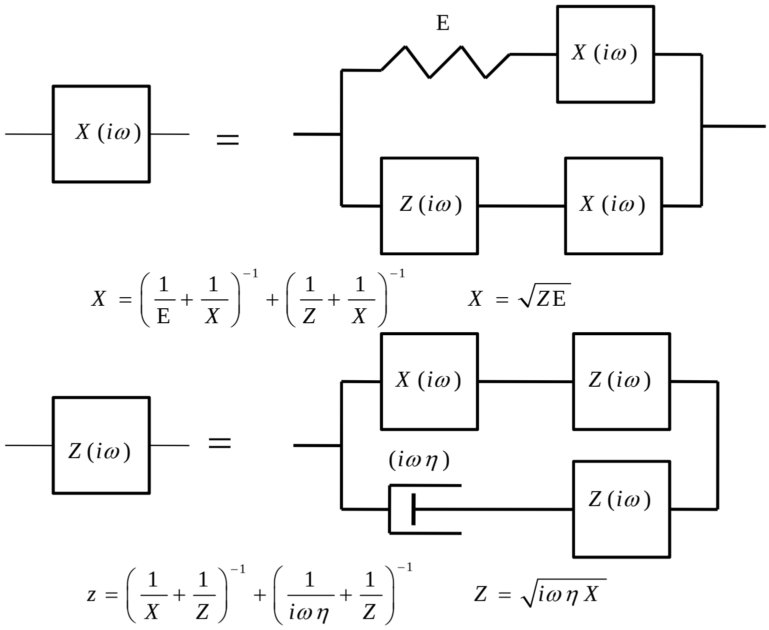


Figure-4 Further generalization of Bouwen's fractal model

Here in figure-4 the pure viscous element is replaced by a fractal visco-elastic element composed as transfer function $Z(i\omega)$. We have from calculations $Z(i\omega) = \sqrt{i\omega\eta X}$, and the transfer function of the main element as $X(i\omega) = \sqrt{Z\mathrm{E}}$; placing the transfer function of Z here we obtain

$$X = \sqrt{Z\mathrm{E}} = \sqrt{\mathrm{E}\sqrt{i\omega\eta X}} = \mathrm{E}^{1/2} X^{1/4} (i\omega\eta)^{1/4}$$

$$X^{3/4} = (i\omega\eta)^{1/4}\mathrm{E}^{1/2}$$

$$X(i\omega) = (i\omega\eta)^{1/3}\mathrm{E}^{2/3} = \frac{\hat{\sigma}(i\omega)}{\hat{\varepsilon}(i\omega)} \qquad X(s) = s^{1/3}\eta^{1/3}\mathrm{E}^{2/3} = s^{1/3}\left(\frac{\eta}{\mathrm{E}}\right)^{1/3}\mathrm{E}^{2/3}\mathrm{E}^{1/3} \tag{40}$$

$$X(s) = \frac{\hat{\sigma}(s)}{\hat{\varepsilon}(s)} = s^{1/3}\tau^{1/3}\mathrm{E}$$

Taking Laplace inverse we obtain

$$\sigma(t) = \eta^{1/3}\mathrm{E}^{2/3}\left(\mathrm{d}_t^{1/3}\varepsilon\right)(t) \qquad \alpha = \frac{1}{3}; \quad \mathrm{E}_\alpha = \tau^{1/3}\mathrm{E}, \quad \tau = \frac{\eta}{\mathrm{E}}; \quad \sigma(t) = \mathrm{E}_\alpha\left(\mathrm{d}_t^{\alpha}\varepsilon\right)(t) \tag{41}$$

From figure-3 and figure-4 we can have a generalized transfer function as $X = \hat{\sigma}(i\omega)/\hat{\varepsilon}(i\omega) = (i\omega\eta)^{\alpha}\mathrm{E}^{1-\alpha}$; with α expressing behavior intermediate between purely elastic ($\alpha = 0$) or linear viscous behavior ($\alpha = 1$).

There is another model as 'ladder' model of Marvin-Oser, depicted in figure-5

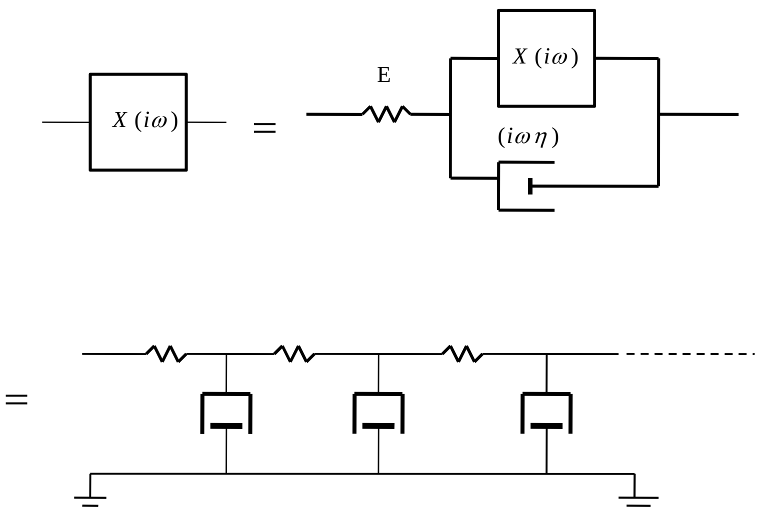


Figure-5 Ladder model

The figure-5 gives the transfer function expression as

$$\frac{1}{X}=\frac{1}{X+i\omega\eta}+\frac{1}{\mathrm{E}} \qquad X=\frac{-i\omega\eta\pm\sqrt{(i\omega\eta)^2+4i\omega\eta\mathrm{E}}}{2} \tag{42}$$

We select the positive value and write

$$X=-\frac{i\omega\eta}{2}+\sqrt{\frac{(i\omega\eta)^2}{4}+i\omega\eta\mathrm{E}} \qquad \tau=\frac{\eta}{\mathrm{E}} \qquad X=-\frac{i\omega\eta}{2}+\sqrt{\frac{(i\omega\eta)^2}{4}+i\omega\tau\mathrm{E}^2} \tag{43}$$

The ladder and the tree models are equivalent in the continuum limit, since then the viscous element then acts on an infinitely short segment; thus η becomes vanishingly small, and E remains constant. In this case ladder expression becomes tree expression as

$$X=\lim_{\eta\to 0}\left(-\frac{i\omega\eta}{2}+\sqrt{\frac{(i\omega\eta)^2}{4}+i\omega\tau\mathrm{E}^2}\right)=\mathrm{E}\sqrt{i\omega\tau} \tag{44}$$

The ladder model with identical values of E and η for the entire self-similar elements yield only semi-derivative ($\alpha=1/2$); like infinite transmission line impedance when denoted by series resistance R ohm/length, and shunt capacity C farad/length, and repeating this infinitely.

8. Time Response of fractional model

We have thus a generalized model (from above fractal models), with fractional order $0<\alpha<1$ as

$$\sigma=\mathrm{E}\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\varepsilon}{\mathrm{d}t^{\alpha}} \quad \text{or} \quad \varepsilon=\frac{1}{\mathrm{E}\tau^{\alpha}}\frac{\mathrm{d}^{-\alpha}\sigma}{\mathrm{d}t^{-\alpha}}. \tag{45}$$

We have generalized differ-integral of order q for a monomial function $f(x)=x^p$ formula as

$$\frac{\mathrm{d}^q}{\mathrm{d}x^q}x^p=\frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q}, \quad p>-1, \quad q\in\mathbb{R} \tag{46}$$

The positive order in fractional derivative and negative order is fractional integration. The above fractional differ-integral of order α for stress strain expressions are not equivalent for general function, however, they are equivalent for functions which are expandable in terms of differ-integrable units, in particular for functions which can be expressed as a series of powers of t^{α} and which are bounded for $t=0$.

In, creep strain is found by fractional integration of the constant stress (σt^{p} ; $p=0$) to order α . So applying above differ-integration formula with $q=-\alpha$, and $p=0$, we get

$$\varepsilon(t)=\frac{\sigma_0}{\mathrm{E}}\frac{1}{\Gamma(1+\alpha)}\left(\frac{t}{\tau}\right)^{\alpha} \tag{47}$$

Here we see if and how the fractional derivative is related to ordinary derivative. The strain response got from the equation $\sigma=\mathrm{E}\,\tau^{\alpha}(\mathrm{d}^{\alpha}\varepsilon/\mathrm{d}t^{\alpha})$, is like a power law $\varepsilon(t/\tau)\sim(t/\tau)^{\alpha}$.

The above is response to a visco-elastic element with parameters time constant τ , elastic-constant E and order $0<\alpha<1$; to which if we add in series a pure elastic element (a spring) of constant E_M .(i.e. a type of generalized Maxwell model); whose behavior is now is:

$$\mathrm{E}\,\tau^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\varepsilon=\sigma+\frac{\mathrm{E}\,\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{48}$$

The creep response is

$$\varepsilon=\frac{\sigma_0}{\mathrm{E}_M}+\frac{\sigma_0}{\mathrm{E}}\frac{1}{\Gamma(1+\alpha)}\left(\frac{t}{\tau}\right)^{\alpha} \tag{49}$$

This we got a standard model where instantaneous creep compliance is added to the retarded creep-compliance (of bare visco-elastic block). Note in above without loss of generality one can have E equal to E_M by a suitable adjustment of time constant.

The relaxation modulus solution is slightly difficult, i.e. response function $\sigma(t)$ to a constant strain (step input) ε_0 applied at $t=0$. The above generalized Maxwell equation we consider a different fractional order derivative in time (β) for the starin variable, and rewrite the equation, and without loss of generality we can also have $\mathrm{E}=\mathrm{E}_M$

$$\mathrm{E}\,\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon=\sigma+\frac{\mathrm{E}\,\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \qquad \mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon=\sigma+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{50}$$

We put $\varepsilon(t)=\varepsilon_0$ and assume that relaxation response will be a power series as $\sigma(t)=(t/\tau)^{\delta}\sum a_k(t/\tau)^{\alpha}$; $\delta>-1$.

$$\begin{aligned}
&\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon(t)=\sigma(t)+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma(t)}{\mathrm{d}t^{\alpha}}\\
&\mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon_0=\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}
\end{aligned} \tag{51}$$

Using the formula $\left(D_x{}^{q}x^{p}\right)=\Gamma(p+1)x^{p-q}/\Gamma(p+1-q)$; $p>-1$ in above we obtain

$$\begin{aligned}
&\mathrm{E}_M\varepsilon_0\tau^{\beta}\frac{t^{-\beta}}{\Gamma(1-\beta)}=\left(\frac{t}{\tau}\right)^{\delta}\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k}+\tau^{\alpha}\sum a_k\left(\frac{\Gamma(\alpha k+\delta+1)}{\Gamma(\alpha k+\delta+1-\alpha)}\right)\frac{1}{\tau^{\alpha k+\delta}}t^{\alpha k+\delta-\alpha}\\
&\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}\left(\frac{t}{\tau}\right)^{-\beta}=\sum a_k\left(\frac{t}{\tau}\right)^{\alpha k+\delta}+\sum a_k b_k\left(\frac{t}{\tau}\right)^{\alpha k+\delta-\alpha}\quad;\quad b_k=\frac{\Gamma(\alpha k+\delta+1)}{\Gamma(\alpha k+\delta+1-\alpha)}
\end{aligned} \tag{52}$$

Put $x = (t/\tau)$ in above to get

$$\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta)} x^{-\beta} = \sum a_k x^{\alpha k+\delta} + \sum a_k b_k x^{\alpha k+\delta-\alpha}$$

$$= (a_0 x^{\delta} + a_1 x^{\alpha+\delta} + a_2 x^{2\alpha+\delta} + ...) + (a_0 b_0 x^{\delta-\alpha} + a_1 b_1 x^{\delta} + a_2 b_2 x^{\alpha+\delta} + a_3 b_3 x^{2\alpha+\delta} + ...)$$

$$= a_0 b_0 x^{\delta-\alpha} + (a_0 + a_1 b_1) x^{\delta} + (a_1 + a_2 b_2) x^{\alpha+\delta} + (a_2 + a_3 b_3) x^{2\alpha+\delta}$$

(53)

Comparing the coefficient of RHS and LHS we can write

$$\delta - \alpha = -\beta; \quad \delta = \alpha - \beta; \quad a_0 b_0 = \mathrm{E}_M \varepsilon_0 / \Gamma(1-\beta); \quad a_k + a_{k+1} b_{k+1} = 0$$

$$b_k = \frac{\Gamma[\alpha(k+1) - \beta + 1]}{\Gamma[\alpha k + 1 - \beta]}, \quad b_0 = \frac{\Gamma(\alpha - \beta + 1)}{\Gamma(1-\beta)}; \quad a_0 = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(1-\beta) b_0} = \frac{\mathrm{E}_M \varepsilon_0}{\Gamma(\alpha - \beta + 1)}$$

$$b_1 = \frac{\Gamma(2\alpha - \beta)}{\Gamma(\alpha - \beta + 1)}, \quad a_1 = -\frac{a_0}{b_1} = -\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(\alpha - \beta + 1)} \frac{\Gamma(\alpha - \beta + 1)}{\Gamma(2\alpha - \beta)} = -\frac{\mathrm{E}_M \varepsilon_0}{\Gamma(2\alpha - \beta)}$$

In general therefore

$$a_k = \frac{(-1)^k \mathrm{E}_M \varepsilon_0}{\Gamma(\alpha k + \alpha - \beta + 1)}$$

Inserting these values we have the relaxation function as

$$\sigma(t) = \mathrm{E}_M \varepsilon_0 (x)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha - \beta + 1)} (x)^{\alpha k}; \quad x = \frac{t}{\tau} \tag{54}$$

This above treatment gives idea that relaxation function is in terms of power-series of time of the order $(k+1)\alpha - \beta$. This can be obtained as Laplace transformation too. The Laplace of basic generalized Maxwell equation is as described in following expression

$$\mathrm{E}_M \tau^{\beta} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} \varepsilon = \sigma + \tau^{\alpha} \frac{\mathrm{d}^{\alpha} \sigma}{\mathrm{d}t^{\alpha}} \qquad \mathrm{E}_M \tau^{\beta} s^{\beta} \hat{\varepsilon}(s) = \hat{\sigma}(s) + \tau^{\alpha} s^{\alpha} \hat{\sigma}(s) \tag{55}$$

For a step strain ε_0 at $t = 0$ we have $\hat{\varepsilon}(s) = \varepsilon_0 / s$, putting this value in the above equation we get

$$\hat{\sigma}(s) = \frac{\mathrm{E}_M \tau^{\beta} s^{\beta}}{1 + (\tau s)^{\alpha}} \frac{\varepsilon_0}{s} = \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{1 + (\tau s)^{\alpha}} \tag{56}$$

For this function no tabulated transform exists and we therefore must determine another way to arrive at solution. We can use term wise inverse Laplace if we break the above as power series of s. Since the function $\left[1 + (\tau s)^{-\alpha}\right]^{-1}$ can be expanded in a series with decreasing powers, will have radius of convergence different from zero.

$$\hat{\sigma}(s) = \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{1 + (\tau s)^{\alpha}} = \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{(\tau s)^{\alpha}\left[1 + (\tau s)^{-\alpha}\right]} = \mathrm{E}_M \varepsilon_0 \tau \frac{(\tau s)^{\beta-1}}{(\tau s)^{\alpha}} \left[1 + (\tau s)^{-\alpha}\right]^{-1} =$$

$$= \mathrm{E}_M \varepsilon_0 \tau (\tau s)^{\beta-1-\alpha} \left[1 + (\tau s)^{-\alpha}\right]^{-1} = \mathrm{E}_M \varepsilon_0 \tau (\tau s)^{\beta-1-\alpha} \left[1 - (\tau s)^{-\alpha} + (\tau s)^{-2\alpha} - (\tau s)^{-3\alpha} + ...\right] \tag{57}$$

$$\hat{\sigma}(s) = \mathrm{E}_M \varepsilon_0 \left[\tau \sum_{k=0}^{\infty} (-1)^k (\tau s)^{\beta-1-\alpha-\alpha k}\right]; \quad \beta - \alpha - 1 < 0 \tag{58}$$

We have used $(1+\mathrm{x})^{-1} = 1 - \mathrm{x} + \mathrm{x}^2 - \mathrm{x}^3 + ...$

For the series $\hat{\sigma}(s) = \mathrm{E}_M \varepsilon_0 \left[\tau \sum_{k=0}^{\infty} (-1)^k (\tau s)^{\beta-1-\alpha-\alpha k} \right]; \quad \beta - \alpha - 1 < 0$, use term wise Laplace inverse for $t/\tau > 0$, and get the result as

$$\sigma(t) = \mathrm{E}_M \varepsilon_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha - \beta + 1)} \left(\frac{t}{\tau} \right)^{\alpha k + \alpha - \beta} = \mathrm{E}_M \varepsilon_0 (x)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k + \alpha - \beta + 1)} (x)^{\alpha k} \qquad (59)$$

$$x > 0: \quad x = \frac{t}{\tau}; \quad \alpha - \beta > -1$$

The retarded response of the generalized Maxwell model is found by Laplace transform as

$$\mathrm{E}_M \tau^{\beta} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} \varepsilon = \sigma + \tau^{\alpha} \frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \qquad \mathrm{E}_M \tau^{\beta} s^{\beta} \hat{\varepsilon}(s) = \hat{\sigma}(s) + \tau^{\alpha} s^{\alpha} \hat{\sigma}(s) \qquad (60)$$

For a step input stress σ_0 applied at $t = 0$, the Laplace is $\hat{\sigma}(s) = \sigma_0 / s$, putting this above we get

$$\hat{\varepsilon}(s) = \frac{\left[1 + (\tau s)^{\alpha}\right]}{\mathrm{E}_M \tau^{\beta} s^{\beta}} \frac{\sigma_0}{s} = \frac{\sigma_0}{\mathrm{E}_M} \tau \frac{\left[1 + (\tau s)^{\alpha}\right]}{(\tau s)^{\beta+1}} = \frac{\sigma_0}{\mathrm{E}_M} \left[\tau (\tau s)^{-(1+\beta)} + \tau (\tau s)^{\alpha-\beta-1} \right] \qquad (61)$$

Using known Laplace pair $\left(1/s^{1-q}\right) \leftrightarrow t^{-q}/(-q)! = t^{-q}/\Gamma(1-q)$ we invert above to get creep-compliance or retarded response as

$$\varepsilon(t) = \frac{\sigma_0}{\mathrm{E}_M} \left[\left(\frac{(t/\tau)^{\beta}}{\Gamma(1+\beta)} + \frac{(t/\tau)^{\beta-\alpha}}{\Gamma(\beta - \alpha + 1)} \right) \right] \qquad (62)$$

For case when $\alpha = \beta$, we have

$$\varepsilon(t) = \frac{\sigma_0}{\mathrm{E}_M} \left[\left(\frac{(t/\tau)^{\beta}}{\Gamma(1+\beta)} + 1 \right) \right] \qquad (63)$$

This we derived earlier too.

Above obtained retardation function or creep compliance is easy to interpret, and show that the retardation function is solely a power law type. It is easily seen that for the whole α and β parameter range the retardation function is strongly increasing if and only if $\beta - \alpha \geq 0$ is fulfilled. The behavior of the relaxation function is not so obvious as we obtained from above.

We take both the fractional orders to be same to have our generalized fractional order Maxwell model as

$$\mathrm{E} \tau^{\beta} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} \varepsilon = \sigma + \frac{\mathrm{E} \tau^{\alpha}}{\mathrm{E}_M} \frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \qquad \mathrm{E} = \mathrm{E}_M \qquad \alpha = \beta \qquad \mathrm{E}_M \tau^{\beta} \frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}} \varepsilon = \sigma + \tau^{\beta} \frac{\mathrm{d}^{\beta}\sigma}{\mathrm{d}t^{\beta}} \qquad (64)$$

Then for relaxation function for a constant strep input strain ε_0, we have power law function as $\sigma(t) = \sum a_k x^{\beta k}; \quad x = t/\tau, \quad \beta k > -1$, and then we repeat the earlier steps to get

$$\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}x^{-\beta}=\sum a_k x^{\beta k}+\sum a_k b_k x^{\beta k-\beta} \qquad b_k=\frac{\Gamma(\beta k+1)}{\Gamma(\beta k+1-\beta)},\quad b_0=\frac{1}{\Gamma(1-\beta)}$$

$$\begin{aligned}&=(a_0+a_1x^{\beta}+a_2x^{2\beta}+...)+(a_0b_0x^{-\beta}+a_1b_1+a_2b_2x^{\beta}+a_3b_3x^{2\beta}+...)\\&=(a_{-1}+a_0b_0)x^{-\beta}+(a_0+a_1b_1)+(a_1+a_2b_2)x^{\beta}+(a_2+a_3b_3)x^{2\beta}+.....\\&=\sum\left(a_j+a_{j+1}b_{j+1}\right)(x)^{j\beta} \qquad b_{j+1}=\frac{\Gamma\left[(j+1)\beta+1\right]}{\Gamma\left[j\beta+1\right]}\end{aligned} \tag{65}$$

$$\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}x^{-\beta}=\sum\left(a_j+a_{j+1}b_{j+1}\right)(x)^{j\beta} \qquad b_{j+1}=\frac{\Gamma\left[(j+1)\beta+1\right]}{\Gamma\left[j\beta+1\right]} \tag{66}$$

From above comparing the RHS and LHS we get, for all $j\neq-1$

$$a_j+a_{j+1}b_{j+1}=0 \qquad a_{j+1}=-a_j/b_{j+1} \qquad a_{j+1}=-a_j\frac{\Gamma\left[1+j\beta\right]}{\Gamma\left[1+(j+1)\beta\right]} \tag{67}$$

We have by comparing $j=-1$ term the following condition

$$a_{-1}+a_0b_0=\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)} \qquad a_{-1}+a_0\frac{1}{\Gamma(1-\beta)}=\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)} \qquad a_0+a_{-1}\Gamma(1-\beta)=\mathrm{E}_M\varepsilon_0 \tag{68}$$

For short time response we can choose $a_{-1}=0$ (actually it is so) and have $a_0=\mathrm{E}_M\varepsilon_0$ i.e. at $x=(t/\tau)\approx 0$ with

$$a_1=-a_0\frac{\Gamma(1)}{\Gamma(1+\beta)}=-\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1+\beta)} \qquad a_2=-a_1\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)}=-\left(-\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1+\beta)}\right)\frac{\Gamma(1+\beta)}{\Gamma(1+2\beta)}=\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1+2\beta)}$$

$$a_j=(-1)^j\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1+j\beta)} \tag{69}$$

Therefore the relaxation function is

$$\sigma(t)=\mathrm{E}_M\varepsilon_0\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(1+k\beta)}\left(\frac{t}{\tau}\right)^{\beta k} \tag{70}$$

Earlier obtained result says $\sigma(t)=\mathrm{E}_M\varepsilon_0\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k+\alpha-\beta}$, here we put $\alpha=\beta$, to get the above obtained result. Well the instantaneous relaxation modulus is nevertheless E_M and at short times the function is 'some type of exponential' decay function. The long time relaxation-function we choose by making $a_0=0$ (expected that function at long times will have a very small value of initial relaxation function). From the $j=-1$ term of comparison we get

$$a_{-1}=\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)} \qquad a_{-j}=(-1)^{j+1}\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1+j\beta)} \tag{71}$$

The expansion at late times thus contains negative powers essentially limited to $j\beta<1$ (we have put condition above that $j\beta>-1$), since the fractional derivative is not defined for larger power of x^{-1}. Thus for large times $t\gg\tau$ the response is power law of exponent β; i.e.

$$\sigma(t)\approx a_{-1}(t/\tau)^{-\beta}=\frac{\mathrm{E}_M\varepsilon_0}{\Gamma(1-\beta)}\left(\frac{t}{\tau}\right)^{-\beta} \tag{72}$$

It should be mentioned that the response of this discussed model can be retrieved from the basic fractional order model that is

$$\sigma = \mathrm{E}\tau^{\beta}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}} \qquad \varepsilon = \frac{1}{\mathrm{E}\tau^{\beta}}\frac{\mathrm{d}^{-\beta}\sigma}{\mathrm{d}t^{-\beta}} \tag{73}$$

by replacing ε in above by $\varepsilon - \sigma/\mathrm{E}$, thus giving retarded deformation under the relaxing stress $\sigma(t)$.

The generalized Maxwell model is got by replacing the elastic element E and the viscous element η (in figure-1) by visco-elastic elements with exponents respectively equal to μ and β, with no loss of generality one may also assume that $\mu \le \beta$. So we get constitutive equation for the generalized model as

$$\hat{\varepsilon}(s) = \hat{\varepsilon}_1(s) + \hat{\varepsilon}_2(s) = \hat{\sigma}(s)\left(\frac{1}{X_1} + \frac{1}{X_2}\right) \quad \text{or} \quad X_2\hat{\varepsilon}(s) = \left(1 + \frac{X_2}{X_1}\right)\hat{\sigma}(s)$$
$$X_2(s) = \tau^{\beta}\mathrm{E}s^{\beta} \qquad X_1(s) = \tau^{\mu}\mathrm{E}s^{\mu} \tag{74}$$

$$\tau^{\beta}\mathrm{E}s^{\beta}\hat{\varepsilon}(s) = \left(1 + \tau^{\beta-\mu}s^{\beta-\mu}\right)\hat{\sigma}(s) \qquad \text{put} \quad \alpha = \beta - \mu \qquad \tau^{\beta}\mathrm{E}\frac{\mathrm{d}^{\beta}\varepsilon}{\mathrm{d}t^{\beta}} = \sigma + \tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{75}$$

9. The decay law and the relaxation-response with analytical Laplace inversion "without contour integration"

In this section we invert the time domain response of decay of stress, to get 'distribution of relaxation rates', for a complex decay process-governed by several rates of exponential decay. A generally complex decay in time is in figure-6 and figure-7 denoted by Mittag-Leffler functions, but cannot be like figure-8. The complex decay may be expressed as following with several rate constants $k_1, k_2, k_3, \ldots$ with weights $a_1, a_2, a_3, \ldots$

$$\sigma(t) = a_1 e^{-k_1 t} + a_2 e^{-k_2 t} + a_3 e^{-k_3 t} + \ldots = \sum a_n e^{-k_n t} \tag{76}$$

In continuum limit we may write the above as

$$\sigma(t) = f(t) = \int_0^{\infty} H(k)e^{-kt}\mathrm{d}k \qquad f(0) = 1 \tag{77}$$

Where $H(k)$ is distribution function of the rate of the relaxation decay process, $f(t)$ depicted in figure-9. While the discrete rates are shown here the rate distribution function would be

$$H(k) = a_1\delta(k - k_1) + a_2\delta(k - k_2) + a_3\delta(k - k_3) + \ldots = \sum a_n\delta(k - k_n) \tag{78}$$

Here we will generalize the decay curves observed our relaxation function. This section is new one involving Berberan-Santos method of obtaining Laplace inversion without the use of contour integration. Thus we have a numerical friendly method to get Laplace inverse. The Laplace transform $F(s)$ of a function in time domain $f(t)$ is defined as

$$F(s) \triangleq \int_0^{\infty} f(t)e^{-st}\mathrm{d}t \qquad F(s) = 0 \quad \text{for} \quad s < 0 \tag{79}$$

This is standard integral transform from a time domain to a complex frequency domain $s = \mathrm{Re}\{s\} + i\omega$; where real part is significant in transient response and the imaginary part of the frequency corresponds to 'steady-state' response. Here $f(t)$ is inverse Laplace transform of $F(s)$. Compare the two as follows:

$$f(t) = \int_0^{\infty} H(k)e^{-tk}\mathrm{d}k \qquad F(s) = \int_0^{\infty} f(t)e^{-st}\mathrm{d}t \tag{80}$$

Both above are Laplace transform expressions, the first one transforming the function $H(k)$ from k domain to t domain; while the second one is transforming $f(t)$ from t domain to s domain. However, both are Laplace transformation-change of variable symbol! Therefore we can say $H(k)$ is inverse Laplace of $f(t)$ in the first expression as $f(t)$ is inverse Laplace of $F(s)$ in the second expression. Therefore in order to get the rate distribution-function $H(k)$ from the decay curve (or relaxation-function), we need to have inverse Laplace of the time function $f(t)$ like conventional definition if inverse Laplace

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}\mathrm{d}s \qquad H(k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(t)e^{tk}\mathrm{d}t \tag{81}$$

Refer figure-10 describes this process of inversion of Laplace transformation. In the above expression c is real number larger than c_0, where c_0 being such that $f(t)$ has some form of singularity on the real line $\mathrm{Re}\{t\}=c_0$ but is analytic in the complex plane to the right of that line, i.e. for $\mathrm{Re}\{t\}>c_0$. The Laplace inversion is usually carried out by contour integration.

We describe Berberan-Santos method of evaluation of the Laplace inversion without going for contour integration. First is change of variable i.e. from real time to complex time variable as $t=c+i\varpi$. The symbol ϖ is different from imaginary part of frequency in the usual Laplace variable ω in complex frequency parameter s .With this change we have

$$H(k)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(t)e^{kt}\mathrm{d}t=\frac{e^{ck}}{2\pi}\int_{-\infty}^{+\infty}f(c+i\varpi)e^{ik\varpi}\mathrm{d}\varpi$$
$$H(k)=\frac{e^{ck}}{2\pi}\left[\int_{-\infty}^{+\infty}f(c+i\varpi)\cos(k\varpi)\mathrm{d}\varpi+i\int_{-\infty}^{+\infty}f(c+i\varpi)\sin(k\varpi)\mathrm{d}\varpi\right] \tag{82}$$

Write $f(c+i\varpi)=\mathrm{Re}\left[f(c+i\varpi)\right]+i\,\mathrm{Im}\left[f(c+i\varpi)\right]$ and place above, we get

$$H(k)=\frac{e^{ck}}{2\pi}\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(c+i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\}$$
$$+\frac{e^{ck}}{2\pi}i\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi)+\mathrm{Im}[f(c+i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} \tag{83}$$

Given that $H(k)$ is a real function, we get

$$\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi)+\mathrm{Im}[f(c+i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\}=0 \tag{84}$$

And the above expression (82) reduces to

$$H(k)=\frac{e^{ck}}{2\pi}\left\{\int_{-\infty}^{+\infty}\left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(c+i\varpi)]\sin(k\varpi)\right]\mathrm{d}\varpi\right\} \tag{85}$$

But we have $f(t)=\int_0^{\infty}H(k)e^{-kt}\mathrm{d}k$ putting $t=c+i\varpi$ we get therefore

$$f(c+i\varpi)=\int_0^{\infty}H(k)e^{-k(c+i\varpi)}\mathrm{d}k=\int_0^{\infty}e^{-ck}H(k)\cos(k\varpi)\mathrm{d}\varpi-i\int_0^{\infty}e^{-ck}H(k)\sin(k\varpi)\mathrm{d}\varpi \tag{86}$$

$$\mathrm{Re}\left[f(c+i\varpi)\right]=\int_0^{\infty}e^{-ck}H(k)\cos(k\varpi)\mathrm{d}k \qquad \mathrm{Im}\left[f(c+i\varpi)\right]=-\int_0^{\infty}e^{-ck}\sin(k\varpi)\mathrm{d}k \tag{87}$$

Using this in obtained expression for $H(k)$, we observe that integrand is even function for $k > 0$ therefore we re-write the formula as

$$H(k) = \frac{e^{ck}}{\pi} \int_0^{\infty} \left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi) - \mathrm{Im}[f(c+i\varpi)]\sin(k\varpi) \right] \mathrm{d}\varpi \tag{88}$$

Using $\frac{\pi}{e^{ck}} H(k) = \int_0^{\infty} \left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi) - \mathrm{Im}[f(c+i\varpi)]\sin(k\varpi) \right] \mathrm{d}\varpi$ and the relation

$0 = \int_0^{+\infty} \left[\mathrm{Re}[f(c+i\varpi)]\cos(k\varpi) + \mathrm{Im}[f(c+i\varpi)]\sin(k\varpi) \right] \mathrm{d}\varpi$; adding and subtracting these we get

$$H(k) = \frac{2e^{ck}}{\pi} \int_0^{\infty} \mathrm{Re}\left[f(c+i\varpi) \right] \cos(k\varpi) \mathrm{d}\varpi \qquad H(k) = -\frac{2e^{ck}}{\pi} \int_0^{\infty} \mathrm{Im}\left[f(c+i\varpi) \right] \sin(k\varpi) \mathrm{d}\varpi \tag{89}$$

Write $f(c+i\varpi) = \rho(\varpi) e^{i\theta(\varpi)}$ $\qquad \rho(\varpi) = \left| f(c+i\varpi) \right|$ $\qquad \theta(\varpi) = \angle f(c+i\varpi)$ to get following formulas

$$\begin{aligned} H(k) &= \frac{e^{ck}}{\pi} \int_0^{\infty} [\rho(\varpi)\cos\theta(\varpi)\cos(k\varpi) - \rho(\varpi)\sin\theta(\varpi)\sin(k\varpi)] \mathrm{d}\varpi \\ &= \frac{e^{ck}}{\pi} \int_0^{\infty} \rho(\varpi) \cos\left[k\varpi + \theta(\varpi) \right] \mathrm{d}\varpi \end{aligned} \tag{90}$$

$$H(k) = \frac{2e^{ck}}{\pi} \int_0^{\infty} \rho(\varpi)\cos\left[\theta(\varpi)\right]\cos(k\varpi)\mathrm{d}\varpi = -\frac{2e^{ck}}{\pi} \int_0^{\infty} \rho(\varpi)\sin\left[\theta(\varpi)\right]\sin(k\varpi)\mathrm{d}\varpi \tag{91}$$

10.Few examples of Laplace inversion without contour integrations

The above obtained relations allow direct calculation of $H(k)$ the rate distribution of relaxation function from $f(t)$ without contour integration. Consider a very simple case of decay function
$f(t) = (t-a)^{-1}$; we know from standard Laplace pair that $F(s) = (s \pm a)^{-1} \leftrightarrow f(t) = e^{\mp at}$.
Thus for $f(t) = (t-a)^{-1}$ we should get via inverse Laplace the rate distribution function $H(k) = e^{ak}$. The application of the formula with $c > a$ yields

$$\begin{aligned} H(k) &= \frac{2e^{ck}}{\pi} \int_0^{\infty} \mathrm{Re}\left[\frac{1}{(c-a)+i\varpi} \right] \cos(k\varpi)\mathrm{d}\varpi = \frac{2e^{ck}}{\pi} \int_0^{\infty} \frac{(c-a)}{(c-a)^2+\varpi^2} \cos(k\varpi)\mathrm{d}\varpi \\ &= \frac{2(c-a)e^{ck}}{\pi} \int_0^{\infty} \frac{\cos(k\varpi)}{(c-a)^2+\varpi^2} \mathrm{d}\varpi = e^{ak} \end{aligned} \tag{92}$$

Let the decay function be $f(t) = (t)/(t^2+1)$. Well if $F(s) = (s)/(s^2+1)$ its inverse is $\cos(t)$. Thus we should have the distribution function $H(k) = \cos(k)$ as $\mathcal{L}^{-1}\left[(t)/(t^2+1) \right]$ with use of above formula with setting $c = 1$

$$H(k)=\frac{2e^{k}}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{1+i\varpi}{(1+i\varpi)^{2}+1}\right]\cos(k\varpi)\mathrm{d}\varpi=\frac{2e^{k}}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{1+i\varpi}{2-\varpi^{2}+2i\varpi}\right]\cos(k\varpi)\mathrm{d}\varpi$$
$$=\frac{2e^{k}}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[\frac{(1+i\varpi)(2-\varpi^{2}-2i\varpi)}{(2-\varpi^{2})^{2}+4\varpi^{2}}\right]\cos(k\varpi)\mathrm{d}\varpi=\frac{2e^{k}}{\pi}\int_{0}^{\infty}\frac{(\varpi^{2}+2)\cos(k\varpi)}{\varpi^{4}+4}\mathrm{d}\varpi=\cos(k)$$

(93)

These definite integrals are difficult to solve in closed form, even in simple cases. But, they allow obtaining results that are not so direct with contour integration, and are suited for numerical integration.

11. Some examples of standard decay functions and evaluation of rate distribution via Laplace inversion

In reality of decay functions, we can take $c=0$; as decay function will not expected to have singularity at time $t>0$. For a case of 'exponential-decay' i.e. $f(t)=e^{-t/\tau_0}$, obviously this function has only one decay rate i.e. $1/\tau_0$. As per procedure discussed above we do Laplace inversion by taking complex time with $c=0+i\varpi$; making it

$$f(i\varpi)=e^{-i\varpi/\tau_0}=\cos(\varpi/\tau_0)-i\sin(\varpi/\tau_0)\qquad \mathrm{Re}[f(i\varpi)]=\cos(\varpi/\tau_0)\tag{94}$$

$$H(k)=\frac{1}{\pi}\int_{0}^{\infty}\{\mathrm{Re}[f(i\varpi)]\cos(k\varpi)-\mathrm{Im}[f(i\varpi)]\sin(k\varpi)\}\mathrm{d}\varpi$$
$$=\frac{1}{\pi}\int_{0}^{\infty}[\cos(\varpi/\tau_0)\cos(k\varpi)+\sin(\varpi/\tau_0)\sin(k\varpi)]\mathrm{d}\varpi=\frac{1}{\pi}\int_{0}^{\infty}\cos\left[\varpi(k-1/\tau_0)\right]\mathrm{d}\varpi=\delta\left(k-\frac{1}{\tau_0}\right)$$

(95)

For a decay function stretched exponential $f(t)=e^{-(t/\tau_0)^{\beta}}$, in complex time variable we get

$$f(i\varpi)=e^{-\left(\frac{i\varpi}{\tau_0}\right)^{\beta}}=e^{-\left(\frac{\varpi}{\tau_0}\right)^{\beta}(i)^{\beta}}=e^{-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\left[\cos\left(\frac{\beta\pi}{2}\right)+i\sin\left(\frac{\beta\pi}{2}\right)\right]}=e^{\left[-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}e^{\left[-i\left(\frac{\varpi}{\tau_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]}\tag{96}$$
$$=\rho(\varpi)e^{i\theta(\varpi)}$$

$$\left|f(i\varpi)\right|=\rho(\varpi)=e^{\left[-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\qquad \angle f(i\varpi)=\theta(\varpi)=-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\tag{97}$$

Therefore;

$$H(k)=\frac{e^{ck}}{\pi}\int_{0}^{\infty}\rho(\varpi)\cos\left[k\varpi+\theta(\varpi)\right]\mathrm{d}\varpi=\frac{1}{\pi}\int_{0}^{\infty}e^{\left[-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left[k\varpi-\left(\frac{\varpi}{\tau_0}\right)^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]\mathrm{d}\varpi\tag{98}$$

Doing change of variable $u=\varpi/\tau_0$ we obtain

$$H(k)=\frac{\tau_0}{\pi}\int_{0}^{\infty}e^{\left[-u^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left[k\tau_0 u-u^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]\mathrm{d}u\tag{99}$$

Using other formulas we will get

$$H(k)=\frac{2\tau_0}{\pi}\int_0^{\infty} e^{\left[-u^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\cos\left[u^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]\cos(k\tau_0 u)\mathrm{d}u$$

$$H(k)=\frac{2\tau_0}{\pi}\int_0^{\infty} e^{\left[-u^{\beta}\cos\left(\frac{\beta\pi}{2}\right)\right]}\sin\left[u^{\beta}\sin\left(\frac{\beta\pi}{2}\right)\right]\sin(k\tau_0 u)\mathrm{d}u \tag{100}$$

Any other linear combination is also valid for getting solution $H(k)$.

The radioactive decay we write as pure exponential decay, however, Becquerel used compressed hyperbola function to describe this as

$$f(t)=\frac{1}{\left[1+(1-\beta)\left(\frac{t}{\tau_0}\right)\right]^{1/(1-\beta)}} \tag{101}$$

We have following steps

$$f(i\varpi)=\frac{1}{\left[1+(1-\beta)\left(\frac{i\varpi}{\tau_0}\right)\right]^{1/(1-\beta)}}$$

$$\left|f(i\varpi)\right|=\rho(\varpi)=\left[1+\left(\frac{(1-\beta)\varpi}{\tau_0}\right)^2\right]^{-1/[2(1-\beta)]}\quad;\quad \angle f(i\varpi)=\theta(\varpi)=-\frac{\tan^{-1}\left(\frac{(1-\beta)\varpi}{\tau_0}\right)}{1-\beta} \tag{102}$$

$$H(k)=\frac{e^{ck}}{\pi}\int_0^{\infty}\rho(\varpi)\cos\left[k\varpi+\theta(\varpi)\right]\mathrm{d}\varpi$$

$$=\frac{1}{\pi}\int_0^{\infty}\left[1+\left(\frac{(1-\beta)\varpi}{\tau_0}\right)^2\right]^{-1/[2(1-\beta)]}\cos\left[k\varpi-\frac{\tan^{-1}\left(\frac{(1-\beta)\varpi}{\tau_0}\right)}{1-\beta}\right]\mathrm{d}\varpi \tag{103}$$

With change of variable $u=\frac{(1-\beta)\varpi}{\tau_0}$, we get

$$H(k)=\frac{\tau_0}{\pi(1-\beta)}\int_0^{\infty}(1+u^2)^{-1/[2(1-\beta)]}\cos\left(\frac{k\tau_0 u-\tan^{-1}u}{1-\beta}\right)\mathrm{d}u \tag{104}$$

Using other formulas we get

$$H(k)=\frac{2\tau_0}{\pi(1-\beta)}\int_0^{\infty}(1+u^2)^{-1/[2(1-\beta)]}\cos\left(\frac{\tan^{-1}u}{1-\beta}\right)\cos(k\tau_0 u)\mathrm{d}u$$

$$H(k)=\frac{2\tau_0}{\pi(1-\beta)}\int_0^{\infty}(1+u^2)^{-1/[2(1-\beta)]}\sin\left(\frac{\tan^{-1}u}{1-\beta}\right)\sin(k\tau_0 u)\mathrm{d}u \tag{105}$$

The rate distribution function $H(k)$ for a simple power law as $f(t)=\left[1+(t/\tau_0)^{\alpha}\right];\quad \alpha<1$, will be expressed via same rule as above

$$f(i\varpi)=\frac{1}{1+\left(\frac{i\varpi}{\tau_0}\right)^{\alpha}}=\frac{1}{1+\left(\frac{\varpi}{\tau_0}\right)^{\alpha}(i)^{\alpha}}$$
$$=\frac{1}{1+\left(\frac{\varpi}{\tau_0}\right)^{\alpha}e^{i\frac{\alpha\pi}{2}}}=\frac{1}{1+\left(\frac{\varpi}{\tau_0}\right)^{\alpha}\left[\cos\left(\frac{\alpha\pi}{2}\right)+i\sin\left(\frac{\alpha\pi}{2}\right)\right]} \tag{106}$$

The real part of the complex function is

$$\mathrm{Re}\left[f(i\varpi)\right]=\frac{\left(\frac{\varpi}{\tau_0}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\varpi}{\tau_0}\right)^{2\alpha}+2\left(\frac{\varpi}{\tau_0}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1} \tag{107}$$

$$H(k)=\frac{2e^{ck}}{\pi}\int_0^{\infty}\mathrm{Re}\left[f(c+i\varpi)\right]\cos(k\varpi)\mathrm{d}\varpi$$
$$=\frac{2}{\pi}\int_0^{\infty}\frac{\left(\frac{\varpi}{\tau_0}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{\left(\frac{\varpi}{\tau_0}\right)^{2\alpha}+2\left(\frac{\varpi}{\tau_0}\right)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(k\varpi)\mathrm{d}\varpi \tag{108}$$
$$=\frac{2\tau_0}{\pi}\int_0^{\infty}\frac{(u)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}{(u)^{2\alpha}+2(u)^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)+1}\cos(k\tau_0 u)\mathrm{d}u \qquad u=\frac{\varpi}{\tau_0}$$

12.The relaxation-response with Mittag-Leffler function

The obtained relaxation function for a generalized Maxwell model

$$\mathrm{E}\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon=\sigma+\frac{\mathrm{E}\tau^{\alpha}}{\mathrm{E}_M}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \qquad \mathrm{E}_M\tau^{\beta}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}\varepsilon=\sigma+\tau^{\alpha}\frac{\mathrm{d}^{\alpha}\sigma}{\mathrm{d}t^{\alpha}} \tag{109}$$

is the following power series

$$\sigma(t)=\mathrm{E}_M\varepsilon_0\left(\frac{t}{\tau}\right)^{\alpha-\beta}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)}\left(\frac{t}{\tau}\right)^{\alpha k} \tag{110}$$

Putting $\alpha=\beta=0.5$ gives $\sigma(x)=\mathrm{E}_M\varepsilon_0\exp(x)[\mathrm{erfc}(x^{1/2})]$, $x=(t/\tau)$, where erfc is the ‘complementary error function’. Putting $\alpha=\beta=1$ we obtain relaxation response of ordinary Maxwell model as $\sigma(x)=E_M\varepsilon_0\exp(-x)$.

The sum contained in the relaxation function is the generalized Mittag-Leffler function (GML), which reads as follows

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{(x)^k}{\Gamma(\alpha k+\beta)} \qquad \alpha,\beta>0 \qquad E_{\alpha,1}(x)=E_{\alpha}(x) \tag{111}$$

Note the notation of Mittag-Leffler function uses $E_{*,**}$, whereas E_M, E denotes elastic modulus (not to be confused with GML notation): as one is in italics the other is not. For negative x, we have the following expressions for GML function

$$E_{\alpha,\beta}(-x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\beta)} x^k \qquad E_{\alpha,\beta}(-x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\beta)} (x)^{\alpha k} \qquad (112)$$

Figure-6, 7, 8 gives plots of $E_{\alpha,\beta}(-x)$ for $\beta = 1$ with $\alpha = 0.25, 1.75$ and 2.25. Thus for relaxation function we restrict $\alpha < 2$

$$\sigma(t) = \mathrm{E}_M \varepsilon_0 \left(\frac{t}{\tau}\right)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\alpha-\beta+1)} \left(\frac{t}{\tau}\right)^{\alpha k} \qquad \left(\frac{t}{\tau}\right) = x$$
$$= \mathrm{E}_M \varepsilon_0 \left(x\right)^{\alpha-\beta} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha k+\gamma)} \left(x\right)^{\alpha k} \qquad \gamma = \alpha - \beta + 1 \qquad (113)$$
$$= \mathrm{E}_M \varepsilon_0 (x)^{\alpha-\beta} E_{\alpha,\gamma}(-x^{\alpha})$$

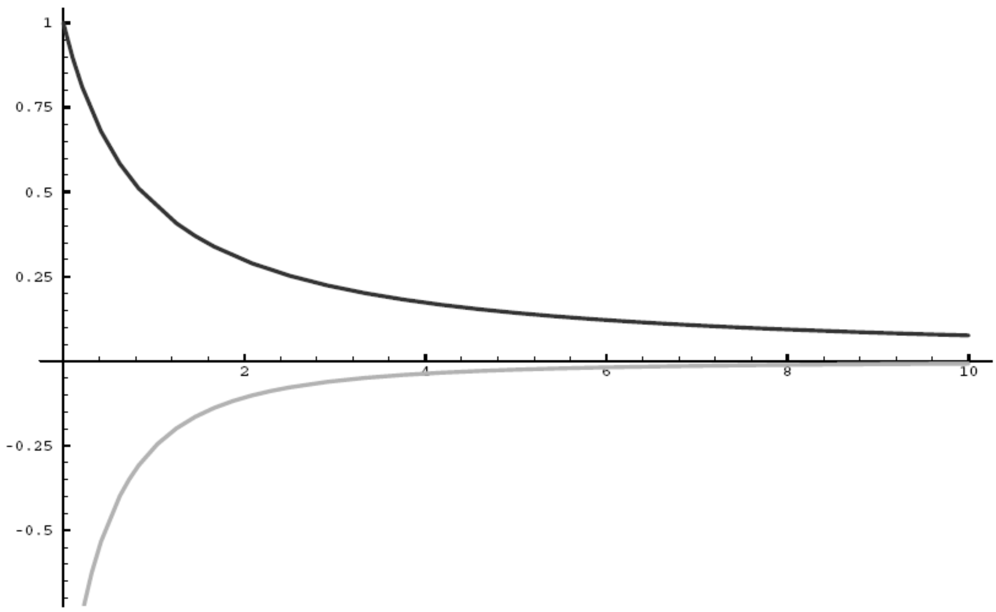


Figure-6: Plot of Function $E_{\alpha,\beta}(-t)$, $\alpha = 0.25$, $\beta = 1$ **and its derivative**

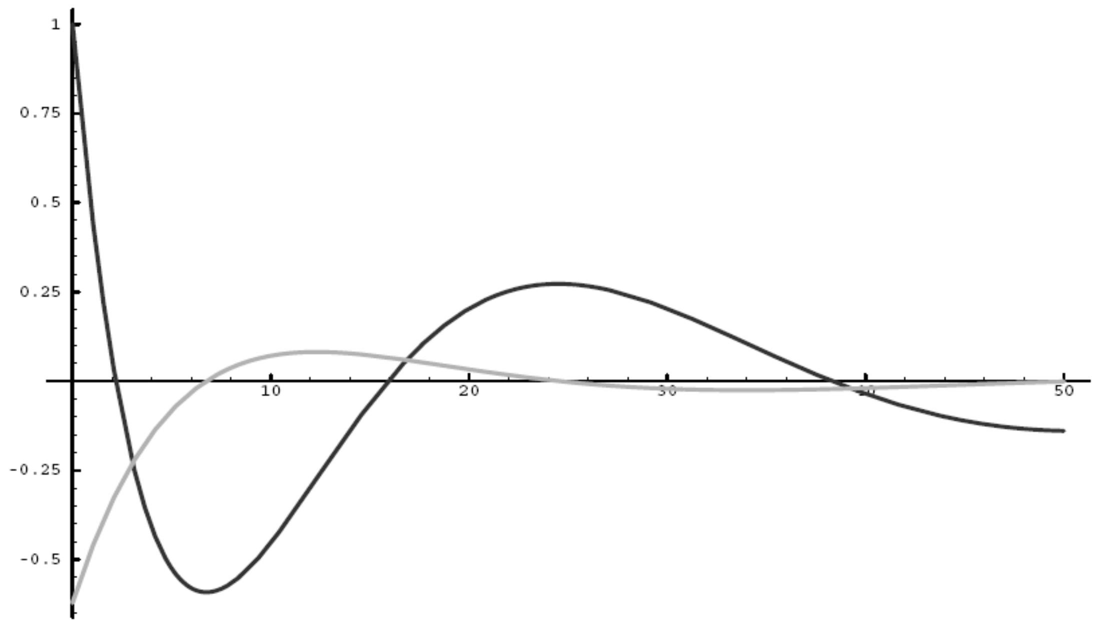


Figure-7: Plot of Function $E_{\alpha,\beta}(-t)$, $\alpha = 1.75$, $\beta = 1$ **and its derivative**

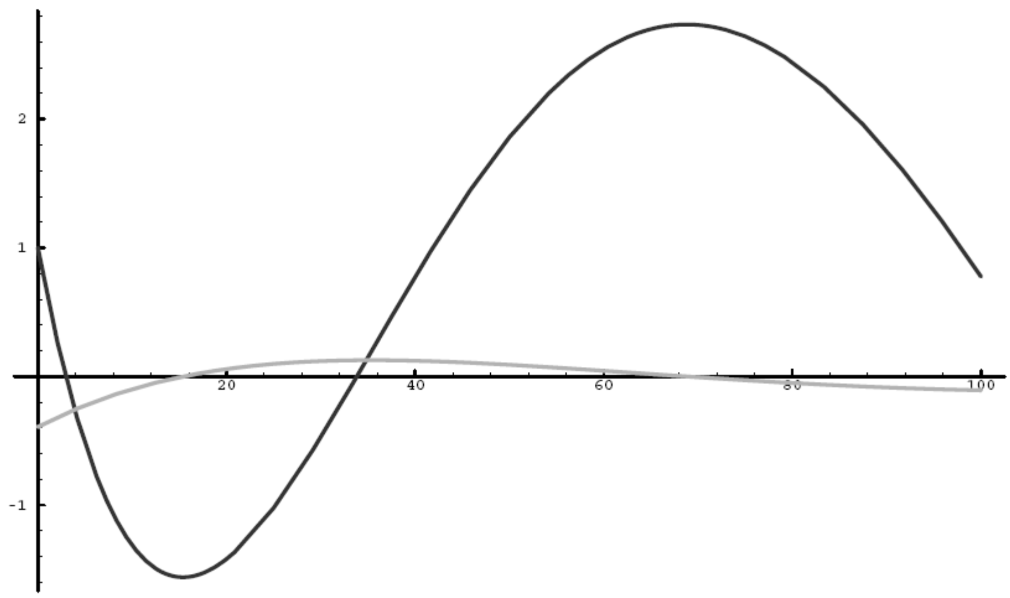


Figure-8: Plot of Function $E_{\alpha,\beta}(-t)$, $\alpha = 2.25$, $\beta = 1$ **and its derivative**

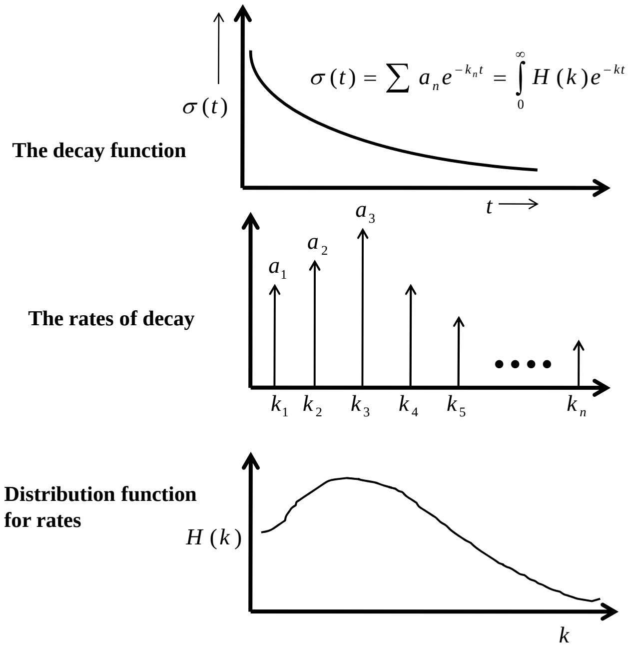


Figure-9: Rate of decay function

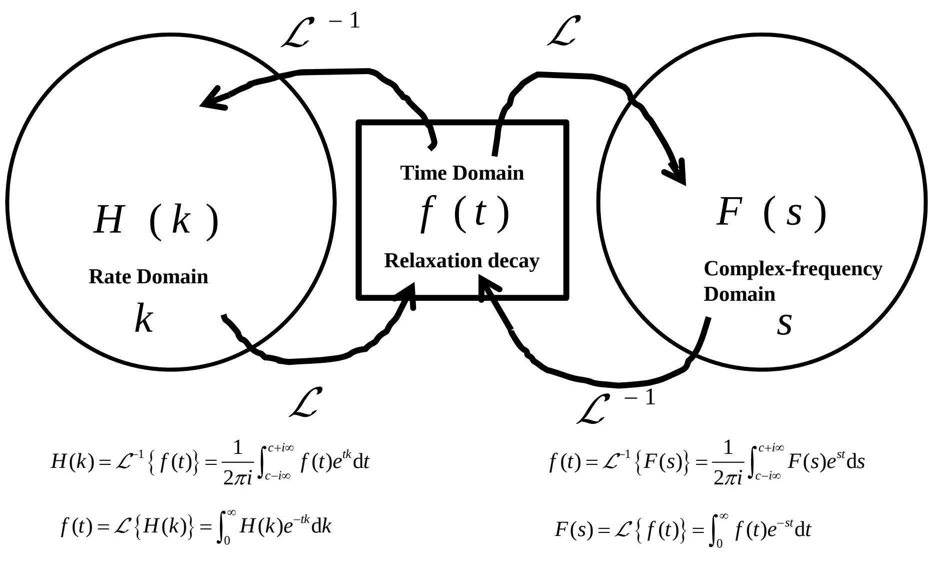


Figure-10: Laplace inversion and Laplace transformation

The asymptotic expansion for the Mittag-Leffler function for negative arguments at $x \to \infty$ is the following:

$$E_{\alpha,\alpha}(-x) \sim \frac{\alpha}{\Gamma(1-\alpha)} x^{-2}, \quad \alpha \neq 1 \qquad E_{\alpha,\gamma}(-x) \sim \frac{1}{\Gamma(\gamma-\alpha)} x^{-1}, \quad \gamma \neq \alpha$$

$$E_{\alpha,\alpha}(-x^\alpha) \sim \frac{\alpha}{\Gamma(1-\alpha)} x^{-2\alpha}, \quad \alpha \neq 1 \qquad E_{\alpha,\gamma}(-x^\alpha) \sim \frac{1}{\Gamma(\gamma-\alpha)} x^{-\alpha}, \quad \gamma \neq \alpha \tag{114}$$

With these approximations we express the asymptotic behavior of the relaxation function for short and long times. The relaxation function is $\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-\beta} E_{\alpha,\gamma}(-x^{\alpha})$ with $\gamma = \alpha - \beta + 1$. For case $\beta = 1$, i.e. order of differentiation of strain as unity and $0 < \alpha \le 1$, we have

$$\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-1} E_{\alpha,\alpha} = \begin{cases} \mathrm{E}_M \varepsilon_0 \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{as} \quad x \to 0 \\ \mathrm{E}_M \varepsilon_0 \dfrac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)} & \text{as} \quad x \to \infty \end{cases} \tag{115}$$

For the other case we have

$$\sigma(x) = \mathrm{E}_M \varepsilon_0 x^{\alpha-\beta} E_{\alpha,\gamma} = \begin{cases} \mathrm{E}_M \varepsilon_0 \dfrac{x^{\alpha-\beta}}{\Gamma(\gamma)} & \text{as} \quad x \to 0 \\ \mathrm{E}_M \varepsilon_0 \dfrac{1}{\Gamma(1-\beta)} x^{-\beta} & \text{as} \quad x \to \infty \end{cases} \qquad \gamma = \alpha - \beta + 1 \tag{116}$$

The figure-11 and figure-12 gives the plot of $E_{\alpha}(-t^{\alpha}), \quad 0 \le \alpha \le 1, \quad t > 0$ in linear scale and log scale respectively. For $\alpha = 1$, the curve is pure exponential curve. For other cases for large t we get a power law behavior visible from figure-12 after about $t = 1$, we get a constant slope.

13.Properties of Mittag-Leffler function

We write some of the important properties of Mittag-Leffler function as

$$E_{\alpha}(-x) = E_{2\alpha}(x^2) - xE_{2\alpha,1+\alpha}(x^2) \tag{117}$$

$$E_{2\alpha}(x^2) = \frac{E_{\alpha}(x) + E_{\alpha}(-x)}{2} \tag{118}$$

$$E_{\alpha}(-i\varpi) = E_{2\alpha}(-\varpi^2) - i\varpi E_{2\alpha,1+\alpha}(-\varpi^2) \qquad \mathrm{Re}\left[E_{\alpha}(-i\varpi)\right] = E_{2\alpha}(-\varpi^2) \tag{119}$$

We can extract the rate distribution function i.e. $H_{\alpha}(k)$ for Mittag-Leffler decay $f(t) = E_{\alpha}(-x)$, $x = t/\tau$, with the Laplace inversion formula derived in earlier section, to expand it as Laplace transform, as follows:

$$E_{\alpha}(-x) = \int_0^{\infty} H_{\alpha}(k) e^{-kx} \mathrm{d}k \tag{120}$$

Put $x = i\varpi$, thus we have $\mathrm{Re}\left[E_{\alpha}(-i\varpi)\right] = E_{2\alpha}(-\varpi^2)$; and write

$$H_{\alpha}(k) = \frac{2}{\pi}\int_0^{\infty} \mathrm{Re}\left[f(i\varpi)\right]\cos(k\varpi)\mathrm{d}\varpi = \frac{2}{\pi}\int_0^{\infty} E_{2\alpha}(-\varpi^2)\cos(k\varpi)\mathrm{d}\varpi \tag{121}$$

For various $\alpha, \quad 0 < \alpha < 1, \quad k > 0$, we have following integral representations for $H_{\alpha}(k)$

$$H_1(k) = \frac{2}{\pi}\int_0^{\infty} \cosh(i\varpi)\cos(k\varpi)\mathrm{d}\varpi = \frac{2}{\pi}\int_0^{\infty} \cos(\varpi)\cos(k\varpi)\mathrm{d}\varpi = \delta(k-1) \tag{122}$$

$$H_{1/2}(k) = \frac{2}{\pi}\int_0^{\infty} e^{-\varpi^2}\cos(k\varpi)\mathrm{d}\varpi = \frac{1}{\sqrt{\pi}} e^{-\frac{k^2}{4}} \tag{123}$$

$$H_{1/4}(k) = \frac{2}{\pi}\int_0^{\infty} e^{\varpi^2}\mathrm{erfc}(\varpi^2)\cos(k\varpi)\mathrm{d}\varpi \tag{124}$$

$$H_0(k)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos(k\varpi)}{1+\varpi^2}\mathrm{d}\varpi=e^{-k} \tag{125}$$

It is known that $E_\alpha(-x)$ is complete monotonic for $x\ge 0$ and $0\le\alpha\le 1$, i.e.

$$(-1)^n\frac{\mathrm{d}^n E_\alpha(-x)}{\mathrm{d}x^n}\ge 0 \qquad x\ge 0 \qquad 0\le\alpha\le 1 \tag{126}$$

We describe monotonic behavior of the Mittag-Leffler function $E_\alpha(-x)$ by its Laplace transformed relation $E_\alpha(-x)=\int_0^\infty H_\alpha(k)e^{-kx}\mathrm{d}k$; and write the following relation for $n-$th differentiation of $E_\alpha(-x)$. The above result follow immediately from Laplace transform relation i.e.

$$(-1)^n\frac{\mathrm{d}^n E_\alpha(-x)}{\mathrm{d}x^n}=\int_0^\infty k^n H_\alpha(k)e^{-kx}\mathrm{d}k \tag{127}$$

Noting that $H_\alpha(k)>0$ is probability density function (rate distribution-what we call) for $k>0$. The behavior near the origin for Mittag-Leffler function $E_\alpha(-x)$ can be obtained by recognizing that any decay function $f(x)$ is written as $f(x)=\exp\left(-\int_0^x k(u)\mathrm{d}u\right)$, where $k(x)$ is the time (x) dependent rate coefficient. When the relaxation is pure exponential, one has $k(x)$ as constant say k_0 described as $k(x)=k_0\delta(x)$. For Mittag-Leffler relaxation function $f(x)=E_\alpha(-x)$, we have

$$k(x)=-\frac{\mathrm{d}}{\mathrm{d}x}\ln[E_\alpha(-x)]=-\frac{1}{E_\alpha(-x)}\frac{\mathrm{d}E_\alpha(-x)}{\mathrm{d}x}=-\frac{1}{E_\alpha(-x)}\sum_{n=0}^{\infty}\frac{(n+1)(-x)^n}{\Gamma(1+\alpha+\alpha n)} \tag{128}$$

Near origin $E_\alpha(0)=1$, thus we get rate near origin as $k(0)=-\dfrac{1}{\Gamma(1+\alpha)}$.

For asymptotic behavior we can use the Laplace transformation, as done above and expanding in the power-series described below.

$$H_\alpha(k)=\frac{2}{\pi}\int_0^\infty E_{2\alpha}(-\varpi^2)\cos(k\varpi)\mathrm{d}\varpi=\frac{2}{\pi}\sum_{n=0}^{\infty}a_n(\alpha)k^n \qquad a_0(\alpha)=\int_0^\infty E_{2\alpha}(-\varpi^2)\mathrm{d}\varpi \tag{129}$$

We take Laplace transformation of the above to get

$$E_\alpha(-x)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{a_n(\alpha)}{x^{n+1}} \tag{130}$$

From above we say at $n=0$ the $E_\alpha(-x)$ has asymptotic decay as x^{-1} and $E_\alpha(-x^2)$ decays as x^{-2}, and $E_\alpha(-x^\alpha)$ decays as $x^{-\alpha}$.

The integral representation of the Mittag-Leffler function we will obtain from the described method of Berberan-Santos as we described. The start point is Laplace transform of $E_\alpha(-x^\alpha)$ with $x=t/\tau$, to complex frequency $s=\mathrm{Re}[s]+i\omega$ i.e.

$$\mathcal{L}\left\{E_\alpha(-x^\alpha)\right\}=\int_0^\infty E_\alpha(-x^\alpha)e^{-sx}\mathrm{d}x=\frac{s^{\alpha-1}}{1+s^\alpha} \tag{131}$$

This can be got via representing $E_\alpha(-x^\alpha)$ as series and taking term by term $s-$domain transform. Here now we apply the Berberan-Santo technique on

$$F(s)=\frac{s^{\alpha-1}}{1+s^{\alpha}} \qquad F(i\omega)=\frac{(i\omega)^{\alpha-1}}{1+(i\omega)^{\alpha}}=\frac{\omega^{\alpha-1}\left[\cos[(\alpha-1)\pi/2]+i\sin[(\alpha-1)\pi/2]\right]}{1+\omega^{\alpha}\cos(\alpha\pi/2)+i\omega^{\alpha}\sin(\alpha\pi/2)} \tag{132}$$

$$\mathrm{Re}\left[F(i\omega)\right]=\frac{\omega^{\alpha-1}\sin(\alpha\pi/2)}{1+2\omega^{\alpha}\cos(\alpha\pi/2)+\omega^{2\alpha}} \tag{133}$$

Use inverse Laplace technique in this case from $s=i\omega$ domain to x domain, by following

$$\begin{aligned} E_{\alpha}(-x)&=\mathcal{L}^{-1}\left\{F(i\omega)\right\}=\frac{2}{\pi}\int_{0}^{\infty}\mathrm{Re}\left[F(i\omega)\right]\cos(x\omega)\mathrm{d}\omega \\ &=\frac{2}{\pi}\sin(\alpha\pi/2)\int_{0}^{\infty}\frac{\omega^{\alpha-1}\cos(x\omega)\mathrm{d}\omega}{1+2\omega^{\alpha}\cos(\alpha\pi/2)+\omega^{2\alpha}} \end{aligned} \tag{134}$$

From here we write integral representation of $E_{\alpha}(-x)$ as

$$E_{\alpha}(-x)=\frac{2}{\pi}\sin\left(\frac{\alpha\pi}{2}\right)\int_{0}^{\infty}\frac{\omega^{\alpha-1}\cos(x^{1/\alpha}\omega)\mathrm{d}\omega}{1+2\omega^{\alpha}\cos(\alpha\pi/2)+\omega^{2\alpha}} \tag{135}$$

14. Thermodynamic considerations for realistic model & relaxation and retardation criteria

We have seen stress relaxation function i.e. response to a step strain, and strain retardation function i.e. strain function response to a step stress. Now we give here criteria to select the describing fractional differential equation. This is because the second law of thermodynamics imposes restrictions on physically realizable process, it is thus necessary to consider the thermodynamics compatibility of a given rheological constitutive equation. That means we have to look for conditions under which the constitutive equation in general or the parameter function or parameters in particular guarantee non-negative rates of mechanical energy dissipation δ_m. This can be formulated as

$$\delta_m=\Phi-(Q)(\dot{f})\geq 0 \qquad \Phi=\sigma_{12}\dot{\varepsilon}_{12}{}^{2} \tag{136}$$

Where Φ is the stress power which is, in the case of relaxation after a step-strain is zero; $\dot{f}$ is the rate of change of free energy, and Q is the density. It can be seen that in case of relaxation, the demand for non-negative mechanical dissipation rates is equivalent to the non-positive rate of free energy. This implies relaxation is associated with release of the free energy stored during the stress-jump. Investigations concerning the energy storage during harmonic excitations or arbitrary deformation histories demand that relaxation and retardation functions should be positive definite functions, which means:

The relaxation and retardation functions must be greater than or equal to zero for all time.
The relaxation function is monotonically non-increasing
The retardation function is monotonically non-decreasing.

For retardation function

$$\varepsilon(t)\sim\frac{(t/\tau)^{\beta}}{\Gamma(\beta+1)}+\frac{(t/\tau)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)} \qquad \text{for} \quad \alpha=\beta \qquad \varepsilon(t)\sim\frac{(t/\tau)^{\beta}}{\Gamma(\beta+1)}+1 \tag{137}$$

The conditions are fulfilled if the relation $\beta\geq\alpha>0,\quad \tau>0$ exists.

For relaxation function $\sigma(t)\sim x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-x^{\alpha})$, the analysis is more complicated. It follows from the power law term that $\tau>0$ and $\alpha-\beta\leq 0$. However for the GML function $E_{\alpha,\gamma}(-x^{\alpha}),\quad \gamma=\alpha-\beta+1$ it can be stated from monotonic behavior of Mittag-Leffler function $E_\alpha(-x^\alpha)$, $0 \le \alpha \le 1$, that this is non-increasing, if $\beta \ge \alpha > 0$. Thus the constitutive equation

$$\mathrm{E}\tau^\beta \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\varepsilon = \sigma + \frac{\mathrm{E}\tau^\alpha}{\mathrm{E}_M}\frac{\mathrm{d}^\alpha\sigma}{\mathrm{d}t^\alpha} \qquad \mathrm{E}_M\tau^\beta \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}\varepsilon = \sigma + \tau^\alpha \frac{\mathrm{d}^\alpha\sigma}{\mathrm{d}t^\alpha} \tag{138}$$

is thermodynamically possible with condition $\beta \ge \alpha > 0$, $\tau > 0$

Here we mention that in some cases of starch samples (arrowroot on glass figure-14) the retarded response though rises but there is oscillatory behavior-this is one case of anomalous response obtained. This particular material show oscillatory non-decreasing nature of retarded response, though overall the thermodynamic compatibility is maintained-yet locally at small time scales that is not-this may be due to some local disorder in the material! Whereas the figure-13 showing non-decreasing curve for castor oil sample.

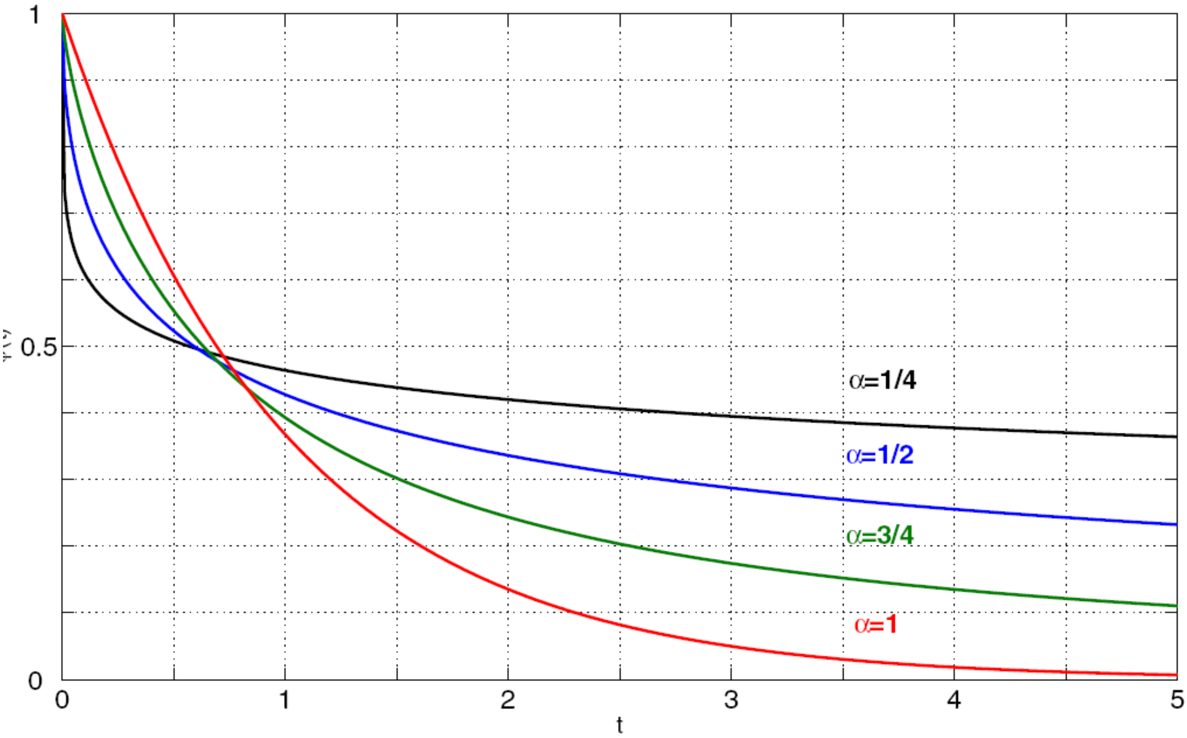


Figure-11: Plot of $E_\alpha(-t^\alpha)$, $0 \le \alpha \le 1$, $t > 0$ **in linear scale**

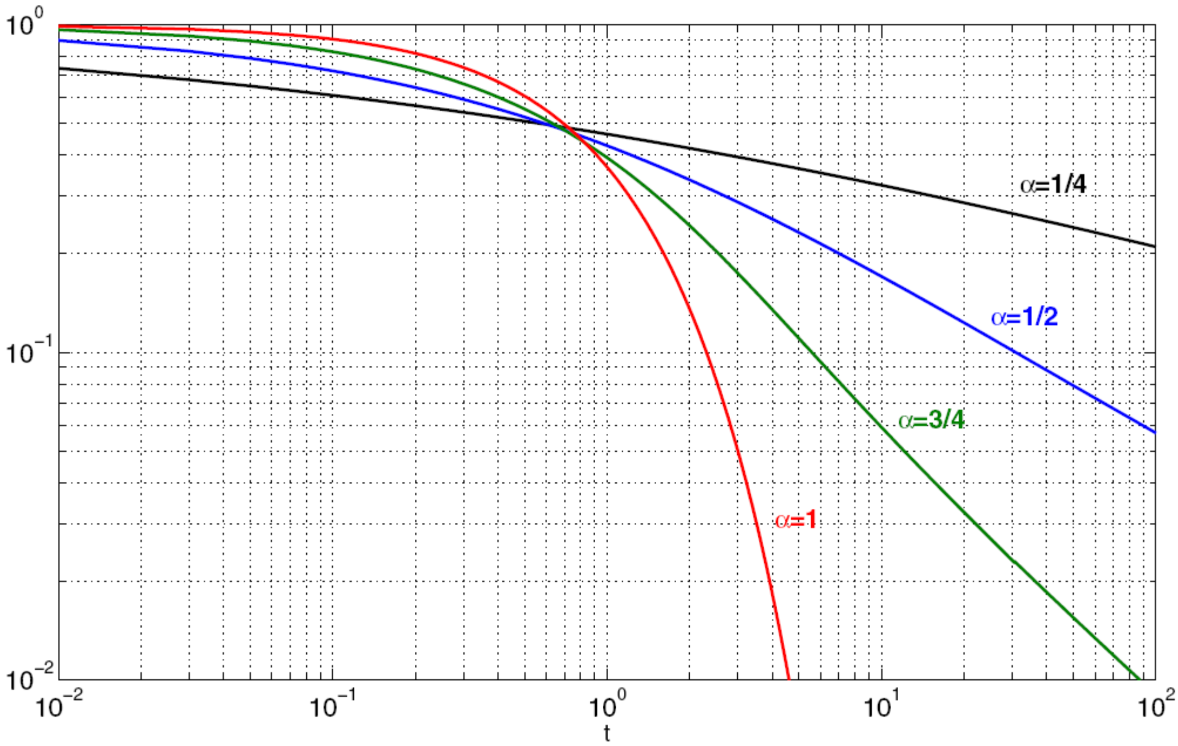


Figure-12: Plot of $E_\alpha(-t^\alpha), \quad 0 \le \alpha \le 1, \quad t > 0$ **in log scale**

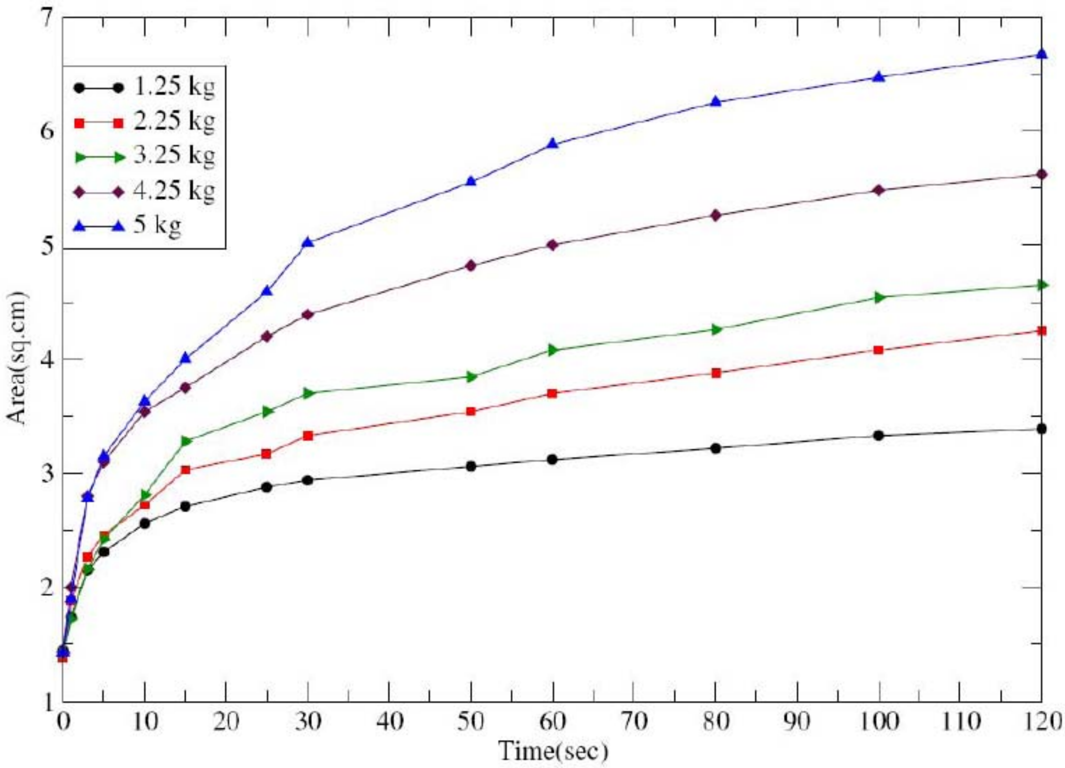


Figure-13 Plot showing normal non-decreasing retarded response in castor oil

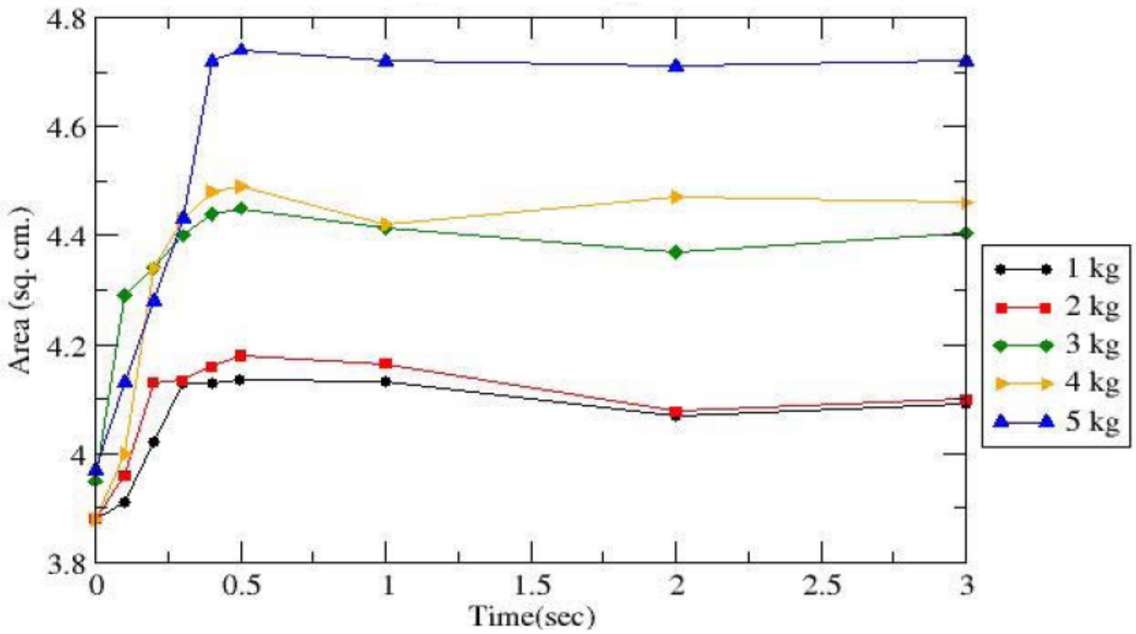


Figure-14 Showing small oscillatory retarded response for arrowroot on glass

15. Using fractional order differential equation to get generalized visco-elastic model

Another way to generalize the integer order model as stated above is via use of several fractional derivatives operating on stress as well as strain as shown below.

$$\sigma^{\alpha_n}(t)+\sum_{k=1}^{p}a_k D^{\alpha_k}\sigma(t)=\sum_{k=0}^{p}b_k D^{\beta_k}\varepsilon(t) \qquad a_k, b_k, \alpha_k, \beta_k \in \mathbb{R} \qquad k=0,1,2,\ldots,p \tag{139}$$

$D^{\alpha}; D^{\beta}$ represents appropriate fractional derivative; such that the frequency transformed equation for above generalized visco-elastic model is

$$\left[s^{\alpha_n}+\sum_{k=1}^{p}a_k s^{\alpha_k}\right]\hat{\sigma}(s)=\left[\sum_{k=1}^{p}b_k s^{\beta_k}\right]\hat{\varepsilon}(s) \tag{140}$$

The steady state frequency domain equation is by using $s=i\omega$, we get

$$\left[(i\omega)^{\alpha_n}+\sum_{k=1}^{p}a_k (i\omega)^{\alpha_k}\right]\hat{\sigma}(i\omega)=\left[\sum_{k=0}^{p}b_k (i\omega)^{\beta_k}\right]\hat{\varepsilon}(i\omega) \tag{141}$$

From above we have reduced parameter model of visco-elasticity comprising of four parameters as

$$\begin{gathered}\left[1+bD^{\alpha}\right]\sigma(t)=\left[\mathrm{E}_0+\mathrm{E}_1 D^{\alpha}\right]\varepsilon(t)\\ 0<\alpha<1; \quad \mathrm{E}_0\ge 0; \quad \mathrm{E}_1>0; \quad b\ge 0; \quad \mathrm{E}_1\ge b\mathrm{E}_0\end{gathered} \tag{142}$$

It should be noted here that the fractional derivative operator though is considered as RL fractional derivative with sole consideration that

$$\mathcal{F}\left(D^{\alpha}f(t)\right)(\omega)=(i\omega)^{\alpha}\hat{f}(i\omega) \qquad \hat{f}(i\omega)=\mathcal{F}\left(f(t)\right)(\omega) \tag{143}$$

With this sole consideration there are several definitions of fractional derivatives which can be of usage, like Grunwald-Letnikov ${}_{-\infty}^{G}D_{t,+}^{\alpha}$ or Liouvelli type forward derivative, Caputo-Liouvelli type forward derivatives-over infinite interval $(-\infty,t]$; or that of RL type ${}_{0}^{RL}D_{t,+}^{\alpha}f$ if $f(t)=0$ for $t<0$ or RL type ${}_{a}^{RL}D_{t,+}^{\alpha}f$; if $f(t)=0$ for $t<a$; or those corresponding to Caputo type ${}_{0}^{C}D_{t,+}^{\alpha}$ or ${}_{a}^{C}D_{t,+}^{\alpha}$.

16. Sequential fractional derivative Miller-Ross ${}_{a}\mathscr{D}_{x}^{k\alpha}$ & Sequential Fractional Differential Equations (SFDE)

Sequential linear fractional differential equation of order $n\alpha$; $n \in \mathbb{N}$ is represented as

$$b_0(x)y(x) + b_1(x)\left[{}_{a}\mathscr{D}_{x}^{\alpha}y(x)\right] + b_2(x)\left[{}_{a}\mathscr{D}_{x}^{2\alpha}y(x)\right] + ... + b_n(x)\left[{}_{a}\mathscr{D}_{x}^{n\alpha}y(x)\right] = f(x) \quad (144)$$

${}_{a}\mathscr{D}_{x}^{k\alpha}$ is fractional sequential derivative operator, of commensurate order α, ${}_{a}\mathscr{D}_{x}^{\alpha}y(x) = {}_{a}^{*}D_{x}^{\alpha}y(x)$ where ${}_{a}^{*}D_{x}^{\alpha}$ is RL or Caputo operator (denoted via *), and ${}_{a}\mathscr{D}_{x}^{k\alpha}y(x) = {}_{a}\mathscr{D}_{x}^{\alpha}\,{}_{a}\mathscr{D}_{x}^{(k-1)\alpha}y(x)$; $k = 2,3,....$ For $k = 2$; $0 < \alpha < 1/2$ we have

$${}_{a}\mathscr{D}_{x}^{2\alpha}y(x) = {}_{a}D_{x}^{2\alpha}y(x) - {}_{a}I_{x}^{1-\alpha}y(a)\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \quad (145)$$

Comes from the fact that for $\alpha > 0, \beta > 0$ generally $D^{\alpha}D^{\beta}f(x) \neq D^{\alpha+\beta}f(x)$

Representing in matrix form, the SFDE

$$\mathscr{D}^{n\alpha}y(x) + \sum_{k=0}^{(n-1)} a_k(x)\mathscr{D}^{k\alpha}y(x) = f(x); \quad n \in \mathbb{N} \quad (146)$$

reduces to

$${}^{*}D^{\alpha}\mathrm{Y}(x) = \mathrm{A}(x)\mathrm{Y}(x) + \mathrm{B}(x) \quad (147)$$

by changing the variables with $y_1(x) = y(x)$; ${}^{*}D^{\alpha}y_j(x) = y_{j+1}(x)$; $j = 1,2,...,(n-1)$, where

$$\mathrm{A}(x) = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ -a_0 & -a_1 & ... & ... & -a_{n-1} \end{pmatrix}, \quad \mathrm{B}(x) = \begin{pmatrix} 0 \\ 0 \\ ... \\ ... \\ f(x) \end{pmatrix}; \quad \mathrm{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ ... \\ ... \\ y_n(x) \end{pmatrix} \quad (148)$$

As an example

$$y''(t) + 3\left({}_{0}D_{t}^{3/2}y(t)\right) + y(t) = f(t) \quad (149)$$

set $y''(t) = {}_{0}\mathscr{D}_{t}^{4\frac{1}{2}}y(t)$ gives SFDE as

$${}_{0}\mathscr{D}_{t}^{4\alpha}y(t) + 3{}_{0}\mathscr{D}_{t}^{3\alpha}y(t) + y(t) = f(t); \quad \alpha = 1/2 \quad (150)$$

reduced to

$${}_{0}D_{t}^{\alpha}\mathrm{Y}(t) = \mathrm{A}\mathrm{Y}(x) + \mathrm{B}(t) \quad (151)$$

with

$$\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -3 \end{pmatrix}; \quad \mathrm{B}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{pmatrix}; \quad \mathrm{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix} \quad (152)$$

17. Solution of ordinary integer order differential equation with state transition matrices with exponential function

Let us take general differential equation $\frac{\mathrm{d}}{\mathrm{d}t}x(t)+a(t)x(t)=bu(t)$ with $x(t_0)=x_0$ multiply both the sides by $e^{\int a(t)\mathrm{d}t}$ and get $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t}x(t)\right]=e^{\int a(t)\mathrm{d}t}bu(t)$, and then integrate $\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[e^{\int a(\tau)\mathrm{d}\tau}x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}e^{\int a(\tau)\mathrm{d}\tau}bu(\tau)\mathrm{d}\tau$. Using $\Phi(t)=e^{\int a(t)\mathrm{d}t}$ we write the following

$$\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\Phi(\tau)x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau$$

$$\Phi(t)x(t)-\Phi(t_0)x(t_0)=\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau \tag{153}$$

$$x(t)=[\Phi(t)]^{-1}\Phi(t_0)x_0+[\Phi(t)]^{-1}\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau$$

For a constant $a(t)=a$ we have $\Phi(t)=e^{at}$ and $\Phi(t_0)=e^{at_0}$ and thus we have

$$\begin{aligned}x(t)&=e^{-at}e^{at_0}x_0+e^{-at}\int_{t_0}^{t}e^{a\tau}bu(\tau)\mathrm{d}\tau=e^{a(t_0-t)}x_0+\int_{t_0}^{t}e^{a(\tau-t)}bu(\tau)\mathrm{d}\tau\\&=\Phi(t_0-t)x_0+\int_{t_0}^{t}\Phi(\tau-t)bu(\tau)\mathrm{d}\tau=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)bu(\tau)\mathrm{d}\tau\end{aligned} \tag{154}$$

Integer order system-solution classical multivariate system defined as Ω below

$$\Omega:\qquad (D_t x)(t)=\mathrm{A}(t)x(t)+\mathrm{B}(t)u(t)\qquad\qquad y(t)=\mathrm{C}(t)x(t)+\mathrm{D}(t)u(t) \tag{155}$$

$x(t)\in\mathbb{R}^{n\times 1}$; $u(t)\in\mathbb{R}^{p\times 1}$; $y(t)\in\mathbb{R}^{q\times 1}$; $\mathrm{A}\in\mathbb{R}^{n\times n}$; $\mathrm{B}\in\mathbb{R}^{n\times p}$; $\mathrm{C}\in\mathbb{R}^{q\times n}$; $\mathrm{D}\in\mathbb{R}^{q\times p}$ whose entries are continuous function of time. The solution is as follows

$$\begin{aligned}x(t)&=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau=\Phi(t,t_0)\left[x_0+\int_{t_0}^{t}\Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau\right]\\y(t)&=\mathrm{C}(t)\Phi(t,t_0)\left[x_0+\int_{t_0}^{t}\Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau\right]+\mathrm{D}(t)u(t)\end{aligned} \tag{156}$$

$\Phi(t_0,t)=e^{\int_{t_0}^{t}\mathrm{A}(\tau)\mathrm{d}\tau}$ is state the transition matrix (Green's function) of homogeneous system, that is $(D_t x)(t)=\mathrm{A}(t)x(t)$.

For (LTI) system $\Phi(t_0,t)=\Phi(t-t_0)=e^{\mathrm{A}(t-t_0)}=G(t-t_0)$ the homogeneous solution is given by $G(t-t_0)=\Phi(t-t_0)=e^{\mathrm{A}(t-t_0)}$ & particular solution is convolution by convolution as follows

$$x_p(t)=(G*^{t_0}\mathrm{B}u)(t)=\int_{t_0}^{t}G(t-\tau)\mathrm{B}u(\tau)\mathrm{d}\tau=\int_{t_0}^{t}e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau \tag{157}$$

We get state trajectory and output trajectory as solutions to Ω as depicted in following expressions

$$
\begin{aligned}
x(t) &= e^{\mathrm{A}t}x_0 + \int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau = e^{\mathrm{A}t}\left[x_0 + \int_0^t e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau\right] \\
y(t) &= \mathrm{C}e^{\mathrm{A}t}\left[x_0 + \int_0^t e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau\right] + \mathrm{D}u(t)
\end{aligned} \tag{158}
$$

For a constant $a(t) = a$, we have $\Phi(t) = e^{at}$ and $\Phi(t_0) = e^{at_0}$, and thus we have

$$
\begin{aligned}
x(t) &= e^{-at}e^{at_0}x_0 + e^{-at}\int_{t_0}^t e^{a\tau}bu(\tau)\mathrm{d}\tau = e^{a(t_0-t)}x_0 + \int_{t_0}^t e^{a(\tau-t)}bu(\tau)\mathrm{d}\tau \\
&= \Phi(t_0-t)x_0 + \int_{t_0}^t \Phi(\tau-t)bu(\tau)\mathrm{d}\tau = \Phi(t,t_0)x_0 + \int_{t_0}^t \Phi(t,\tau)bu(\tau)\mathrm{d}\tau
\end{aligned} \tag{159}
$$

If $a(t)$ is matrix $\mathrm{A}(t) \in \mathbb{R}^{n\times n}$, there are several ways to represent $e^{\int \mathrm{A}(t)\mathrm{d}t}$; which will be elucidated in this section with examples. Where we define state transition matrix; $\Phi(t_0-t) = e^{\int_t^{t_0} a(t)\mathrm{d}t} = \Phi(t,t_0)$, is also a Green's function of the homogeneous part of the system of differential equation.

One way is Matrix exponential as for constant A, as

$$
e^{\int \mathrm{A}(t)dt} = e^{\mathrm{A}t} \cong \mathrm{I} + \mathrm{A}t + \frac{\mathrm{A}^2t^2}{2} + \frac{\mathrm{A}^3t^3}{3!} + \dots. \tag{160}
$$

The other way is, via inverse Laplace that is

$$
e^{\mathrm{A}t} = \mathcal{L}^{-1}\{(s\mathrm{I}-\mathrm{A})^{-1}\}. \tag{161}
$$

For example as for illustration sake take the following linear time variant (LTV) system, represented as following state space form

$$
\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t) \tag{162}
$$

$$
\mathrm{A} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, x(0) = x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, x(1) = x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{163}
$$

given $u(t) = 5.8384e^{-2t} - 0.3026e^t$, we need to find $\overline{x}(t)$. Calculation of state transition Matrix Φ for this Linear Time Variant (LTV) system is demonstrated as follows from the two sets of coupled differential equations for states are;

$$
\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_1 + e^{-t}x_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}x_2 = -x_2 + u(t) \tag{164}
$$

Let us take the second state and solve the 'homogeneous' system that is $\dot{x}_2 = -x_2$. To solve this let us take a general differential equation $\dot{x}(t) + a(t)x(t) = y(t)$. Multiplying both sides by $e^{\int a(t)\mathrm{d}t}$; and with manipulation we get the form as $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t}x\right] = e^{\int a(t)\mathrm{d}t}y$. In our case that is $\dot{x} + x = 0$ we have $a(t) = 1$ and $y = 0$, so $e^{\int a(t)\mathrm{d}t} = e^t$. We write the above equation, and integrating from t_0 to t for our case for x_2 as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^t x_2\right] = e^t y \qquad \int_{t_0}^t \frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^\theta x_2(\theta)\right]\mathrm{d}\theta = \int_{t_0}^t e^\theta y(\theta)\mathrm{d}\theta \qquad e^t x_2(t) - e^{t_0}x_2(t_0) = \int_{t_0}^t e^\theta y(\theta)\mathrm{d}\theta \tag{165}
$$

Giving

$$x_2(t) = e^{-t}e^{t_0}x_2(t_0) + e^{-t}\int_{t_0}^{t} e^{\theta}y(\theta)\mathrm{d}\theta = e^{-(t-t_0)}x_2(t_0) \qquad y(\theta)=0 \tag{166}$$

Putting the solution of the second state's homogeneous equation into the first state equation we obtain the following;

$$\dot{x}_1 = x_1 + e^{-t}e^{-(t-t_0)}x_2(t_0) = x_1 + e^{-2t}\left(e^{t_0}x_2(t_0)\right) \qquad \dot{x}_1 - x_1 = e^{-2t}\left[e^{t_0}x_2(t_0)\right] \tag{167}$$

Comparing to $\dot{x} + a(t)x = y$, we get $a(t) = -1$, $e^{\int a(t)\mathrm{d}t} = e^{-t}$ and $y(t) = e^{-2t}\left(e^{t_0}x_2(t_0)\right)$. Using the procedure as done for state x_2, we write similarly for state x_1, the following expressions;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t}x_1\right] = e^{-t}y \qquad \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{-\theta}x_1(\theta)\right]\mathrm{d}\theta = \int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta \qquad e^{-t}x_1(t) - e^{-t_0}x_1(t_0) = \int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta$$

$$x_1(t) = e^{t-t_0}x_1(t_0) + e^{t}\int_{t_0}^{t}e^{-\theta}\left(e^{-2\theta}e^{t_0}x_2(t_0)\right)\mathrm{d}\theta = e^{(t-t_0)}x_1(t_0) + \left[\frac{1}{3}e^{(t-2t_0)} - \frac{1}{3}e^{(t_0-2t)}\right]x_2(t_0) \tag{168}$$

In matrix form the solution of homogeneous system as we obtained for the two state variables is expressed as following;

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{(t-t_0)} & \frac{1}{3}e^{(t-2t_0)} - \frac{1}{3}e^{(t_0-2t)} \\ 0 & e^{-(t-t_0)} \end{pmatrix}\begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \Phi(t,t_0)\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \tag{169}$$

Thus from the homogeneous system's solution we obtain the state transition matrix as

$$\Phi(t,\tau) = \begin{pmatrix} e^{t-\tau} & \frac{1}{3}\left(e^{t-2\tau} - e^{-2t+\tau}\right) \\ 0 & e^{-t+\tau} \end{pmatrix}; \qquad \Phi(0,\tau) = \begin{pmatrix} e^{-\tau} & \frac{1}{3}\left(e^{-2\tau} - e^{\tau}\right) \\ 0 & e^{\tau} \end{pmatrix} \tag{170}$$

The state trajectory due to above obtained control input is obtained as

$$\begin{aligned}\overline{x}(t) &= \begin{pmatrix} e^{t} & \frac{1}{3}\left(e^{t} - e^{-2t}\right) \\ 0 & e^{-t} \end{pmatrix}\left[\int_{0}^{t}\begin{pmatrix} \frac{1}{3}\left(e^{-2\tau} - e^{\tau}\right) \\ e^{\tau} \end{pmatrix}\left(5.8384e^{-2t} - 0.3026e^{t}\right)\mathrm{d}\tau\right] \\ &= \begin{pmatrix} 0.3856e^{t} - 0.1513 - 1.9966e^{-2t} + 1.4596e^{-2t} \\ -0.1513e^{t} + 5.9897e^{-t} - 5.8384e^{-2t} \end{pmatrix}\end{aligned} \tag{171}$$

Above example elucidates in detail the calculation of the state transition matrix for LTV system. The system when is Linear Time Invariant with matrix A a constant matrix, then simpler method is invoked via inverse Laplace transform to get to $\Phi(t)$. The equation $\dot{x} = \mathrm{A}x + \mathrm{B}u$ has fundamental solution for homogeneous system $\dot{x}(t) = \mathrm{A}x(t)$ as $\Phi(t) = e^{\mathrm{A}t} = \mathcal{L}^{-1}\left\{(s\mathrm{I} - \mathrm{A})^{-1}\right\}$; where s is complex frequency (Laplace variable), I is identity matrix .Let us elaborate with following example where, LTI system matrix is

$$\mathrm{A} = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\Phi(t)=\mathcal{L}^{-1}\left\{(s\mathrm{I}-\mathrm{A})^{-1}\right\}=\mathcal{L}^{-1}\left\{\left(\begin{bmatrix}s&0\\0&s\end{bmatrix}-\begin{bmatrix}-3&-2\\1&0\end{bmatrix}\right)^{-1}\right\}=\mathcal{L}^{-1}\left\{\begin{pmatrix}s+3&2\\-1&s\end{pmatrix}^{-1}\right\}$$

$$=\mathcal{L}^{-1}\left[\frac{1}{s(s+3)+2}\begin{pmatrix}s&-2\\1&s+3\end{pmatrix}\right]$$

$$=\quad\mathcal{L}^{-1}\begin{pmatrix}\frac{2}{s+2}-\frac{1}{s+1}&\frac{2}{s+2}-\frac{2}{s+1}\\-\frac{1}{s+2}+\frac{1}{s+1}&-\frac{1}{s+2}+\frac{2}{s+1}\end{pmatrix}\tag{172}$$

$$=\begin{pmatrix}2e^{-2t}-e^{-t}&2e^{-2t}-2e^{-t}\\-e^{-2t}+e^{-t}&-e^{-2t}+2e^{-t}\end{pmatrix}$$

This method will be used in the following section to get state transition matrix for the fractional order differential equation system as $\Phi_\alpha(t)=\mathcal{L}^{-1}\left\{(s^\alpha\mathrm{I}-\mathrm{A})^{-1}\right\}$, $\alpha\in[0,1)$

18. The Alpha-exponential functions

In the integer order calculus, the function $e^{\lambda t}$ plays an important role in solution of ordinary differential equations LTI systems; as it satisfies $(\mathrm{d}e^{\lambda t}/\mathrm{d}t)=\lambda e^{\lambda t}$. Similarly the alpha exponential function-1 satisfies; $x(t)=e_\alpha^{\lambda(t-a)}$, satisfies ${}_aD_t^\alpha x(t)=\lambda x(t)$, with RL derivative; and the alpha exponential function-2 $x(t)=\tilde{e}_\alpha^{\lambda(t-a)}$ satisfies ${}_a^CD_t^\alpha x(t)=\lambda x(t)$ with Caputo derivative. This we develop in this section. In the next section we used the notation Φ as 'state transition matrix' (associated Green's function) which is $\Phi(t)=e^{\mathrm{A}t}$ for LTI system of integer order differential equation system. For fractional order system we can (similarly) define state transition matrix as $\Phi_\alpha(t)=e_\alpha^{\mathrm{A}t}$, and $\tilde{\Phi}_\alpha(t)=\tilde{e}_\alpha^{\mathrm{A}t}$ where the notation, $e_\alpha^{\mathrm{A}t}$ is alpha-exponential function-1, and $\tilde{e}_\alpha^{\mathrm{A}t}$ is alpha-exponential function-2; which are also Green's function and also eigenvectors for RL and Caputo derivatives based homogeneous linear differential equations. Here we make use of the previous section's development, to get solution of the fractional order sequential linear differential equations (SFDE) in terms of these alpha-exponential functions.

The alpha-exponential functions follow from the higher transcendental functions of basic types as Mittag-Leffler function .The two parameter Mittag-Leffler function is defined by series as following;

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\qquad\alpha>0\qquad\beta>0\qquad z\in\mathbb{C}\tag{173}$$

$\Gamma(.)$ is Euler Gamma function. For matrix $\mathrm{A}\in\mathbb{R}^{n\times n}$ the above is extended for matrix case as following;

$$E_{\alpha,\beta}(\mathrm{A}t^\alpha)=\sum_{k=0}^{\infty}\mathrm{A}^k\frac{t^{k\alpha}}{\Gamma(k\alpha+\beta)}\tag{174}$$

Put $\beta=\alpha$ in above and we define alpha-exponential function-1 as

$$e_\alpha^{\mathrm{A}t}=(t^{\alpha-1})E_{\alpha,\alpha}(\mathrm{A}t^\alpha)=(t^{\alpha-1})\sum_{k=0}^{\infty}\mathrm{A}^k\frac{t^{k\alpha}}{\Gamma[(k+1)\alpha]}=\sum_{k=0}^{\infty}\mathrm{A}^k\frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\tag{175}$$

For $\alpha = 1$, we have $E_1(\mathrm{A}t) = e_1^{\mathrm{A}t} = e^{\mathrm{A}t} = \Phi(t)$, is state transition matrix of integer order LTI system with initial time $t_0 = 0$, as described in previous section. Put $\beta = 1$ in above we have one parameter Mittag-Leffler function as following

$$E_{\alpha,1}(\mathrm{A}t^{\alpha}) = E_{\alpha}(\mathrm{A}t^{\alpha}) = \sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{176}$$

Define second alpha-exponential function-2 as

$$\tilde{e}_{\alpha}^{\mathrm{A}t} = E_{\alpha,1}(\mathrm{A}t^{\alpha}) = \sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{177}$$

The alpha-exponential function-1; $e_{\alpha}^{\mathrm{A}t}$ is useful for solving sequential fractional differential equations (SFDE) with Riemann-Liouvelli (RL) fractional derivative, while the alpha-exponential-2; $\tilde{e}_{\alpha}^{\mathrm{A}t}$ is useful for solution of SFDE with Caputo derivative.

Since $e_{\alpha}^{\mathrm{A}t} = t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})$ and each Mittag-Leffler function $E_{\alpha,\alpha}(\lambda z^{\alpha})$, $\lambda > 0$ $z \in \mathbb{C}$ is an entire function on the complex plane, we can have an uniquely determined function $f(t) = t^{1-\alpha}F(t)$ such that $e_{\alpha}^{\mathrm{A}t} f(t) = E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})F(t) = \mathrm{I}$ for $t \neq 0$ and $\lim_{t\to 0} e_{\alpha}^{\mathrm{A}t} f(t) = \mathrm{I}$. For $\mathrm{A} = 0$;

$$e_{\alpha}^{\mathrm{A}t} = t^{\alpha-1}\left(\sum_{k=0}^{\infty} \frac{\mathrm{A}^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right) = t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^{\alpha}}{\Gamma(2\alpha)} + \ldots\right) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{178}$$

Therefore for this particular case, $f(t) = t^{1-\alpha}\Gamma(\alpha)$, so that identity condition gets satisfied.

19. Fractional Derivatives Riemann-Liouvelli and Caputo and their relation

Let us define a convolution kernel as a power law function and its Laplace depicted as

$$K_{\alpha}(t) \triangleq \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \qquad \alpha > 0 \qquad \mathcal{L}\{K_{\alpha}(t)\} = s^{-\alpha} \qquad \operatorname{Re} s > 0 \tag{179}$$

When $\alpha \to 1$, then $K_{\alpha} \to H(t)$ the Heaviside step function. When $\alpha \to 0$, then $K_{\alpha} \to \delta(t)$, the Dirac's delta function. The alpha exponential functions are then

$$e_{\alpha}^{\lambda t} = (t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \lambda^k K_{(1+k)\alpha} = t_+^{\alpha-1}\sum_{k=0}^{\infty} \frac{(\lambda t_+^{\alpha})^k}{\Gamma[(1+k)\alpha]} \qquad \mathcal{L}\left\{e_{\alpha}^{\lambda t}\right\} = (s^{\alpha} - \lambda)^{-1} \tag{180}$$

The alpha exponential function-1 is same as Robotnov-Hartley function; $F_{\alpha}(\lambda, t)$

$$\tilde{e}_{\alpha}^{\lambda t} = E_{\alpha}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \lambda^k K_{(1+k\alpha)} = \sum_{k=0}^{\infty} \frac{(\lambda t_+^{\alpha})^k}{\Gamma(1+k\alpha)} \qquad \mathcal{L}\left\{\tilde{e}_{\alpha}^{\lambda t}\right\} = s^{\alpha-1}(s^{\alpha} - \lambda)^{-1} \tag{181}$$

The alpha-exponential function-2 is one parameter Mittag-Leffler function. The symbol $(*)$ we use for convolution operation, and find interesting convolution link between these two alpha-exponential functions; that is

$$\tilde{e}_{\alpha}^{\lambda t} = E_{\alpha}(\lambda t^{\alpha}) = \{K_{(1-\alpha)}(t)\} * \{(t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^{\alpha})\} = \{K_{(1-\alpha)}\}(t) * \{e_{\alpha}^{\lambda t}\} \tag{182}$$

The two alpha exponential functions are same for $\alpha = 1$, that is exponential function

$$\tilde{e}_1^{\lambda t} = E_1(\lambda t) = \{e_1^{\lambda t}\} * \{K_0(t)\} = e_1^{\lambda t} = e^{\lambda t}; \qquad K_0(t) = \delta(t) \tag{183}$$

The causal convolution then defines fractional integration, as ${}_{0+}I_t^{\alpha} f \triangleq K_{\alpha} * f$. The kernel is Heaviside step function for $\alpha = 1$ and Dirac delta function for $\alpha = 0$ (in limit). Above is fractional integral of order α of a continuous causal function f. Interesting observation is that if f, is delta function then we have the convolution as $K_{\alpha} * \delta = K_{\alpha}$; meaning that ${}_{0+}I_t^{\alpha}\delta(t) = K_\alpha = t_+^{\alpha-1}/\Gamma(\alpha)$; the fractional integration of Dirac-delta function. From this definition of fractional integration, and above convolution relation of two alpha exponentials, we see that ${}_{0+}I_t^{(1-\alpha)}e_\alpha^{\lambda t} = \tilde{e}_\alpha^{\lambda t}$.

While we reverse the sign of α then $K_{-\alpha}$ gets defined as causal distribution as 'convolute inverse' of $K_{+\alpha}$ in 'convolution algebra' $D'_+(\mathbb{R})$ with the use of δ-Dirac distribution-which is neutral element of convolution operation; reads as $K_{+\alpha} * K_{-\alpha} = \delta$. The Laplace of $K_{-\alpha}(t)$ is s^α for $\operatorname{Re} s > 0$. With this preliminaries and notations we tend to define Fractional Derivative of order α of continuous causal function as $D^\alpha f \triangleq K_{-\alpha} * f$. In order to make this above definition tractable from a analytical point of view it proves useful to define 'smooth' fractional derivative operator for continuous f with first derivative f', for $0 < \alpha \le 1$, is as following;

$$d^\alpha f \triangleq D^\alpha f - f(0^+)K_{1-\alpha} = D^\alpha f - f(0^+)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \int_0^t K_{1-\alpha}(t-\tau) f'(\tau)\mathrm{d}\tau \qquad (184)$$

The difference between D^α and d^α is exactly the same as the one between derivation in sense of distribution (D^1) and classical derivation (d^1); namely $D^1 f = d^1 f + f(0^+)\delta$ .First we define the fractional integrals for f; as Riemann-Liouvelli fractional integration as follows;

$${}_{t_0+}I_t^\alpha = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t f(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau \qquad t > t_0 \qquad \alpha > 0 \qquad (185)$$

The above is causal integration (left sided integration), and below we write non causal integration (Weyl's integration-or right sided integration)

$${}_tI_{t_1-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^{t_1} f(\tau)(\tau-t)^{\alpha-1}\mathrm{d}\tau \qquad t < t_1 \qquad \alpha > 0 \qquad (186)$$

Identity operation (I) is defined as $I := {}_{t_0+}I_t^0 = {}_{t_1-}I_t^0$

The left sided (causal) Riemann-Liouvelli (RL) fractional derivative, for fractional order $\alpha \in \mathbb{R}^+$, $\alpha > 0$ and natural number, $n \in \mathbb{N}$ and $n = [\alpha] + 1$; where $[\alpha]$ denoting integer part of fractional number α; is as follows

$${}_{t_0+}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^t f(\tau)(t-\tau)^{n-\alpha-1}\mathrm{d}\tau \qquad (187)$$

Similarly the right sided (non-causal) RL fractional derivative is

$${}_tD_{t_1-}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^{t_1} f(\tau)(\tau-t)^{n-\alpha-1}\mathrm{d}\tau \qquad (188)$$

Let $\alpha \ge 0$, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$; and $n = \alpha$ if $\alpha \in \mathbb{N}_0$. $AC[t_0, t_1]$ is the space of functions that are absolutely continuous on $[t_0, t_1]$ and $AC^n[t_0, t_1]$ denote space of functions $f(t)$ that have continuous derivatives up to order $n-1$ on $[t_0, t_1]$ and such that $f^{(n-1)} \in AC[t_0, t_1]$. If $f \in AC^n[t_0, t_1]$; then Caputo derivatives are for $\alpha \notin \mathbb{N}_0$;

$${}_{t_0+}^{\ C}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t f^{(n)}(\tau)(t-\tau)^{n-\alpha-1}\mathrm{d}\tau = {}_{t_0+}I_t^{n-\alpha}\left(f^{(n)}(t)\right) \qquad (189)$$

$$ {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^{t_1} f^{(n)}(\tau)(\tau - t)^{n-\alpha-1} \mathrm{d}\tau = (-1)^n \, {}_{t}I_{t_1-}^{n-\alpha} \left(f^{(n)}(t) \right) \tag{190} $$

The relation between RL and Caputo's derivatives are following:

$$ {}_{t_0+}^{C}D_{t}^{\alpha} f(t) = {}_{t_0+}D_{t}^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)(t-t_0)^k}{k!} \right) \tag{191} $$

$$ {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = {}_{t}D_{t_1-}^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_1)(t_1-t)^k}{k!} \right) \tag{192} $$

For $0 < \alpha < 1, [\alpha] = 0, n = 1$ we get

$$ {}_{t_0+}^{C}D_{t}^{\alpha} f(t) = {}_{t_0+}D_{t}^{\alpha} \left(f(t) - f(t_0) \right) \qquad {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = {}_{t}D_{t_1-}^{\alpha} \left(f(t) - f(t_1) \right) \tag{193} $$

Expressions above in a way too gives Caputo derivative once RL fractional derivatives are defined as above, for a several manifolds differentiable function. Here we note that the ${}^{C}D^{\alpha}$ the Caputo derivative is same as d^{α} (derivation in classical sense) and RL derivative that is D^{α} same as derivation in distribution sense, as discussed previously.

20. Fractional Derivatives of alpha exponential functions

The Caputo derivative of $\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})$ we find as follows with Euler expression of Caputo derivative (and RL derivative) of the power function denoted as following. (Note that Caputo derivative of constant function is zero-but for RL derivative it is not)

$$ {}_{t_0+}^{C}D_{t}^{\alpha}(t-t_0)^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha} & \beta \neq 0 \\ 0 & \beta = 0 \end{cases} \tag{194} $$

We apply the above to the series-expression and get

$$ \begin{aligned} {}_{t_0+}^{C}D_{t}^{\alpha}\left[\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}\right] &= {}_{t_0+}^{C}D_{t}^{\alpha}\{E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})\} = {}_{t_0+}^{C}D_{t}^{\alpha} \sum_{k=0}^{\infty} \mathrm{A}^k \frac{(t-t_0)^{\alpha k}}{\Gamma(k\alpha+1)} \\ &= \sum_{k=1}^{\infty} \mathrm{A}^k \frac{(t-t_0)^{(k-1)\alpha}}{\Gamma[(k-1)\alpha+1]} = \mathrm{A}E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha}) = \mathrm{A}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} \end{aligned} \tag{195} $$

Therefore we have useful relation, from above calculations i.e. ${}_{t_0+}^{C}D_{t}^{\alpha}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{A}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}$; similar to exponential function in integer order differential equation where we have; $(\mathrm{d}e^{\lambda t}/\mathrm{d}t) = \lambda e^{\lambda t}$. This alpha-exponential $\tilde{e}_{\alpha}^{\mathrm{A}t}$ function-2 therefore is useful in solving fractional differential equation with Caputo derivative formulation. It follows that $d^{\alpha}\tilde{e}_{\alpha}^{\lambda t} = \lambda\tilde{e}_{\alpha}^{\lambda t}$ or ${}_{0+}^{C}D_{t}^{\alpha}[\tilde{e}_{\alpha}^{\lambda t}] = \lambda\tilde{e}_{\alpha}^{\lambda t}$, that is $f(t) = \tilde{e}_{\alpha}^{\lambda t}$ a fundamental solution (eigenvector) of the Caputo system ${}_{0+}^{C}D_{t}^{\alpha} f(t) = \lambda f(t)$, with λ as eigenvalue.

The RL derivative of $e_{\alpha}^{\mathrm{A}(t-t_0)}$ is evaluated by applying term by term the Euler formula and also using $\lim_{\alpha\to 0}(\Gamma(\alpha))^{-1} = 0$, to the series expression and we get the following

$$ \begin{aligned} {}_{t_0+}D_{t}^{\alpha} e_{\alpha}^{\mathrm{A}(t-t_0)} &= {}_{t_0+}D_{t}^{\alpha} \sum_{k=0}^{\infty} \mathrm{A}^k \frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \\ &= \sum_{k=1}^{\infty} \mathrm{A}^k \frac{(t-t_0)^{k\alpha-1}}{\Gamma(k\alpha)} = \mathrm{A}e_{\alpha}^{\mathrm{A}(t-t_0)} \end{aligned} \tag{196} $$

Using Weyl derivative formulas that is as below

$$
\begin{aligned}
&{}_tD_{T-}^{\alpha}(T-\tau)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(T-\tau)^{\beta-\alpha-1} \\
&\lim_{\beta\to\alpha} {}_tD_{T-}^{\beta}\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)} = \frac{\Gamma(\alpha)(T-\tau)^{\alpha-1-\beta}}{\Gamma(\beta-\alpha)\Gamma(\alpha)} = 0
\end{aligned}
\tag{197}
$$

We get useful eigenvector for Riemann-Liouvelli operator as below

$$
\begin{aligned}
{}_tD_{T-}^{\alpha}\left[e_{\alpha}^{\mathrm{A}(T-\tau)}\right] &= {}_tD_{T-}^{\alpha}\left(I\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \mathrm{A}\frac{(T-\tau)^{2\alpha-1}}{\Gamma(2\alpha)} + \dots\right) \\
&= \mathrm{A}\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)} + \mathrm{A}^2\frac{(T-\tau)^{2\alpha-1}}{\Gamma(2\alpha)} + \dots\dots\dots\dots\dots \\
&= (T-\tau)^{\alpha-1}\mathrm{A}\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(T-\tau)^{k\alpha}}{\Gamma[(k+1)\alpha]} = \mathrm{A}\left[e_{\alpha}^{\mathrm{A}(T-\tau)}\right]
\end{aligned}
\tag{198}
$$

Thus we have useful property, ${}_{t_0+}D_t^{\alpha}e_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{A}e_{\alpha}^{\mathrm{A}(t-t_0)}$, which states that for fractional differential equation involving RL derivative has solution in terms of alpha-exponential function-1 ($e_{\alpha}^{\mathrm{A}t}$).This alpha-exponential $e_{\alpha}^{\mathrm{A}t}$ function-1 therefore is useful in solving fractional differential equation with Riemann-Liouvelli derivative formulation. It follows that $D^{\alpha}[e_{\alpha}^{\lambda t}] = \lambda[e_{\alpha}^{\lambda t}]$ or ${}_{0+}D_t^{\alpha}[e_{\alpha}^{\lambda t}] = \lambda e_{\alpha}^{\lambda t}$, that is $f(t) = e_{\alpha}^{\lambda t}$ a fundamental solution (eigenvector) of the RL system ${}_{0+}D_t^{\alpha}f(t) = \lambda f(t) + \delta(t)$, with λ as eigenvalue.

Following interesting relation between the two alpha exponential functions is derived,

$$
\begin{aligned}
\int_{t_0}^{t}\mathrm{A}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{d}\tau &= \int_{t_0}^{t}\sum_{k=0}^{\infty}\mathrm{A}^{k+1}\frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\mathrm{d}\tau \\
&= \sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} = E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha}) - \mathrm{I} = \tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} - \mathrm{I}
\end{aligned}
\tag{199}
$$

$$
\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{I} + \int_{t_0}^{t}\mathrm{A}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{d}\tau \qquad \tilde{\Phi}_{\alpha}(t-t_0) = \mathrm{I} + \int_{t_0}^{t}\Phi_{\alpha}(t-\tau)\mathrm{A}\mathrm{d}\tau \tag{200}
$$

Denoting $\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t) = \tilde{e}_{\alpha}^{\mathrm{A}t}$ as state transition matrices or the Green's functions for homogeneous fractional multivariate dynamics with Riemann-Liouvelli and Caputo derivative formulations respectively, we have useful expression (200).

21. General Solution to Sequential Fractional Differential Equation with use of Alpha-exponential

The sequential fractional order differential equation we defined in earlier section and again we write a homogeneous SFDE as

$$
\left[{}_a\mathscr{D}_t^{n\alpha} + \sum_{k=0}^{n-1}a_k(t)\,{}_a\mathscr{D}_t^{k\alpha}\right]x(t) = 0 \qquad a\in\mathbb{R} \qquad a_k(t)\in C[a,b] \qquad {}_a\mathscr{D}_t^{k\alpha} = {}_a\mathscr{D}_t^{\alpha}\left({}_a\mathscr{D}_t^{(k-1)\alpha}\right) \tag{201}
$$

For $k = 2,3,.....n$. For constant coefficients, $a_k(t) = a_k$ we get LTI system, and the above system has indicial polynomial (characteristic polynomial) as $p(\lambda) = \lambda^n + \sum_{k=0}^{n-1}a_k\lambda^k$. Also we reiterate that $\mathscr{D}_t^{\alpha}$ can be of RL fractional derivative D_t^{α}; or Caputo type fractional derivative ${}^{C}D_t^{\alpha}$. This is one important fact about SFDE is that the indicial polynomial are integer order polynomial just as we get indicial polynomial for an integer order differential equation (linear, quadratic, cubic etc). Assume that $x_1(t), x_2(t),...x_n(t)$ are n functions, defined on $[a,b]$ they are called linearly dependent in $[a,b]$ if there exists constants $c_1, c_2,..., c_n$ that are not zero simultaneously, such that $c_1x_1(t)+c_2x(t)+...+c_nx_n(t)\equiv 0$, for $a\le t\le b$; else $x_1(t), x_2(t),...x_n(t)$ are called linearly independent in $[a,b]$. To check the linear dependence in fractional calculus context we use generalized Wronskian defined as

$$W_\alpha(t)=W_\alpha(x_1,x_2,...x_n)=\det\begin{pmatrix} x_1(t) & x_2(t) & ... & x_n(t) \\ {}_aD_t^{\alpha}x_1(t) & {}_aD_t^{\alpha}x_2(t) & ... & {}_aD_t^{\alpha}x_n(t) \\ ... & ... & ... & ... \\ {}_aD_t^{(n-1)\alpha}x_1(t) & {}_aD_t^{(n-1)\alpha}x_2(t) & ... & {}_aD_t^{(n-1)\alpha}x_n(t) \end{pmatrix} \qquad (202)$$

Let us assume that the fractional derivatives in SFDE are of RL type. If $x_1(t), x_2(t),...x_n(t)$ were solution to our SFDE, then we have ${}_aD_t^{\alpha}W_\alpha(t)+a_{n-1}W_\alpha(t)=0 \qquad a\le t\le b$. The general solution is thus $W_\alpha(t)=ce_\alpha^{-a_{n-1}(t-a)}=c(t-a)^{\alpha-1}E_{\alpha,\alpha}(-a_{n-1}(t-a)^n)$, with c as constant. This we have got from previous sections development. Therefore, the solutions $x_1(t), x_2(t),...x_n(t)$, of SFDE are linearly dependent in $[a,b]$ if and only if there is a $t_0\in[a,b]$ for which $W_\alpha(t_0)=0$.

If the indicial polynomial $p(\lambda)=\lambda^n+\sum_{k=0}^{n-1}a_k\lambda^k$, has n different $\lambda_1,\lambda_2,...\lambda_n$ simple roots, then the corresponding SFDE with RL formulation will have $x_1(t)=e_\alpha^{\lambda_1(t-a)}$ $x_2(t)=e_\alpha^{\lambda_2(t-a)}$ .... $x_n(t)=e_\alpha^{\lambda_n(t-a)}$ corresponding particular solutions, and the general solution to the SFDE is $x(t)=c_1e_\alpha^{\lambda_1(t-a)}+c_2e_\alpha^{\lambda_2(t-a)}+.....+c_ne_\alpha^{\lambda_n(t-a)}$. The $c_1, c_2,...c_n$ are arbitrary constants. If the indicial polynomial $p(\lambda)=\lambda^n+\sum_{k=0}^{n-1}a_k\lambda^k$, has repeated roots with multiplicity $l,(1<l\le n)$ then $e_\alpha^{\lambda(t-a)},\frac{\partial}{\partial\lambda}e_\alpha^{\lambda(t-a)},\frac{\partial^2}{\partial\lambda^2}e_\alpha^{\lambda(t-a)},..........,\frac{\partial^{l-1}}{\partial\lambda^{l-1}}e_\alpha^{\lambda(t-a)}$ are l linearly independent solutions of SFDE.

Let $\lambda_1,\lambda_2,...,\lambda_k$ be different roots with the multiplicities $l_1, l_2,....,l_k$ respectively, with $l_1+l_2+...+l_n=n$, then general solution of the SFDE is the linear combinations of the following fundamental solutions:

$$\begin{matrix} e_\alpha^{\lambda_1(t-a)}, & \frac{\partial}{\partial\lambda_1}e_\alpha^{\lambda_1(t-a)}, & \frac{\partial^2}{\partial{\lambda_1}^2}e_\alpha^{\lambda_1(t-a)}, & ... & \frac{\partial^{l_1-1}}{\partial{\lambda_1}^{l_{1-1}}}e_\alpha^{\lambda_1(t-a)} \\ e_\alpha^{\lambda_2(t-a)}, & \frac{\partial}{\partial\lambda_2}e_\alpha^{\lambda_2(t-a)}, & \frac{\partial^2}{\partial{\lambda_2}^2}e_\alpha^{\lambda_2(t-a)}, & ... & \frac{\partial^{l_2-1}}{\partial{\lambda_2}^{l_{2-1}}}e_\alpha^{\lambda_2(t-a)} \\ ... & ... & ... & ... & ... \\ e_\alpha^{\lambda_k(t-a)} & \frac{\partial}{\partial\lambda_k}e_\alpha^{\lambda_k(t-a)} & \frac{\partial^2}{\partial{\lambda_k}^2}e_\alpha^{\lambda_k(t-a)} & ... & \frac{\partial^{l_k-1}}{\partial{\lambda_k}^{l_{k-1}}}e_\alpha^{\lambda_k(t-a)} \end{matrix} \qquad (203)$$

If the SFDE is formed with Caputo derivative then we get similar solutions as in above cases with alpha exponential function-2, that is $\tilde{e}_\alpha^{\lambda(t-a)}=E_{\alpha,1}(\lambda(t-a)^\alpha)$.

To illustrate let us take example of equation of motion with fractional damping term as

$\ddot{x}(t)+\mu\,{}_0D_t^{\alpha}x(t)+x(t)=0 \qquad \mu>0 \qquad \alpha=1/2 \quad \text{or} \quad \alpha=3/2$

The above homogeneous equation is called Torvik-Bagley equation, is studied extensively in visco-elasticity. The parameter μ is fractional viscous coefficient. Clearly this equation can be casted into SFDE as

$$ {}_0\mathscr{D}_t^{4\alpha}x(t)+\mu\,{}_0\mathscr{D}_t^{\alpha}x(t)+x(t)=0 \text{ for } \alpha=1/2 \qquad (204)$$

$$ {}_0\mathscr{D}_t^{4\beta}x(t)+\mu\,{}_0\mathscr{D}_t^{3\beta}x(t)+x(t)=0 \text{ for } \alpha=3/2 \text{ and } \beta=1/2=\alpha/3 \qquad (205)$$

The indicial polynomial reads as $p(\lambda)=\lambda^4+\mu\lambda^k+1$ with $k=1$ for $\alpha=1/2$, and $k=3$ for $\alpha=3/2$. We take case for $k=1$; then $p(\lambda)=0=\lambda^4+\mu\lambda+1$, has four roots (real or complex, distinct or repeated) depending on the value of μ. For a specific case of value μ, we have repeated real roots $\lambda_1=\lambda_2$ and a pair of complex call them $\lambda_{3.4}$. For this combination we write general solution as

$$x(t)=c_1x_1(t)+c_2x_2(t)+c_3x_3(t)+\bar{c}_3\bar{x}_3(t) \qquad (206)$$

With, $c_1,c_2\in\mathbb{R}$, $c_3\in\mathbb{C}$ as arbitrary constants. For case of SFDE with $\alpha=1/2$ and with Caputo derivative, $\mathscr{D}^{\alpha}\equiv{}^{C}D^{\alpha};\alpha=1/2$ we take $x_i(t)=\tilde{e}_{1/2}^{\lambda_i t},(i=1,3)$. Then

$$x_2(t)=\frac{\partial}{\partial\lambda_1}\tilde{e}_{1/2}^{\lambda_1 t}=\frac{t^{1/2}}{\Gamma(3/2)}+\frac{2\lambda_1 t}{\Gamma(2)}+\frac{3\lambda_1^2t^2}{\Gamma(5/2)}+\frac{4\lambda_1^3t^2}{\Gamma(3)}+\ldots \qquad (207)$$

To find the general solution $x(t)$ for the SFDE of Torvik-Bagley system, it is required to study the asymptotic behavior of $x(t)$ as $t\to0^+$. For this mechanical systems it holds $|\dot{x}(0)|<\infty$ and $|\ddot{x}(0)|<\infty$ definitely, or equivalently the terms involving $t^{1/2}$ and involving $t^{3/2}$ vanishes in $x(t)$; namely

$$\begin{cases} c_1\lambda_1+c_2+c_3\lambda_3+\bar{c}_3\bar{\lambda}_3=0 \\ c_1\lambda_1^3+3c_2\lambda_1^2+c_3\lambda_3^3+\bar{c}_3\bar{\lambda}_3^3=0 \end{cases} \qquad \mathrm{Re}(c_3)=\frac{\sqrt[4]{3}}{6}c_2 \qquad \mathrm{Im}(c_3)=\frac{-\sqrt{2}}{12}(3c_1-4\sqrt[4]{3}c_2) \qquad (208)$$

With such complex number c_3 the arbitrary real constants c_1,c_2 in the general solution $x(t)$ of the Torvik-Bagley SFDE equation are determined by initial condition $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$.

For case of SFDE with $\alpha=1/2$ and with RL derivative, $\mathscr{D}^{\alpha}\equiv D^{\alpha};\alpha=1/2$ we take $x_i(t)=e_{1/2}^{\lambda_i t},(i=1,3)$. Then

$$x_2(t)=\frac{\partial}{\partial\lambda_1}e_{1/2}^{\lambda_1 t}=\frac{1}{\Gamma(1)}+\frac{2\lambda_1t^{1/2}}{\Gamma(3/2)}+\frac{3\lambda_1^2t}{\Gamma(2)}+\frac{4\lambda_1^3t^{3/2}}{\Gamma(5/2)}+\ldots \qquad (209)$$

To find the general solution $x(t)$ for the SFDE of Torvik-Bagley system, it is required to study the asymptotic behavior of $x(t)$ as $t\to0^+$. In order that $|x(0)|<\infty$ and $|\dot{x}(0)|<\infty$ the two terms appearing in $x(t)$ that are

$(c_1+c_3+\bar{c}_3)\dfrac{t^{-1/2}}{\Gamma(1/2)} \qquad (c_1\lambda_1^2+2c_2\lambda_1+c_3\lambda_3^2+\bar{c}_3\bar{\lambda}_3^2)\dfrac{t^{1/2}}{\Gamma(3/2)}$ should be zero. Therefore,

$$\begin{cases} c_1+c_3+\bar{c}_3=0 \\ c_1\lambda_1^2+2c_2\lambda+c_3\lambda_3^2+\bar{c}_3\bar{\lambda}_3^2=0 \end{cases} \qquad \mathrm{Re}(c_3)=-\frac{1}{2}c_1 \qquad \mathrm{Im}(c_3)=\frac{1}{2\sqrt{2}}(c_1-\sqrt[4]{3}c_2) \qquad (210)$$

With such complex number c_3 the arbitrary real constants c_1,c_2 in the general solution $x(t)$ of the Torvik-Bagley SFDE equation are determined by initial condition $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$.

22. Solution of fractional order differential equation with R-L derivative and Caputo derivative with state transition matrix with alpha-exponential functions 1 & 2

A non-homogeneous fractional differential equation with RL fractional derivative and with

$$0<\alpha\le 1;\ (D^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)+\mathrm{B}(t);\ \mathrm{Y}_0=\mathrm{Y}(t_0) \tag{211}$$

General Solution

$$\begin{aligned}\mathrm{Y}(t)&=e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Y}_0+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\mathrm{d}\tau\\&=\Phi_{\alpha}(t-t_0)+\int_{t_0}^{t}\Phi_{\alpha}(t-\tau)\mathrm{B}(\tau)\mathrm{d}\tau\end{aligned} \tag{212}$$

Where $G_{\alpha}(t-\tau)=\Phi_{\alpha}(t-\tau)=e_{\alpha}^{\mathrm{A}(t-\tau)}$ is Green's function for RL-derivative $\mathrm{Y}_p(t)=(G_{\alpha}*^{t_0}\mathrm{B})(t)=\int_{t_0}^{t}G_{\alpha}(t-\tau)\mathrm{B}(\tau)\mathrm{d}\tau$ is particular solution

Fractional differential equation with Caputo derivative for $0<\alpha\le 1$,

$$({}_{t_0+}^{\;C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t) \tag{213}$$

with, $\mathrm{Y}(t_0)=b$ has general solution

$$\mathrm{Y}(t)=b+\int_{t_0}^{t}\left(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b\right)\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right) \tag{214}$$

$({}_{t_0}^{C}D_t^{\alpha}f)(t)={}_{t_0}D_t^{\alpha}[f(t)-f(t_0)]$ is RL-Caputo relation for $0<\alpha\le 1$. Using this we get

$$({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)\qquad {}_{t_0}D_t^{\alpha}[\mathrm{Y}(t)-b]=\mathrm{AY}(t) \tag{215}$$

Put $\mathrm{Y}(t)=\mathrm{Z}(t)+b$ thus we have $\mathrm{Z}(t_0)=0=\mathrm{Z}_0$. Thus the equivalent equation in RL derivative based fractional differential equation is;

$$({}_{t_0}D_t^{\alpha}\mathrm{Z})(t)=\mathrm{A}[\mathrm{Z}(t)+b]=\mathrm{AZ}(t)+\mathrm{A}b \tag{216}$$

whose solution we know from just previous derivation with RL, and we thus write the solution as

$$\mathrm{Z}(t)=e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Z}_0+\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=\mathrm{Y}(t)-b \tag{217}$$

We have thus the result now that is;

$$\mathrm{Y}(t)=b+\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right) \tag{218}$$

Thus FOS with Caputo requires-two Green's functions in solution
Similarly for system

$$({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)+\mathrm{B}(t) \tag{219}$$

with $\mathrm{Y}(t_0)=b$ as constant the solution we write as;

$$\mathrm{Y}(t)=b+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\left[\mathrm{B}(\tau)+\mathrm{A}b\right]\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right)+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\mathrm{d}\tau \tag{220}$$

which is also $Y(t)=b\tilde{\Phi}_\alpha(t)+\int_{t_0}^{t}\Phi_\alpha(t-\tau)B(\tau)d\tau$ Where state transition matrices are, $\Phi_\alpha(t)=e_\alpha^{At}$ and $\tilde{\Phi}_\alpha(t)=\tilde{e}_\alpha^{At}$ .The solution with Caputo's formulation requiring thus two Green's functions (state transition matrices)!

$$:\qquad {}_{0+}^{C}D_t^\alpha x(t)=u(t);\quad \alpha\in(0,1];\quad x(0)=a\in\mathbb{R};\quad x(T)=b\in\mathbb{R};\quad T>0 \tag{221}$$

In terms of system matrix equation, we have

$$^{C}D^\alpha x(t)=Ax(t)+Bu(t) \tag{222}$$

in this case $A=0;\quad B=1$. Here

$$\Phi_\alpha(t)=e_\alpha^{At}=t^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right)=t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+A\frac{t^\alpha}{\Gamma(2\alpha)}+...\right)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\qquad ;A=0 \tag{223}$$

The integral $\tilde{\Phi}_\alpha(t)=I+\int_0^t e_\alpha^{A\tau}Ad\tau=I=1\,\text{for}\,A=0$. Therefore the state trajectory of system-the solution is

$$x(t)\Big|_0^T=x(0)\tilde{\Phi}_\alpha(t)+\int_0^T\Phi_\alpha(t-\tau)u(\tau)d\tau=a+\frac{1}{\Gamma(\alpha)}\int_0^T(T-t)^{\alpha-1}u(t)dt \tag{224}$$

Here we formalize what we did in above, and use the results, with alpha exponential functions. Consider a linear time invariant control system denoted by $\sum$ of fractional commensurate order α, where $0<\alpha\le 1$.

$$\sum:\qquad {}_{0+}^{C}D_t^\alpha x(t)=Ax(t)+Bu(t)\qquad y(t)=Cx(t) \tag{225}$$

By commensurate order means, for each component the same fractional order of α is used. For a function $x:[0.T]\to\mathbb{R}^{n\times 1}$

$${}_{0+}^{C}D_t^\alpha x(t)={}_{0+}^{C}D_t^\alpha\begin{pmatrix}x_1(t)\\ \cdot\\ \cdot\\ x_n(t)\end{pmatrix}=\begin{pmatrix}{}_{0+}^{C}D_t^\alpha x_1(t)\\ \\ \\ {}_{0+}^{C}D_t^\alpha x_n(t)\end{pmatrix} \tag{226}$$

Where, $x(t)\in\mathbb{R}^{n\times1}$ $u(t)\in\mathbb{R}^{m\times1}$ matrix $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{p\times n}$. The Caputo fractional derivative $^{C}D_t^\alpha$ is used. The control is $u(t)\in\mathbb{R}^{m\times1}$, the state is $x(t)\in\mathbb{R}^{n\times1}$ the output (or observation) is $y(t)\in\mathbb{R}^{p\times1}$. The forward trajectory of the system $\sum$ starting at $t_0=0$ and evaluated at $t\ge 0$ is initial value problem of Fractional Differential equation ${}_{0+}^{C}D_t^\alpha x(t)=Ax(t)+Bu(t)$, given $x(0)=a$, where $a\in\mathbb{R}^{n\times1}$ is given as, described in above, is expressed as

$$x(t)=\left(I+\int_0^t\Phi_\alpha(\tau)Ad\tau\right)a+\int_0^t\Phi_\alpha(t-\tau)Bu(\tau)d\tau \tag{227}$$

In above $\Phi_\alpha(t)=e_\alpha^{At}$. Another way to write above expression is given below as

$$x(t)=\tilde{\Phi}_\alpha(t)a+\int_0^t\Phi_\alpha(t-\tau)Bu(\tau)d\tau\ \ ;\ \tilde{\Phi}_\alpha(t)=E_\alpha(At^\alpha)=I+\int_0^t\Phi_\alpha(\tau)Ad\tau \tag{228}$$

$$y(t)=Cx(t)=C\left(I+\int_0^t\Phi_\alpha(\tau)Ad\tau\right)a+C\int_0^t\Phi_\alpha(t-\tau)Bu(\tau)d\tau \tag{229}$$

Let system $\sum 1$ defined as

$$\sum 1: \quad \begin{cases} {}^{C}D^{0.5}_{0+}x_1(t) = x_2(t) \\ {}^{C}D^{0.5}_{0+}x_2(t) = u(t) \end{cases} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{230}$$

Take initial point $a = (1,0)^*$, final point as origin $b = (0,0)^*$.

$$\begin{aligned} \Phi_\alpha(t) &= \mathcal{L}^{-1}\left\{(s^{0.5}\mathrm{I}-\mathrm{A})^{-1}\right\} = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{s} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} \sqrt{s} & -1 \\ 0 & \sqrt{s} \end{pmatrix}^{-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s}\begin{pmatrix} \sqrt{s} & 1 \\ 0 & \sqrt{s} \end{pmatrix}\right\} = \mathcal{L}^{-1}\begin{pmatrix} \frac{1}{\sqrt{s}} & \frac{1}{s} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix} \end{aligned} \tag{231}$$

Using $\mathcal{L}^{-1}\{s^{-\alpha}\} = t^{\alpha-1}/\Gamma(\alpha)$; $\quad \mathcal{L}^{-1}\{s^{-0.5}\} = 1/\sqrt{\pi t}$, and $\mathcal{L}\left\{\begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}\right\} = \frac{1}{s}$, we write

$$\Phi_\alpha(t) = \begin{pmatrix} \frac{1}{\sqrt{\pi t}} & 1 \\ 0 & \frac{1}{\sqrt{\pi t}} \end{pmatrix} \tag{232}$$

$$\tilde{\Phi}_\alpha(t) = \mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi\tau}} & 1 \\ 0 & \frac{1}{\sqrt{\pi\tau}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathrm{d}\tau = \begin{pmatrix} 1 & 2\sqrt{\frac{t}{\pi}} \\ 0 & 1 \end{pmatrix} \tag{233}$$

Using formulas as obtained above we get state trajectory as

$$x(t) = \begin{pmatrix} 1 & \frac{2\sqrt{t}}{\sqrt{\pi}} \\ 0 & 1 \end{pmatrix} a + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi(t-\tau)}} & 1 \\ 0 & \frac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix} \mathrm{B}u(\tau)\mathrm{d}\tau \tag{234}$$

Taking $u(t) \equiv 1$, then for a given a, $x(t) = \begin{pmatrix} 1+t & 2\sqrt{\frac{t}{\pi}} \end{pmatrix}^*$. Meaning for a constant $u(.) \equiv 1$ for $t > 0$ we cannot steer the given initial condition a to the origin.

For $0 < \alpha < 1$ a system in $\mathbb{R}^3$ that is $\sum 2$ is described as

$$\sum 2: \quad \begin{cases} {}^{C}D^{\alpha}_{0+}x_1(t) = x_2(t) \\ {}^{C}D^{\alpha}_{0+}x_2(t) = -x_1(t) + u(t) \end{cases} \qquad \mathrm{A}^0 = \mathrm{I} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathrm{A}^2 = -\mathrm{I} \tag{235}$$

The system matrix is skew symmetric; hence $\mathrm{A}^k = \mathrm{I}$ for $k = 0,4,8,..$, $\mathrm{A}^k = \mathrm{A}$ for $k = 1,5,9,...$ and $\mathrm{A}^k = -\mathrm{I}$ for $k = 3,7,11,...$. Also

$$\begin{aligned} \Phi_\alpha(t) &= t^{\alpha-1}\left(\mathrm{I}\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)} - \mathrm{I}\frac{t^{2\alpha}}{\Gamma(3\alpha)} - \mathrm{A}\frac{t^{3\alpha}}{\Gamma(4\alpha)} + ...\right) \\ &= \mathrm{I}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + ..\right) + \mathrm{A}\left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{4\alpha-1}}{\Gamma(4\alpha)} + ...\right) \end{aligned} \tag{236}$$

Use the following notation, to simplify above expression as following

$$\sin_\alpha t = \left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{4\alpha-1}}{\Gamma(4\alpha)} + ... \right) \qquad \cos_\alpha t = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} + ... \right) \tag{237}$$

And write the solution as

$$\Phi_\alpha(t) = \mathrm{I}\sin_\alpha t + \mathrm{A}\cos_\alpha t = \begin{pmatrix} \sin_\alpha t & \cos_\alpha t \\ -\cos_\alpha t & \sin_\alpha t \end{pmatrix} \tag{238}$$

23. Solution of fractional order differential modeling visco-elasticity

Proceeding for the solution of the four parameter generalized model

$$\begin{gathered} \sigma(t) + b\left(D^\alpha \sigma\right)(t) = \mathrm{E}_0 \varepsilon(t) + \mathrm{E}_1\left(D^\alpha \varepsilon\right)(t) \\ 0 < \alpha \le 1, \quad b \ge 0, \quad \mathrm{E}_0 \ge 0, \quad \mathrm{E}_1 > 0, \quad b \le \mathrm{E}_1 / \mathrm{E}_0 \end{gathered} \tag{239}$$

We will obtain solution for strain for known stress and then solution for stress for known strain

1. Solution for a known tension $\sigma(t)$

$$\left(D^\alpha z\right)(t) + \lambda z(t) = f(t) \tag{240}$$

With $z(t) = \frac{b}{\mathrm{E}_1}\sigma(t) - \varepsilon(t)$; $\lambda = \frac{\mathrm{E}_0}{\mathrm{E}_1}$; $f(t) = A\sigma(t)$; $A = \frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}$. The general solution of the above, when $D^\alpha \equiv {}_a D^\alpha_{t,+}$ if $\sigma(t)$ is continuous and integrable in (a,t), is

$$z(t) = C e_\alpha^{-\lambda(t-a)} + A\int_a^t e_\alpha^{-\lambda(t-\xi)}\sigma(\xi)\mathrm{d}\xi \tag{241}$$

Where C is arbitrary real constant. Therefore the general solution to the above model (with change of variables to original one), we have

$$\varepsilon(t) = \frac{b}{\mathrm{E}_1}\sigma(t) - C e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-a)} - \frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}\int_a^t e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-\xi)}\sigma(\xi)\mathrm{d}\xi \tag{242}$$

Moreover if $C = 0$ meaning $z(a) = 0$ i.e. $\varepsilon(a) = \frac{b}{\mathrm{E}_1}\sigma(a)$. In particular $\varepsilon(a) = \sigma(a) = 0$ and while for the case in the $\lim_{t\to a+}(t-a)^{1-\alpha}\sigma(t) = k_1$ and $\lim_{t\to a+}(t-a)^{1-\alpha}\varepsilon(t) = k_2$, the solution is :

$$\varepsilon(t) = \frac{b}{\mathrm{E}_1}\sigma(t) - C e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-a)} - \frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}\int_a^t e_\alpha^{-\frac{\mathrm{E}_0}{\mathrm{E}_1}(t-\xi)}\sigma(\xi)\mathrm{d}\xi \quad \text{with} \quad C = \Gamma(\alpha)\left[\frac{b}{\mathrm{E}_1}k_1 - k_2\right] \tag{243}$$

In addition we can write the general solution of the equation $\left(D^\alpha z\right)(t) + \lambda z(t) = f(t)$, with Caputo derivative $D^\alpha z \equiv {}^C_a D^\alpha_{t,+} z$, from the derived case as:

$$z(t) = k + \int_a^t e_\alpha^{-\lambda(t-\xi)}\left[A\sigma(\xi) - \lambda k\right]\mathrm{d}\xi \qquad z(a) = k \tag{244}$$

The corresponding solution to

$$\begin{gathered} \sigma(t) + b\left({}^C D^\alpha \sigma\right)(t) = \mathrm{E}_0 \varepsilon(t) + \mathrm{E}_1\left({}^C D^\alpha \varepsilon\right)(t) \\ 0 < \alpha \le 1, \quad b \ge 0, \quad \mathrm{E}_0 \ge 0, \quad \mathrm{E}_1 > 0, \quad b \le \mathrm{E}_1 / \mathrm{E}_0 \end{gathered} \tag{245}$$

is

$$\varepsilon(t) = k + \frac{b}{\mathrm{E}_1}\sigma(t) - \int_a^t e_\alpha^{-\lambda(t-\xi)}\left(\frac{b\mathrm{E}_0 - 1}{\mathrm{E}_1^2}\sigma(\xi) - \lambda k\right)\mathrm{d}\xi \tag{246}$$

2. Solution for known unit strain (deformation) $\varepsilon(t)$

$$\left(D^{\alpha}y\right)(t)+\gamma y(t)=g(t)$$
$$y(t)=\frac{\mathrm{E}_1}{b}\varepsilon(t)-\sigma(t);\quad \gamma=\frac{1}{b};\quad g(t)=B\varepsilon(t);\quad B=\frac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2} \tag{247}$$

If $\varepsilon(t)$ is continuous and integrable in (a,t) then the general solution of above with RL derivative is:

$$y(t)=Ce_{\alpha}^{-\gamma(t-a)}+B\int_a^t e_{\alpha}^{-\gamma(t-\xi)}\varepsilon(\xi)\mathrm{d}\xi \tag{248}$$

Therefore the general solution to the equation with all RL derivatives is

$$\sigma(t)+b\left(D^{\alpha}\sigma\right)(t)=\mathrm{E}_0\varepsilon(t)+\mathrm{E}_1\left(D^{\alpha}\varepsilon\right)(t)$$
$$0<\alpha\le 1,\quad b\ge 0,\quad \mathrm{E}_0\ge 0,\quad \mathrm{E}_1>0,\quad b\le \mathrm{E}_1/\mathrm{E}_0 \tag{249}$$

$$\sigma(t)=\frac{\mathrm{E}_1}{b}\varepsilon(t)-Ce_{\alpha}^{-\frac{1}{b}(t-a)}-\frac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\int_a^t e_{\alpha}^{-\frac{1}{b}(t-\xi)}\varepsilon(\xi)\mathrm{d}\xi \tag{250}$$

In addition if $C=0$ then $\sigma(a)=\frac{\mathrm{E}_1}{b}\varepsilon(a)$. In particular $\sigma(a)=\varepsilon(a)=0$ if $C=0$. Moreover while for the case $\lim_{t\to a+}(t-a)^{1-\alpha}\sigma(t)=k_1$ and $\lim_{t\to a+}(t-a)^{1-\alpha}\varepsilon(t)=k_2$, the solution is :

$$\sigma(t)=\frac{\mathrm{E}_1}{b}\varepsilon(t)-Ce_{\alpha}^{-\frac{1}{b}(t-a)}-\frac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\int_a^t e_{\alpha}^{-\frac{1}{b}(t-\xi)}\varepsilon(\xi)\mathrm{d}\xi\qquad C=\Gamma(a)\left(\frac{\mathrm{E}_1}{b}k_2-k_1\right) \tag{251}$$

For the case with Caputo derivative for $\left(D^{\alpha}y\right)(t)+\gamma y(t)=g(t)$ with $D^{\alpha}y\equiv {}_a^C D_{t,+}^{\alpha}y$, we write the solution as

$$y(t)=k+\int_a^t e_{\alpha}^{-\gamma(t-\xi)}\left[B\varepsilon(\xi)-\gamma k\right]\mathrm{d}\xi\qquad y(a)=k \tag{252}$$

Corresponding solution to the equation with Caputo formulation is:

$$\sigma(t)+b\left({}^C D^{\alpha}\sigma\right)(t)=\mathrm{E}_0\varepsilon(t)+\mathrm{E}_1\left({}^C D^{\alpha}\varepsilon\right)(t)$$
$$0<\alpha\le 1,\quad b\ge 0,\quad \mathrm{E}_0\ge 0,\quad \mathrm{E}_1>0,\quad b\le \mathrm{E}_1/\mathrm{E}_0 \tag{253}$$

is

$$\sigma(t)=k+\frac{\mathrm{E}_1}{b}\varepsilon(t)-\int_a^t e_{\alpha}^{-\gamma(t-\xi)}\left[\left(\frac{\mathrm{E}_1-\mathrm{E}_0^2}{b^2}\right)\varepsilon(\xi)-\gamma k\right]\mathrm{d}\xi \tag{254}$$

24. Generalization of fractional Berger's model of visco-elasticity with sequential fractional derivative

Case-I

We had the equation

$$\left(D^{\alpha}z\right)(t)+\lambda z(t)=f(t) \tag{255}$$

we generalize this as

$$\left(\mathscr{D}^{n\alpha}z\right)(t)+\sum_{j=0}^{n-1}\lambda_j\left(\mathscr{D}^{j\alpha}z\right)(t)=K\sigma(t)\qquad z(t)=\beta\sigma(t)-\varepsilon(t) \tag{256}$$

With parameters α, K, β and $\lambda_j\in\mathbb{R}\quad (j=1,2,3...n-1)$, or

$$\left(\mathscr{D}^{n\alpha}z\right)(t)+\sum_{j=0}^{n-1}\eta_j\left(\mathscr{D}^{j\alpha}z\right)(t)=\mathrm{E}\varepsilon(t)\qquad z(t)=\gamma\varepsilon(t)-\sigma(t)\tag{257}$$

With parameters α, E, γ and $\eta_j\in\mathbb{R}$ $(j=1,2,3...n-1)$.

The model above with sequential fractional derivative can be expressed as follows:

$$\left(D^{\alpha}\overline{\mathrm{Z}}\right)(t)=\mathrm{A}\overline{\mathrm{Z}}(t)+\overline{\mathrm{B}}(t)\tag{258}$$

where $D^{\alpha}y$ is RL derivative or Caputo derivative, and

$$\mathrm{A}=\begin{pmatrix}0&1&0&.&.&.&0&0\\0&0&1&.&.&.&0&0\\...&.&.&.&.&.&.&.\\...&.&.&.&.&.&.&.\\...&.&.&.&.&.&.&.\\...&.&.&.&.&.&.&.\\0&0&0&.&.&.&0&1\\-\lambda_0&-\lambda_1&-\lambda_2&.&.&.&-\lambda_{n-2}&-\lambda_{n-1}\end{pmatrix}\quad \overline{\mathrm{B}}(t)=K\begin{pmatrix}0\\0\\0\\0\\...\\0\\0\\\sigma(t)\end{pmatrix}\quad \overline{\mathrm{Z}}(t)=\begin{pmatrix}z_1(t)\\...\\...\\...\\...\\...\\...\\z_n(t)\end{pmatrix}\tag{259}$$

With $z_j(t)=\left({}_aD_t^{(j-1)\alpha}z\right)(t);\quad j=1,2,...n$ or $z_j(t)=\left({}_a^CD_t^{(j-1)\alpha}z\right)(t);\quad j=(1,2,...,n)$. The general solution is

$$\overline{\mathrm{Z}}(t)=\overline{\mathrm{C}}e_{\alpha}^{\mathrm{A}(t-a)}+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\overline{\mathrm{B}}(\xi)\mathrm{d}\xi\tag{260}$$

Where $z_1(t)=z(t)=\beta\sigma(t)-\varepsilon(t);\quad \beta\in\mathbb{R}$ and $\overline{\mathrm{C}}$ is arbitrary constant vector.

Case-II

$$\left({}_aD_{t,+}^{\alpha}\overline{\mathrm{Z}}\right)(t)=\mathrm{A}\overline{\mathrm{Z}}(t)+\overline{\mathrm{B}}(t)\tag{261}$$

Where

$$\mathrm{A}=\left(a_{ij}\right)_{i,j=1,...n};\quad \overline{\mathrm{B}}(t)=\begin{pmatrix}\sigma_1(t)\\...\\...\\\sigma_n(t)\end{pmatrix};\quad \overline{Z}(t)=\begin{pmatrix}z_1(t)\\...\\...\\z_2(t)\end{pmatrix}\tag{262}$$

$$z_j(t)=\left({}_aD_t^{(j-1)\alpha}z\right)(t);\quad j=(1,2,...,n-1)$$

The solution is

$$\overline{\mathrm{Z}}(t)=\overline{\mathrm{C}}e_{\alpha}^{\mathrm{A}(t-a)}+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\overline{\mathrm{B}}(\xi)\mathrm{d}\xi\tag{263}$$

Case-III

$$\left({}_a^CD_{t,+}^{\alpha}\overline{\mathrm{Z}}\right)(t)=\mathrm{A}\overline{\mathrm{Z}}(t)+\overline{\mathrm{B}}(t)\tag{264}$$

with parameters as per case-II. The solution is

$$\overline{\mathrm{Z}}(t)=\overline{\mathrm{Z}}(a)+\int_a^t e_{\alpha}^{\mathrm{A}(t-\xi)}\left[\overline{\mathrm{B}}(\xi)+\mathrm{A}\overline{\mathrm{Z}}(a)\right]\mathrm{d}\xi\tag{265}$$

In particular if $\overline{\mathrm{Z}}(a)=\overline{b}$; then

$$\overline{Z}(t) = \overline{b} + \int_a^t e_\alpha^{A(t-\xi)} \left[\overline{B}(\xi) + A\overline{b} \right] d\xi \tag{266}$$

Here we saw generalization method to get response of a fractional order model of visco-elastic material. The criteria shown in this note is of immense practical interest, in the field of study of visco-elastic material which shows anomalous behavior.

End of Part-A