Memory and Hysteresis modeling with Fractional Calculus

Shantanu Das
Scientist; RCSDS E&I Group BARC, Mumbai
Senior Research Professor, (CMPRC) Dept. of Physics, Jadavpur University Kolkata.
Adjunct Professor. DIAT-Pune
UGC Visiting Fellow. Dept. of Appl. Mathematics; Calcutta Univ.
shantanu@barc.gov.in , shantanu.das@live.com, shantanudas.ffc@gmail.com

May-2017

Abstract

Hysteresis and memory are related phenomena that are well known in various physical systems. Fractional derivative has a history of 300 years plus as long as that of classical calculus, but it is much less popular than it should be. What is the physical meaning of fractional derivative? This is still an open problem. In modeling various memory phenomena, we observe that a memory process usually consists of two stages. One is short with permanent retention (called as initial condition), and the other is governed by a simple model of fractional derivative. It is observed that with the numerical method the fractional model perfectly fits the test data of memory phenomena in different disciplines in mechanics (stress strain relaxation), dielectric relaxation, super-capacitor charge-discharge etc. Based on this fractional model, we find that a physical meaning of the fractional order is an index of memory. Fractional derivative is a generalization of integer-order derivative and integration. It originated in the letter about the meaning of ½ order derivative from L'Hospital to Leibnitz in September 30, 1695 and is a promising tool for describing memory phenomena. The kernel function of fractional derivative is called memory function but it does not reflect any physical process. Unclear physical meaning has been a big obstacle that keeps fractional derivative lagging far behind the integer-order calculus. In 1974, the question "what are the physical interpretations of fractional calculus" was put forward as an open problem. In 2002, a physical explanation was proposed in terms of inhomogeneous and changing time scale by analogy reasoning, but the new time scale has not been validated by any experiment. Till now there is still no simple answer to the open problem. In fitting the test data of memory phenomena from different fields, we find that the fractional order can be physically explained as an index of memory or forgetfulness. Memory means retention, and that is cause of hysteresis, that we will discuss, and how fractional calculus plays role in modeling the phenomena of hysteresis, that we will show. The introduction of fractional derivative in dynamic systems makes the system behavior non-local and the hysteresis-loss thus have memory. This note gives some ideas how can we use Fractional Calculus to model hysteresis, and if a system exhibits memory via having fractional order components how the hysteresis curve gets manifested.

Keywords: hysteresis, memory, fractional derivative, fractional integration

Introduction

We observe Hysteresis in magnetic and electric phenomena, generally. Hysteresis means “remaining” in Greek, an effect remains after its cause has disappeared. Hysteresis, a term coined by Sir James Alfred Ewing in 1881, a Scottish physicist and engineer (1855-1935), defined it as: When there are two physical quantities Y and X such that cyclic variations of X cause cyclic variations of Y, then if the changes of X lag behind those of Y, we may say that there is “hysteresis in the relation of X to Y". The most notable example of hysteresis in physics is magnetism. Iron maintains some magnetization (B) or (M) after it has been exposed to and removed from a magnetic field (H), and we have usual BH curve or MH curve. Similarly we observe Hysteresis in Ferro-electric materials, i.e. Polarization P remains after we remove electric field E. We also observe hysteresis in piezo-electric material-and several others, like strain to resistance conversion etc. We also observe hysteresis in super-capacitors, where the voltage (charge) is retained after the charging current is withdrawn. We also observe that how a capacitor memorizes its charging history; which can only be expressed via using fractional calculus.

The hysteresis accounts for losses, the theory of, “movement of domain wall” (Jiles & Atherton 1984). The movement of domain wall can be reversible as well as irreversible-and is causes of dynamic and static friction-which opposes the cause of polarization and or magnetization. The material domain’s motion is having micro spring-dashpot systems, and generally is non-Newtonian in nature; which can be represented by fractional derivative, resisting the motion of domain walls. This introduction of fractional derivative makes the system behavior non-local and the hysteresis-loss thus have memory.

In super-capacitors though there is no movement of domain walls yet the hysteresis is accounted for due to fractal arrangement of distributed resistor capacitors. When the charging current is withdrawn, the internal capacitors redistribute the charges and thus we still have remnant voltage and the super-capacitor follows a self-discharge phenomena. Can this phenomena lead to hysteresis? Can we thus identify the internal parameters of super-capacitors from this hysteresis curve. We will develop mathematical methods with use of fractional differentiation and fractional integration to model those phenomena of hysteresis.

Basic Riemann-Liouville fractional derivative and fractional integration

Fractional calculus allows the definition of the derivative and integral of generalized order. This often helps in creating a compact representation of a system. Fractional derivative D^{α} and fractional integration I^{α} (or $D^{-\alpha}$) of a function, for $0<\alpha<1$ is following

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{-\alpha}f(\tau)\mathrm{d}\tau $$

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{t^{-\alpha}f(0)}{\Gamma(1-\alpha)}+\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-\tau)^{-\alpha}f^{(1)}(\tau)\mathrm{d}\tau $$

$$ {}_0D_t^{-\alpha}\left(f(t)\right)={}_0I_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau=\frac{1}{\alpha\left(\Gamma(\alpha)\right)}\int_0^t\alpha(t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau $$

$$ =\frac{1}{\Gamma(\alpha+1)}\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}t}(t-\tau)^{\alpha}\right)f(\tau)\mathrm{d}\tau $$

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{-\alpha}f(\tau)\mathrm{d}\tau \qquad {}_0I_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(\alpha+1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t(t-\tau)^{\alpha}f(\tau)\mathrm{d}\tau $$

Here $\Gamma(.)$ is the Euler Gamma function. The above definitions are Riemann-Liouville (RL) definition of fractional derivative and RL-fractional integration, for fractional order $0<\alpha<1$, only. We will be using this order range in this note. The sign change from fractional derivative formula to get fractional integration formula is accepted, but converse is untrue. We note the integral has a kernel i.e. $k_{t-\tau}=(t-\tau)^{-\alpha}$ appearing in above formulas signifies memory. Actually all the above formula containing integrals are convolution integrals; that we will take in detail.

A constant function x_0 decays sharply while operated by fractional derivative and increases with time at a decaying rate on being operated by fractional integrals as indicated below.

$$ D^{\alpha}\left(x_0\right)=\frac{x_0}{\Gamma(1-\alpha)}t^{-\alpha} \qquad I^{\alpha}\left(x_0\right)=\frac{x_0}{\Gamma(1+\alpha)}t^{\alpha} $$

In classical calculus derivative of constant function is zero, whereas the fractional derivative of a constant gives a decaying function and that tends to zero when $t\uparrow\infty$.

The Laplace Transform and Fourier Transform of Fractional derivative

The Laplace Transform of fractional derivative-integral of order α operation is

$$ \mathcal{L}\left\{{}_0D_x^{\alpha}f(x)\right\}=s^{\alpha}\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^k\left[{}_0D_x^{\alpha-1-k}f(x)\right]_{\text{at}\quad x=0} $$

Where Laplace Transform defined as

$$ \mathcal{L}\left\{f(x)\right\}\overset{\text{def}}{=}\int_0^{\infty}\mathrm{d}x\left(e^{-sx}f(x)\right) $$

In Laplace definition above the order of differ-integration $\alpha\in\mathbb{R}$; and the integer $n\in\mathbb{Z}^+$ such that $(n-1)<\alpha\le n$. In this expression of Laplace $\mathcal{L}\left\{{}_0D_x^{\alpha}f(x)\right\}$ when $\alpha<0$, that is operation is fractional integration, the term involving summation becomes zero for any function, $f(x)$ with available Laplace Transform. Also one can have similar to Laplace Transform of fractional differ-integrals of $f(x)$; a Fourier Transform of fractional differ-integral operation. A function $f(x)$, which is "well-behaved" at $x = -\infty$, we can have

$$\mathcal{F}\left\{ {}_{-\infty}D_x^{\alpha} f(x)\right\} = (i\omega)^{\alpha}\,\mathcal{F}\left\{ f(x)\right\}$$

and therefore we have fractional derivative/integral operation as inverse Fourier transformed one

$${}_{-\infty}D_x^{\alpha} f(x) = \mathcal{F}^{-1}\left\{(i\omega)^{\alpha}\,\mathcal{F}\{ f(x)\}\right\}$$

Where the Fourier and Inverse Fourier Transform is depicted as following

$$\mathcal{F}\left\{ f(x)\right\} = F(\omega) \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \mathrm{d}x\left(e^{i\omega x} f(x)\right) \qquad f(x) = \mathcal{F}^{-1}\left\{F(\omega)\right\} \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega\left(e^{-i\omega x} F(\omega)\right)$$

In some cases (especially for steady state systems with lower terminal of differ-integration $-\infty$) the Fourier Transformation method is another way to find fractional derivative/fractional integration of function $f(x)$. That is

(i) Obtain the Fourier Transform of $f(x)$ as $F(\omega)$.

(ii) Then this transformed $F(\omega)$ in frequency ω domain we multiply by $(i\omega)^{\alpha}$, where $\alpha \in \mathbb{R}$.

(iii) The resulting function $(i\omega)^{\alpha} F(\omega)$ we inverse Fourier transform, to get ${}_{-\infty}D_x^{\alpha} f(x)$
.

In the frequency domain Fourier transform of $f(t)$ and its derivatives and integrals of order $\alpha > 0$ are

$$G_D(\omega) = \left(i\omega\right)^{\alpha} F(\omega) \qquad G_I(\omega) = \left(i\omega\right)^{-\alpha} F(\omega)$$

Where $F(\omega)$, $G_D(\omega)$ and $G_I(\omega)$ are the Fourier transforms of $f(t)$, and its fractional derivatives and integrals respectively.

Fractional Derivative & Fractional Integrals of Sinusoids

Derivatives and integrals of pure sinusoidal functions can be characterized by phase-shift and the modulation amplitude depending on the frequency of the sinusoidal function and order of derivatives and integrals, as following

$${}_0D_t^{\alpha}\left(\sin(\omega t)\right) = \omega^{\alpha} \sin\left(\omega t + \tfrac{\pi\alpha}{2}\right) + \frac{\left(\omega t\right)^{-1-\alpha}}{\omega\left(\Gamma(-\alpha)\right)} - \frac{\left(\omega t\right)^{-3-\alpha}}{\omega^{3}\left(\Gamma(-\alpha-2)\right)} + \ldots.$$

$$ {}_0D_t^{\alpha}\left(\cos(\omega t)\right) = \omega^{\alpha}\cos\left(\omega t + \tfrac{\pi\alpha}{2}\right) + \frac{(\omega t)^{-2-\alpha}}{\omega^2\left(\Gamma(-\alpha-1)\right)} - \frac{(\omega t)^{-4-\alpha}}{\omega^4\left(\Gamma(-\alpha-3)\right)} + \ldots $$

$$ {}_0I_t^{\alpha}\left(\sin(\omega t)\right) = \omega^{-\alpha}\sin\left(\omega t - \tfrac{\pi\alpha}{2}\right) + \frac{(\omega t)^{-1+\alpha}}{\omega\left(\Gamma(\alpha)\right)} - \frac{(\omega t)^{-3+\alpha}}{\omega^3\left(\Gamma(\alpha-2)\right)} + \ldots. $$

$$ {}_0I_t^{\alpha}\left(\cos(\omega t)\right) = \omega^{-\alpha}\cos\left(\omega t - \tfrac{\pi\alpha}{2}\right) + \frac{(\omega t)^{-2+\alpha}}{\omega^2\left(\Gamma(\alpha-1)\right)} - \frac{(\omega t)^{-4+\alpha}}{\omega^4\left(\Gamma(\alpha-3)\right)} + \ldots $$

The first term in both fractional differentiations shows a forward phase-shifting of the periodic function by amount $\frac{\alpha\pi}{2}$; and for fractional integration shows phase lagging by $\frac{\alpha\pi}{2}$. We will use this observation later, for modeling hysteresis.

Discrete Time Numerical Evaluation of Riemann-Liouville Fractional Derivative and Fractional Integration formula for fractional order zero to one

We try to get numerical scheme to evaluate the Riemann-Liouville (RL) Fractional Derivative formula; with known values of $f(t)$ at $m+1$ evenly spaced points in the range 0 to t. We designate symbols $f_m \equiv f(0)$, $f_{m-1} \equiv f\left(\frac{t}{m}\right)$, … $f_j \equiv f\left(x - j\frac{t}{m}\right)$, … $f_0 \equiv f(t)$. The RL fractional derivative formula for $0 \le \alpha < 1$

$$ {}_0D_t^{\alpha}\left(f(t)\right) = \frac{t^{-\alpha}f(0)}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} f^{(1)}(\tau)\mathrm{d}\tau $$

This above expression is nothing but a relation between RL and Caputo derivative for $0 \le \alpha < 1$ with $f^{(1)}$ the one-whole derivative of function $f(t)$ assumed to exist and with initial value $f(0)$ as finite. We re-write the above in following form, with change of variables of integration and writing m discrete integration per interval $\Delta t = \frac{t}{m}$ and then summing them up

$$ {}_0D_t^{\alpha}\left(f(t)\right) = \frac{t^{-\alpha}f(0)}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^t \left(\frac{\mathrm{d}\ f(\tau)}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}\tau}{(t-\tau)^{\alpha}}\right) $$

$$ = \frac{t^{-\alpha}f(0)}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\left(\int_0^{\Delta t}\mathrm{F} + \int_{\Delta t}^{2\Delta t}\mathrm{F} + \ldots\int_{(m-1)\Delta t}^{m\Delta t}\mathrm{F}\right) \qquad \mathrm{F} = \left(\frac{\mathrm{d}\ f(\tau)}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}\tau}{(t-\tau)^{\alpha}}\right) $$

$$ = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(0)}{t^{\alpha}} + \sum_{j=0}^{m-1}\int_{jt/m}^{(j+1)t/m}\left(\frac{\mathrm{d}\ f(t-\tau)}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}\tau}{\tau^{\alpha}}\right)\right) $$

We use approximation $\frac{\mathrm{d}\ f(t-\tau)}{\mathrm{d}\tau} \approx \frac{f(t-j\Delta t) - f\left(t-(j+1)\Delta t\right)}{\Delta t}$ and use this in following derivation

$$\int_{jt/m}^{(j+1)t/m}\left(\frac{\mathrm{d}\ f(t-\tau)}{\mathrm{d}\tau}\right)\frac{\mathrm{d}\tau}{\tau^{\alpha}} \approx \left(\frac{f\left(t-j\Delta t\right)-f\left(t-(j+1)\Delta t\right)}{\left(\Delta t\right)}\right)\int_{jt/m}^{(j+1)t/m}\frac{\mathrm{d}\tau}{\tau^{\alpha}}$$

$$=\left(\frac{f\left(t-\frac{jt}{m}\right)-f\left(t-\frac{(j+1)t}{m}\right)}{\left(\frac{t}{m}\right)}\right)\int_{jt/m}^{(j+1)t/m}\frac{\mathrm{d}\tau}{\tau^{\alpha}}$$

$$=\frac{1}{1-\alpha}\left(\frac{t}{m}\right)^{-\alpha}\left(f_j-f_{j+1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)$$

Using the above expression we get the following

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{1}{\Gamma(1-\alpha)}\left(\frac{f(0)}{t^{\alpha}}+\sum_{j=0}^{m-1}\int_{jt/m}^{(j+1)t/m}\left(\frac{\mathrm{d}\ f(t-\tau)}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}\tau}{\tau^{\alpha}}\right)\right); \qquad f(0)=f_m$$

$$=\frac{1}{\Gamma(1-\alpha)}\left(f_m t^{-\alpha}+\frac{1}{(1-\alpha)}\left(\frac{t}{m}\right)^{-\alpha}\sum_{j=0}^{m-1}\left(\left(f_j-f_{j+1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)\right)\right)$$

$$=\frac{\left(\frac{t}{m}\right)^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\left((1-\alpha)m^{-\alpha}f_m+\sum_{j=0}^{m-1}\left(\left(f_j-f_{j+1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)\right)\right)$$

$$=\frac{\left(\frac{t}{m}\right)^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{(1-\alpha)}{m^{\alpha}}f_m+\sum_{j=0}^{m-1}\left(\left(f_j-f_{j+1}\right)\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)\right)\right)$$

We have following final formula

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{\left(\Delta t\right)^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{(1-\alpha)}{m^{\alpha}}f_m+\sum_{j=0}^{m-1}\left(\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)\left(f_j-f_{j+1}\right)\right)\right); \qquad 0\le\alpha<1$$

We change the sign of the α to $-\alpha$ and get RL fractional integration formula for numerical evaluation as follows

$$ {}_0I_t^{\alpha}\left(f(t)\right)=\frac{\left(\Delta t\right)^{\alpha}}{\Gamma(2+\alpha)}\left(\frac{(1+\alpha)}{m^{-\alpha}}f_m+\sum_{j=0}^{m-1}\left(\left((j+1)^{1+\alpha}-j^{1+\alpha}\right)\left(f_j-f_{j+1}\right)\right)\right); \qquad 0\le\alpha<1$$

We can also have the above formula with different indices so that $f(0)=f_0$ and $f_n=f(t)$

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\frac{\left(\Delta t\right)^{-\alpha}}{\Gamma(2-\alpha)}\left(\frac{(1-\alpha)}{n^{\alpha}}f_0+\sum_{j=0}^{n-1}\left(\left((j+1)^{1-\alpha}-j^{1-\alpha}\right)\left(f_{n-j}-f_{n-(j+1)}\right)\right)\right); \qquad 0\le\alpha<1$$

$$ {}_0I_t^{\alpha}\left(f(t)\right)=\frac{\left(\Delta t\right)^{\alpha}}{\Gamma(2+\alpha)}\left(\frac{(1+\alpha)}{n^{-\alpha}}f_0+\sum_{j=0}^{n-1}\left(\left((j+1)^{1+\alpha}-j^{1+\alpha}\right)\left(f_{n-j}-f_{n-(j+1)}\right)\right)\right); \qquad 0\le\alpha<1$$

We have Grunwald-Letnikov (GL) method described as following

$$ {}_0D_t^{\alpha}\left(f(t)\right)=\left(\lim_{\Delta t\downarrow 0}\sum_{k=0}^{\left\lfloor k=\frac{t}{\Delta t}\right\rfloor}(-1)^k\binom{\alpha}{k}f(t-k\Delta t)\right) \qquad \binom{\alpha}{k}=\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} $$

In the GL method we need to evaluate generalized binomial coefficients that appear as weight $\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$; that involves use of Gamma function values at different k and also evaluation of $k!$; that sometimes are cumbersome. Whereas, in the RL computation methods, we do not require this type of evaluation involving Gamma function and factorial.

Order of Fractional Derivative is indicative of Memory or Forgetfulness

First we take example of visco-elastic system. The Scott-Blair's model originally for a material model; can be also a formula for memory phenomena in various disciplines. The model takes the form

$$ {}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right) $$

where ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)$ is the fractional derivative which depends on the strain history from 0 to *t*, and μ is a positive constant. Riemann-Liouville derivative is one of the most popular definitions, described by the formulas stated in earlier section.

The key point of our observation is that a memory process usually consists of two stages: the fresh stage and the working stage. The former is short with permanent retention at the beginning (call as initial retention) and it cannot be neglected in general, while the latter is governed by the fractional model of ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$. The critical point between the fresh stage and the working stage is usually not the origin at $t=0$-rather we cannot initialize by use of delta function at origin. The retention value at the start point gets manifested as a spread that we call a fresh stage, and at a point $t>0$ working stage starts governed by fractional derivative.

This observation is quite different from the traditional fractional models of one stage. For example, the fractional Maxwell model is a one-stage model. As the combination of two simple models, it has a more complicated expression than the equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$.

We also find that someway the order of fractional derivative is an index of memory. This is a probable answer to the open problem: i.e. 'what is the physical meaning of fractional derivative'. The test data in various disciplines fit ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ very well, as shown in Figure-1 to 4. In the recovering or forgetting stage ($T>1$), the dimensionless form of the solution of equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ is following

$$ E=T^{\alpha}-(T-1)^{\alpha} $$

where $T = t/t_M$, $E(T) = \varepsilon(t)/\varepsilon_M$ and ε_M is the strain at the end instant of creeping $t = t_M$. This we will describe and derive shortly. The dimensionless retention E increases with an increase of order α . The higher the index α , the slower the forgetting is. In particular if $E = 0$ then nothing memorized that is if $\alpha = 0$, and if $E = 1$ the nothing is forgotten that is if $\alpha = 1$. Therefore, we define the fractional-order α as the index of memory (or forgetfulness).

When a student is in class room learns he is in stress period and after that till next day while he is not learning he is in relaxing period where he tends to forget. In the next day depending on α of the student, at the start of the class; his retention E gets manifested. He may forget all, or he may remember all or he may retain partially! If he retains partially he has to spend less time in the subsequent class to re-learn to the level of previous day's level of learning. This will repeat the next day-for this repeated learning cases-while spending less and still less time for re-learning (to the original learnt condition).

Firstly, we consider an example in mechanics. The behaviors of viscoelastic materials with memory are usually described by Kevin model, Voigt model, Maxwell model, and so on, in terms of the strain, the stress, and their integer-order derivatives or integrals. Comparing with fractional models, such integer-order models reflect memory effects much less accurately. Early observations show that viscoelastic materials behave between elasticity and viscosity. It is reasonable, hence, to assume that models of viscoelasticity take the form of equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right) = \mu\left(\sigma(t)\right)$ and it covers two extremes: $\alpha = 0$ for Hooke's Law of elasticity, and $\alpha = 1$ for Newton's Law of viscosity.

For a standard creep and recovery test, the specimen is usually loaded under a constant stress $\sigma(t) = \sigma_0$ from $t = 0$ to $t = t_M$ and the load is removed at the instant $t = t_M$ then $\sigma(t) = 0$ for $t \geq t_M$. Let $u(t)$ be the Heaviside function, then equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right) = \mu\left(\sigma(t)\right)$ takes the following simple form, with stress excitation written as $\sigma(t) = \sigma_0 u(t) - \sigma_0\left(u(t - t_M)\right)$; i.e.

$$ {}_0D_t^{\alpha}\left(\varepsilon(t)\right) = \mu\left(\sigma_0\left(u(t) - u(t - t_M)\right)\right) $$

where ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)$ is Riemann-Liouville's fractional-order derivative with zero initial condition, $\varepsilon(t)\big|_{t=0} = 0$. Heaviside step is $u(t-\tau) = 1$ for $t \geq \tau$ and $u(t-\tau) = 0$ for $t < \tau$. The superposition method results in the solution of equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right) = \mu\left(\sigma_0\left(u(t) - u(t - t_M)\right)\right)$ as follows using fractional integration of constant ${}_aD_t^{-\alpha}\left(x_0\right) = \frac{x_0}{\Gamma(1+\alpha)}(t-a)^{\alpha}$.

$$ \varepsilon(t) = \frac{\mu\sigma_0}{\Gamma(1+\alpha)}\left(t^{\alpha}u(t) - (t - t_M)^{\alpha}u(t - t_M)\right) $$

The above is working stage expression starts with $t = 0$ having zero initial state $\varepsilon(t)\big|_{t=0} = 0$. The above can also be obtained using Laplace technique that we will show in different case.

The Figure-1 shows the black plot as experimental values and red one with above equation (where the red dot is taken as origin). This sample had initial retention of strain-and that initial retention quickly relaxed to red dot in small time (but non-zero time) and then the working stage follows the experimental curve nicely. Therefore we remarked earlier that the "fresh-stage" need not start from origin; this is due to presence of retention, at $t=0$. Thus in a way we can say the initial retention (or already learned stuffed) relaxes very fast towards red-dot initially and then slowly goes to maximum-when the step load of stress $\sigma(t)=\sigma_0$ is applied at time $t=0$. This red dot is called critical point, before that point we call the "fresh stage" and after that point we call "working stage". We note that fresh stage is spread on a very small time span.

From here we observe few interesting facts. While we have stressed the sample initially the time taken to reach ε_M is short, and also we observe that there is prompt jump (from origin to the critical point) due to initial retention of strain. This we call as initialization function that we add to the uninitialized solution i.e. $\varepsilon(t)=\frac{\mu\sigma_0}{\Gamma(1+\alpha)}\left(t^\alpha u(t)-(t-t_M)^\alpha u(t-t_M)\right)$, and get the graph of Figure-1.

Then when at time t_M when the system is de-stressed with $\sigma(t)=0$, the strain relaxes quickly initially and then followed by lingering tail. This relaxing period $t>t_M$ is "forgetting phase"; and period $0<t<t_M$ is learning phase (stressed period).

This is analogous to learning process. While we are learning we are stressed, and at start with fresh mind we quickly learn then the learning becomes slow till our learning time (period) stops-when we have learned fully (say ε_M). In the de-stressed period $\sigma(t)=0$ we are not learning instead we are forgetting. We tend to forget at maximum rate initially and then forgetting rate gradually becomes slow. That is after some large time, $t\gg t_M$ we tend to remember few things (forget most of them) and retain few of them longer, as compared to early times $t\approx t_M$, where we tend to forget at faster rate but have greater things learnt or retained.

We write the above again and do the manipulations as indicated below

$$\begin{aligned}
\varepsilon(t) &= \frac{\mu\sigma_0}{\Gamma(1+\alpha)}\left(t^\alpha u(t)-(t-t_M)^\alpha u(t-t_M)\right)\\
&= \frac{\mu\sigma_0}{\Gamma(1+\alpha)} t_M^\alpha \left(\frac{t^\alpha}{t_M^\alpha} u\left(\frac{t}{t_M}\right) - \frac{1}{t_M^\alpha}\left(t-t_M\right)^\alpha u\left(\frac{t}{t_M}-1\right)\right)\\
&= \frac{\mu\sigma_0 t_M^\alpha}{\Gamma(1+\alpha)}\left(\left(\frac{t}{t_M}\right)^\alpha u\left(\frac{t}{t_M}\right) - \left(\frac{t-t_M}{t_M}\right)^\alpha u\left(\frac{t}{t_M}-1\right)\right), \qquad \varepsilon_M=\frac{\mu\sigma_0 t_M^\alpha}{\Gamma(1+\alpha)},\quad T=\frac{t}{t_M}\\
&= \varepsilon_M\left(T^\alpha u(T)-(T-1)^\alpha u(T-1)\right), \qquad E=\frac{\varepsilon(t)}{\varepsilon_M}
\end{aligned}$$

$$E = T^\alpha - (T-1)^\alpha$$

The loading period is from $t=0$ to $t=t_M$, is analogous to learning period when a student learning is stressed. The period after t_M is relaxation period, when stress is relaxed to zero, is analogous to forgetting period of student's relaxing period. This is in agreement with the early observations of the behaviors of some viscoelastic materials.

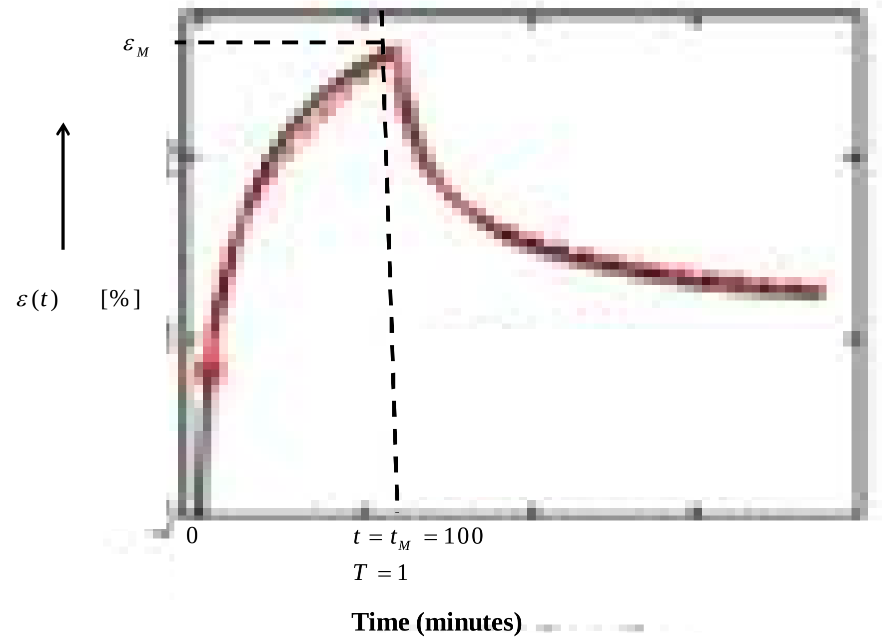


Figure 1 : Fitting of the test data of specimen of VISCOELASTIC Material

Secondly, we show that ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ works not only in modeling viscoelastic materials, but also in modeling biological kinetics with memory. For example, ${}_0D_t^{\alpha}\left(\rho_s(t)\right)=\mu\left(c(t)\right)$ for the protein, 'adsorption kinetics', if the symbols of concentration is c and the surface density is ρ_s of the sample protein, then with $c(t)=c_0\left(u(t)-u(t-t_M)+u(t-t_N)\right)$ the absorbed density is found to be

$$\rho_s(t)=\frac{\mu c_0}{\Gamma(1+\alpha)}\left(t^{\alpha}u(t)-(t-t_M)^{\alpha}u(t-t_M)+(t-t_N)^{\alpha}u(t-t_N)\right)$$

as shown in Figure-2 . Here too a short period of "fresh-stage" due to initial retention is adjusted with the above given working stage expression.

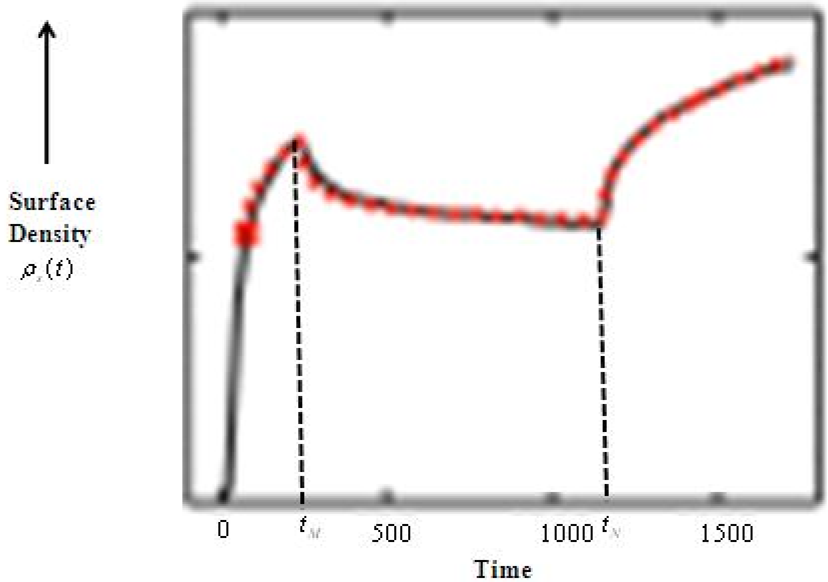


Figure 2: Fitting of the test data of protein adsorption kinetics.

Thirdly, we show that ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ works well even for the cognitive dynamics in psychology, by fitting the memorizing test data performed by Hermann Ebbinghaus and reported in 1885. The average time of learning is $t_M = 944\,\text{sec}$. The retention ratios of learning $\varepsilon(t)$ in % were 58.2, 44.2, 35.8, 33.7, 27.8, 25.4, and 21.1, respectively after 0.33, 1, 8.8, 1 × 24, 2 × 24, 6 × 24, and 7 × 24 hours, where the retention ratio was obtained by using the ratio of the time of relearning to the original.

By drawing an analogy between learning (forgetting) and loading (unloading), and regarding the retention as the strain use ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ to fit Ebbinghause's test data. Here we have $\sigma(t)=1$ in the given learning stage from 0 to t_M which means that for the series of syllables to be learned, the tested person is fully occupied, and $\sigma(t)=0$ when the learning stops after the instant t_M .

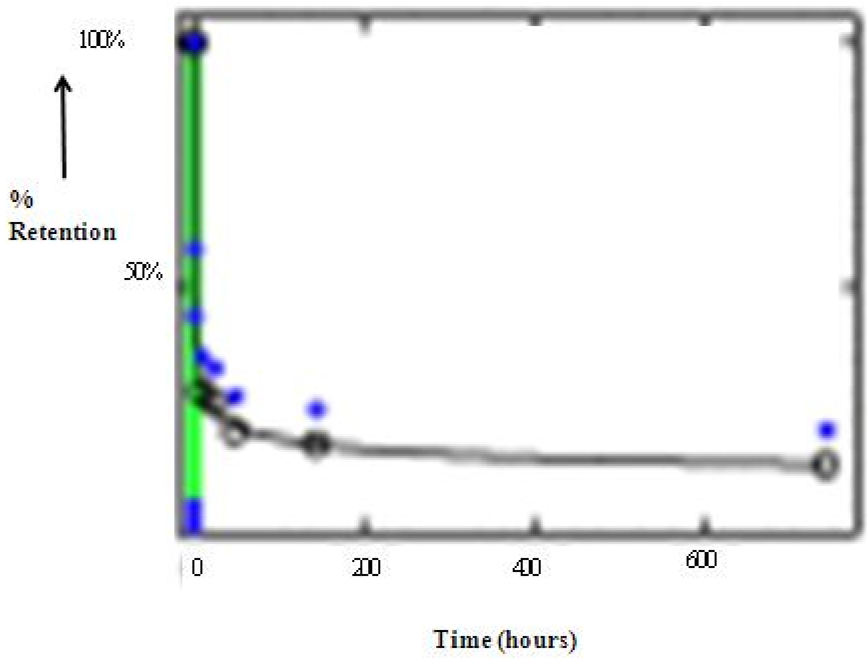


Figure 3: Forgetting curve

Here since the learning time is 944seconds the curve from $t = 0$ to $t = t_M$ in the timescale of hours is making the learning period as similar to an impulse function. Similar to the material tests, at the beginning of learning, one can keep a few syllables quickly in mind and memorize them very well (this gives sudden jump). This is the fresh stage of learning. In the working stage, something is gradually forgotten after the learning stops. We assume that the fresh stage is recovered after one day. Under this assumption, we use the corresponding values of ε minus the vertical value of the critical point respectively to fit the fractional model. Here too solution is

$$\varepsilon(t) = \frac{\mu\sigma_0}{\Gamma(1+\alpha)}\left(t^{\alpha}u(t) - (t - t_M)^{\alpha}u(t - t_M)\right)$$

The forgetting curve is as shown in Figure-3.

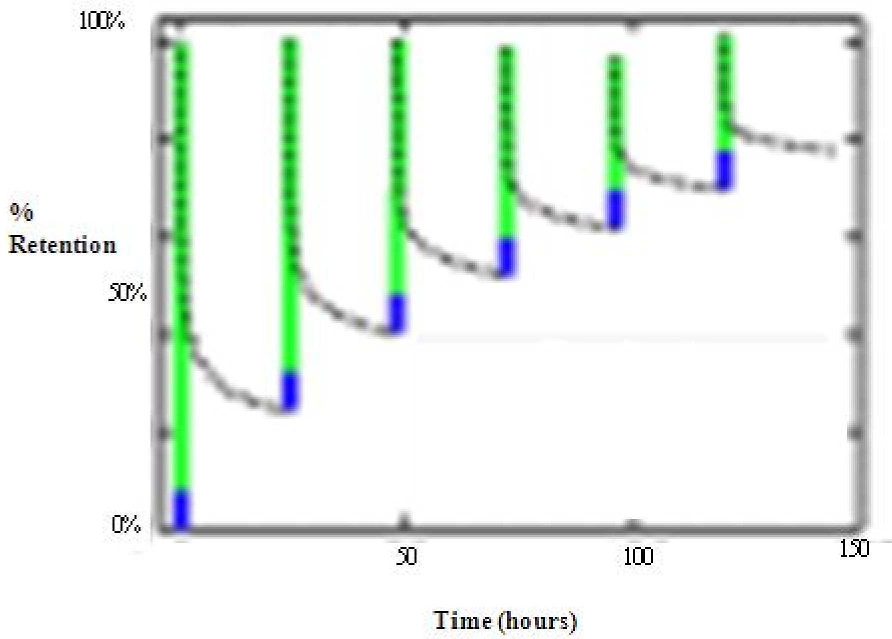


Figure 4: Retention with respect to repeated learning

Furthermore, we consider Ebbinghause's repeated learning test, where 9 series of 12 syllables were learned, and then in the same hour on successive days for next six days, relearned to the point of the first possible reproduction. In successive 6 days $n=1,2,...6$ the times of repetitions for each series averagely are $m_n=16.5$, 11, 7.5, 5, 3, and 2.5, in order to get to the full learning. As one syllable takes $0.4\sec$, the learning time equals $t_n=m_n\times 9\times 12\times\left(\frac{0.4}{3600}\right)$ hours. Thus, the learning time for six consecutive days are; $t_1=0.198\text{hour}$, $t_2=0.132\text{hour}$, $t_3=0.09\text{hour}$, $t_4=0.06\text{hour}$, $t_5=0.036\text{hour}$, $t_5=0.03\text{hour}$ -this shows as the days pass the learning period (stress period decrease). The solution of ${}_0D_t^\alpha\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ for the working stage takes the form following for six days

$$\varepsilon(t)=\frac{\mu\sigma_0}{\Gamma(1+\alpha)}\begin{pmatrix} t^\alpha u(t)-(t-t_1)^\alpha u(t-t_1)+ \\ (t-24)^\alpha u(t-24)-(t-t_2-24)^\alpha u(t-t_2-24)+ \\ (t-2\times 24)^\alpha u(t-2\times 24)-(t-t_3-2\times 24)^\alpha u((t-t_3-2\times 24)+ \\ (t-3\times 24)^\alpha u(t-3\times 24)-(t-t_4-3\times 24)^\alpha u(t-t_4-3\times 24)+ \\ (t-4\times 24)^\alpha u(t-4\times 24)-(t-t_5-4\times 24)^\alpha u(t-t_5-4\times 24)+ \\ (t-5\times 24)^\alpha u(t-5\times 24)-(t-t_6-5\times 24)^\alpha u(t-t_6-5\times 24) \end{pmatrix}$$

For the working stages and taking the fresh stage into account, the repeated learning and forgetting curves are plotted in Figure- 4, for six days $n=1,2,...6$.

The equation ${}_0D_t^{\alpha}\left(\varepsilon(t)\right)=\mu\left(\sigma(t)\right)$ is valid for a fractional capacitor observed in super-capacitor system. The constant charging and discharging is given by summation of step currents as given to a capacitor with zero initial charge.

$$i(t)=I_0\left(u(t)-u(t-T_c)\right)$$

Comparing to $\sigma(t)=\sigma_0\left(u(t)-u(t-t_M)\right)$, we see $i(t)\equiv\sigma(t)$, $I_0\equiv\sigma_0$ and $t_M\equiv T_c$. The fractional capacitor is having relation ${}_0D_t^{\alpha}\left(v(t)\right)=\mu\left(i(t)\right)$ with $\mu=\frac{1}{C_\alpha}$ the fractional capacity having units of $C_\alpha\equiv \text{Farad}/\text{sec}^{1-\alpha}$, with $v(t)\big|_{t=0}=0$ Thus we get voltage across fractional capacitor as

$$v(t)=\frac{I_0}{C_\alpha\Gamma(1+\alpha)}\left(t^{\alpha}u(t)-(t-T_c)^{\alpha}u(t-T_c)\right)$$

The plot is similar to Figure-1. We now use Laplace transform techniques to get above solution

$$i(t)=I_0u(t)-I_0u(t-T_c)$$

The Laplace of the above is

$$I(s)=\frac{I_0}{s}-\frac{I_0e^{-sT_c}}{s}$$

The expression ${}_0D_t^{\alpha}\left(v(t)\right)=\frac{1}{C_\alpha}\left(i(t)\right)$, has Laplace $s^{\alpha}V(s)=\frac{1}{C_\alpha}I(s)$, for $\left(v(t)\right)_{t=0}=0$ i.e. no initial charge is retained in the super-capacitor, when we stress the same by a constant current stress. The impedance is defined as $Z(s)=V(s)/I(s)$, which is $Z(s)=\left(s^{\alpha}C_\alpha\right)^{-1}$. Thus voltage across the super-capacitor is

$$\begin{aligned} V(s)&=Z(s)I(s)\\ &=\left(\frac{1}{C_\alpha s^{\alpha}}\right)\left(\frac{I_0}{s}-\frac{I_0}{s}e^{-sT_c}\right)\\ &=\frac{I_0}{C_\alpha s^{\alpha+1}}-\frac{I_0e^{-sT_c}}{C_\alpha s^{\alpha+1}}\end{aligned}$$

Applying inverse Laplace to the above obtained expression we get

$$\begin{aligned} v(t)&=\frac{I_0t^{\alpha}}{C_\alpha\Gamma(1+\alpha)}u(t)-\frac{I_0(t-T_c)^{\alpha}}{C_\alpha\Gamma(1+\alpha)}u(t-T_c)\\ &=\frac{I_0}{C_\alpha\Gamma(1+\alpha)}\left(t^{\alpha}u(t)-(t-T_c)^{\alpha}u(t-T_c)\right)\end{aligned}$$

In summary, we divide memory phenomena into fresh stage and working stage, and show that the fractional order is an index of memory by fitting test data of memory phenomena from different fields.

Now we say in all the above we have seen that fractional order α which we say that is representative of memory (or forgetfulness), is a constant (a fractional value between zero and one). We may have condition where the fractional order α is time varying function, giving notion of variable memory. We may even have condition that for learning (loading) stage governed by say fractional order α_1 and forgetting (unloading) stage governed by α_2; may be constant or time varying.

Is the memory due to power law of relaxation process dynamics?

This part we will explore. In the previous sections we saw that the response functions that we always get is in the form of $\sim t^{\alpha}$, with $0<\alpha<1$. We start with Curie law of 1889, where in dielectric relaxation, the relaxation of current to an impressed constant voltage stress U_0 i.e. a step voltage applied at $t=0$ to an initial uncharged system the current is by following law

$$I(t)=\frac{U_0}{h_\alpha t^{\alpha}} \qquad t>0 \qquad 0<\alpha<1$$

The h_α and α are constants for particular dielectric material. The above law is from experimental measurements of several insulating materials. For example the log-log plot of current to a voltage i.e. $\log\left(I(t)\right)=K-\alpha\log\left(t\right)$ 100V connected at say time $t=0$ to a 0.47μF metalized paper dielectric capacitor, show a linear fall with a slope of -0.86, thus here we get $\alpha=0.86$.

We now get Transfer Function of a capacitor via Laplace Transform $I(s)=\mathcal{L}\left\{I(t)\right\}=\mathcal{L}\left\{\frac{U_0}{h_\alpha}t^{-\alpha}\right\}$. Use Laplace pair $\frac{n!}{s^{n+1}}\leftrightarrow t^{n}$ and $n!=\Gamma(1+n)$ to get the following

$$I(s)=U_0\frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha+1}}=\left(\frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha}}\right)\left(\frac{U_0}{s}\right), \qquad 0<\alpha<1$$

$$=\left(\frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha}}\right)U(s) \qquad U(s)=\frac{U_0}{s}$$

Note that the voltage excitation is a constant step input at time zero is $U(t)=U_0\left(u(t)\right)$

$$u(t)=\begin{cases}1 & t>0\\ 0 & t\le 0\end{cases} \qquad \mathcal{L}\left\{u(t)\right\}=\frac{1}{s}$$

Laplace is $U(s)=\frac{U_0}{s}$. Thus we have the following transfer functions in the term of admittance $Y(s)$ and impedance $Z(s)$ defined as $Y(s)=I(s)/U(s)$ and $Z(s)=U(s)/I(s)$

$$
\begin{aligned}
Y(s) &= \frac{I(s)}{U(s)} \\
&= \frac{\Gamma(1-\alpha)}{h_\alpha s^{-\alpha}} = \frac{\Gamma(1-\alpha)}{h_\alpha} s^\alpha = C_\alpha s^\alpha \qquad\qquad C_\alpha = \frac{\Gamma(1-\alpha)}{h_\alpha} \\
Z(s) &= \frac{U(s)}{I(s)} = \frac{1}{C_\alpha s^\alpha}
\end{aligned}
$$

Unit of fractional capacity C_α is $\mathrm{F/sec}^{1-\alpha}$ in this new relation of fractional impedance. This gives current-voltage expression for capacitor as

$$
Z(s)=\frac{U(s)}{I(s)}=\frac{1}{C_\alpha s^\alpha} \qquad 0<\alpha<1
$$

$$
I(s)=\left(C_\alpha s^\alpha\right)U(s)=C_\alpha\left(s^\alpha U(s)\right)
$$

As we have for zero initial condition with similarity to known relation $s\left(F(s)\right)=s\left(\mathcal{L}\left\{f(t)\right\}\right)\leftrightarrow\frac{\mathrm{d}\, f(t)}{\mathrm{d}t}$ we have $s^\alpha\left(F(s)\right)=s^\alpha\left(\mathcal{L}\left\{f(t)\right\}\right)\leftrightarrow\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}$; that is generalized Laplace transform. Our new-capacitor expression is having fractional order derivative and fractional order integration that is

$$
I(t)=C_\alpha\frac{\mathrm{d}^\alpha U(t)}{\mathrm{d}t^\alpha}=C_\alpha\left({}_0D_t^\alpha\left(U(t)\right)\right) \qquad U(t)=\frac{1}{C_\alpha}\left({}_0D_t^{-\alpha}\left(I(t)\right)\right)=\frac{1}{C_\alpha}\int_0^t I(\tau)\left(\mathrm{d}\tau\right)^\alpha
$$

Contrary to classical expression is following.

$$
I(t)=C\frac{\mathrm{d}\left(U(t)\right)}{\mathrm{d}t} \qquad U(t)=\frac{1}{C}\int_0^t I(\tau)\left(\mathrm{d}\tau\right)
$$

A capacitor is charged from time $-T$ to t with a constant voltage U_0 the charging current is $I_c(t)$ that is

$$
\begin{aligned}
I_c(t) &= C_\alpha\left.\frac{\mathrm{d}^\alpha U_0}{\mathrm{d}t^\alpha}\right|_{-T}^{t} = \left(\frac{\Gamma(1-\alpha)}{h_\alpha}\right)\left.\frac{\mathrm{d}^\alpha U_0}{\mathrm{d}t^\alpha}\right|_{-T}^{t} = \frac{\Gamma(1-\alpha)}{h_\alpha}\left(\frac{U_0}{\Gamma(1-\alpha)}(t-(-T))^{-\alpha}\right) \\
&= \frac{U_0}{h_\alpha(t+T)^\alpha} \qquad 0<\alpha<1 \qquad (t+T)>0
\end{aligned}
$$

We have used the formula ${}_{a}D_{x}^{\alpha}\mathbf{C}=\mathbf{C}\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$, where $\mathbf{C}$ is a constant. At $t=0$ it is kept in open-circuit; there will be self discharge thus a discharge current will appear depending on decaying of the terminal voltage $U(t)$. When we remove the voltage source and leave the capacitor open, a discharge current $I_d(t)$ starts to flow.

$$I_d(t)=C_\alpha\left.\frac{\mathrm{d}^\alpha U(t)}{\mathrm{d}t^\alpha}\right|_0^t=\frac{\Gamma(1-\alpha)}{h_\alpha}\left.\frac{\mathrm{d}^\alpha U(t)}{\mathrm{d}t^\alpha}\right|_0^t \qquad t>0$$

Superposition requires this discharge current when we open the capacitor since charging currents must be balanced so the sum of the two currents is zero. The equation from which the voltage $U(t)$ is determined is then $I_c(t)+I_d(t)=0$, i.e.

$$\frac{U_0}{h_\alpha(t+T)^\alpha}+\frac{\Gamma(1-\alpha)}{h_\alpha}\frac{\mathrm{d}^\alpha U(t)}{\mathrm{d}t^\alpha}=0; \qquad t>0$$

Do fractional integration of order α for above expression, to write the following

$${}_{0}D_{t}^{-\alpha}\left[\frac{U_0}{h_\alpha(t+T)^\alpha}\right]+\frac{\Gamma(1-\alpha)}{h_\alpha}\left[U(t)-U_0\right]=0$$

We apply the formula for fractional integration i.e. ${}_{0}D_{t}^{-\alpha}\left[f(t)\right]=\frac{1}{\Gamma(\alpha)}\int_0^t(t-x)^{\alpha-1}f(x)\mathrm{d}x$, to get

$$\frac{U_0}{h_\alpha\Gamma(\alpha)}\int_0^t\frac{1}{(T+x)^\alpha}\frac{\mathrm{d}x}{(t-x)^{1-\alpha}}+\left[\frac{\Gamma(1-\alpha)}{h_\alpha}U(t)-\frac{\Gamma(1-\alpha)}{h_\alpha}U_0\right]=0$$

Rearranging the above, we write

$$U(t)=U_0-\frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\frac{\mathrm{d}x}{(T+x)^\alpha(t-x)^{1-\alpha}}$$

Now we see when $\alpha=1$ the second term of above is zero as $\Gamma(0)=\infty$, makes $U(t)=U_0$ that is what ever voltage is charged to; is remaining or holding at ideal capacitor. When $\alpha=1$ the charging current is

$$I_c(t)=C\frac{\mathrm{d}\left(U_0u(t+T)\right)}{\mathrm{d}t}=CU_0\left(\delta(t+T)\right)$$

That is delta function at $t=-T$. This is due to step function at $t=-T$, i.e. $U_0u(t+T)$, that is capacitor is kept on charge at time $t=-T$. We see in the ideal case at $t=0$ the charging current

$I_c(t) = 0$, and when switch is opened at time $t = 0$ we have by super-position, the discharge current $I_d(t)$ for $t > 0$ is $I_c(t) + I_d(t) = 0$, that is $I_d(t) = 0$. This implies the discharge voltage $I_d(t) = C\frac{dU(t)}{dt} = 0$ means $U(t) = U_0$ a constant. Thus ideal capacitor with $\alpha = 1$ and with no leakage current holds the charged voltage.

Now we take the self discharge expression $U(t) = U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{dx}{(T+x)^{\alpha}(t-x)^{1-\alpha}}$ and see that if the charging time T becomes $2T$ then the droop given by second term is less, and thus $U(t)$ will decay slowly. In a way due to power law of relaxation process we find that the charging time T is getting memorized! Experimental evidence is in Figure-5 (Courtesy CMET-Thrissur).

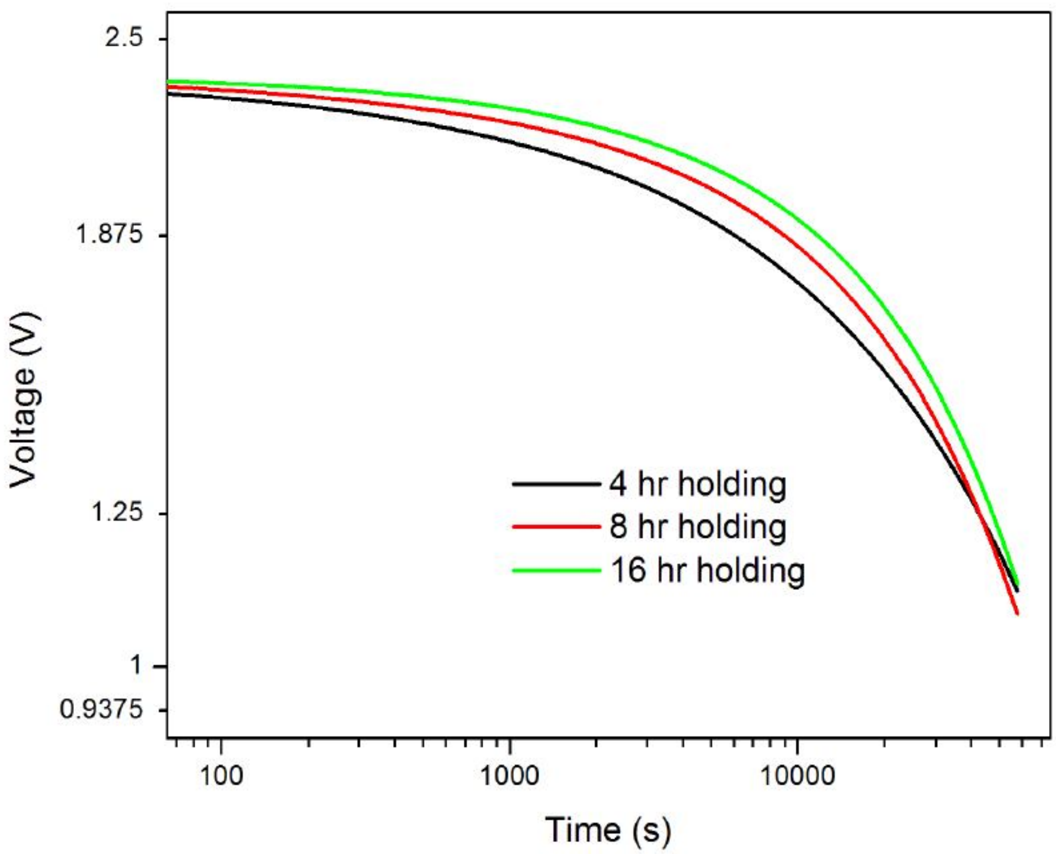


Figure-5: Self discharge of super-capacitor memorizing its time of charging

We can have say different α during discharge call it α_d and while charging we have say α_c. Considering $h_{\alpha_c} = h_{\alpha_d}$ then we will have a self-discharge equation that is

$$U(t) = U_0 - \frac{U_0}{\Gamma(\alpha_d)\Gamma(1-\alpha_d)}\int_0^t \frac{dx}{(T+x)^{\alpha_c}(t-x)^{1-\alpha_d}}; \qquad t > 0$$

Now we may have situation while discharge α_d is varying then we may evaluate the above by placing $\alpha_d(t)$ as some function of time, and evaluate the following via iteration at different times with different values of α_d

$$U(t) = U_0 - \frac{U_0}{\Gamma\left(\alpha_d(t)\right)\Gamma\left(1-\alpha_d(t)\right)}\int_0^t \frac{\mathrm{d}x}{(T+x)^{\alpha_c}(t-x)^{1-\alpha_d(t)}}; \qquad t>0$$

Now we further reduce the obtained expression $U(t) = U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t \frac{\mathrm{d}x}{(T+x)^{\alpha}(t-x)^{1-\alpha}}$ i.e. for self discharge, in following steps.

Put $T+x=\tau$, $\mathrm{d}x=\mathrm{d}\tau$ so for $x=0$ we have $\tau=T$ and for $x=t$ we have $\tau=T+t$, with this we get

$$\begin{aligned} U(t) &= U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_T^{T+t} \frac{\mathrm{d}\tau}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}}, \qquad t>0 \\ &= U_0 - \frac{U_0}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_T^{T+t} F(\tau)\mathrm{d}\tau, \qquad F(\tau) = \frac{1}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}} \end{aligned}$$

Now we break $\int_T^{T+t} F(\tau)\mathrm{d}\tau$ as

$$\int_T^{T+t} F(\tau)\mathrm{d}\tau = \int_T^{0} F(\tau)\mathrm{d}\tau + \int_0^{T+t} F(\tau)\mathrm{d}\tau$$

and call the second term as $\mathcal{I}(t)$. We write in terms of convolution of two functions $f*g$ defined as $\left(f(t)\right)*\left(g(t)\right) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$ with substitution $T+t=\overline{t}$

$$\begin{aligned} \mathcal{I}(t) = \int_0^{T+t} F(\tau)\mathrm{d}\tau &= \int_0^{T+t} \frac{\mathrm{d}\tau}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}} \\ &= \int_0^{\overline{t}} \frac{\mathrm{d}\tau}{\tau^{\alpha}(\overline{t}-\tau)^{1-\alpha}} \\ &= \left(\frac{1}{t^{\alpha}}\right)*\left(\frac{1}{t^{1-\alpha}}\right) \end{aligned}$$

Using Laplace pair $t^{\alpha} \leftrightarrow \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ we write $\mathcal{L}\left\{\mathcal{I}(t)\right\} = \mathcal{I}(s) = \left(\mathcal{L}\left\{t^{-\alpha}\right\}\right)\left(\mathcal{L}\left\{t^{-(1-\alpha)}\right\}\right)$ as follows

$$\mathcal{I}(s) = \left(\frac{\Gamma(-\alpha+1)}{s^{-\alpha+1}}\right)\left(\frac{\Gamma\left(-(1-\alpha)+1\right)}{s^{\left(-(1-\alpha)+1\right)}}\right)$$
$$= \frac{\left(\Gamma(1-\alpha)\right)\left(\Gamma(\alpha)\right)}{s}$$

Extracting $\mathcal{I}(t)$ by inverse Laplace of obtained $\mathcal{I}(s)$ we get

$$\begin{aligned}\mathcal{I}(t) &= \mathcal{L}^{-1}\left\{\mathcal{I}(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\left(\Gamma(1-\alpha)\right)\left(\Gamma(\alpha)\right)\left(\tfrac{1}{s}\right)\right\}, \qquad \mathcal{L}^{-1}\left\{\tfrac{1}{s}\right\} = 1 \\ &= \left(\Gamma(1-\alpha)\right)\left(\Gamma(\alpha)\right)\end{aligned}$$

Thus we derived the expression $\int_0^{T+t} F(\tau)\mathrm{d}\tau = \int_0^{T+t} \frac{1}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}}\mathrm{d}\tau = \left(\Gamma(1-\alpha)\right)\left(\Gamma(\alpha)\right)$. Using this we write the following steps

$$\begin{aligned}U(t) &= U_0 - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\left[\int_0^{T+t} F(\tau)\mathrm{d}\tau + \int_T^{0} F(\tau)\mathrm{d}\tau\right] \\ &= U_0 - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\left(\Gamma(1-\alpha)\Gamma(\alpha)\right) - \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_T^{0} F(\tau)\mathrm{d}\tau \\ &= -\frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_T^{0} F(\tau)\mathrm{d}\tau \\ &= \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T} F(\tau)\mathrm{d}\tau \\ &= \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T} \frac{\mathrm{d}\tau}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}}\end{aligned}$$

Therefore $U(t) = \frac{U_0}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^{T} \frac{\mathrm{d}\tau}{\tau^{\alpha}(T+t-\tau)^{1-\alpha}}$ is the voltage over open capacitor at self discharge. This function of time T depends on the total time the capacitor has been on the voltage source. In a way this capacitor is memorizing its charging history. This explanation was possible only by usage of fractional derivative; taking into the observation of power law in relaxation process. In the above formulation putting $\alpha = 1$ to get $U(t) = U_0$ is not direct.

Fractional Derivative in Hysteresis-from curve fitting point of view

The lag or delay of a magnetic or electric material known commonly as Magnetic Hysteresis, or Ferroelectric Hysteresis. Say this normal phase lag or lead is present in Capacitor or Inductor component. The relation of current and voltage is

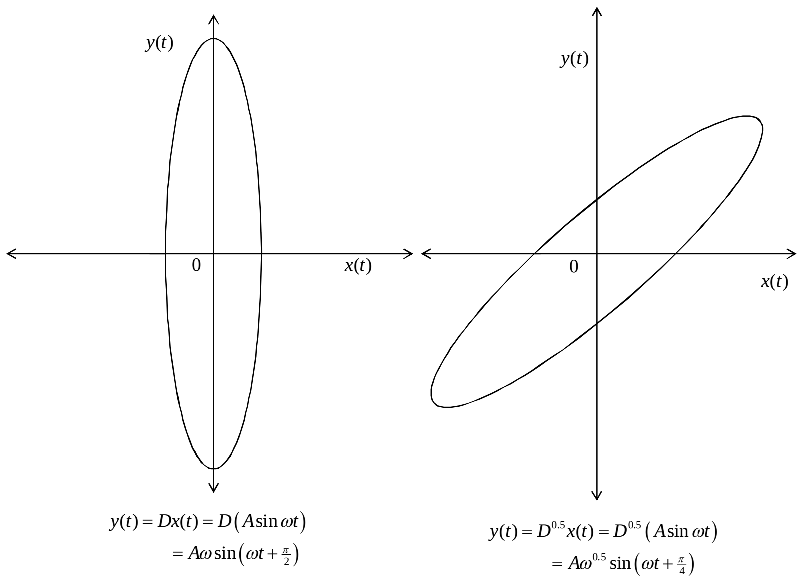
$$v_C(t) = \frac{1}{C}\int_0^t i_C(t)\mathrm{d}t \qquad i_C(t) = C\frac{\mathrm{d}v_C(t)}{\mathrm{d}t}$$

$$v_L(t) = L\frac{\mathrm{d}i_L(t)}{\mathrm{d}t} \qquad i_L(t) = \frac{1}{L}\int_0^t v_L(t)\mathrm{d}t$$

In above if we denote by parameters $x(t)$ and $y(t)$, then we have phase-lead/lag as

$$y(t) = k\frac{\mathrm{d}}{\mathrm{d}t}x(t)$$

Let us take $x(t) = A\sin\omega t$, then we have $y(t) = A\omega\sin\left(\omega t + \frac{\pi}{2}\right)$. This is how we measure Hysteresis practically, and the Lissajous figure, $y = f(x)$ is following, in Figure-6 (a); is a Hysteresis curve too.



(a) **(b)**

Figure-6: Lissajious figure showing lead/lag (a) Integer order derivative; (b) Fractional Order Derivative

The plot (a) shows that it resembles a Hysteresis curve given by following figure

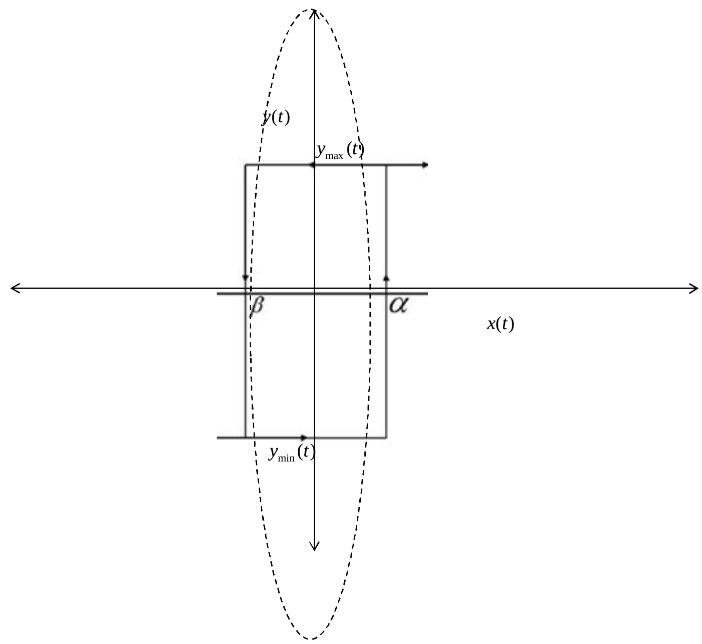


Figure-7: A rectangular hysteresis curve

Generally this type of steep slope from the plus saturation point to minus saturation point is not possible, except in electronic circuit comparator. However, the classical Preisach operator (1935) used to model hysteresis is like above figure-7 (this we will not be discussing). Now looking at the fractional derivative relation,

$$y(t) = k\left(D^{\alpha}\left(x(t)\right)\right)$$

With $x(t) = A\sin\omega t$ we have $y(t) = D^{\alpha}\left(A\sin\omega t\right) \approx A\omega^{\alpha}\sin\left(\omega t + \alpha\frac{\pi}{2}\right)$, the Lissajous figure is represented in the Figure-6 (b), with $\alpha = 0.5$. These slanted ellipses are visible in hysteresis experiments at low input signals when saturation aspects are not prominent. The corresponding Hysteresis plot-with saturation is in following Figure-8, in actual MH curve of magnetic material.

Therefore, we have some usefulness of using fractional derivative in modeling Hysteresis. We note that normal derivative gives a phase lead/lag of 90^0; but this is point/local operator, and the fractional derivative gives a phase lead lag of $\alpha \times 90^0$ and it is non-local operator with memory. The above is true for fractional integration operator i.e. $D^{-\alpha}x(t)$, $0 < \alpha < 1$ too.

This plot figure-6 (b) of $y(t) = f(x(t))$, when $x(t)$ is periodic gives a typical hysteresis plot for a particular frequency ω. Therefore with different frequency of $x(t)$ we will get different plots.

This figure-6 (b) is also for a particular phase shift for $\alpha = \frac{1}{2}$ that is 45^0. In general thus the hysteresis curve will have different phase shift (may be given by different α .

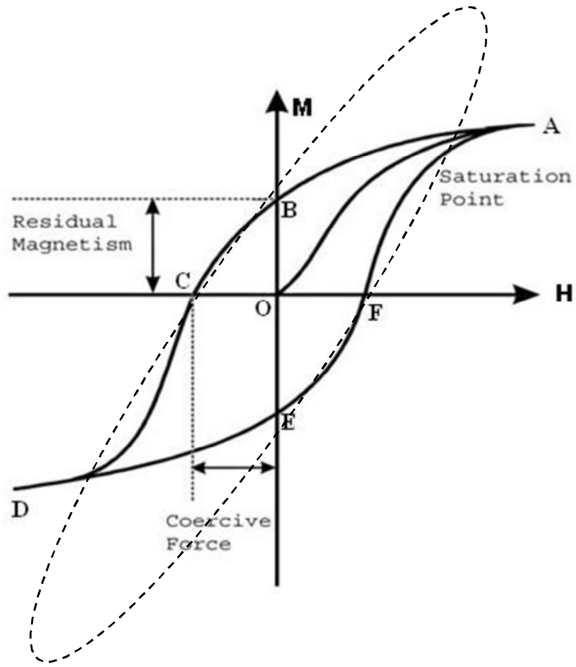


Figure-8: The Hysteresis curve of Magnetic material (Inductor coil)

One can thus say a hysteresis curve may thus be obtained by fitting a general fractional differential-integro equation as following

$$y(t) + \sum_{k=1}^{L} A_k D^{\alpha_k} y(t) = A_{L+1} x(t) + \sum_{k=L+2}^{M} A_k D^{\alpha_k} x(t) + \sum_{k=M+1}^{N} A_k D^{-\alpha_k} x(t)$$
$$0 < \alpha_k < 1$$

The above is showing generalization of using fractional derivative and integral operator to have curve fitting process; without going into physical aspect. Now we turn to physics.

Friction or Viscoelasticty is the cause of Hysteresis

The losses are often described as being in three categories, for Magnetic material are follows

Hysteresis losses: When the magnetic field through the core changes, the magnetization of the core material changes by expansion and contraction of the tiny magnetic domains it is composed of, due to movement of the domain walls. This process causes losses, because the domain walls get "snagged" on defects in the crystal structure and then "snap" past them, dissipating energy as heat (irreversible loss). This is called hysteresis loss. It can be seen in the graph of the *B* field versus the *H* field for the material, which has the form of a closed loop. The amount of energy lost in the material in one cycle of the applied field is proportional to the area inside the hysteresis loop. Since the energy lost in each cycle is constant, hysteresis power losses increase proportionally with frequency. This loss is at low frequency. This domain wall movement exists in ferroelectric material too, and we get P E hysteresis.

Eddy-current losses: At higher frequencies, we have additional eddy current losses. If the core is electrically conductive, the changing magnetic field induces circulating loops of current in it, called eddy currents, due to electromagnetic induction. The loops flow perpendicular to the magnetic field axis. The energy of the currents is dissipated as heat in the resistance of the core material. The power loss is proportional to the area of the loops and inversely proportional to the resistivity of the core material. Eddy current losses can be reduced by making the core out of thin laminations which have an insulating coating, or alternatively, making the core of a nonconductive magnetic material, like ferrite.

Anomalous losses: By definition, this category includes any losses in addition to eddy-current and hysteresis losses. This can also be described as broadening of the hysteresis loop with frequency. Physical mechanisms for anomalous loss include localized eddy-current effects near moving domain walls.

Therefore the point is regarding "moving-wall-domain"; is due to magneto-mechanical response in BH curve or electro-mechanical response in PE curve, giving hysteresis. These electro or magneto mechanical effects are significant in presence of parallel and anti-parallel domains which switch at various field strengths. The movement of domain walls exhibit "effective viscocity" and this provides reversible and irreversible component to the polarization or magnetization-where loss takes place as heat.

Modeling the non-linear hysteresis curve via fractional derivative

Below the Curie point temperature, we have a non linear relation that can be used to plot the an-hysteric curve $y = f(x)$ as

$$x = f^{-1}(y) = \tan(\kappa y)$$

Many curves Hysteresis curves can be plotted by translation of the above as following

$$x - x_c\left(\operatorname{sgn}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\right) = \tan(\kappa y)$$

The point x_c is coercive field like E_c in case of ferroelectric hysteresis of P-E plot, and H_c in magnetic hysteresis in M-H plot (figure-8). The function $f^{-1}(y)$ is an odd monotonic function that saturates at high value of $y(t)$ that is at high Polarization P field, or high magnetization M (or B) value. The parameter κ is to be fitted as per material under consideration-from its first and basic an-hysteric curve, where $x = f^{-1}(y) = 0$ at $y = 0$.

For P-E curves and B-H curves we have following

$$E - E_c \operatorname{sgn}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) = f^{-1}(P) \qquad H - H_c \operatorname{sgn}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) = f^{-1}(B)$$

The above equation is very close to static friction mechanical equation used for dry-friction. The friction always opposes the motion. The dynamic friction is usually introduced by adding a resistive term to the above static operator. Commonly ohmic dependence of this dynamic friction term that leads to an increase in the high frequency components of the polarization or magnetization is added and we get the following

$$E - E_c \operatorname{sgn}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) + \rho\frac{\mathrm{d}P}{\mathrm{d}t} = f^{-1}(P) \qquad H - H_c \operatorname{sgn}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) + \gamma\frac{\mathrm{d}B}{\mathrm{d}t} = f^{-1}(B)$$

With the usage of fractional derivative, we write "damped hysteresis" operator as follows with $0 < \alpha < 1$

$$E - E_c \operatorname{sgn}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) + \rho\left(D_t^{\alpha} P(t)\right) = f^{-1}(P) \qquad H - H_c \operatorname{sgn}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) + \gamma\left(D_t^{\alpha} B(t)\right) = f^{-1}(B)$$

The above evolution gave the last expression, where the damping due to visco-elastic movement of domain walls is having memory due to non-local operator, the fractional derivative with $0 < \alpha < 1$

$$E - E_c \operatorname{sgn}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) + \rho\left(\lim_{\Delta t \downarrow 0} \sum_{k=0}^{\left\lfloor k=\frac{t}{\Delta t}\right\rfloor} (-1)^k \binom{\alpha}{k} P(t - k\Delta t)\right) = f^{-1}(P)$$
$$H - H_c \operatorname{sgn}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) + \gamma\left(\lim_{\Delta t \downarrow 0} \sum_{k=0}^{\left\lfloor k=\frac{t}{\Delta t}\right\rfloor} (-1)^k \binom{\alpha}{k} B(t - k\Delta t)\right) = f^{-1}(B)$$

Where the generalized binomial coefficients are in terms of Gamma function, which generalizes the factorial as $\alpha! = \Gamma(\alpha + 1)$.

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{k!\Gamma(\alpha - k + 1)}$$

The function i.e. $y = f(x)$, have different forms too, like Langevin model gives $y = y_S\left(\coth(ax) - \frac{1}{ax}\right)$ and Ising spin model gives $y = y_S \tanh(ax)$, where y_S is saturation level.

Hysteresis due to Fractional Differ-integration of Periodic Function

We have following formulas

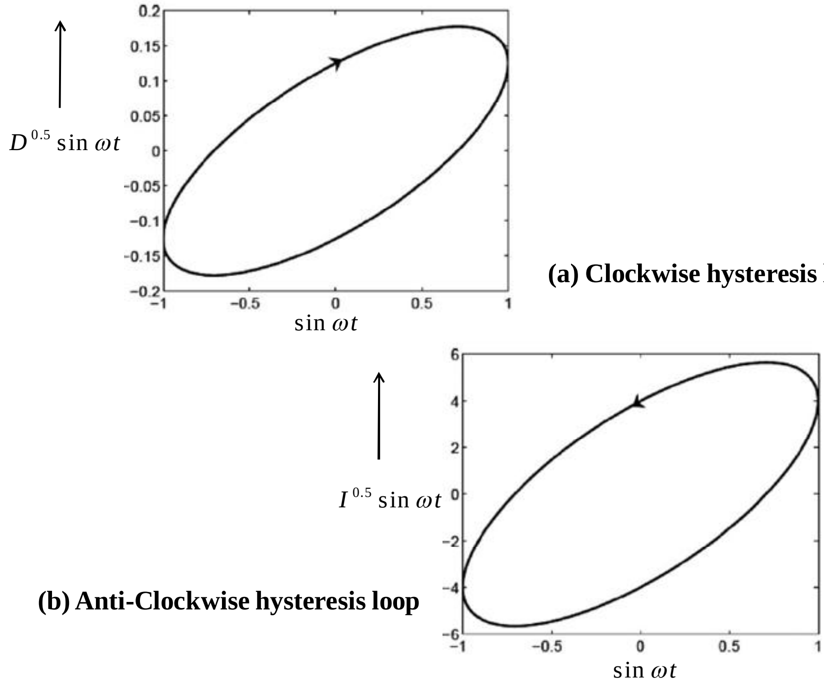
$$ {}_0D_t^{\alpha}\left(\sin(\omega t)\right) = \omega^{\alpha}\sin\left(\omega t + \tfrac{\pi\alpha}{2}\right) + \frac{\left(\omega t\right)^{-1-\alpha}}{\omega\left(\Gamma(-\alpha)\right)} - \frac{\left(\omega t\right)^{-3-\alpha}}{\omega^{3}\left(\Gamma(-\alpha-2)\right)} + ... $$

$$ {}_0D_t^{\alpha}\left(\cos(\omega t)\right) = \omega^{\alpha}\cos\left(\omega t + \tfrac{\pi\alpha}{2}\right) + \frac{\left(\omega t\right)^{-2-\alpha}}{\omega^{2}\left(\Gamma(-\alpha-1)\right)} - \frac{\left(\omega t\right)^{-4-\alpha}}{\omega^{4}\left(\Gamma(-\alpha-3)\right)} + ... $$

$$ {}_0I_t^{\alpha}\left(\sin(\omega t)\right) = \omega^{-\alpha}\sin\left(\omega t - \tfrac{\pi\alpha}{2}\right) + \frac{\left(\omega t\right)^{-1+\alpha}}{\omega\left(\Gamma(\alpha)\right)} - \frac{\left(\omega t\right)^{-3+\alpha}}{\omega^{3}\left(\Gamma(\alpha-2)\right)} + ... $$

$$ {}_0I_t^{\alpha}\left(\cos(\omega t)\right) = \omega^{-\alpha}\cos\left(\omega t - \tfrac{\pi\alpha}{2}\right) + \frac{\left(\omega t\right)^{-2+\alpha}}{\omega^{2}\left(\Gamma(\alpha-1)\right)} - \frac{\left(\omega t\right)^{-4+\alpha}}{\omega^{4}\left(\Gamma(\alpha-3)\right)} + ... $$

The first term in both fractional differentiation shows a forward phase-shifting of the periodic function by amount $\frac{\alpha\pi}{2}$; and for fractional integration shows phase lagging by $\frac{\alpha\pi}{2}$. Thus when derivative of the periodic function is plotted against actual periodic function, then we get ‘clock-wise’ hysteresis loop, due to this forward phase shifting. The above expressions also generate transient terms as $t^{-(k\pm\alpha)}$, with $k = 1, 2, 3, 4, ...$ and $0 < \alpha < 1$, they all will decay as t grows. Similarly when the fractional integration of the periodic function is plotted against actual periodic function, then we get ‘anti-clockwise’ hysteresis loop. Both the hysteresis is depicted in Figure-9.



(a) Clockwise hysteresis loop

(b) Anti-Clockwise hysteresis loop

<u>Figure-9: Clockwise and anti-clockwise hysteresis loop</u>

A constant function x_0 decays sharply while operated by fractional derivative and increases with time at a decaying rate on being operated by fractional integrals as indicated below.

$$D^{\alpha}\left(x_0\right)=\frac{x_0t^{-\alpha}}{\Gamma(1-\alpha)} \qquad\qquad I^{\alpha}\left(x_0\right)=\frac{x_0t^{\alpha}}{\Gamma(1+\alpha)}$$

Hence under the operation of fractional differential operator the non-harmonic components of a function dies out shortly and its harmonic components give rise to a closed loop clockwise hysteresis and on the other hand the same function shows an anti-clockwise hysteresis with a hysteresis loop moving upward with a number of cycles when operated by a fractional integral operator.

Hysteresis experiment with super-capacitor to determine presence of fractional capacity

We give triangular current $I(t)$ to the super-capacitor as shown in Figure-10, and record the voltage across the same, depicted as plot of $V(t)$.

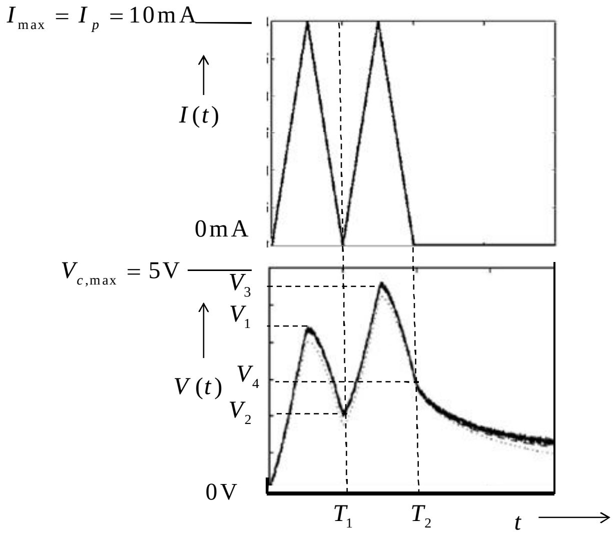


Figure-10: Current excitation and voltage measurement of super-capacitor.

For $0 < t < (T_1/2)$, the current is $I(t) = \frac{2I_p}{T_1}t$, and the $V(t)$ is $\frac{1}{C_\alpha}\left({}_0D_t^{-\alpha}\left(I(t)\right)\right) + V(0)$ that gives (considering $V(0) = 0$) the following

$$V(t) = \left(\frac{2I_p t^{1+\alpha}}{C_\alpha T_1\left(\Gamma(2+\alpha)\right)}\right)$$

$$V\left(T_1/2\right) = \left(\frac{2I_p\left(\frac{T_1}{2}\right)^{1+\alpha}}{C_\alpha T_1\left(\Gamma(2+\alpha)\right)}\right) = \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_1}{2}\right)^\alpha$$

Writing $\frac{T_1}{2} = T_{1/2}$ we get

$$V_1 = V\left(T_1/2\right) = \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(T_{1/2}\right)^\alpha$$

For $(T_1/2) < t < T_1$, the current is $I(t) = -\left(\frac{2I_p}{T_1}\right)\left(t - \frac{T_1}{2}\right) + I_p$ and the $V(t)$ is $\frac{1}{C_\alpha}\left({}_{T_1/2}D_t^{-\alpha} I(t)\right) + V\left(T_1/2\right)$ that is

$$V(t) = -\frac{2I_p\left(t - {}^{T_1}\!/_2\right)^{1+\alpha}}{C_\alpha T_1\left(\Gamma(2+\alpha)\right)} + \frac{I_p\left(t - {}^{T_1}\!/_2\right)^{\alpha}}{C_\alpha\left(\Gamma(\alpha+1)\right)} + \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha}$$

We note that $V\left({}^{T_1}\!/_2\right) = \frac{I_p}{C_\alpha(\Gamma(2+\alpha))}\left(\frac{T_1}{2}\right)^\alpha$ and $V\left(T_1\right)$ is

$$\begin{aligned}
V\left(T_1\right) &= -\frac{2I_p\left(T_1 - {}^{T_1}\!/_2\right)^{1+\alpha}}{C_\alpha T_1\left(\Gamma(2+\alpha)\right)} + \frac{I_p\left(T_1 - {}^{T_1}\!/_2\right)^{\alpha}}{C_\alpha\left(\Gamma(\alpha+1)\right)} + \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} \\
&= -\frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} + \frac{I_p}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} + \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} \\
&= \frac{I_p}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} \\
V_2 = V\left(T_1\right) &= \frac{I_p}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(T_{1/2}\right)^{\alpha}
\end{aligned}$$

We have used ${}_aD_t^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}(t-a)^{n-\alpha}$, in above derivations. Now for the region $T_1 < t < \frac{T_1+T_2}{2}$ we write $I(t) = \frac{2I_p}{T_2-T_1}\left(t - T_1\right)$, then $V(t)$ is $\frac{1}{C_\alpha}\left({}_{T_1}D_t^{-\alpha} I(t)\right) + V\left(T_1\right)$, gives

$$V(t) = \frac{2I_p\left(t - T_1\right)^{1+\alpha}}{C_\alpha\left(T_2 - T_1\right)\left(\Gamma(2+\alpha)\right)} + \frac{I_p}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha}$$

From above we write $V\left(\frac{T_1+T_2}{2}\right)$ as

$$\begin{aligned}
V\left(\tfrac{T_1+T_2}{2}\right) &= \frac{I_p}{C_\alpha\left(\Gamma(2+\alpha)\right)}\left(\frac{T_2 - T_1}{2}\right)^{\alpha} + \frac{I_p}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(\frac{T_1}{2}\right)^{\alpha} \\
&= \frac{I_p\left(\frac{T_2-T_1}{2}\right)^{\alpha}}{C_\alpha\left(1+\alpha\right)\left(\Gamma(1+\alpha)\right)} + \frac{I_p\left(\frac{T_1}{2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)}
\end{aligned}$$

Let us call $\frac{T_2-T_1}{2} = \frac{T_1}{2} = T_{1/2}$ then

$$\begin{aligned}
V_3 = V\left(\tfrac{T_1+T_2}{2}\right) &= \frac{I_p\left(T_{1/2}\right)^{\alpha}}{C_\alpha\left(1+\alpha\right)\left(\Gamma(1+\alpha)\right)} + \frac{I_p\left(T_{1/2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)} \\
&= \frac{I_p\left(T_{1/2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(\frac{1}{1+\alpha} + 1\right) = \frac{(2+\alpha)}{(1+\alpha)\left(\Gamma(1+\alpha)\right)C_\alpha} I_p\left(T_{1/2}\right)^{\alpha}
\end{aligned}$$

In the region $\frac{T_1+T_2}{2} < t < T_2$ we have $I(t) = -\frac{2I_p}{T_2-T_1}\left(t - \frac{T_1+T_2}{2}\right) + I_p$ and $V(t)$ is $\frac{1}{C_\alpha}\left({}_{(T_1+T_2)/2}D_t^{-\alpha} I(t)\right) + V\left(\frac{T_1+T_2}{2}\right)$, which we write as

$$V(t) = -\frac{2I_p\left(t-\frac{T_1+T_2}{2}\right)^{1+\alpha}}{C_\alpha\left(T_2-T_1\right)\left(\Gamma(2+\alpha)\right)} + \frac{I_p\left(t-\frac{T_1+T_2}{2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)}$$

At $t = T_2$, we write $V\left(T_2\right)$ as

$$\begin{aligned} V\left(T_2\right) &= -\frac{I_p\left(\frac{T_2-T_1}{2}\right)^{\alpha}}{C_\alpha\left(\Gamma(2+\alpha)\right)} + \frac{I_p\left(\frac{T_2-T_1}{2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)} \\ &= \frac{I_p\left(T_{1/2}\right)^{\alpha}}{C_\alpha\left(\Gamma(1+\alpha)\right)}\left(1-\frac{1}{(\alpha+1)}\right) \\ V_4 = V\left(T_2\right) &= \frac{\alpha}{(\alpha+1)\left(\Gamma(\alpha+1)\right)}\frac{I_p}{C_\alpha}\left(T_{1/2}\right)^{\alpha} \end{aligned}$$

The ramp current rising from zero to peak current I_p i.e. 10 mA, at $t = T_1/2$ will give a rising voltage $V(t)$ from zero to a peak value $V(t) = V_1$ (less than maximum limit of the super-capacitor), then the falling ramp current from I_p at $t = T_1/2$ till time $t = T_1$ will show a decrement in voltage $V(t)$ to a value not as zero, i.e. V_2 Volts at $t = T_1$. Again after the time $t = T_1$; the ramp current $I(t)$ rises from zero to I_p, this again gives a voltage rise, from V_2, to V_3 .The falling current $I(t)$, to zero at time $t = T_2$ gives a falling voltage curve, from V_3 to V_4 . After the time $t > T_2$, when the current is zero, the retained voltage V_4 in the super-capacitor holds, and the decay of voltage is due to self discharge and redistribution phenomena, as shown in Figure-10. The corresponding Hysteresis curve is depicted in Figure-11.

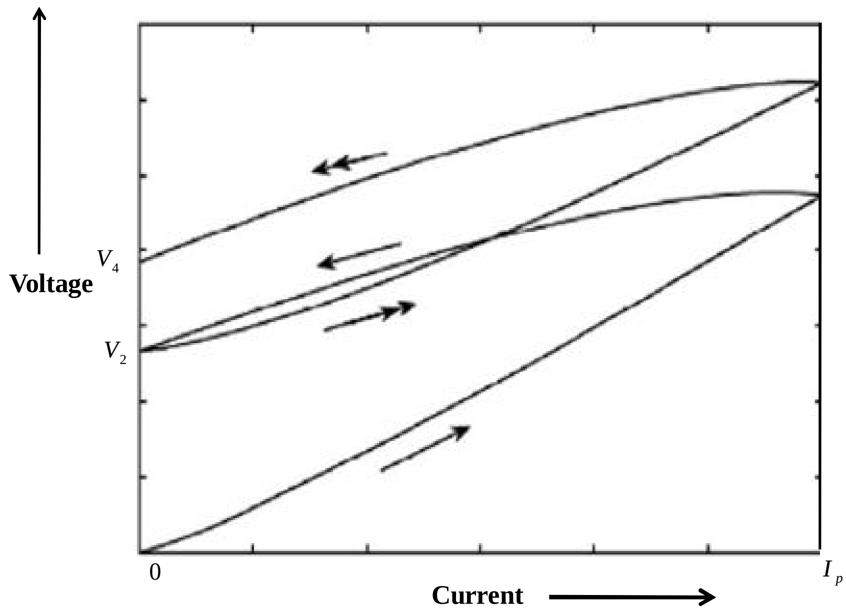


Figure-11: Hysteresis curve for super-capacitor

Hysteresis Model with Fractional Integration for super-capacitor

If we take the Impedance of super-capacitor as series combination of ESR (R) in Ohms, R several fractional capacity C_k in units of Farad / $\sec^{1-\alpha}$ (also termed as Constant Phase Element CPE), with $k=1,...N$ (our choice) as

$$Z(s)=R+\sum_{k=1}^{N}\frac{1}{s^{\alpha_k}C_k};\qquad 0<\alpha_k<1\qquad k=1,2,...N$$

$$V(t)=RI(t)+\sum_{k=1}^{N}\frac{1}{C_k}D^{-\alpha_k}\left[I(t)\right]$$

Using Numerical scheme the above can be expressed as

$$V(t_m)=R\left(I(t_m)\right)+\sum_{k=1}^{N}\frac{\Delta t^{\alpha_k}}{C_k\Gamma(2+\alpha_k)}\left[\frac{(1+\alpha_k)}{m^{-\alpha_k}}I(t_0)+\sum_{j=0}^{m-1}\left((j+1)^{(1+\alpha_k)}-(j)^{(1+\alpha_k)}\right)\left(I(t_{m-j})-I(t_{m-j-1})\right)\right]$$

The parameters C_k for $k=1,...N$, R and α_k for $k=1,..N$ can be obtained by minimizing error defined as

$$e=\sum_{m=0}^{n}\left(V_{\exp}(t_m)-V(t_m)\right)^2$$

i.e. between experimental $V(t_m)$ denoted by $V_{\exp}(t_m)$ and numerically calculated $V(t_m)$ from above formula.

A generalized Hysteresis Model with fractional derivative with phase shifting of input Fourier components with convolution integral

In this model the change in output variable $y(t)$ is assumed to be combination of a linear function of input variable $x(t)$ fractional order derivatives of order α_k with $0<\alpha_k<1$ of $x_{Ph}(t)$ i.e. with each frequency component of $x(t)$ shifted backwards and integer order integral of different powers of $x(t)$, multiplied by an exponentially decaying function. It can be expressed as

$$y(t)=A_0x(t)+\sum_{k=1}^{N}A_kD_{ph}^{\alpha_k}\left[x(t)\right]+\sum_{l=1}^{N-M}\left(A_{l+M}\int_0^t e^{-p_l(t-\tau)}\left(x(\tau)\right)^{q_l}\mathrm{d}\tau\right)$$

$D_{Ph}^{\alpha_k}$ is a fractional derivative of order α_k with a phase of the each frequency component of $x(t)$ shifted by $-\pi\left(\beta_k/2\right)$, where $\beta_k>0$. The term $\sum_{l=1}^{N-M}A_{l+M}\int_0^t e^{-p_l(t-\tau)}\left(x(\tau)\right)^{q_l}\mathrm{d}\tau$ is sum of convolution integrals i.e. $\sum_{l=1}^{N-M}A_{l+M}\left(\exp(-p_lt)*(x(t))^{q_l}\right)$. The convolution is defined as $h(t)*f(t)=\int_{-\infty}^{t}h(t-\tau)f(\tau)\mathrm{d}\tau$, where $h(t)=e^{-p_lt}$; $p_l>0$, in our case-i.e. exponentially decaying function, with decay rate as p_i . Causal functions mean that no convolution response can obtained before the function $f(t)$ is applied, in our case it is $h(t)*f(t)=\int_0^t h(t-\tau)f(\tau)\mathrm{d}\tau$.

Figure-12 demonstrates the convolution process. The function is $f(t)=\cos\left(\frac{2\pi}{5}t\right);\quad t\geq 0$, and $f(t)=0;\quad t<0$. The figure 12- (a) shows $h(\tau)$ vs. τ. The figure 12 (b) shows $f(t);\quad t\geq 0$. The figure-12 (c) shows $h(-\tau)$. The shifted curve $h(t-\tau)$ is obtained for the value $t=5$, and the figure-12 (d) shows the plot of $h(t-\tau)$ for $t=5$ i.e. $h(5-\tau)$ vs. τ. The figure-12 (e) shows the full integrand for $t=5$ i.e. $h(5-\tau)f(\tau)$. The shaded portion the area under the curve $h(5-\tau)f(\tau)$ that is $\int_0^5 h(5-\tau)f(\tau)\mathrm{d}\tau$ Now moving this $h(t-\tau)$ for several continuous values of t from 0-10, repeating the graphs (d) and (e) and obtaining the value of the integral of the product (for several values of t the final graph figure-12 (f) is obtained. In the figure-12 (f), the point X is $\int_0^5 f(\tau)h(5-\tau)\mathrm{d}\tau$, definite value of the integration.

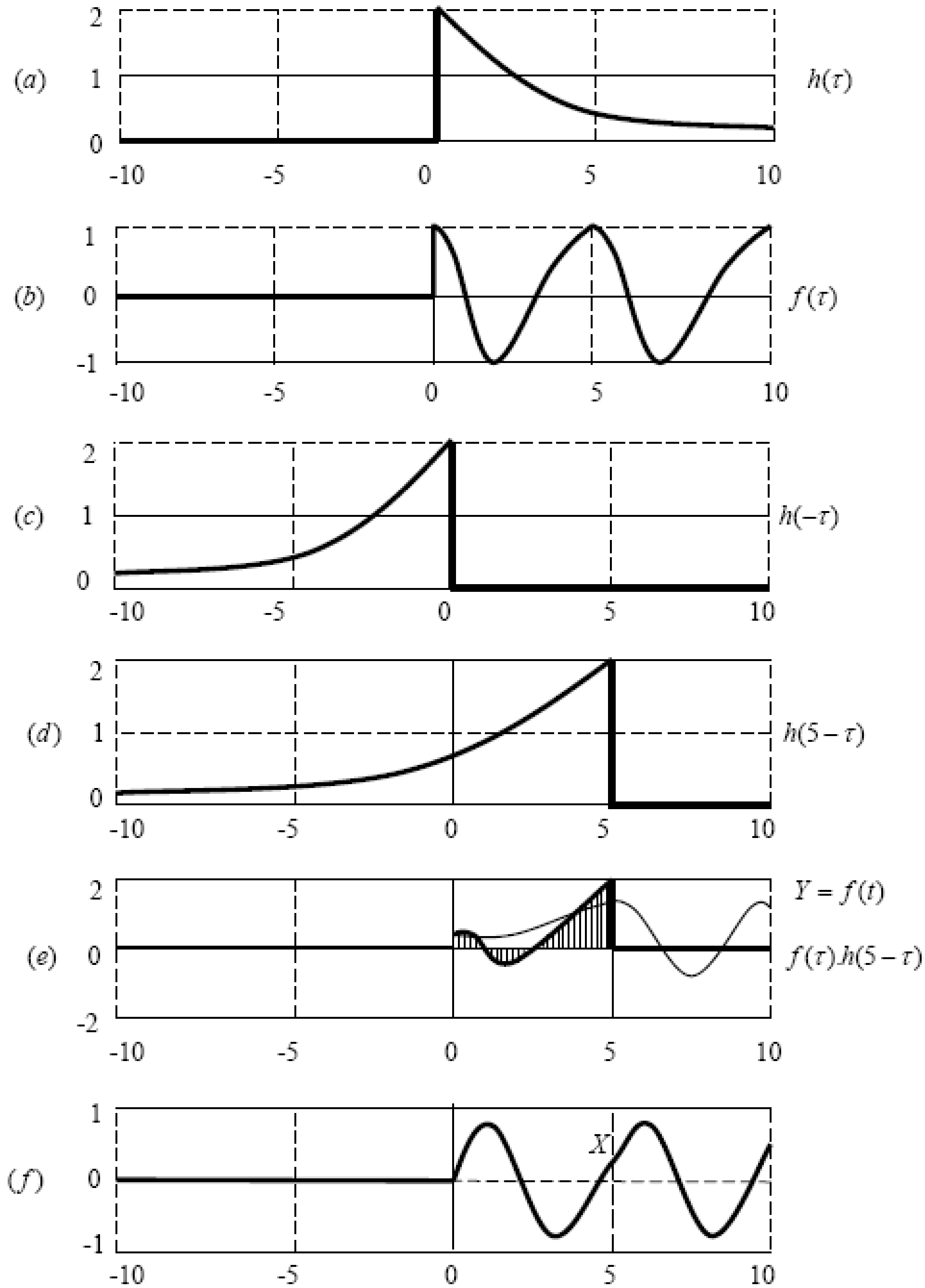


Figure-12: Convolution process

If $f(t)$ constant function say x_0 for $t \ge 0$, (take $q_l = 1$) then the convolution with $h(t) = e^{-p_i t}$ $p_i > 0$ is $\int_0^t e^{-p_i(t-\tau)} x_0 \mathrm{d}\tau$ is $\frac{x_0}{p_i}\left(1 - e^{-p_i t}\right)$. This convolution with an exponentially decaying function with a constant function gives monotonically increasing function with t .

Now we take formula Riemann-Liouville for fractional integration i.e. $\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{-(1-\alpha)} f(\tau)\mathrm{d}\tau$. The kernel of integration is $h(t-\tau) = (t-\tau)^{-(1-\alpha)}$, and we note that $h(t) = t^{-(1-\alpha)}$ is also a decaying function for $0 < \alpha < 1$ like in above example with $h(t) = e^{-p_i t}$, $p_i > 0$. Thus the convolution integral that is used in the formulation gives notion of use of fractional integration. The constant function say $f(t) = x_0$, for $t \ge 0$ has fractional integration as $\frac{1}{\Gamma(1+\alpha)} x_0 t^{\alpha}$ i.e. monotonically increasing function; like one $\frac{x_0}{p_i}\left(1 - e^{-p_i t}\right)$, obtained for $h(t) = e^{-p_i t}$. Thus if the input function $x(t)$ has any DC value say x_0, then the convolution process with a decaying function will generate increasing function with time.

If $x(t)$ can be expressed as Fourier components as following (only the periodic part)

$$x(t) = \sum_{i=0}^{N} x_i \sin(\omega_i t) + \sum_{i=N+1}^{2N} x_i \cos(\omega_{i-N} t)$$

$D_{Ph}^{\alpha_k}$ can be defined as

$$D_{Ph}^{\alpha_k}\left[x(t)\right] \overset{\text{def}}{=} D^{\alpha_k}\left[x_{Ph}(t)\right]$$

Where $x_{Ph}(t)$ is given by the following equation (i.e. each component phase shifted)

$$x_{Ph}(t) = \sum_{i=0}^{N} x_i \sin\left(\omega_i t - \frac{\pi\beta_k}{2}\right) + \sum_{i=N+1}^{2N} x_i \cos\left(\omega_{i-N} t - \frac{\pi\beta_k}{2}\right)$$

In the frequency domain the Fourier transform of $x_{Ph}(t)$ and $G_{D_{Ph}^{\alpha_k}}(t)$ can be related to the Fourier transform of $x(t)$ by the following

$$x_{Ph}(\omega) = i^{-\beta_k} x(\omega)$$
$$G_{D_{Ph}^{\alpha_k}}(\omega) = (i\omega)^{\alpha_k}\left(i^{-\beta_k} x(\omega)\right) = \omega^{\alpha_k}\left(i^{(\alpha_k - \beta_k)}\right) x(\omega)$$

The phase shifted fractional order derivatives of $x(t)$ impart the anti-clockwise hysteresis nature of the loop to the variation in $y(t)$ with $x(t)$.

The convolution term i.e. $\sum_{l=1}^{N-M} A_{l+M} \int_0^t e^{-p_l(t-\tau)} \left(x(\tau)\right)^{q_l} \, \mathrm{d}\tau$ are functions of $x(t)$, which keep increasing with time at a decaying rate for positive values of $x(t)$. These terms are used to take care of the properties of system, which causes its hysteresis loop to move upwards with number of cycles of applied $x(t)$.

By using the numerical formula as above for $D^{\alpha}\left(f(t)\right)$ the fractional derivative and trapezoidal rule for integration the instantaneous value of $y(t)$ at time t_m is

$$y(t_m) = y_1(t_m) + y_2(t_m)$$

Where

$$y_1(t_m) = A_0\left(x(t_m)\right) + \sum_{k=1}^{N} \frac{A_k(\Delta t)^{-\alpha_k}}{\Gamma(2-\alpha_k)} \left[\frac{(1-\alpha_k)}{m^{(\alpha_k)}} x_{ph}(t_0) + \sum_{j=0}^{N-1} \left((j+1)^{(1-\alpha_k)} - j^{(1-\alpha_k)} \right) \left(x_{Ph}(t_{m-j}) - x_{Ph}(t_{m-j-1}) \right) \right]$$

$$y_2(t_m) = \sum_{l=1}^{N-M} \frac{A_{l+M}}{2} \left(e^{-p_l(t_m-t_0)} \left(x(t_0)\right)^{q_l} + 2\sum_{j=0}^{m-1} e^{p_l(t_m-t_j)} \left(x(t_j)\right)^{q_l} + \left(x(t_m)\right)^{q_l} \right) \Delta t$$

Here we take $t_0 = 0$. The $x_{Ph}(\omega)$ can be calculated by using equation of $x_{Ph}(t)$ that is $x_{Ph}(t) = \sum_{i=0}^{N} x_i \sin\left(\omega_i t - \frac{\pi\beta_k}{2}\right) + \sum_{i=N+1}^{2N} x_i \cos\left(\omega_{i-N} t - \frac{\pi\beta_k}{2}\right)$ after obtaining the $x(\omega)$ by performing FFT (Fast Fourier Transform) of $x(t)$. Then by inverse i.e. IFFT of $x_{Ph}(\omega)$ the $x_{Ph}(t)$ is obtained. The $x_{Ph}(t)$ depends on unknown parameter β_k. Unknown parameters involved are $A_k(k=0,M)$, $\alpha_k(k=1,N)$, $\beta_k(k=1,N)$, $p_k(k=1,M-N)$ and $q_k(k=1,M-N)$. These parameters can be found out by minimizing error e between experimental readings and calculated values

$$e = \sum_{m=0}^{n} \left(y_{\exp}(t_m) - y(t_m) \right)^2$$

With constraints $0 < \alpha_k(k=1,N) < 1$ and $\beta_k(k=1,N) > 0$.

Modified Hysteresis Model with fractional integration of phase shifted input Fourier components of input current for super-capacitors

We use the above method to modify the super-capacitor model i.e.

$$V(t) = RI(t) + \sum_{k=1}^{N} \frac{1}{C_k} D^{-\alpha_k}\left[I(t)\right]$$

The modification involves frequency domain model as

$$V(\omega_j) = RI(\omega_j) + \sum_{k=1}^{N} \frac{1}{C_k} (i\omega_j)^{-\alpha_k} \left(I(\omega_j) \right)$$

In the above frequency model the multiplication of $i^{-\alpha_k}$ with each Fourier component of current causes a phase shift and imparts a hysteric nature in voltage-current variations and the multiplication of $\omega^{-\alpha_k}$ with each Fourier component of current causes the voltage-current variation to be dependent on the rate of applied current. Thus this modified model aims to take care these two effects independently.

Hence the above gets modified as

$$V(\omega_j) = RI(\omega_j) + \sum_{k=1}^{N} \frac{1}{C_k} (i\omega_j)^{(-\alpha_k)} i^{-\beta_k} I(\omega_j)$$

In time domain this modification above leads to

$$V(t) = RI(t) + \sum_{k=1}^{N} \frac{1}{C_k} D_{Ph}^{-\alpha_K} \left[I(t) \right]$$

Where $D_{Ph}^{-\alpha_K} \left[I(t) \right] = D^{-\alpha_K} \left[I_{Ph}(t) \right]$

and phase shifted input is following

$$I_{Ph}(t) = I_{DC} + \sum_{i=0}^{N} I_i \sin\left(\omega_i t - \tfrac{\pi\beta_k}{2} \right) + \sum_{i=N+1}^{2N} I_i \cos\left(\omega_{i-N} t - \tfrac{\pi\beta_k}{2} \right)$$

$$V(t_m) = RI(t_m) + \sum_{k=1}^{N} \frac{(\Delta t)^{\alpha_k}}{C_k \Gamma(\alpha_k + 2)} \left(\frac{1+\alpha_k}{m^{(-\alpha_k)}} \sum_{j=0}^{m-1} \left((j+1)^{(1+\alpha_k)} - j^{(1+\alpha_k)} \right) \left(I_{Ph}(t_{m-j}) - I_{Ph}(t-(j+1)) \right) \right)$$

These parameters $C_k (k=1,N)$ R, $\alpha_k (k=1,N)$, $\beta_k (k=1,N)$ can be found out by minimizing error e between experimental readings and calculated values

$$e = \sum_{m=0}^{n} \left(V_{\exp}(t_m) - V(t_m) \right)^2$$

With constraints $0 < \alpha_k (k=1,N) < 1$ and $\beta_k (k=1,N) > 0$. Here we are not using the convolution term, as we note that the fractional integration operation is giving that effect of convolution operation.

Conclusions

We tried to find simple answer to the open problem i.e. fractional order relating to memory. In fitting the test data of memory phenomena from different fields, we find that the fractional order can be physically explained as an index of memory or forgetfulness. Memory means retention, and that is cause of hysteresis, that we discussed, and how fractional calculus plays role in modeling the phenomena of hysteresis that we showed. The introduction of fractional derivative in dynamic systems makes the system behavior non-local and the hysteresis-loss thus have memory. This note developed some ideas how can we use Fractional Calculus to model hysteresis, and if a system exhibits memory via having fractional order components how the hysteresis curve gets manifested. The link between memory and hysteresis is established in this note. The methods described here can be seen to model hysteresis of any nature. The path dependent behavior of the material and system for that particular input rate is taken care by the hysteresis models with fractional derivative/integration with phase shift. The model involving fractional integral is more realistic, than fractional derivative model. In this fractional integral model, the effect of phase shift and rate dependent independently is a better option.

References

1. Podlubny I. Fractional Differential Equations (Academic Press, San Diego, 1999).
2. Oldham K. B. & Spanier J. The Fractional Calculus (Academic Press, New York, London, 1974).
3. Miller K. S. & Ross B. An Introductory to the Fractional Calculus and Fractional Differential Equations (John Wiley & Sons Inc., New York, 1993).
4. Rossikhin A. & Shitikova M. V. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. Appl. Mech. Rev. 63, 1010801-1-51 (2010).
5. Stiassnie M. On the application of fractional calculus for formulation of viscoelastic models. Appl. Math. Model. 3, 300–302 (1979).
6. Lundstrom B. N., Higgs M. H., Spain W. J. & Fairhall A. L. Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. 11, 1335–1342 (2008Caputo M. Linear models of dissipation whose Q is almost frequency independent-II. Geophys. J. R. Astr. Soc. 13, 529–539 (1967).
7. Bagley R. L. & Torvik P. J. Fractional calculus-a different approach to the analysis of viscoelastically damped structures. AIAA J. 21, 741–748 (1983).
8. Mathai M. & Haubold H. J. Special Functions for Applied Scientists (Springer, New York, 2008).
9. Kneller G. R. & Hinsen K. Fractional Brownian dynamics in proteins. J. Chem. Phys. 121, 10278–10283 (2004).
10. Podlubny I. Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. 5, 367–386 (2002).
11. Mainardi F. Fractional Calculus and Waves in Linear Viscoelasticity (London, Imperial College Press and Singapore, World Scientific, 2010).
12. Besson J. L., Streicher, Chartier E. T. & Goursat P. Viscoelastic creep of nitrogen ceramics. J. Mater. Sci. Lett. 5, 803–805 (1986).
13. Calonder C., Tie Y. & Van Tassel P. R. History dependence of protein adsorption kinetics. PNAS. 98, 10664–10669 (2001).
14. Ebbinghaus H. Memory: A contribution to experimental psychology (1885), Translated by Henry A. Ruger and Clara E. Bussenius in 1913 (http://psychclassics.yorku.ca/Ebbinghaus/).
15. Zhang C. Y. Viscoelastic Fracture Mechanics (Science Press, Beijing, 2006).
16. Brinson H. F. & Brinson L. C. Polymer Engineering Science and Viscoelasticity: An Introduction (Springer, New York, 2008).

17. Du M. L. & Wang Z. H. Initialized fractional differential equations with Riemann-Liouville fractional-order derivative. Eur. Phys. J. Special Topics. 193, 49–60 (2011).
18. Nutting P. G. A general stress-strain-time relationship formula. J. Franklin Inst. 235, 513–524 (1943).
19. Nutting P. G. A new general law of deformation. J. Franklin Inst. 191, 679–685 (1921).
20. Chaplain R. A. Simple viscoelastic model for the stress relaxation of rubber vulcanizates. Nature. 220, 1028–1029 (1968).
21. Engelhardt H., Gaub H. & Sackmann E. Viscoelastic properties of erythrocyte membranes in high-frequency electric fields. Nature. 307, 378–380 (1984).
22. D C Jiles, Introduction to magnetism and Magnetic Material, Chapman and Hall, New-York 1991.
23. D C Jiles, D L Atherton, Theory of magnetization process in ferromagnets and its application to the magnetic field effect, Journal of Physics D, Applied Physics, 17, pp. 1265-1281, 1984.
24. D C Jiles and Atherton, Theory of ferromagnetic hysteresis, Journal of Magnetism and Magnetic Materials, 61, pp. 48-60, 1986.
25. D C Jiles, J B Thoelke, M K Devine, Numerical determination of hysteresis parameters for the modeling of magnetic properties using the theory of ferromagnetic hysteresis, IEEE Transactions on Magnetics, 28 (1), pp. 27-35, 1992.
26. J C Anderson, Dielectrics, Reinhold Publishing Corporation, New-York, 1961.
27. H T Banks et al, Identification of hysteresis control influence operators representing smart actuators, Part-I: Formulation-Mathematical Problems in Engineering-3, pp 287-328, 1997.
28. J W F Brown, Magnetoelastic Interactions, Springer-Verlag, 1966.
29. E Callen et al, Initial magnetization remenance and coercivity of the random anisotropy amorphous ferromagnets, Physical Review B, 16 (1), pp 263-270, 1977.
30. Preisach F, Uber Die Magneticische, Z. Phys, 94, pp 277-302 1935.
31. A Schmidt et al, On numerical evaluation of fractional derivative in Multi-Degree-of Freedom Systems, Signal Process 86 (10) pp 2592-2601.
32. D H Evertett et al, A general approach to hysteresis, Trans Faraday Soc, 48 pp 749-757, 1952.
33. S Das, 'Kindergarten of Fractional Calculus' (A book of lecture notes in limited prints by Dept. of Physics Jadavpur University Kolkata)
34. S Das. Mechanism of wave dissipation via Memory Integral vis-a-vis Fractional derivative, International Journal of Mathematics & Computation Vol-19, Issue-2, 2013, pp732-83.
35. S. Das. Functional Fractional Calculus 2nd Edition, Springer-Verlag 2011.
36. Du M, Wang et al Measuring memory with the order of fractional derivative, Scientific Report 3, Article No. 3431, 2013.
37. Weterlund S, et al, Capacitor Theory, IEEE Transactions on Dielectrics and Electrical Insulation Vol. 1 No. 5, 1994.