Bode loop-shaping with Fractional and Complex Order Differ-integration for robust controls

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Abstract

Here proposal is made to use fractional order and fractional complex order controller design strategy to cope up with uncertain transfer function, that arise in various non-linear system dynamics, especially Fuel Cells. In such systems the operating point shifts under various conditions, thus giving rise to spread values of poles and zero and the gains-giving uncertain transfer function. This discrepancy is needed to be tackled by a robust controller-thus a Fractional Order and Fractional Complex Order Controller is described. This paper describes the derivation of the Bode loop shaping, where the open-transfer function is a fractional integrator and also a case with fractional complex order, and its usage in dealing with robustness.

Key words

Uncertain Transfer-function, Fractional Order Controller, Complex Order Controller, Robust Control, Fractional Calculus, Nichols' chart, Bode-loop shaping

Introduction

Due to simplicity and its popularity in industry, Proportional Integral Derivative (PID) is the most used controller for plant. Thus the academic research around this is very important, and development of newer design methods is developing. Various optimizations on this PID have been carried out, but PID controller suffers lack of 'stability robustness' towards parameter uncertainties. Many approaches are developed to overcome this drawback and most are based on numerical optimization [1, 2]. Even after optimization, the robustness towards parametric uncertainties regarding PID controllers' remains. Therefore new form of PID namely Fractional Order PID or called $\mathrm{PI}^{\alpha}\mathrm{D}^{\beta}$, employing fractional order integration of non-integer order α and fractional order differentiation of non-integer order β, are developing; which give flexibility in design and its robustness characteristics [3, 4, 5]. There exist two main methodologies to design fractional order controller. Firstly, a time-domain specification such as dominant pole placement tuning or methods based on integral performance index optimization [7]. The second one is Frequency domain specifications such as modified Ziglers-Nichols empirical rules [6] and optimization based methods. Various research works on Fractional Order Controller and its tuning and robustness aspects are listed in [11, 14].

We will be using frequency domain design because it gives a global and simpler framework especially when it proposes design requirement in terms of Gain Margin, Phase Margin, and bandwidth criteria. Also, when fractional operator is used it is simple and more efficient to use frequency domain with Fractional Laplace variable, where robustness characteristics are well established. We shall be also using the optimization in time domain technique where performance criteria in terms of performance index are to be minimized, and will be bringing out the basic rules. However, both the methods are interlinked.

A fractional integrator in Laplace domain is $s^{-\upsilon}$, with $\upsilon \in \mathbb{R}^{+}, \upsilon > 0$; is considered ideal open-loop transfer function. This was Bode's dream [25] to have circuits doing fractional differentiation and integration. He remarked in 1949-'wish I have these circuits'. Making this as Bode-loop one gets a constant phase of $-\upsilon \frac{\pi}{2}$, and this is why we get robust iso-damped reactions, while there are spread (uncertainty) in gain of plant transfer function. Complex order differentiation is an extension of regular fractional order one concerning complex power. Here for complex order the term υ is a complex number say $\upsilon = \alpha + i\beta$. Here the real ($\alpha$) and imaginary ($\beta$) provide the capability of independent design of the phase and gain indices, respectively. The imaginary part (β), also develop the ability to deal with uncertainty of the phase of plant transfer function, arising due to uncertain poles and uncertain delays.

Fractional Integration

The fractional integration was developed by Liouvelli (1832) and Riemann (1876), who developed logical definitions of this fractional operation when in above formula n is non-integer, but arbitrary positive or negative real number. This above definite integral is Riemann-Liouvelli integral. Let $-\infty \le a < x \le b \le \infty$, then the Riemann- Liouvelli (RL) fractional integration of order $\alpha > 0$ defined for a locally integrable function $f(x)$, as $f:[a,b] \to \mathbb{R}$, are following [15, 18]

$$ {}_aI_x^\alpha\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)\mathrm{d}y $$

$$ {}_bI_x^\alpha\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1} f(y)\mathrm{d}y $$

The first one is integration in Left sense, and the second one is integration in the right sense. We may call the first expression as causal one and the second expression as anti-causal one. From above, for $x < b$ and for $\alpha = 0$, we write the following identity

$$ {}_aI_x^0\left[f(x)\right] = {}_xI_b^0\left[f(x)\right] = f(x) $$

Fractional Derivatives Riemann-Liouville (RL) Left Hand Definition (LHD)

The formulation of this definition is:

Select an integer *m* greater than fractional number α

(i) Integrate the function $(m-\alpha)$ folds by RL integration method.

(ii) Differentiate the above result by *m*.

Expression is given as:

$$ {}_aD_t^\alpha\left[f(t)\right] = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau\right] $$

In this LHD the limit of integration if is from 0 to t, we denote the derivative by notation ${}_0D_t^\alpha f(t)$. In fractional calculus we find limit of derivative-i.e. derivatives are taken in interval. We call this as ‘forward derivative’. Now if the limits of integration are changed to (t to 0) the derivative is denoted as ${}_tD_0^\alpha\left[f(t)\right]$ the ‘backward derivative’. The backward derivative is related to forward derivative by

$$ {}_{t}D_{0}^{\alpha}\left[f(t) \right] = (-1)^{m} \frac{\mathrm{d}^{m}}{\mathrm{d}t^{m}} {}_{t}I_{0}^{m-\alpha}\left[f(t) \right] $$

Therefore in order to obtain fractional derivative of a function at a point (say 0) we should have the values of these two derivatives same: forward derivative should equal the backward derivative. This implies not only one should know the function from past to the point of interest (say 0) but also the function should be known into the future-in order to have point fractional derivative at a point!

Fractional Derivatives Caputo Right Hand Definition (RHD)

The formulation is exactly opposite to LHD.

Select an integer *m* greater than fractional number

(i) Differentiate the function *m* times.

(ii) Integrate the above result $(m-\alpha)$ fold by RL integration method.

In LHD and RHD the integer selection is made such that $(m-1) < \alpha < m$. For example differentiation of the function by order 3.6 will select $m = 4$. The formulation of RHD Caputo is as follows

$$ {}_{a}^{C}D_{t}^{\alpha}\left[f(t) \right] = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} \frac{\frac{\mathrm{d}^{m} f(\tau)}{\mathrm{d}\tau^{m}}}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} \mathrm{d}\tau $$

Laplace Transformation of Fractional Differ-integrals

Here we form Laplace Transforms of all $\frac{\mathrm{d}^{\alpha}[f(x)]}{\mathrm{d}x^{\alpha}}$ for all α and differ-integrable function $f(x)$, which is

$$ \mathcal{L}\left\{ \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} \right\} = \int_{0}^{\infty} e^{-sx} \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} \mathrm{d}x $$

and we wish to relate the above with $\mathcal{L}\left\{ f(x) \right\}$ that is Laplace transform of $f(x)$ defined as

$$ \mathcal{L}\left\{ f(x) \right\} = \int_{0}^{\infty} e^{-sx} f(x) \mathrm{d}x $$

We note that $\mathcal{L}\left\{f(x)\right\}$ is function in complex frequency s, sometimes it is also expressed as $\mathcal{L}\left\{f(x)\right\}=F(s)$. In terms of engineering science the variable x is time variable t. For inverse Laplace we have a contour integration as

$$\mathcal{L}^{-1}\left\{\mathcal{L}\left\{f(x)\right\}\right\}=\mathcal{L}^{-1}\left\{F(s)\right\}\overset{\text{def}}{=}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(s)e^{sx}\mathrm{d}s$$

From classical Laplace transform of integer order calculus we write following for multiple derivative operations

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}-s^{n-1}f(0)-s^{n-2}\frac{\mathrm{d}}{\mathrm{d}x}f(x)\bigg|_{x=0}-\ldots s^0\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}f(x)\bigg|_{x=0}$$

$$=s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^{n-1-k}\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\bigg|_{x=0}\qquad n=1,2,3,\ldots$$

and for multiple iterated integrals as

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}\qquad n=0,-1,-2,\ldots$$

We note that both the above formulas are similar and can be formulated as common formula which is following

$$\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\}=s^n\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^{k}\frac{\mathrm{d}^{n-1-k} f(x)}{\mathrm{d}x^{n-1-k}}\bigg|_{x=0}\qquad n=0,\pm1,\pm2,\ldots$$

Here in the above formulation, the upper summation limit is written as $n-1$, may be replaced by any integer larger than $n-1$ and even ∞. Here the effect is to add terms containing $\frac{\mathrm{d}^{-1}f(x)}{\mathrm{d}x^{-1}}\Big|_{x=0}$, $\frac{\mathrm{d}^{-2}f(x)}{\mathrm{d}x^{-2}}\Big|_{x=0}$ etc. for $n=0,-1,-2,\ldots$; such terms are necessarily zero for any function $f(x)$ whose Laplace transform exists. If n is non-integer, we generalize the above formula and write

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right\}=s^\alpha\mathcal{L}\left\{f(x)\right\}-\sum_{k=0}^{n-1}s^{k}\frac{\mathrm{d}^{\alpha-1-k} f(x)}{\mathrm{d}x^{\alpha-1-k}}\bigg|_{x=0}\qquad \text{for all}\quad \alpha$$

where, n is the largest integer such that $(n-1)<\alpha\le n$. Notice that the sum is zero when $\alpha\le 0$. In satisfying the above generalization, first consider $\alpha<0$ and the RL formula is

$$\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}} = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(y)\mathrm{d}y}{(x-y)^{\alpha+1}} \qquad \alpha < 0$$

We apply convolution theorem that is $\mathcal{L}\left\{\int_0^x f_1(x-y) f_2(y)\mathrm{d}y\right\} = \mathcal{L}\{f_1(x)\}\mathcal{L}\{f_2(x)\}$ to the above formula and write

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}}\right\} = \frac{1}{\Gamma(-\alpha)} \mathcal{L}\left\{x^{-1-\alpha}\right\}\mathcal{L}\{f(x)\} = s^{\alpha}\mathcal{L}\{f(x)\} \qquad \alpha < 0$$

We used known Laplace transform $\mathcal{L}\left\{x^m\right\} = \frac{\Gamma(m+1)}{s^{m+1}}$. We see that for $\alpha < 0$, that is negative integers the generalized formula described above remains unchanged. For the positive α, we have RL composition as

$$\frac{\mathrm{d}^{\alpha}[f(x)]}{\mathrm{d}x^{\alpha}} = \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\frac{\mathrm{d}^{\alpha-n}[f(x)]}{\mathrm{d}x^{\alpha-n}}\right]$$

Here, n is positive integer such that $(n-1) < \alpha < n$, now on application of the formula as obtained $\mathcal{L}\left\{\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right\} = s^n \mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^k \sum_{k=0}^{n-1} s^k \left.\frac{\mathrm{d}^{n-1-k} f(x)}{\mathrm{d}x^{n-1-k}}\right|_{x=0}$ for $n = 0, \pm 1, \pm 2, \ldots$, we find

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}}\right\} = \mathcal{L}\left\{\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left[\frac{\mathrm{d}^{\alpha-n} f(x)}{\mathrm{d}x^{\alpha-n}}\right]\right\}$$

$$= s^n \mathcal{L}\left\{\frac{\mathrm{d}^{\alpha-n} f(x)}{\mathrm{d}x^{\alpha-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{\mathrm{d}^{n-1-k}}{\mathrm{d}x^{n-1-k}}\left[\frac{\mathrm{d}^{\alpha-n} f(x)}{\mathrm{d}x^{\alpha-n}}\right]_{x=0}$$

The difference $\alpha - n < 0$ that is negative, the first term of RHS is evaluated using $\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha-n} f(x)}{\mathrm{d}x^{\alpha-n}}\right\} = s^{\alpha-n}\mathcal{L}\{f(x)\}$ as obtained above for $\alpha - n < 0$. Since $\alpha - n < 0$ the composition rule may be applied to the second term, within summation sign and we write

$$\mathcal{L}\left\{\frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}x^{\alpha}}\right\} = s^{\alpha}\mathcal{L}\{f(x)\} - \sum_{k=0}^{n-1} s^k \left(\frac{\mathrm{d}^{\alpha-1-k} f(x)}{\mathrm{d}x^{\alpha-1-k}}\right)_{x=0} \qquad 0 < \alpha \neq 1, 2, 3, \ldots$$

So we have proved our generalization. The above Laplace formula is a very simple generalization of the Laplace transform of classical integral calculus that is of the Laplace transform of derivative or the integral of function $f(x)$.

In summary we write Laplace transform of a fractional integration is

$$\mathcal{L}\left\{D^{-\alpha} f(x)\right\} = s^{-\alpha} F(s)$$

Laplace transform of the fractional differentiation is

$$\mathcal{L}\left\{ f(x)\right\} = s^{\alpha} F(s) - D^{\alpha-1} f(x)\Big|_{x=0}$$

About uncertain Transfer Functions

As mentioned earlier the uncertain transfer function deviates from the nominal with respect to the bandwidth and gain. The transfer function can be defined as interval transfer function as

$$G(s) = \frac{\left[k_L, k_U\right]}{1+\left[a_{1-L}, a_{1-U}\right]s + \dots\dots + \left[a_{n-L}, a_{n-U}\right]s^n} e^{-[\delta_L, \delta_U]s}$$

Where a_{i-L} and a_{i-U} are lower and upper-bounds of each parameters, poles gains and delay. The uncertainty in gain k gives uncertain transfer function in Gain magnitude plot, the uncertainty in the poles and the delays give uncertainty in the phase angle plot (or say phase distortion). The above representation is uncertain plant (transfer function) with uncertain gains, uncertain poles, and with uncertain delay-having lower and upper bound. The fuel cell [10] dynamic is non-linear and time varying. The operating points changes [15], shown in Table-1.

Points	1	2	3	4	5	6	7	8
I_{stack}	90	120	150	170	180	195	210	250
V_{Comp}	90	100	130	130	145	160	165	190

Table 1: Operating points versus different load conditions

In this regards an approach is to linearize the non-linear system of fuel cell for likely operating points, while the model variations are regarded as uncertainties. During sudden changes in the load current the transient behavior of fuel cell needs to be controlled. The higher λ_{O_2} (that is excess oxygen ratio), decreases the efficiency while lower ratio may cause damage to the cell membrane. The control system thus adjusts the ratio λ_{O_2} to the specific amount-that is the set-point. The error signal of this excess oxygen ratio is given to the controller $C(s)$ which gives V_{Comp} as output that is compressor voltage. This V_{Comp} is input to the 'non-linear stack of fuel cell' the plant having transfer function $G(s) = \lambda_{O_2} / V_{Comp}$. The stack current of fuel cell I_{Stack}, causes the operating point to shift. The following example is taken from [10] regarding Transfer Function of a Fuel Cell. The detailed transfer function is tabulated below.

$$G_U(s) = \frac{29(s+291)(s+72)(s+26)(s+4)(s+2.5)(s+0.8)}{(s+291)(s+92)(s+50)(s+26)(s+15)(s+4)(s+2.5)}$$
$$G_N(s) = \frac{20(s+324)(s+69)(s+25)(s+3)(s+2.8)(s+1.2)}{(s+324)(s+93)(s+47)(s+25)(s+16)(s+3.4)(s+3)(s+1.85)}$$
$$G_L(s) = \frac{19(s+374)(s+68)(s+23)(s+3)(s+2.1)(s+1.5)}{(s+374)(s+95)(s+46)(s+23)(s+20)(s+3)(s+2.6)(s+1.56)}$$

The $G_U(s)$ is upper-limit transfer function while operating point is at $I_{Stack} = 90$, $V_{Comp} = 90$. The nominal transfer function is $G_N(s)$ for operating condition $I_{Stack} = 150$, $V_{Comp} = 130$; and lower-limit transfer function is $G_L(s)$ with $I_{Stack} = 250$ and $V_{Comp} = 190$. See there is a spread in Gain from 19 to 29; and also there is spread in poles and zeros of the G, the plant transfer function. The eight such operating-points related transfer function is plotted in Figure-1. [10]. Here we are not discussing the derivation for that refers [10, 15, and 16]. This is to show the 'uncertain transfer function' of a system. Since the response of the fuel cell is adjusted using the nominal plant, the uncertainty degrades the performance. Therefore an elegant robust controller strategy is required to regulate the effect of the disturbance. Thus this controller $C(s)$ must be capable to counter the gain (magnitude) variation and phase angle distortion simultaneously.

To counteract the gain variation, the control strategy of 'iso-damping' is employed, where fractional order controller is so designed to make the open-loop characteristic as fractional integrator $L_{int}(s) = G(s)C(s) = \left(\omega_u / s\right)^{\upsilon}$, $\upsilon > 0$; with unity gain or 0dB at $\omega = \omega_u$, the 'gain-cross over frequency', and a constant flat phase of $-\upsilon\left(\frac{\pi}{2}\right)$; for a wide band of frequency around ω_u; or we also say $\frac{d}{d\omega}\angle L_{int}(s) = 0$ For a required design phase margin of $\varphi_m = 45^0$, we will have $\angle\left(L_{\text{int}}(s)\right)=-\pi+\varphi_m=-135^0$, that gives $\upsilon=\frac{3}{2}$, and the open-loop transfer function is a fractional integrator of order $\upsilon=1.5$, also called 'constant phase element' (CPE). In above $C(s)$ fractional order PID which comes in feed forward path in front of plant $G(s)$, with transfer function as $C(s)=\left(k_p+k_i s^{-\upsilon}+k_d s^{\mu}\right)$, with υ as fractional order of integral and μ as fractional order of derivative, with k_p, k_i and k_d as proportional, integral and derivative constants for the fractional order PID, derived from the loop shaping criteria of iso-damping.

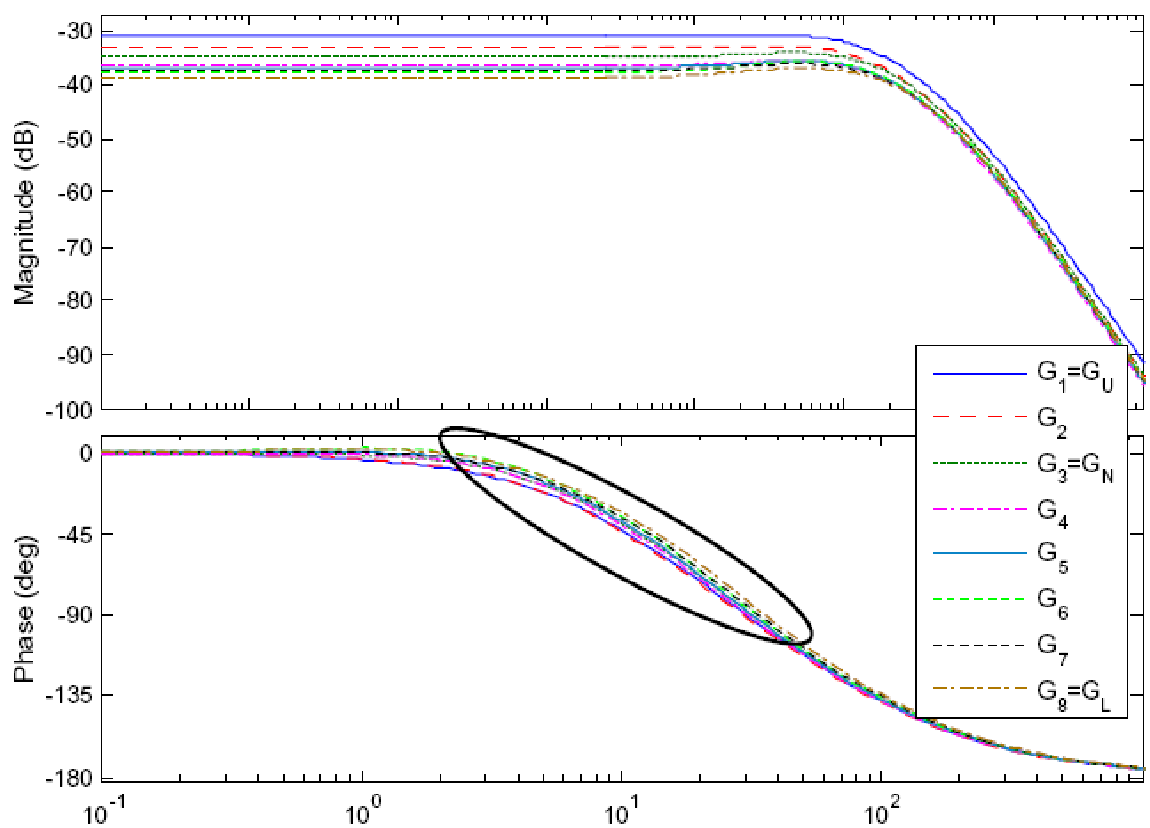


Figure-1: Bode-plot of the linearized plant transfer function with eight different operating points

Isodamping robustness

This is a basic fractional order control, and the concept of iso-damping makes robust system against parametric spreads in the gain magnitude. The open loop transfer function is $L(s) = G(s)C(s) \sim (\omega_u / s)^{\upsilon}$, $\upsilon > 0$. This is demonstrated in [11-14, 17]. This gives robustness towards gain-uncertainty. Say for example a plant $G(s) = \frac{k}{s^2 + 2\xi\omega_0 s + s^2}$, with nominal $k = 1$, with uncertainty in k say as $[0.2, 5]$, that is wide spread, when controlled via fractional order PID with perfect tuning should have robustness as iso-damped case as depicted in the figure-2, where the peak overshoot remains constant for the wide spread in gain [17].

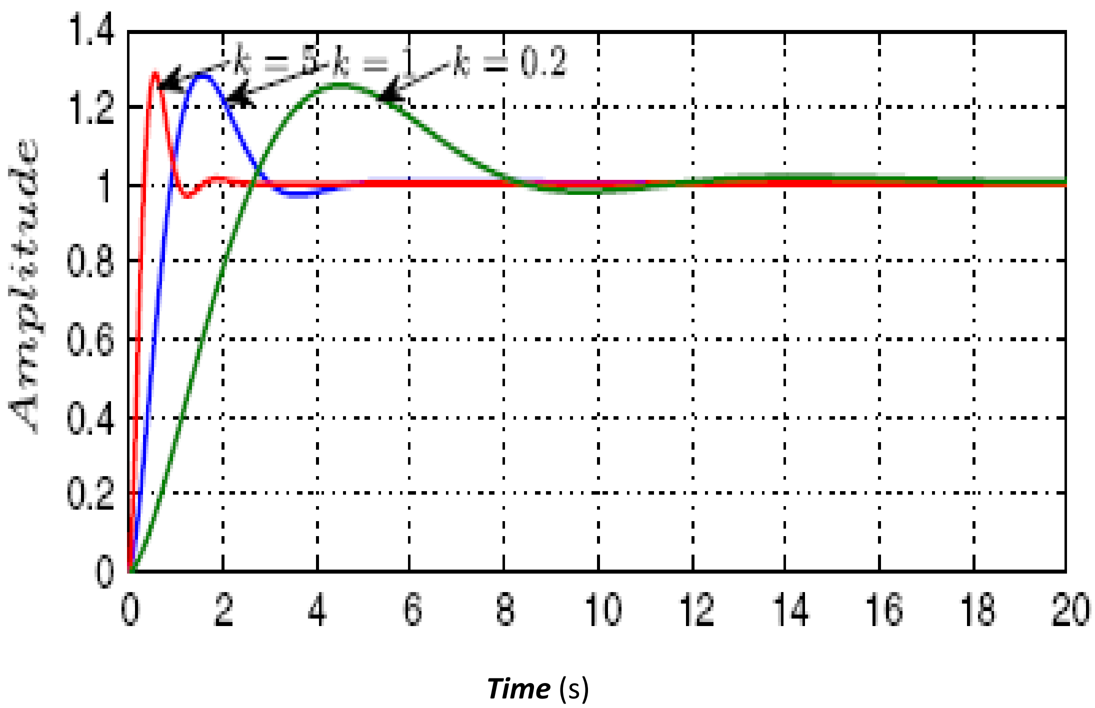


Figure-2: Step response of a controlled system showing iso-damping

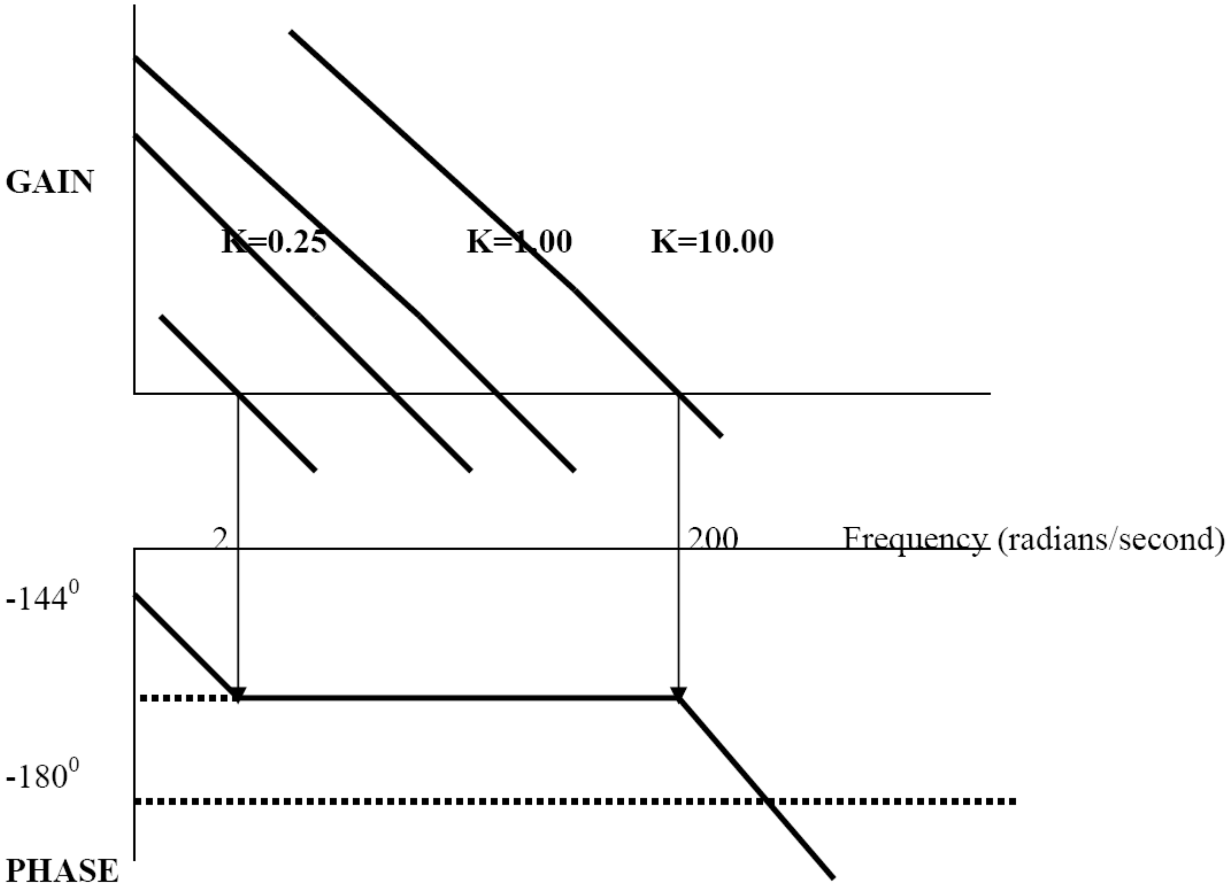


Figure-3: Constant phase element doing iso-damping

The figure-3 demonstrate that if we make $L(s) = G(s)C(s) \sim (\omega_u / s)^{\upsilon}$ a phase shaping, then the $\angle L(s)$ will be a constant phase element, say we have that angle as -144^0 from ω 2 rad/s to 200 rad/s. Then as shown in the Gain plot above, the several Gain plots will have several gain cross over frequencies ω_u, and corresponding phase angle will still be at -144^0. Therefore we can operate the plant with wide gain variations with same phase margin φ_m as 46^0 in this figure-3. This constant φ_m gives desired constant damping for a wide range of gain spread. That is what is iso-damping.

This robustness (of iso-damping) cannot be got via classical PID systems. This strategy though robust does not address the phase distortion-happening due to uncertainty in the poles and due to uncertainty in the delays, of the plant transfer function. Therefore we shall be also studying the open loop transfer function $L(s) = G(s)C(s)$ of following complex order

$$L(s) = \left(\frac{\omega_u}{s}\right)^{\alpha+i\beta} = \left(\frac{\omega_u}{s}\right)^{\alpha}\left(\frac{\omega_u}{s}\right)^{i\beta}$$

$\beta = 0$ gives us iso-damping case, as discussed above.

Complex order Differ-integrations

We have discussed differ-integral operation $g(t) = {}_0D_t^{\alpha}\left[f(t)\right]$ with α as arbitrary real number. Say if $\alpha = u + iv$, with u and v real numbers, then we have complex order differ-integration as $g(t) = \frac{\mathrm{d}^{u+iv}}{\mathrm{d}t^{u+iv}}\left[f(t)\right]$, is analytical continuation of differ-integral operator, what we discussed in earlier section. Assuming that our initial values are zero $f(t) = 0$ at $t = 0$; then we can use our generalized Laplace transforms to write

$$\begin{aligned} G(s) &= \mathcal{L}\left\{g(t)\right\} = s^{u+iv}F(s) \\ &= s^u s^{iv}F(s) = s^u e^{\ln s^{iv}}F(s) \\ &= s^u e^{iv\ln s}F(s) \\ &= s^u\left(\cos(v\ln s) + i\sin(v\ln s)\right)F(s) \end{aligned}$$

We write invert of above as $f(t) = {}_0D_t^{-(u+iv)}g(t)$ and it's Laplace as $F(s) = s^{-(u+iv)}G(s)$. Using $\mathcal{L}^{-1}\left\{s^{-\alpha}\right\} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and taking $G(s) = 1$ or $g(t) = \delta(t)$, we get impulse response that is following

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{s^{-(u+iv)}G(s)\right\} \\ &= \mathcal{L}^{-1}\left\{s^{-(u+iv)}\right\} \\ &= \frac{t^{u+iv-1}}{\Gamma(u+iv)} = \frac{t^{u-1}}{\Gamma(u+iv)}e^{iv\ln t} \\ &= \frac{t^{u-1}\left(\cos(v\ln t) + i\sin(v\ln t)\right)}{\Gamma(u+iv)} \end{aligned}$$

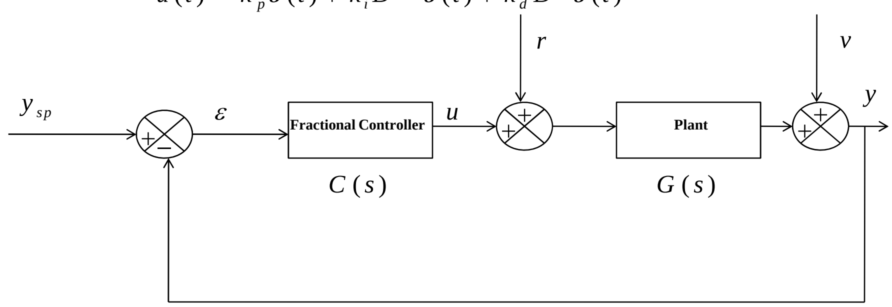
The above derivation says we get imaginary response component along with real response component. However, we will not be dealing in time domain, but in Laplace frequency (s) domain.

Fractional Order loop-shaping-taking into account plant uncertainties

The fractional Laplace operators, that is s^{α} overcomes the drawback such as the suddenness changes in the gain and phase plots of the counterpart integer order Laplace operator. The gain magnitude plot of fractional Laplace operator is represented by straight line with slope as 20α dB/decade, while the phase angle diagram is a line parallel to X-axis and its ordinate equal to $\alpha\frac{\pi}{2}$, the constant phase element (CPE). The figure-4 gives the block diagram

$$C(s) = k_p + \frac{k_i}{s^{\alpha}} + k_d s^{\beta} \qquad \alpha > 0 \qquad \beta > 0$$

$$u(t) = k_p \varepsilon(t) + k_i D^{-\alpha} \varepsilon(t) + k_d D^{\beta} \varepsilon(t)$$



y_{sp} : **Set-point**

ε : **Error**

u : **Controller output**

r : **Load disturbance**

v : **Measurement Noise**

y : **Out-put**

Figure-4: Feedback loop scheme using controller

The fractional order PID as depicted has five parameters that are to be obtained via tuning, those are k_p, k_i, k_d, α and β. The fractional order α and β lead to a more flexible controller that take into account of robustness and disturbance rejections performances. We use a design method to tune fractional order PID [22].

The most commonly used frequency parameters specifications are phase margin φ_m, gain margin M_g, unity gain cross over frequency ω_u and the phase cross over frequency ω_{cp}. The use of this specification ensures robustness ant time specification such as overshoot and settling time. Then the fractional controllers must achieve the robustness property and provide

- A desired phase margin, which is related to the desired damping
- A desired unity-gain cross-over frequency ω_u, which is related to settling time.
- A desired rejection to the load disturbance and un-modeled system dynamics at low-frequency.
- A good set point tracking and good measurement noise rejection (attenuation) at high frequency.

Consider $G(s)$ the transfer function of the plant and $C(s)$ the transfer function of the fractional controller that is

$$C(i\omega) = k_p + \frac{k_i}{(i\omega)^{\alpha}} + k_d (i\omega)^{\beta}$$

To satisfy the five parameters as-unity gain cross-over frequency ω_u, phase margin φ_m, robustness to plant uncertainties, load disturbance and high frequency noise rejection, the fractional PID should be tuned by finding five parameters k_p, k_i, k_d, α and β, subjected to following constraints

- Unity gain-cross over frequency ω_u is the frequency at which the open-loop that is $L(s) = C(s)G(s)$ has unity gain or zero-dB, meaning

$$\left|C(i\omega_u)G(i\omega_u)\right|_{\mathrm{dB}} = 0$$

- The phase margin φ_m represents the difference between the open-loop phase at ω_u and angle $-\pi$. We represent $\arg(F(i\omega))$ as $\angle(F(i\omega))$ and write

$$\angle\left(C(i\omega_u)G(i\omega_u)\right) = -\pi + \varphi_m$$

- Robustness to the gain uncertainty represented by a constant phase angle of open-loop transfer function near gain-cross over frequency ω_u, that is

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\angle\left(C(i\omega_u)G(i\omega_u)\right)\right)\Big|_{\omega=\omega_u} = 0$$

- The high-frequency noise attenuation is got by imposing a small magnitude (in dB) of the complimentary sensitivity function $T(i\omega)$ at some specified frequency ω_t

$$|T(i\omega)| = \left|\frac{C(i\omega)G(i\omega)}{1+C(i\omega)G(i\omega)}\right|_{\mathrm{dB}} \le A \qquad \omega \ge \omega_t$$

Where A is the desired attenuation.

- Output disturbance rejection at low frequency is obtained by imposing a small magnitude of the sensitivity function $S(i\omega)$ at the low frequency before some pre-defined frequency ω_s.

$$|S(i\omega)| = \left|\frac{1}{1+C(i\omega)G(i\omega)}\right|_{\mathrm{dB}} \le B \qquad \omega \le \omega_s$$

Where B is desired value of disturbance rejection (attenuation).

Therefore the optimization problem is

Minimize $|C(i\omega_u)G(i\omega_u)|_{\mathrm{dB}} = 0$, under the constraints

1. $\angle\left(C(i\omega_u)G(i\omega_u)\right) = -\pi + \varphi_m$
2. $\left.\frac{\mathrm{d}\left(\angle\left(C(i\omega_u)G(i\omega_u)\right)\right)}{\mathrm{d}\omega}\right|_{\omega=\omega_u} = 0$
3. $|T(i\omega)| = \left|\frac{C(i\omega)G(i\omega)}{1+C(i\omega)G(i\omega)}\right|_{\mathrm{dB}} \le A, \quad \omega \ge \omega_t$
4. $|S(i\omega)| = \left|\frac{1}{1+C(i\omega)G(i\omega)}\right|_{\mathrm{dB}} \le B, \quad \omega \le \omega_s$.

Considering a second order plant

$$G(s) = \frac{k}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

The uncertain gain bounds are supposed to be known, but nominal value of k is say 1, and $\xi = 0.5$ the damping factor, and natural frequency $\omega_0 = 1$ rad/s. The Bode-diagram of $G(s)$ is in figure-5

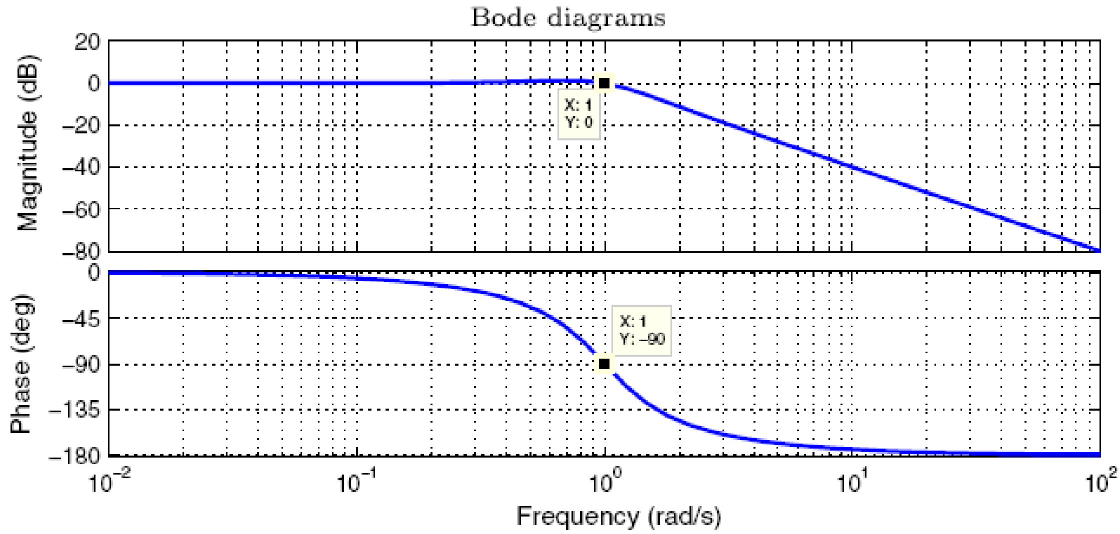


Figure-5: Bode diagram of a nominal second order system

The desired performances are selected from the Bode diagram of $G(s)$. In fact to get an appropriate time-domain response (short settling time and acceptable overshoot) the frequency specification must be with $\omega_u \geq 1$ rad/s; to incase rapidity and a good phase margin to increase stability robustness. The values are $\omega_u = 2$ rad/s, $\varphi_m = 45^0$, $\omega_t = 9$ rad/s, $\omega_s = 0.4$ rad/s, $A = -20$ dB and $B = -20$ dB

With the above procedure the controller is able to provide good response, but we further shape the phase angle of the open-loop response and extend the $\left.\frac{\mathrm{d}(\angle(C(i\omega_u)G(i\omega_u)))}{\mathrm{d}\omega}\right|_{\omega=\omega_u} = 0$ to several other frequencies around ω_u to make better constant phase element CPE, depicted in figure- 6, [17]. With the constraint $\left.\frac{\mathrm{d}(\angle(C(i\omega_u)G(i\omega_u)))}{\mathrm{d}\omega}\right|_{\omega=\omega_u} = 0$, in figure-6, we have open-loop phase as shown in black, by; $\left.\frac{\mathrm{d}(\angle(C(i\omega_u)G(i\omega_u)))}{\mathrm{d}\omega}\right|_{\omega=\omega_u} = 0$ criteria and that we have to make it as blue.

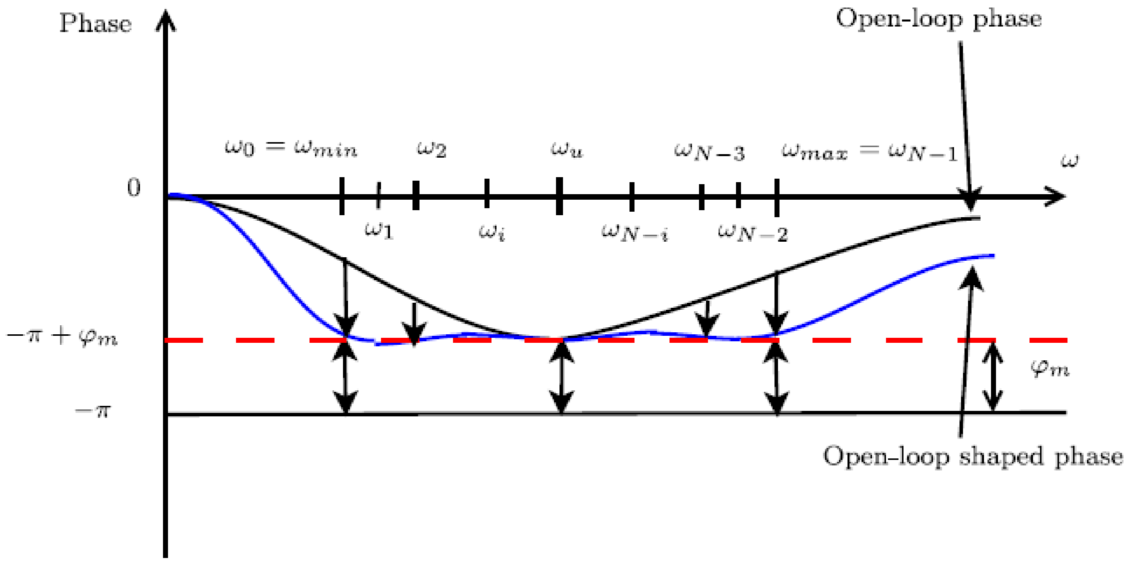


Figure-6: Making the phase of open loop phase angle constant at wide range of frequencies.

To achieve the further phase angle constant, open loop phase angle must be equal to $-\pi + \varphi_m$ (where φ_m is desired phase margin) for several frequencies ω_i in the band $[\omega_{\min}, \omega_{\max}]$. To ensure that the phase still is flat (almost) in this range around ω_u, is got by [17]

- Ensuring a desired phase margin for several frequencies $\omega_i \in [\omega_0, \omega_1, ..., \omega_{N-2}, \omega_{N-1}]$ including the unity gain crossover frequency ω_u

$$\angle\left(C(i\omega_i)G(i\omega_i)\right) = -\pi + \varphi_m \qquad \omega_i \in [\omega_{\min}, \omega_1, ...\omega_{N-2}, \omega_{\max}]$$

- The angle of open-loop transfer functions derivative

$$\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\angle\left(C(i\omega_i)G(i\omega_i)\right)\right)\Big|_{\omega_i \in [\omega_{\min}, \omega_1, \omega_2, ...\omega_{N-1}, \omega_{\max}]} = 0$$

We can modify the above constraints and reframe as following

$$\sum_{i=0}^{N-1}\left(\angle\left(C(i\omega_i)G(i\omega_i)\right) + \pi - \varphi_m\right)^2 = 0 \qquad \omega_i \in [\omega_{\min}, ..., \omega_{\max}]$$

$$\sum_{i=0}^{N-1}\left(\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\angle\left(C(i\omega_i)G(i\omega_i)\right)\right)\right)^2\Bigg|_{\omega_i \in [\omega_{\min}, \omega_1, \omega_2, ..., \omega_{N-2}, \omega_{\max}]} = 0$$

Here, N is the number of frequencies belonging to range $[\omega_{\min}, \omega_{\max}]$ in which the angle of open loop transfer function is maintained constant at $-\pi + \varphi_m$.

Now the optimization is minimize $|C(i\omega_u)G(i\omega_u)|_{\mathrm{dB}} = 0$ under the constraints

1. $\sum_{i=0}^{N-1} \left(\angle\left(C(i\omega_i)G(i\omega_i) \right) + \pi - \varphi_m \right)^2 = 0 \qquad \omega_i \in [\omega_{\min}, \ldots, \omega_{\max}]$
2. $\sum_{i=0}^{N-1} \left(\frac{\mathrm{d}}{\mathrm{d}\omega_i} \left(\angle\left(C(i\omega_i)G(i\omega_i) \right) \right) \right)^2 \Big|_{\omega_i \in [\omega_{\min}, \omega_1, \omega_2, \ldots, \omega_{N-2}, \omega_{\max}]} = 0$
3. $|T(i\omega)| = \left| \frac{C(i\omega)G(i\omega)}{1 + C(i\omega)G(i\omega)} \right|_{\mathrm{dB}} \le A, \quad \omega \ge \omega_t$
4. $|S(i\omega)| = \left| \frac{1}{1 + C(i\omega)G(i\omega)} \right|_{\mathrm{dB}} \le B, \quad \omega \le \omega_s$

This revised method guarantees a constant phase element, in a desired frequency band and then therefore guarantees the robustness of the closed loop controlled system but only in case of gain uncertainty. In case of the pole uncertainty i.e. the plant having phase distortion (phase uncertainty) or even in the presence of time delays, the above method is not so robust, and thus can lead to unstable loops. In the transfer function of plant that is $G(s) = \frac{k}{s^2 + 2\xi\omega_0 s + \omega_0^2}$ with $\xi = 0.5$, $\omega_0 = 1$ and nominal gain $k = 1$, (of figure-5) the CPE may considered for $[0.5, 20]$ rad/s as $[\omega_{\min}, \omega_{\max}]$ with the chosen parameters as indicated above as $\omega_u = 2$, $\varphi_m = 45$, $\omega_t = 9$, $\omega_s = 0.4$ $A = -20$ and $B = -20$.

Bode Envelope Tuning for Robustness

Knowing that the time delay and the pole uncertainty have a great effect on the open loop phase angle, the idea is to consider Bode envelopes (i.e. the minimum and the maximum of the gain and the phase angle) of the uncertain plants and then design the fractional order controller. Bode envelops are obtained by four transfer functions those are G_{gain-U}, G_{gain-L}, $G_{phase-U}$ and $G_{phase-L}$. These are obtained from the modeled plant, as depicted in Table-1, from the obtained $G_U(s)$ and $G_L(s)$. To guarantee stability robustness of uncertain plant, the following is the proposed method [17]

- Unity-gain cross-over frequency for the upper bound of gain is

$$\left|C(i\omega_u)G_{gain-U}(i\omega_u)\right|_{\text{dB}} = 0$$

- Desired phase margin for the lower bound of the phase

$$\left.\sum_{i=0}^{N-1}\left(\angle\left(C(i\omega_i)G_{phase-L}(i\omega_i)\right)+\pi-\varphi_m\right)^2\right| = 0 \qquad \omega_i \in \left[\omega_{\min},.....\omega_{\max}\right]$$

- Robustness to the gain uncertainty for lower bound of phase

$$\sum_{i=0}^{N-1}\left(\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\angle\left(C(i\omega_i)G_{phase-L}(i\omega_i)\right)\right)\right)_{\omega_i\in\left[\omega_{\min},\omega_1,...\omega_{N-2},\omega_{\max}\right]} = 0$$

- High frequency noise attenuation constraint for the upper-bound of the gain

$$\left|T(i\omega)\right| = \left|\frac{C(i\omega)G_{gain-U}(i\omega)}{1+C(i\omega)G_{gain-U}(i\omega)}\right|_{\text{dB}} \le A \qquad \omega \ge \omega_t$$

- Output disturbance rejection for the lower bound of gain is

$$\left|S(i\omega)\right| = \left|\frac{1}{1+C(i\omega)G_{gain-L}(i\omega)}\right|_{\text{dB}} \le B \qquad \omega \le \omega_s$$

The optimization problem is minimize $\left|C(i\omega_u)G_{gain-U}(i\omega_u)\right|_{\text{dB}} = 0$, under constraints

1. $\left.\sum_{i=0}^{N-1}\left(\angle\left(C(i\omega_i)G_{phase-L}(i\omega_i)\right)+\pi-\varphi_m\right)^2\right| = 0 \qquad \omega_i \in \left[\omega_{\min},.....\omega_{\max}\right]$
2. $\sum_{i=0}^{N-1}\left(\frac{\mathrm{d}}{\mathrm{d}\omega_i}\left(\angle\left(C(i\omega_i)G_{phase-L}(i\omega_i)\right)\right)\right)_{\omega_i\in\left[\omega_{\min},\omega_1,...\omega_{N-2},\omega_{\max}\right]} = 0$
3. $\left|T(i\omega)\right| = \left|\frac{C(i\omega)G_{gain-U}(i\omega)}{1+C(i\omega)G_{gain-U}(i\omega)}\right|_{\text{dB}} \le A \qquad \omega \ge \omega_t$
4. $\left|S(i\omega)\right| = \left|\frac{1}{1+C(i\omega)G_{gain-L}(i\omega)}\right|_{\text{dB}} \le B \qquad \omega \le \omega_s$

This approach will ensure the phase and the gain shaping for uncertain plant when pole uncertainty corresponds to lower bound values denoted by a_{i-L} for $i = 0,1,...,n$ and/or time delay uncertainty corresponding to δ_L (i.e. the parametric uncertainty leads to lower bound of system phase $G_{phase-L}$), in the plant uncertain transfer function represented by

$$G(s) = \frac{\left[k_L, k_U\right]}{1+\left[a_{1-L}, a_{1-U}\right]s+........+\left[a_{n-L}, a_{n-U}\right]s^n} e^{-[\delta_L,\delta_U]s}$$

For example, if we have a second order with delay uncertain transfer function for a plant as

$$G(s) = \frac{ke^{-\delta s}}{s^2 + 2\xi\omega_0 s + \omega_0^2} \qquad \xi = 0.5 \qquad \omega_0 = 1 \qquad k = [0.25, 4] \qquad \delta = [0, 0.4]$$

For the simple Bode-shaping tuning only nominal values of the gain $k = 1$ and time delay $\delta = 0.2$ are used to get nominal transfer function. For Bode-envelope based tuning the four transfer functions is

$$G_{gain-U} = \frac{4}{s^2 + s + 1} e^{-0.2s} \qquad G_{gain-L} = \frac{0.25}{s^2 + s + 1} e^{-0.2s}$$
$$G_{phase-U} = \frac{1}{s^2 + s + 1} e^{-0.2s} \qquad G_{phase-L} = \frac{1}{s^2 + s + 1} e^{-0.4s}$$

This is interesting because the designed controller will be stabilizing the system even if the uncertainties have the maximum values, and moreover it can guarantee that a gain variation does not deteriorate the system.

Complex order loop-shaping derivation

We desire that our open-loop transfer function be $L(s) = C(s)G(s) = (\omega_u / s)^{\alpha \pm i\beta}$ or we write the following

$$L(s) = C(s)G(s) = \left(\frac{\omega_u}{s}\right)^{\alpha} \left(\left(\frac{\omega_u}{s}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}$$

The above contains a fractional order integrator defined by the order $\alpha > 0$,that is real part and a complex order differentiator or integrator depending on the sign of β, the imaginary part. Let us tackle the pure imaginary part of above that is $Z(s) = \left((\omega_u / s)^{i\beta}\right)^{-\operatorname{sgn}\beta}$ first, with the findings of previous section regarding complex order Differ-integrations; we write the following

$$Z(s) = \left(\left(\frac{\omega_u}{s}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}$$
$$= \left(\cos\left(\beta \ln\left(\tfrac{\omega_u}{s}\right)\right) + i \sin\left(\beta \ln\left(\tfrac{\omega_u}{s}\right)\right)\right)^{-\operatorname{sgn}\beta}$$

Put $s = i\omega$, in above to write

$$Z(i\omega) = \left(\cos\left(\beta \ln\left(\tfrac{\omega_u}{i\omega}\right)\right) + i\sin\left(\beta \ln\left(\tfrac{\omega_u}{i\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}$$

$$\left(\cos\left(\beta \ln \omega_u - \beta \ln i\omega\right) + i\sin\left(\beta \ln \omega_u - \beta \ln i\omega\right)\right)^{-\operatorname{sgn}\beta}$$

We use

$$\ln(u+iv) = \ln\left(\left|(u+iv)\right| e^{i\arg(u+iv)}\right) = \ln\left|u+iv\right| + \ln\left(e^{i\arg(u+iv)}\right) = \ln\left|u+iv\right| + i\angle(u+iv)$$

For $u+iv = i\omega$, we have $\left|i\omega\right| = \omega$, and $\angle(i\omega) = \frac{\pi}{2}$, for $\omega > 0$ .Thus we get

$$\ln i\omega = \ln\omega + \ln e^{i(\pi/2)} = \ln\omega + i\left(\tfrac{\pi}{2}\right)$$

Using this we write

$$\begin{aligned} Z(i\omega) &= \left(\cos\left(\beta \ln \omega_u - \beta \ln i\omega\right) + i\sin\left(\beta \ln \omega_u - \beta \ln i\omega\right)\right)^{-\operatorname{sgn}\beta} \\ &= \left(\cos\left(\beta \ln \omega_u - \beta \ln \omega - i\tfrac{\beta\pi}{2}\right) + i\sin\left(\beta \ln \omega_u - \beta \ln \omega - i\tfrac{\beta\pi}{2}\right)\right)^{-\operatorname{sgn}\beta} \\ &= \left(\cos\left(\beta \ln\left(\tfrac{\omega_u}{\omega}\right) - i\tfrac{\beta\pi}{2}\right) + i\sin\left(\beta \ln\left(\tfrac{\omega_u}{\omega}\right) - i\tfrac{\beta\pi}{2}\right)\right)^{-\operatorname{sgn}\beta} \end{aligned}$$

We use $\cos(u+iv) = \cos u \cosh v - i \sin u \sinh v$ and $\sin(u+iv) = \sin u \cosh v + i\cos u \sinh v$, to get following

$$\begin{aligned} Z(i\omega) &= \left(\cos\left(\beta \ln\left(\tfrac{\omega_u}{\omega}\right) - i\tfrac{\beta\pi}{2}\right) + i\sin\left(\beta \ln\left(\tfrac{\omega_u}{\omega}\right) - i\tfrac{\beta\pi}{2}\right)\right)^{-\operatorname{sgn}\beta} \\ &= \begin{pmatrix} \cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2} + \\ +i\left(\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} - i\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2}\right) \end{pmatrix}^{-\operatorname{sgn}\beta} \\ &= \begin{pmatrix} \cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2} + \\ +\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2} + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} \end{pmatrix}^{-\operatorname{sgn}\beta} \\ &= \begin{pmatrix} \cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} + \cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2} + \\ +i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\sinh\tfrac{\beta\pi}{2} + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\cosh\tfrac{\beta\pi}{2} \end{pmatrix}^{-\operatorname{sgn}\beta} \\ &= \left(\left(\cosh\tfrac{\beta\pi}{2} + \sinh\tfrac{\beta\pi}{2}\right)\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right) + \left(\cosh\tfrac{\beta\pi}{2} + \sinh\tfrac{\beta\pi}{2}\right)\left(i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)\right)^{-\operatorname{sgn}\beta} \\ &= \left(\cosh\tfrac{\beta\pi}{2} + \sinh\tfrac{\beta\pi}{2}\right)^{-\operatorname{sgn}\beta}\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right) + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta} \end{aligned}$$

Now we take magnitude and argument (phase) of the above and write

$$\begin{aligned}|Z(i\omega)| &= \left|\left(\cosh\tfrac{\beta\pi}{2}+\sinh\tfrac{\beta\pi}{2}\right)^{-\operatorname{sgn}\beta}\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)+i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}\right| \\ &= \left(\cosh\tfrac{\beta\pi}{2}+\sinh\tfrac{\beta\pi}{2}\right)^{-\operatorname{sgn}\beta} \\ &= \left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}\end{aligned}$$

Noting that $\left|\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)+i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)\right|=1$, and $\cosh x=\frac{e^x+e^{-x}}{2}$ and $\sinh x=\frac{e^x-e^{-x}}{2}$. The argument or phase is

$$\begin{aligned}\angle\left(Z(i\omega)\right) &= \angle\left[\left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}\right]+\angle\left[\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)+i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}\right] \\ &= (-\operatorname{sgn}\beta)\left(\tan^{-1}\left(\frac{\sin\left(\beta\ln\left(\frac{\omega_u}{\omega}\right)\right)}{\cos\left(\beta\ln\left(\frac{\omega_u}{\omega}\right)\right)}\right)\right)=(-\operatorname{sgn}\beta)\left(\tan^{-1}\left(\tan\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)\right) \\ &= (-\operatorname{sgn}\beta)\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)=(-\operatorname{sgn}\beta)\left(\beta\ln\omega_u-\beta\ln\omega\right)\end{aligned}$$

In above we have used $\angle\left(a+ib\right)=\tan^{-1}\left(\frac{b}{a}\right)$ and $\angle\left((a+ib)^n\right)=n\left(\angle\left(a+ib\right)\right)$ also the trigonometric inverse relation $\tan^{-1}(\tan\theta)=\theta$. Noting that $\left(\frac{\omega_u}{i\omega}\right)^\alpha=(-i)^\alpha\left(\frac{\omega_u}{\omega}\right)$ and we have for $-i^\alpha=e^{-i\alpha(\pi/2)}$, which is also $-i^\alpha=\cos\frac{\alpha\pi}{2}-i\sin\frac{\alpha\pi}{2}$, that give us $\left|-i^\alpha\right|=1$ and $\angle\left(-i^\alpha\right)=-\alpha\frac{\pi}{2}$, we have therefore $\left|\left(\frac{\omega_u}{i\omega}\right)^\alpha\right|=\left(\frac{\omega_u}{\omega}\right)^\alpha$ and $\angle\left(\frac{\omega_u}{\omega}\right)=-\frac{\alpha\pi}{2}$. Using this and from the above derivation we write the overall open loop shaped transfer function as

$$\begin{aligned}L(i\omega) &= \left(\frac{\omega_u}{i\omega}\right)^\alpha\left(\left(\frac{\omega_u}{i\omega}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta} \\ &= \left(\frac{\omega_u}{\omega}\right)^\alpha e^{-i(\alpha\pi/2)}\left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)+i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta} \\ &= \left(\frac{\omega_u}{\omega}\right)^\alpha\left(\cos\tfrac{\alpha\pi}{2}-i\sin\tfrac{\alpha\pi}{2}\right)\left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)+i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}\end{aligned}$$

The magnitude is

$$|L(i\omega)| = \left|\left(\frac{\omega_u}{\omega}\right)^{\alpha}\right| \left|\cos\tfrac{\alpha\pi}{2} - i\sin\tfrac{\alpha\pi}{2}\right| \left|\left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}\right| \left|\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right) + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}\right|$$

$$= \left(\frac{\omega_u}{\omega}\right)^{\alpha} \left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}$$

At the gain cross over frequency we have magnitude of loop transfer function as

$$|L(i\omega)|_{\omega=\omega_u} = \left(e^{\beta(\pi/2)}\right)^{-\operatorname{sgn}\beta}$$

But the need is that we should have $|L(i\omega)|_{\omega=\omega_u} = 1$, thus we need to modify the original $L(i\omega)$, by adding gain (multiplying) $K = \left(e^{\beta(\pi/2)}\right)^{\operatorname{sgn}\beta}$, to make unity gain at gain cross over frequency. Thus our shaped loop transfer function is

$$L_S(s) = C(s)G(s) = K\left(\frac{\omega_u}{s}\right)^{\alpha} \left(\left(\frac{\omega_u}{s}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta} \qquad K = \left(e^{\beta(\pi/2)}\right)^{\operatorname{sgn}\beta}$$

$$L_S(i\omega) = K\left(\frac{\omega_u}{i\omega}\right)^{\alpha} \left(\left(\frac{\omega_u}{i\omega}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}$$

$$= \left(\frac{\omega_u}{\omega}\right)^{\alpha} \left(\cos\tfrac{\alpha\pi}{2} - i\sin\tfrac{\alpha\pi}{2}\right) \left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right) + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}$$

The magnitude of $L_S(i\omega)$ is

$$|L_S(i\omega)| = |K| \left|\left(\frac{\omega_u}{i\omega}\right)^{\alpha}\right| \left|\left(\left(\frac{\omega_u}{i\omega}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}\right|$$

$$= \left|\left(\frac{\omega_u}{\omega}\right)^{\alpha}\right| \left|\left(\cos\tfrac{\alpha\pi}{2} - i\sin\tfrac{\alpha\pi}{2}\right)\right| \left|\left(\cos\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right) + i\sin\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)\right)^{-\operatorname{sgn}\beta}\right|$$

$$= \left(\frac{\omega_u}{\omega}\right)^{\alpha}$$

The magnitude variation is determined by the real part only that is α. Now the phase angle or argument of $L_S(i\omega)$ is

$$\angle\left(L_S(i\omega)\right)=\angle\left(K\right)+\angle\left(\left(\frac{\omega_u}{i\omega}\right)^{\alpha}\right)+\angle\left(\left(\frac{\omega_u}{i\omega}\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}$$
$$=-\frac{\alpha\pi}{2}-\left((-\operatorname{sgn}\beta)\left(\beta\ln\omega_u-\beta\ln\omega\right)\right)$$
$$=-\frac{\alpha\pi}{2}+\left(\operatorname{sgn}\beta\right)\left(\beta\ln\left(\tfrac{\omega_u}{\omega}\right)\right)$$

We find that phase variation is determined by the imaginary part β . From above we have, at the gain cross-over frequency we have the values $\left|L_S(i\omega)\right|_{\omega=\omega_u}=1$ and $\angle\left(L_S(i\omega)\right)_{\omega=\omega_u}=-\alpha\left(\tfrac{\pi}{2}\right)$.

Now we calculate in terms of dB. That is

$$\left|L_S(i\omega)\right|_{\mathrm{dB}}=20\log\left|L_s(i\omega)\right|$$
$$=20\log\left(\left(\tfrac{\omega_u}{\omega}\right)^{\alpha}\right)$$
$$=20\alpha\log\omega_u-20\alpha\log\omega$$

The slope we write as

$$\frac{\mathrm{d}\left[\left|L_S(i\omega)\right|_{\mathrm{dB}}\right]}{\mathrm{d}\left[\log\omega\right]}=-20\alpha$$

This is exactly like pure fractional order integrator, where the slope of log magnitude plot rolls off as -20α dB/ decade. We apply the formula for logarithmic base change that is $\log_b x=\frac{\log_d x}{\log_d b}$, to write $\ln x=\frac{\log x}{\log e}$ and use this to write phase function in logarithm of ω

$$\angle\left(L_S(i\omega)\right)=-\frac{\alpha\pi}{2}-\left((-\operatorname{sgn}\beta)\left(\beta\ln\omega_u-\beta\ln\omega\right)\right)$$
$$=-\frac{\alpha\pi}{2}+\operatorname{sgn}\beta\left(\beta\ln\omega_u-\beta\tfrac{\log\omega}{\log e}\right)$$

From above we get the slope

$$\frac{\mathrm{d}\left[\angle\left(L_S(i\omega)\right)\right]}{\mathrm{d}\left[\log\omega\right]}=-\operatorname{sgn}\beta\left(\frac{\beta}{\log e}\right)$$

This says that tilt from the constant flat phase of CPE (determined by the real part α as fractional order pure-integrator), is depending on the imaginary part β ; and that in logarithmic frequency Bode plot, the slope or tilt of phase is constant, and for fractional order integrator $\beta=0$ the slope is zero.

Diagrammatic view via Nichols chart

In addition to being stable one would like the system to behave well. From root-locus point of view, behave well means the 'dominant poles' are certain distance from imaginary $i\omega$ axis, as defined by 'damping ratio'. From frequency domain point of view well behave means that 'resonance' is not just finite but is limited to a number $M_r = 6$ dB (say). From a second order approximation specifying resonance also specifies damping as

$$M_r = \begin{cases} 1 & \xi > 0.7 \\ \dfrac{1}{2\xi\sqrt{1-\xi^2}} & 0 < \xi < 0.7 \end{cases}$$

So it is just another way of stating. Since the close loop system goes unstable when $L(i\omega) = C(i\omega)G(i\omega) = -1 = 1\angle 180^0$ we can think of this constraint of M_r as being constraint on how close $L(i\omega)$ can get to point -1. Figure-7 denotes a Nichol's chart where constant M circles are plotted. The constant M in dB are locus where $\left|\frac{C(s)G(s)}{1+C(s)G(s)}\right|_{\text{dB}}$ is constant that is also a close loop transfer function, and this tells all points on the M-circle are same distance from -1. This is also the transfer function of $T(i\omega)$ the complimentary sensitive function.

On this Nichols chart the $|L(i\omega)|$ and $\angle L(i\omega)$ are plotted, with Y-axis as $|L(i\omega)|_{\text{dB}}$ and $\angle L(i\omega)$ as the X-axis. Then the M circle with a particular dB, tangent to the $L(i\omega)$ curve gives the M_r in dB for that particular closed loop system, and the frequency at which this tangent occurs gives resonance frequency ω_r. In figure-8 the tangent of M_r curve to $L_S(i\omega)$ is shown

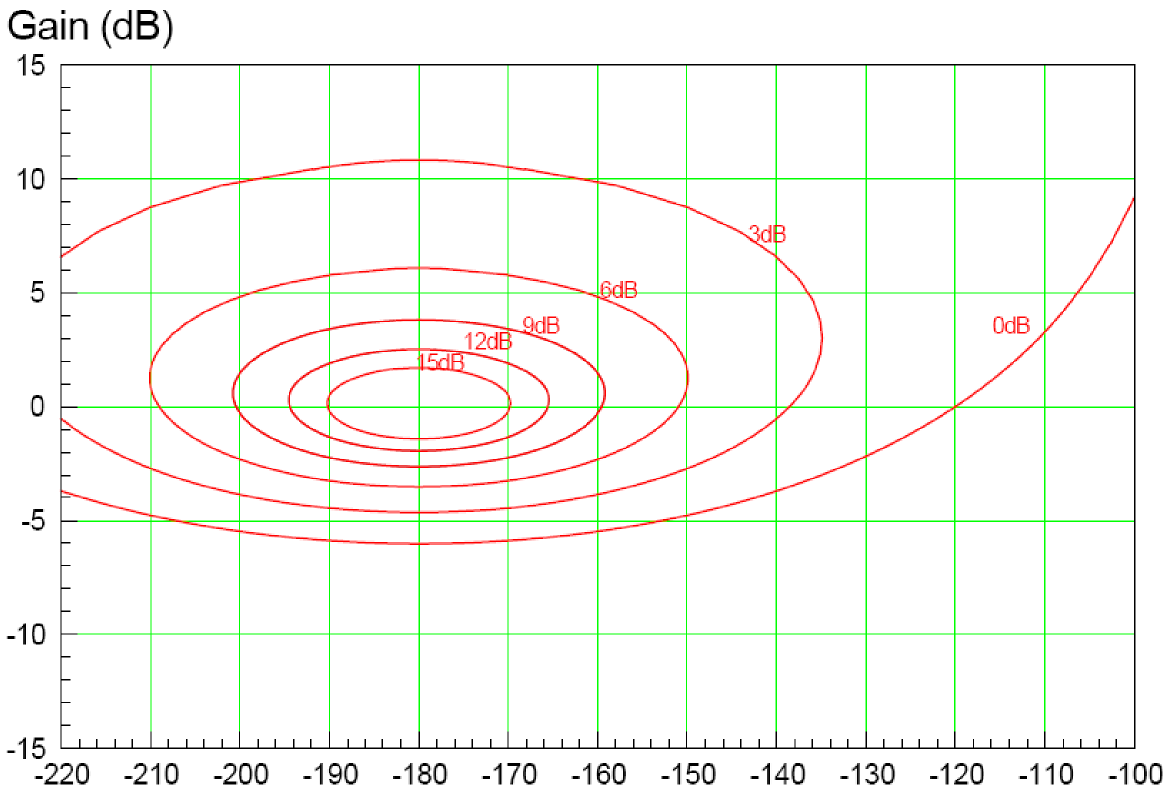


Figure-7: Showing Nichol's chart with M circles

The figure-8 depicts the magnitude and phase angle plot of complex order open loop transfer function. The vertical line (green) passing through the angle $-\alpha\frac{\pi}{2}$ is a constant phase element-used for isodamping robust control. The diagonal nature (blue) as derived earlier is due to imaginary part β of the complex order differ-integral. Ideally this $L_s(\omega)$ is a band-un-limited complex differ-integrator with ω from $-\infty$ to ∞. But we put filters of integer order to limit the band at ω_l to ω_h, by employing two filters as

$$H_{\omega_l}(\omega)=\left(\frac{\omega_l}{i\omega}+1\right)^{n_l} \qquad H_{\omega_h}(\omega)=\frac{1}{\left(1+\frac{i\omega}{\omega_h}\right)^{n_h}} \qquad n_l, n_h \in \mathbb{N}$$

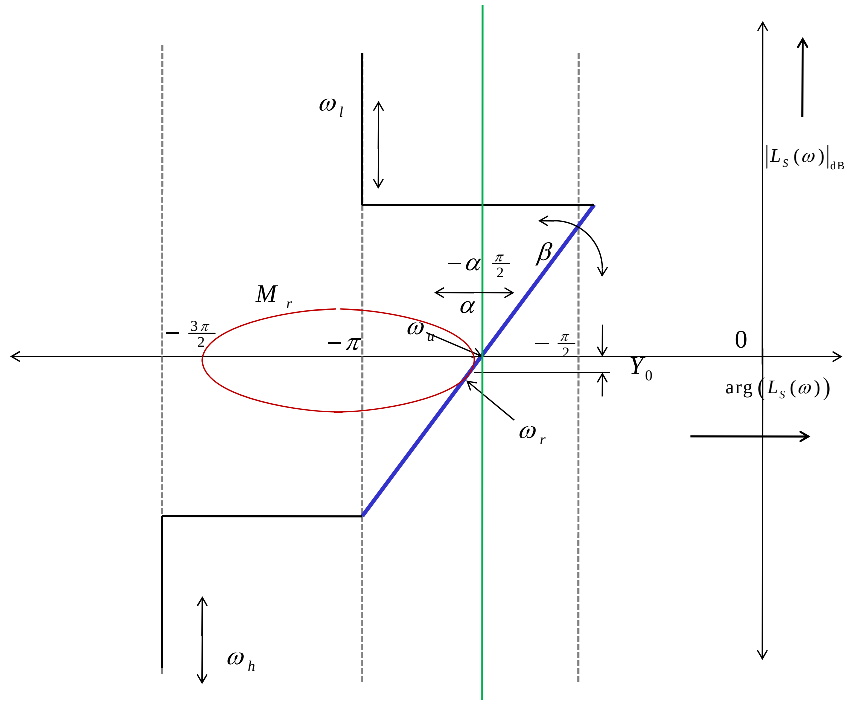


Figure-8: Nichols plot of shaped loop transfer function of complex order

The LF cut-off and HF cut-off of the $L_s(\omega)$ is shown in figure-8. The ω_u gain-cross-over frequency is shown in the figure, at the point where Nichols curve has $|L_S(\omega)| = 0\text{dB}$, and that frequency the phase angle is $-\alpha\frac{\pi}{2}$. The resonance peak M_r contour is having a tangent at an ordinate point of Y_0 dB, at which $\omega = \omega_r$ the resonance frequency of the controlled system.

Band limiting the open-loop complex order transfer function and tuning

As described above, the need is to band-limit the system, between ω_l and ω_h, by employing filters. Also we modify the $\left(\frac{\omega_u}{\omega}\right)$ in terms of ω_l , ω_h and write as follows

$$L_S(s) = K\left(\frac{\omega_l}{s}+1\right)^{n_l}\left(\frac{1+\frac{s}{\omega_h}}{1+\frac{s}{\omega_l}}\right)^{\alpha}\left(\left(C_0\left(\frac{1+\frac{s}{\omega_h}}{1+\frac{s}{\omega_l}}\right)\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}\left(\frac{1}{\left(1+\frac{s}{\omega_h}\right)^{n_h}}\right)$$

$$L_S(i\omega) = K\left(\frac{\omega_l}{i\omega}+1\right)^{n_l}\left(\frac{1+\frac{i\omega}{\omega_h}}{1+\frac{i\omega}{\omega_l}}\right)^{\alpha}\left(\left(C_0\left(\frac{1+\frac{i\omega}{\omega_h}}{1+\frac{i\omega}{\omega_l}}\right)\right)^{i\beta}\right)^{-\operatorname{sgn}\beta}\left(\frac{1}{\left(1+\frac{i\omega}{\omega_h}\right)^{n_h}}\right)$$

Where

$$C_0 = \sqrt{\frac{1+\left(\frac{\omega_r}{\omega_l}\right)^2}{1+\left(\frac{\omega_r}{\omega_h}\right)^2}}$$

We have moved in above re-framing from ideal integrator $\left(\omega_u / s\right)$ to a practical integrator $C_0\left((1+ s/\omega_h)/(1+ s/\omega_l\right)$ with $\omega_h \gg \omega_l$ that becomes $C_0\left(1/(1+ s/\omega_l)\right)$ that is a practical integrator transfer function.

Tuning of Fractional Complex Order Controller

In order to tune the above $L_S(i\omega)$ optimally, we have to use proper cost function. In order to use desired close loop specifications, with respect to percent overshoot minimization, rise time minimization, a convenient cost function

$$J = \left(\sup_{\omega,G}\left|T(i\omega)\right| - M_{r0}\right)^2$$

be minimized. Where '$\sup$'denotes the least upper-bound. The above is cost function that is defined on reducing the variation of 'resonant peak' M_r in the complementary sensitive function $T(i\omega)$; which is also closed loop transfer function, (refer section on Nichols chart). Where complimentary sensitive function and sensitive function are following

$$T(i\omega) = \frac{L_S(i\omega)}{1+L_S(i\omega)} \qquad S(i\omega) = \frac{1}{1+L_S(i\omega)}$$

The $L_S(i\omega)$ is band-limited open-loop shaped transfer function.

The term M_{r0} is the desired nominal closed loop resonant peak of the nominal plant $G_N(i\omega)$, while satisfying the following set of constraints for all plant $G(i\omega)$ for all $\omega \in \mathbb{R}^+$

$$\inf_G \left|T(i\omega)\right| \geq T_l(i\omega) \qquad \sup_G \left|T(i\omega)\right| \leq T_u(i\omega)$$

$$\sup_G \left|S(i\omega)\right| \leq S_u(i\omega) \qquad \sup_G \left|C(i\omega)S(i\omega)\right| \leq C(i\omega)S_u(i\omega)$$

$$\sup_G \left|G(i\omega)S(i\omega)\right| \leq G(i\omega)S_u(i\omega)$$

The best parameters for $L_S(s)$ be obtained via minimizing the J by non-linear optimization.

Basically the resonance peak M_r and the time indices like rise time t_r , settling time t_s and overshoot ($O.V\%$) are dependent on the damping ratio ξ and different indices; the following second order approximations are

$$t_s = \frac{4}{\xi\omega_r} \qquad t_r = \frac{\left(\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)\right)}{\left(\omega_r\sqrt{1-\xi^2}\right)} \qquad O.V\% = 100e^{\left(-\frac{\pi\xi}{\sqrt{1-\xi^2}}\right)} \qquad M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

Indeed these relations still represents essential quantities of higher order systems with two dominant poles. Minimization of above cost function J is also equivalent to following

$$J_\xi = \left(\xi_{\max} - \xi_{\text{desired}}\right)^2 - \left(\xi_{\min} - \xi_{\text{desired}}\right)^2 \qquad O.V\% = \left(O.V_{\max} - O.V_{desired}\right)^2 - \left(O.V_{\min} - O.V_{\text{desired}}\right)^2$$

Since the overshoot and the rise time are required to be small, the above cost functions may be explicitly used instead of J , [19], with the five inequalities listed above as constraints. After the best tuning parameters for $L_S(s)$ is got, the fractional complex order controller $C(s)$ is yielded as

$$C(i\omega) = \frac{L_S(i\omega)}{G_N(i\omega)}$$

Here we can in practice use a rational approximation $C_R(s)$ from obtained $C(s)$ as ratio on rational polynomials [20].

The Fuel cell response is chosen as [10]; reducing the time indices of λ_{O_2} that is $t_r < 0.05$, reducing the peak of λ_{O_2} transient response by $O.V\% \leq 4\%$ and steady state error to as minimum possible are expected parameters. The controller shall be on the linearized model of the Fuel Cell, and steps are

1. Determine nominal plant transfer function $G_N(s)$,

2. Specify the n_l and n_h of open loop transfer functions (the orders $n_l = 1$ and $n_h = 2$ are sufficient), and then

3. Specify the sensitivity functions that guarantee the predefined specifications. For this the time indices parameters t_s, t_r and $O.V\%$ can be translated to required sensitive functions. Although these rules are valid for second order system containing no zeros, these may be used for higher order systems acting under two dominant poles [21], so these time indices evaluation by the formula given above, can approximately be instrumental. This value may be used as initial guess and the optimal parameters may be tuned during the design procedure.

Lower peak overshoot in time of λ_{O_2} necessitates $0.7 \le \xi \le 0.8$. Since settling time and the rise time are inversely proportional to the resonance frequency ω_r (see above noted formulas), the condition on $t_r < 0.05$ s results in initial guess for ω_r as $92 \le \omega_r \le 138$. Furthermore parameter $M_{r0} = 0.0017\text{dB}$ is chosen for $\xi = 0.7$. To provide a required time response of λ_{O_2}, a tradeoff between minimizing t_r and maximizing peak overshoot $O.V\%$ is necessary, which is conceivable for $0.1 \le O.V\% \le 4$. This inequality together with M_r achieves a constraint on T_l and T_u. Meanwhile the relation $S(\omega) + T(\omega) = 1$ derives the output disturbance rejection and also constraint on S_u. A meaningful estimation of maximum of control effort eg at 300V also estimates CS_u. Now finally

4. Use CRONE design module [19] for initial tuning of $L_S(s)$ supposing that J as given is cost function. The aim is to shape the nominal open-loop Nichols locus to provide optimum close loop system as robust as possible in predetermined frequency intervals. The design module also permits to optimize open-loop parameters taking into account the five constraints (inequalities) as listed using 'fmincon MATLAB'. Here in this design module four parameters are defined ω_h, ω_l, ω_r and Y_0 for this system , 290, 1.2, 60 and 6dB suffices.

Conclusion

The fractional order and fractional complex order controller is described here, where the real part and imaginary part of the differ-integration operations play very important role independently, to make the control system robust. It is shown that real part of the fractional complex differ-integrator fixes the roll of slope of the magnitude of the open-loop transfer function, and the imaginary part of the differs-integrator independently manipulates the slope of the phase of the open loop transfer function, independently. This feature adds to the robustness where the plant transfer function has gain as well as phase uncertainty. The optimization scheme to get particular performance criteria in time domain (and frequency domain) could be met via this fractional complex order Bode-shaped open loop transfer function. This becomes active research area in the control of uncertain systems especially fuel cells.

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