DISCUSSION ON NATURAL LAW OF PHYSICS

Module-V

MATHEMATICO-PHYSICS OF GENERALIZED CALCULUS

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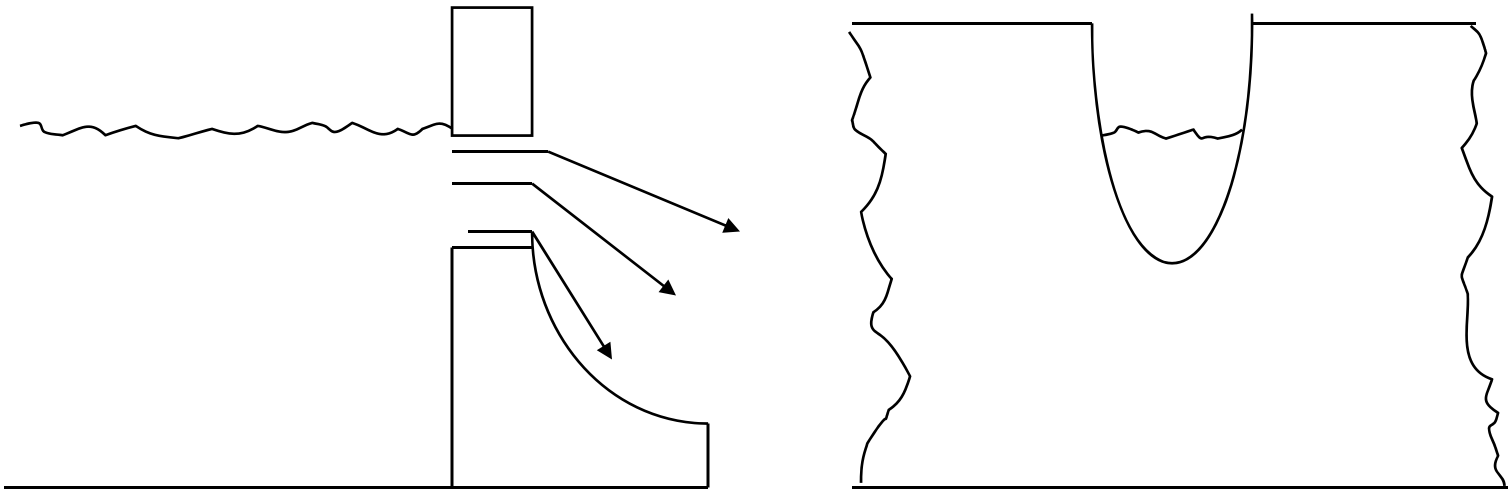
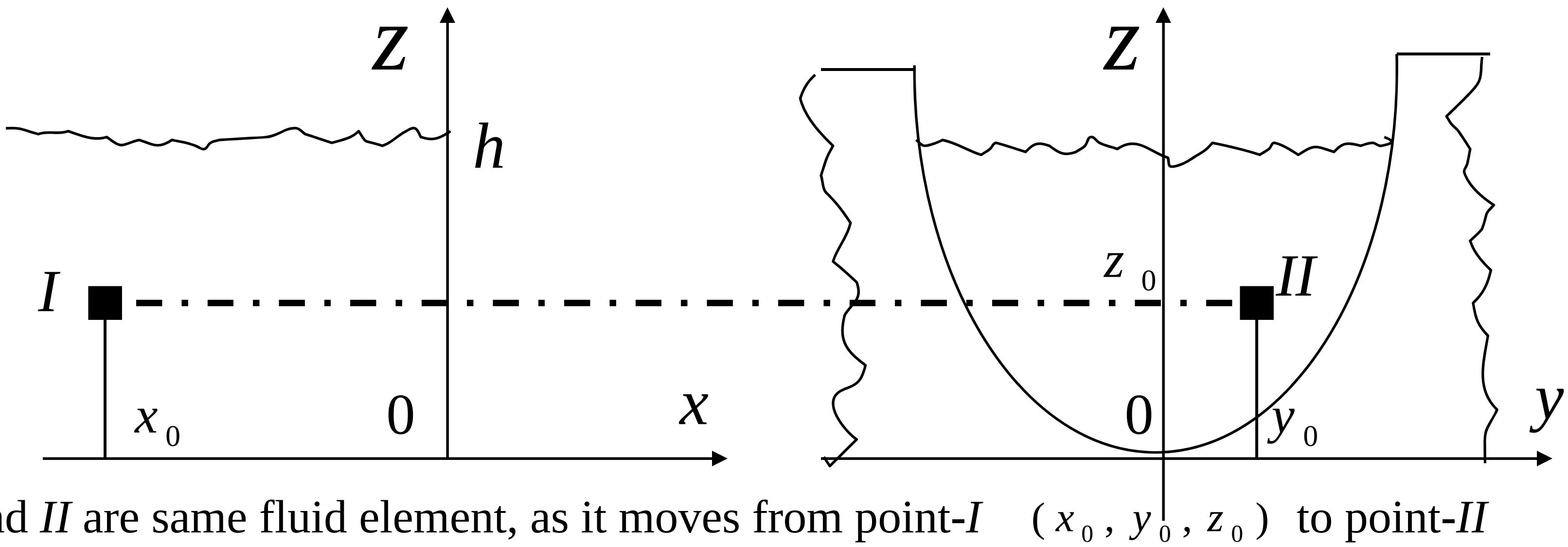
Flow rate vis-à-vis notch shape for flow of water through dam weir.





I and *II* are same fluid element, as it moves from point-*I* (x_0, y_0, z_0) to point-*II* $(0, y_0, z_0)$ along the same “tube of flow”.

Bernoulli's equation for hydrodynamics for point *I* and *II* by head balancing gives:

$$\frac{P_I}{\rho} + g z_0 + \frac{1}{2} V_I^{\,2} = \frac{P_{II}}{\rho} + g z_0 + \frac{1}{2} V_{II}^{\,2}$$

Assuming point-*I* is far enough upstream $V_I << V_{II}; V_I \approx 0$

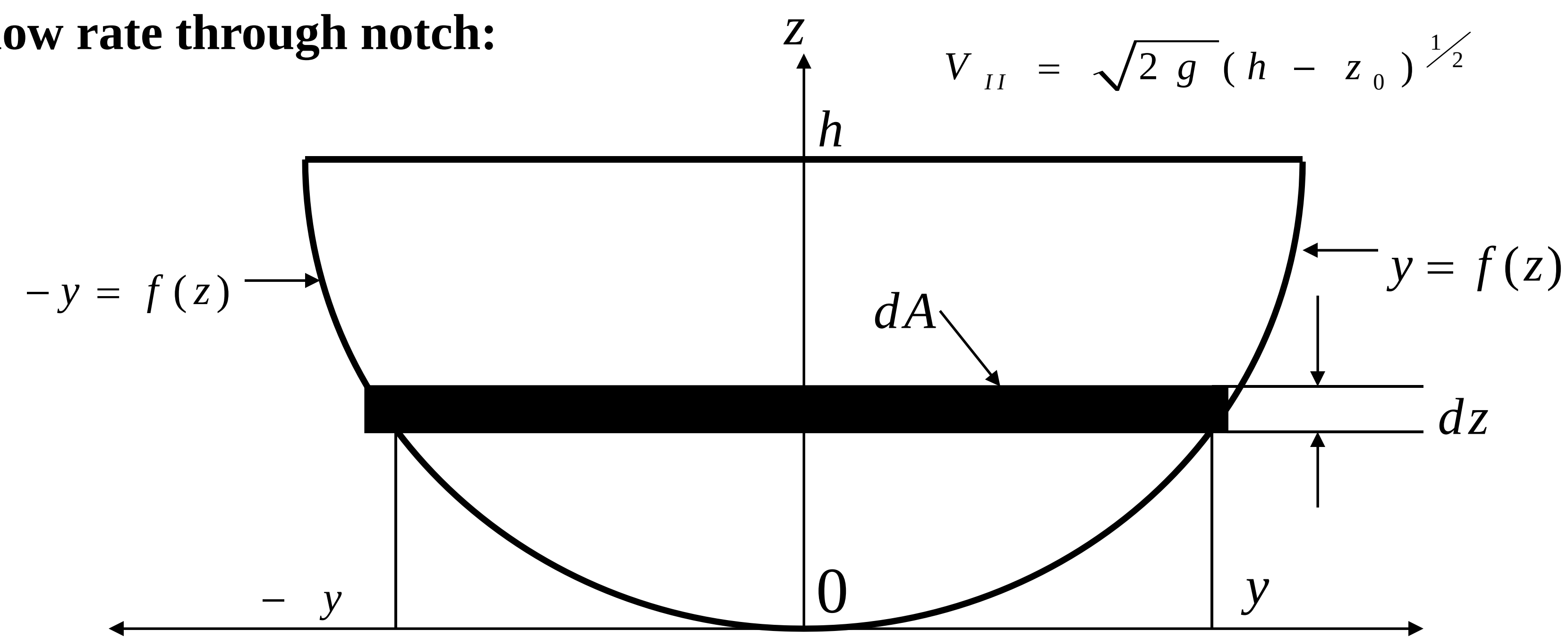
Point-*I* pressure is atmospheric pressure plus the pressure exerted by the column of water of height $(h - z_0)$.Since point-*II* is at the same plane, that of notch, the pressure at point-*II* is atmospheric pressure.

Therefore: $P_I - P_{II} \equiv (\rho g) \times (h - z_0)$

And

$$V_{II} = \sqrt{2g}\,(h - z_0)^{1/2}$$

Flow rate through notch:



$$dA = 2y\,dz$$

$$dA = 2f(z)\,dz$$

The incremental rate of flow of flow of water through '*dA*' is $dQ = V.dA$

$$dQ = 2\sqrt{2g}(h - z)^{1/2} f(z)dz$$

Total flow of water through the notch:

$$Q = \int_0^h dQ = 2\sqrt{2g}\int_0^h (h - z)^{1/2} f(z)dz$$

Flow rate-vis-à-vis notch profile:

The flow rate (total) through weir notch:

$$Q(h) = \int_0^h (h - z)^{1/2} f(z)\,dz$$

When written in notation of FC the formulation is:

$$Q(h) = \sqrt{2g\pi}\, D^{-3/2} f(h)$$

This can be modified as:

$$D^{3/2}\left[Q(h)\right] = \sqrt{2g\pi}\, D^{3/2}\left[D^{-3/2} f(h)\right]$$

Giving:

$$f(h) = \frac{1}{\sqrt{2g\pi}} D^{3/2} Q(h)$$

Let us suppose that flow rate profile we want is: $Q(z) = k z^{\lambda}$

Then the notch profile is:

$$f(z) = \frac{1}{\sqrt{2g\pi}} D^{3/2} Q(z) = \frac{k}{\sqrt{2g\pi}} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - 3/2)} z^{\lambda - 3/2} = \frac{k}{\sqrt{2g\pi}} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - 1/2)} z^{\lambda - 3/2}$$

The existence of the solution is possible if

$$\lambda + 1 > 0;\ \lambda > -1$$

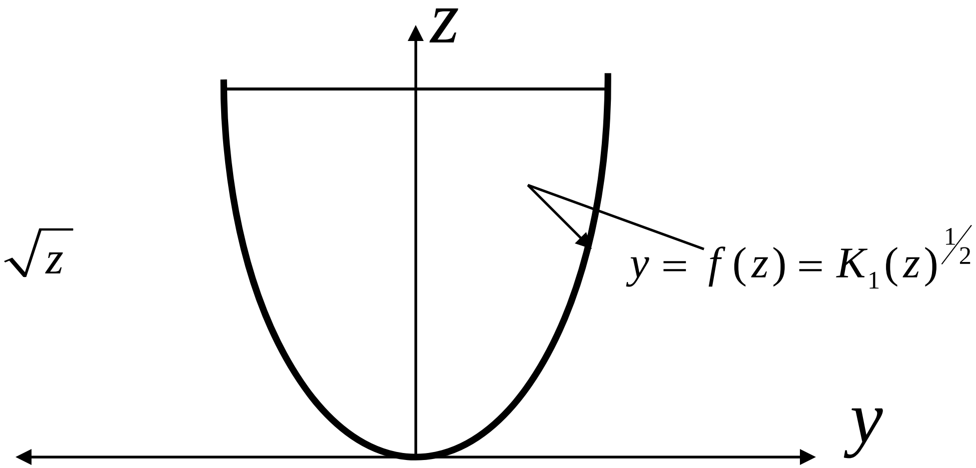
and with added restriction as

$$\lambda + 1 - \frac{3}{2} > 0;\ \lambda > \frac{1}{2}$$

Particular cases for notch profile for particular flow rates function:

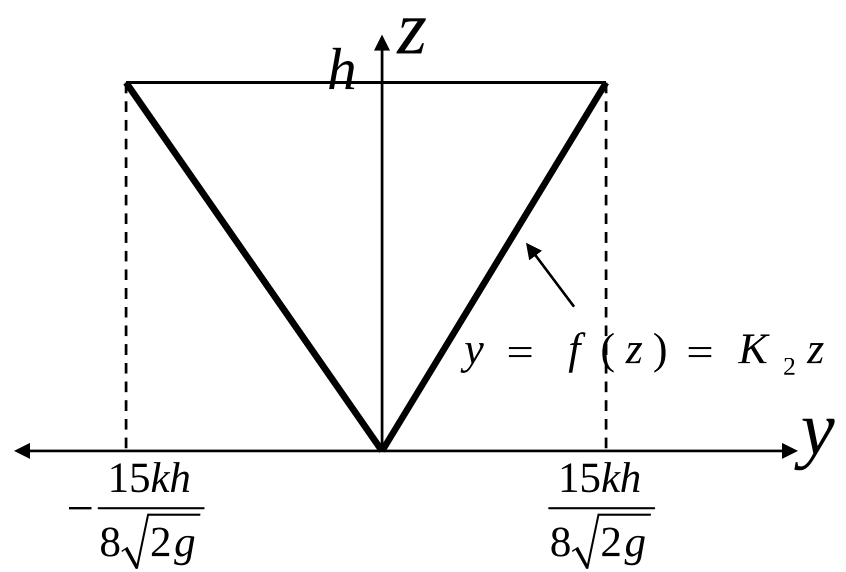
If the flow rate is: $Q(z) = kz^2; \lambda = 2$

Then notch profile is: $y = f(z) = \frac{4k}{\pi\sqrt{2g}}\sqrt{z}$

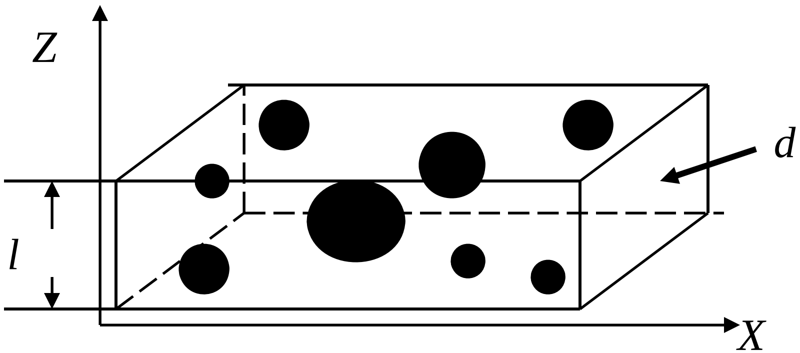


If the flow rate is $Q(z) = kz^{5/2}; \lambda = 5/2$

The notch profile is: $y = f(z) = \frac{15k}{8\sqrt{2g}} z$



Flux through a heterogeneous-porous medium (with fractal support)



Let the largest diameter of the porous media be a_0 ; $l >> a_0$ in elementary dV

The volume of grains dV_{gr} in dV .

$dV_{gr} = (dV)K(x)$ Where $K(x)$ is volume of grain per unit volume dV near x .

Grain free volume is thus: $d\bar{V} = dV - dV_{gr} = dV[1 - K(x)] = \varepsilon(x)dV$

Porosity of the medium at x is: $\varepsilon(x) = 1 - K(x) = \dfrac{d\bar{V}}{dV}$

Let $dS_{gr}(z)$ be the area common to the plane & to the grain. The volume of grains cutting the plane is: $dV_{gr} = \int_0^l dS_{gr}(z)dz$ and the grain free area in this plane is: $d\bar{S} = dS - dS_{gr}(z)$

Averaging over length the grain free area is:

$$d\bar{S} = \frac{1}{l}\int_0^l d\bar{S}(z)dz = \frac{1}{l}\int_0^l [dS - dS_{gr}(z)]dz = dS - \frac{dV_{gr}}{l}$$

$$dV_{gr} = K(x)ldS$$

Diffusion in porous media:

Mass balance in the grain free area/volume $u(x,t)$ is concentration & j is particle flux

$$\frac{\partial}{\partial t}\int_{\bar{V}} u \, d\bar{V} = -\int_{\bar{S}} j \, d\bar{S}$$

$$d\bar{S} = \varepsilon(x)\, dS \; ; \; d\bar{V} = \varepsilon(x)\, dV$$

While the volume shrinks

$$\partial_t (u\varepsilon) + \mathrm{div}(j\varepsilon) = 0$$

Fick's relation is:

$$j = -\mathbb{D}_0 \nabla u$$

$$\partial_t u\varepsilon - \mathbb{D}_0 \nabla . \varepsilon \nabla u = 0$$

Let porosity be a random process with $\varepsilon'(x)$ having zero average which takes the value between 0 and 1. $\varepsilon(x) = \varepsilon_0 e^{-\varepsilon'(x)}$. Substituting this and expanding:

$$\partial_t u - \mathbb{D}_0 \nabla^2 u = -\mathbb{D}_0 (\nabla \varepsilon') . \nabla u$$

Initial condition $u(x,0) = \delta(x)$ and averaging $< u >$ over all possible $< \varepsilon' >$
Fourier-Laplace technique applied with (very involved) Feynman's method gives:

$$\left(\frac{\partial}{\partial t} - \mathbb{D}'_0 \nabla^2 - \eta \frac{\partial^{3/2}}{\partial t^{3/2}} \nabla^2 \right) < u > (x,t) = 0$$

$$b = al ; \mathbb{D}'_0 = \mathbb{D}_0 (2\pi)^{-3}$$

$$\eta = b^3 l^3 (2\pi)^{-3} (1 + 5/3) \pi (\mathbb{D}_0)^{-1/2}$$

Diffusion

Fick's law of continuity (II law of diffusion)

$$\frac{\partial}{\partial t}u(x,t) = -\frac{\partial}{\partial x}j(x,t)$$

Fick's constitutive equation (I law of diffusion like Fourier's law of heat conduction)

$$j(x,t) = -\mathbb{D}\frac{\partial}{\partial x}u(x,t)$$

relating diffusing quantity (temperature, charges, voltage, concentration etc. and its flux)

Combining the two we obtain Fick's law of diffusion as Diffusion Equation

$$\frac{\partial}{\partial t}u(x,t) = \mathbb{D}\frac{\partial^2}{\partial x^2}u(x,t)$$

With initial condition as delta function, and natural BC $u(x,0) = \delta(x); u(|x| \to \infty, t) = 0$

we get Gaussian solution as:

$$u(x,t) = \frac{1}{\sqrt{4\pi\mathbb{D}t}}\exp\left(-\frac{x^2}{4\mathbb{D}t}\right)$$

For a very small time just at the start of diffusion process, there exist finite amount of diffusing quantity at very large distance, meaning diffusing elements posses infinite velocity! Diffusion equation is parabolic equation.

The Fickian diffusion generalized with Euclidian dimension and geometrical parameter:

The Fick's diffusion equation for vector form is thus: $\frac{\partial}{\partial t} u(\overline{X}, t) = \mathbb{D} \nabla^2 u(\overline{X}, t)$

For isotropic case $$\frac{\partial}{\partial t} u(r, t) = \mathbb{D}\, r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} u(r, t)$$

d -is Euclidian dimension of 1, 2, 3. The Laplacian operator is generalized 1- is planar 2- cylindrical, and 3- spherical coordinates

One may too re-write the Laplacian operator with geometrical parameter g with 0- for planar, ½ for cylindrical and 1-for spherical coordinate (geometry), as:

$$\frac{\partial}{\partial t} u(x, t) = \mathbb{D} \frac{\partial^2}{\partial r^2} u(x, t) + \frac{2g\mathbb{D}}{r} \frac{\partial}{\partial r} u(x, t)$$

These generalization of Laplacian operator give integer order diffusion equation. Can these number be arbitrary. Well yes they can and then the Laplacian becomes fractional differential equation-giving fractional order diffusion equation. Well, if the diffusing species is moving in a matrix of well distributed obstacles or traps or attractors can they rate at fast or slow? There is thus possibility of this not following Gaussian or integer order law.

Catteneo diffusion:

allows to have the diffusing flux ‘relaxes’ with some time constant , modifies the first law of Fickean diffusion as:

$$j(x,t) + \tau \frac{\partial}{\partial t} j(x,t) = -\mathbb{D} \frac{\partial}{\partial x} u(x,t)$$

Putting this new law in continuity equation (II law of Fick’s) we get**:**

$$\frac{\partial}{\partial t} u(x,t) + \tau \frac{\partial^2}{\partial t^2} u(x,t) = \mathbb{D} \frac{\partial^2}{\partial x^2} u(x,t)$$

Catteneo diffusion equation (1948), it is like telegrapher’s equation and has finite phase velocity of diffusing quantity, $v_{ph} = \sqrt{\frac{\mathbb{D}}{\tau}}$ and not-infinite! Hyperbolic equation

In a media with MEMORY the flux of diffusing quantity is related to previous history through relaxation function (Memory-Kernel) $K(t)$

$$j(x,t) = -\int_0^t K(t-t') \frac{\partial u(x,t')}{\partial x} dt'$$

This convolution is also Boltzman's superposition law.

Catteneo's diffusion with memory integral: $j(x,t) + \tau \frac{\partial}{\partial t} j(x,t) = -\mathbb{D} \frac{\partial}{\partial x} u(x,t)$

Using this Memory Integral in the Catteneo's flux relaxation equation we get:

$$j(x,t) + \tau \frac{\partial}{\partial t} j(x,t) = -\left(\tau \frac{\partial}{\partial t} + 1 \right) \int_0^t K(t-t') \frac{\partial u(x,t')}{\partial x} dt' = -\mathbb{D} \frac{\partial u(x,t)}{\partial x}$$

By use of Leibniz's rule of differentiation of integral; i.e. $D\left[D^{-1} f\right] = D^{-1}\left[Df\right] + f(0)$ we get:

$$j(x,t) + \tau \frac{\partial}{\partial t} j(x,t) = -\tau K(0) \frac{\partial u(x,0)}{\partial x} - \int_0^t \left[\tau \frac{\partial}{\partial t} K(t-t') + K(t-t') \right] \frac{\partial u(x,t')}{\partial x} dt' = -\mathbb{D} \frac{\partial u(x,t)}{\partial x}$$

We can take (from above)

$$\tau K(0) = \mathbb{D} \quad \text{and} \quad \tau \frac{\partial}{\partial t} K(t) + K(t) = 0$$

Solving for relaxation function or memory kernel we obtain:

$$K(t) = \frac{\mathbb{D}}{\tau} \exp\left(-\frac{t}{\tau} \right)$$

Making non-local (non-Markovian) theory of transport compatible with Catteneo equation!.

Other type of memory kernel for GCE

A memory kernel of the form

$$K(s) = \frac{\mathbb{D}_M}{\tau^{v}} \frac{s^{-1}}{1 + \tau^{-1} s^{-v}}$$ in Laplace form, or $$K(t) = \frac{\mathbb{D}_M}{\tau^{v}} E_{v,1}\left[-\left(\frac{t}{\tau}\right)^{v} \right]$$

$$0 < v < 1$$

Gives fractional diffusion equation as:

$$\frac{\partial}{\partial t} u(x,t) + \tau^{v} \frac{\partial^{v-1}}{\partial t^{v-1}} \frac{\partial^{2}}{\partial t^{2}} u(x,t) = D \frac{\partial^{v-1}}{\partial t^{v-1}} \frac{\partial^{2}}{\partial x^{2}} u(x,t)$$

And corresponding Generalized Catteneo Equation (GCE) is:

$$j(x,t) + \tau^{v} \frac{\partial^{v}}{\partial t^{v}} j(x,t) = -\mathbb{D}_M \frac{\partial^{v-1}}{\partial t^{v-1}} \frac{\partial}{\partial x} u(x,t)$$

A flux relaxation function (memory kernel) of Mittag-Leffler type which for a long time' as a power law behavior $K(t) \approx t^{-v}$ can be associated with GCE Fractional Catteneo Equation or Fractional Diffusion Equation

Integral representation of Fick's law & random walker's survival probabilty

$$\frac{\partial C(x,t)}{\partial t} = A \frac{\partial^2 C(x,t)}{\partial x^2}$$

$$C(x,t) = \delta_{x0} + A\int_0^t \Delta C(x,t')\,dt'$$

Where Laplacian is represented as $\Delta \equiv \partial^2 / \partial x^2$. This is reminiscent of a random walker that starts at origin $x = 0$ at time $t = 0$ and proceeds successfully. The pdf for time interval of length t between two jumps is wait pdf is $w(t)$. The pdf of displacement vector x is in single jump is $\lambda(x)$.

Then CTRW reads as:

$$f(x,t) = \delta_{x0}\chi(t) + \int_0^t w(t-t')\int_{-\infty}^{\infty} \lambda(x-x')\,f(x',t')\,dx'dt'$$

Similar to above!. Where $\chi(t)$ is probability that walker survives at the origin for a time of length t,

The survival probability is thus related to wail times as: $\chi(t) = 1 - \int_0^t w(t')\,dt'$, rearranging we have:

$$f(x,t) - \int_0^t w(t-t')\int_{-\infty}^{\infty} \lambda(x-x')\,f(x',t')\,dx'dt' = \delta_{x0}\left[1 - \int_0^t w(t')\,dt'\right]$$

Fourier Laplace of this is:

$$F(k,s)\left[1 - w(s)\lambda(k)\right] = \frac{1}{s} - \frac{w(s)}{s}$$

$$F(k,s) = \frac{1 - w(s)}{s\left[1 - w(s)\lambda(k)\right]}$$

$$C(k,s) = \frac{1 - w(s)}{s\left[1 - w(s)\lambda(k)\right]}C_0(k)$$

Walking or Diffusion depending on different wait time and jump length statistics

$$N(k,s) = \frac{1-w(s)}{s}\frac{N_0(k)}{1-w(s)\lambda(k)}$$

Finite jump length and finite wait time, returns integer order diffusion equation

$w(t) = \tau^{-1}\exp(-t/\tau)$ Has finite average wait time $<w(t)>=\tau$

$\lambda(x) = (4\pi\sigma^2)^{-1/2}\exp(-x^2/4\sigma^2)$ Has finite MSD as $<\lambda(x)^2>=2\sigma^2$

Using asymptotic expansion as: $w(s)\approx 1-s\tau$ and $\lambda(k)\approx 1-\sigma^2k^2$ putting in above

$$N(k,s) = \frac{1-1+s\tau}{s}\frac{N_0(k)}{1-(1-s\tau)(1-\sigma^2k^2)} \simeq \frac{1}{s+A_1k^2}$$ with $A_1 = \sigma^2/\tau$

$$\frac{\partial N(x,t)}{\partial t} = A_1\frac{\partial^2 N(x,t)}{\partial x^2}$$

Infinite jump length and finite wait time returns fractional spatial diffusion equation

$\lambda(k)\approx 1-\sigma^\mu k^\mu, 1<\mu<2 : \lambda(x)\approx \sigma^\mu x^{-1-\mu}$ Power law asymptotically Levy

$$N(k,s) = \frac{1-1+s\tau}{s}\frac{N_0(k)}{1-(1-s\tau)(1-\sigma^\mu k^\mu)} \simeq \frac{1}{s+A_2k^\mu}; A_2 = \sigma^\mu/\tau$$

$$\frac{\partial N(x,t)}{\partial t} = A_2\left[{}_{-\infty}D_x^\mu N(x,t)\right]$$

Finite jump length and infinite wait time, returns fractional temporal diffusion equation

$w(t) = (t/\tau)^{1+\alpha}; 0<\alpha<1; w(s)\approx 1-(s\tau)^\alpha$

$$N(k,s) = \frac{1-1+(s\tau)^\alpha}{s}\frac{N_0(k)}{1-(1-[s\tau]^\alpha)(1-\sigma^2k^2)} \simeq \frac{N_0(k)/s}{s+A_3s^{-\alpha}k^2}; A_3 = \sigma^2/\tau^\alpha$$

$$N(x,t) - N_0(x,0) = D_t^{-\alpha}A_3\frac{\partial^2 N(x,t)}{\partial x^2}$$

Phase Table for the Fractional Diffusion Equation

$$\frac{\partial}{\partial t}\phi(x,t) = {}_0D_t^{1-\alpha}(\mathbb{D}_{\alpha,\mu})\frac{\partial^{\mu}}{\partial x^{\mu}}\phi(x,t)$$

We are used to $\alpha = 1, \mu = 2$ The fractional order comes as observation of asymptotic behavior in space time relaxation.

Temporal Fractional Order α	Spatial Fractional Order μ	Type of Walk	Average Waiting Time T	Jump-Length Variance σ^2	Nature of Diffusion
$0 < \alpha < 1$	$0 < \mu < 2$	Long-Jump	∞	∞	Non-Markovian
$\alpha \geq 1$	$0 < \mu < 2$	Long-Jump	$< \infty$	∞	Markovian
$0 < \alpha < 1$	$\mu \geq 2$	Sub-diffusion	∞	$< \infty$	Non-Markovian
$\alpha \geq 1$	$\mu \geq 2$	Brownian	$< \infty$	$< \infty$	Markovian

Boltzmann's superposition & stress strain vis-à-vis memory integral:

Formally incorporates memory via causal convolution. Consider arbitrary history of external perturbation of strain. $\varepsilon(t)$ The response of system $\sigma_S(t)$ to a step like input . Then we obtain the response of system

$$\sigma(t) = \int_{-\infty}^{t} dt' \sigma_S(t - t') \frac{d\varepsilon(t')}{dt'}$$

Let $\sigma_S(t) = \frac{C}{\Gamma(1-\alpha)} \left(\frac{t}{\tau} \right)^{-\alpha}$ Then $\sigma(t) = \frac{C\tau^{\alpha}}{\Gamma(1-\alpha)} \int_{-\infty}^{t} dt'(t-t')^{-\alpha} \frac{d\varepsilon(t')}{dt'}$

$0 < \alpha < 1$

$$\sigma(t) = C\tau^{\alpha} \left({}_{-\infty}D_t^{\alpha-1} \frac{d\varepsilon(t)}{dt} \right) = C\tau^{\alpha} \, {}_{-\infty}D_t^{\alpha} \varepsilon(t)$$

$$\varepsilon(t) = \frac{1}{C\tau^{\alpha}} \, {}_{-\infty}D_t^{-\alpha} \sigma(t) = \frac{1}{C\tau^{\alpha}} \, {}_{-\infty}D_t^{-\alpha-1} \frac{d\sigma(t)}{dt}$$

$$\varepsilon(t) = \frac{1}{C\tau^{\alpha}\Gamma(1+\alpha)} \int_{-\infty}^{t} dt'(t-t')^{\alpha} \frac{d\sigma(t')}{dt'} = \int_{-\infty}^{t} dt' \varepsilon_S(t-t') \frac{d\sigma(t')}{dt'}$$

Convolution & Evolution of Process Dynamics:

$$\frac{d}{dt}\Phi(t) = -\int_0^t dt' K(t-t')\Phi(t)$$

Convolution is rolled up condition. Non-exponential relaxation implies memory that is the underlying fundamental relaxation process are non-Markovian. Natural way to incorporate such memory effect is via fractional calculus, via involved convolution integrals in times; the present state is being influenced by all the states the system has been running through at times 0, 1, 2,…, *t*. The power law kernel defining the fractional expression represents a particular long memory.

$$\frac{d}{dt}\Phi(t) = -K_0\Phi(t)$$

Process in this above case at present condition is just entering via present state and not past states.

Fractional Brownian Motion & Fractional Stochastic Difference

For unit interval of time, with down shift operator fractional stochastic process may be modeled as fractional random walk. If ξ_j is random variable used to represent step taken in discrete time j then $(1 - E^{-1})^{\alpha} X_j = \xi_j$ represent analog of Fractional Brownian Motion.

Inverting this we write:

$$X_j = (1 - E^{-1})^{-\alpha} \xi_j$$

$$(1 - E^{-1})^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-1)^k E^{-k}$$

$$\binom{-\alpha}{k} = \binom{k+\alpha-1}{k} = \frac{(k+\alpha-1)!}{k!(\alpha-1)!}$$

$$X_j = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!} E^{-k} \xi_j = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!} \xi_{j-k}$$

$$X_j = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)(\alpha-1)!} \xi_{j-k}$$

Asymptotic behavior of FBM:

We wish to determine asymptotic form of using Stirling's formula for gamma function ratio

$$X_j = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)(\alpha-1)!}\xi_{j-k}$$

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \approx x^{a-b}$$

So we have:

$$X_j \approx \sum_{k=0}^{\infty} \frac{k^{\alpha-1}}{(\alpha-1)!}\xi_{j-k}$$

$$k >> \alpha; k \to \infty$$

Strength of the above decrease asymptotically with increasing time lag k as an 'inverse power law' as long as $\alpha < 1$

Spectrum of FBM

Of fractional time series $X_j = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!}\xi_{j-k}$ D.F.T as $X_\omega = \theta_\omega \xi_\omega$

spectrum is defined as

$$S(\omega) = \left\langle \left| X_\omega \right|^2 \right\rangle$$

Assuming random fluctuations have a 'white-noise' spectrum of constant strength then $S(\omega) = \left\langle \theta_\omega^{\ 2} \right\rangle$

$$\theta_\omega = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!} e^{-ik\omega} = \left(1-e^{-i\omega}\right)^{-\alpha}$$

$$S(\omega) = \frac{1}{\left[2\sin\left(\omega/2\right)\right]^{2\alpha}} \approx \frac{1}{\omega^{2\alpha}}$$

Fractional stochastic process driven by 'white-noise' has inverse power law

Set $\alpha = H - \frac{1}{2}$ so that the spectrum equation becomes $S(\omega) = \frac{1}{\omega^{2\alpha}} = \frac{1}{\omega^{2H-1}}$

Persistence Anti-persistence walks:

$$S(\omega) = \frac{1}{\omega^{2\alpha}} = \frac{1}{\omega^{2H-1}}$$

In the language of random walks the spectrum equation for $1/2 \le H \le 1$ or $0 < \alpha \le 0.5$ implies random walker has tendency to continue in the direction where he/she is going. This means there is persistence to the process, given a step in a particular direction that step is remembered and likelihood of the next step is also in the same direction is greater than reversing the direction. The process is LRD, with 'long range memory', Fractional Brownian Motion. The auto-correlation has slow Power law decay.

Analogy for $0 < H \le 1/2$ or equivalently $-0.5 \le \alpha < 0$ the spectrum increases as power law in frequency. This increase implies that the random walker prefers to change his/her mind after each step. That is anti persistence. In the process a given step is remembered and likelihood that of next step being in the same direction is less than that of reversing the direction. Case of 'short range memory', the process decays monotonically to zero, hyperbolically.

$\alpha = 1$ Indicates process as power spectra $1/\omega^2$ is normal Brownian process

Can be also termed as passing a 'white-noise' through a generalized integrator having T.F as $G(s) = 1/s^{\alpha}$ having output to impulse as $y(t) = t^{\alpha-1}/\Gamma(\alpha)$ and auto-correlation as $R_{yy}(\tau) = \sigma^2 |\tau|^{2\alpha} / 2\Gamma(2\alpha)\cos(\alpha\pi)$

Fractional Brownian Motion & its diffusion equation

FBM is the simplest mathematical model of a Gaussian Stochastic Process (random walk) whose variance does not scale linearly with time. Its pdf in one dimension is:

$$P(x,t) = \frac{1}{(4\mathbb{D}\pi t^{2/d_w})^{1/2}} \exp\left(-\frac{x^2}{4\mathbb{D}t^{2/d_w}}\right)$$

Where $2 \le d_w < \infty$ characterizes the time evolution of the MSD of the FBM $\left\langle x^2(t)\right\rangle = 2\mathbb{D}t^{2/d_w}$ where $\mathbb{D}$ is a constant. The extraordinary fractional diffusion equation corresponding to the solution of the above pdf is:

$$\frac{\partial^{1/d_w} P(x,t)}{\partial t^{1/d_w}} = -A\frac{\partial P(x,t)}{\partial |x|}$$

For $d_w = 2$ the case is simple Brownian Motion.

The above representation is one way to have the FBM represented, with anomalous diffusion exponent

There are other representations of FDE with anomalous diffusion exponent added with fractal dimension and fracton (the spectral density of states), however looks similar, and has diverging MSD. The FBM etc Thus had Fractional Moments as the MSD are diverging.

Disordered Relaxation

Ordered Relaxation: (Intense & Strong)

Standard Maxwell Debye relaxation

Gives pure exponential solution with single relaxation time constant; this is strong relaxation (without-memory).

$$\tau \frac{d}{dt}\Psi(t) = -\Psi(t)$$

$$\frac{d}{dt}\Psi(t) + \lambda\Psi(t) = \delta(t)$$

$$\tau^{-1} = \lambda;\ \Psi(0) = 1;\ \Psi(0^-) = 0$$

$$\Psi(t) = e^{-t/\tau} = e^{-\lambda t}$$

Disordered Relaxation: (Intermittent & weak)

For complex dissipating process we (may) have several time constants and let us have this 'disorder' in a power law representation so, the PDE is, (this is weak relaxation, obtained by modifying above)

$$\frac{d}{dt}\Psi(t) + \lambda\Psi(t) = \delta(t)$$

$$\frac{\partial}{\partial t}\Psi(\lambda, t) + (\lambda)^{1/\phi}\Psi(\lambda, t) = \delta(t)$$

$$0 < \phi < 1$$

Power law is scale free with preferential 'statistics' ;why a random walker prefers to have his/her state maintained; also the law states that 'rich becoming richer' thus preferential

Origination of Fractional Differential Equation in complex ‘disordered' relaxation in condense matter:

$$\frac{\partial}{\partial t}\Psi(\lambda,t)+(\lambda)^{1/\phi}\Psi(\lambda,t)=\delta(t)$$

The solution is $\Psi(\lambda,t)=e^{\left(-\lambda^{\{1/\phi\}}t\right)}$ ‘impulse response function’ $h(\lambda,t)$

On integrating this $h(\lambda,t)$ w.r.t. λ for all $0,\infty$, we obtain the function in time.

$$g(t)=\int_0^\infty h(\lambda,t)\,d\lambda=\int_0^\infty e^{\left\{-\lambda^{(1/\phi)}t\right\}}d\lambda$$

With change of variable and recasting with definition of Gamma function

as $\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}dx$ we get:

‘Impulse response’ of Linear Constant Coefficient System starting from rest: $g(t)=\dfrac{\Gamma(1+\phi)}{t^{\phi}}$

Observed is ‘power-law’ (long-tailed) decay (lingering memory!!)

Response of relaxation to arbitrary forcing function-origin of $D_t^{\phi} f(t)$

$$\frac{\partial}{\partial t}\Psi(\lambda, t) + (\lambda)^{1/\phi}\Psi(\lambda, t) = \dot{f}(t)$$

$$g(t) = \frac{\Gamma(1+\phi)}{t^{\phi}}$$

Response to the arbitrary forcing function is:

$$r(t) = g(t) * \dot{f}(t)$$

$$= \int_0^t d\tau\, g(t-\tau)\, \dot{f}(t)$$

Here put the value of Green's function $g(t)$

$$= \Gamma(1+\phi)\int_0^t \frac{\dot{f}(t)}{(t-\tau)^{\phi}} d\tau = \Gamma(1+\phi)\int_0^t \frac{\dot{f}(t-\tau)}{\tau^{\phi}} d\tau$$

$$= \Gamma(1+\phi)\Gamma(1-\phi)\int_0^t \frac{\tau^{-\phi}}{\Gamma(1-\phi)} \dot{f}(t-\tau)\, d\tau$$

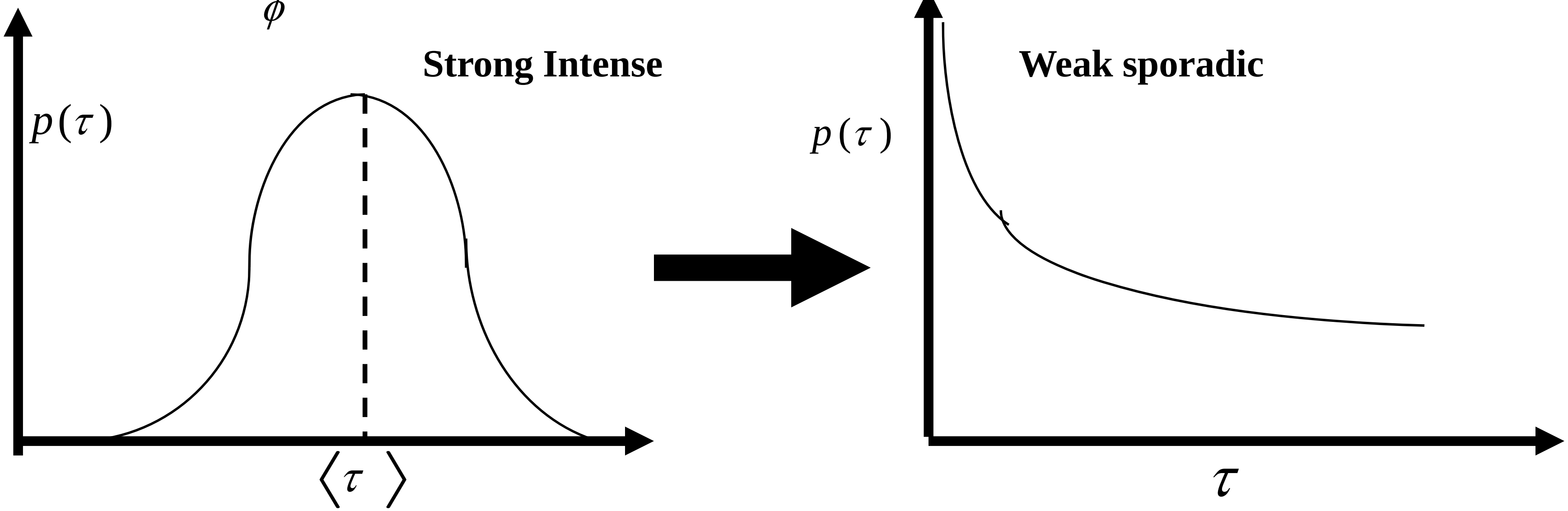
$$= \Gamma(1+\phi)\Gamma(1-\phi)\, I_t^{\phi-1}\left\{\dot{f}(t)\right\}$$

$$= \Gamma(1+\phi)\Gamma(1-\phi)\, D_t^{\phi} f(t)$$

Intermittency of relaxation random normal, and scale-free power law:

$$(\lambda)^{1/\phi}\ ;\ 0 < \phi < 1$$

Let $\alpha = \frac{1}{\phi}; \tau = \lambda^{-1}$ Then Probability Distribution is $p(\tau) \approx \tau^{-\alpha}$



A power law distribution $p(x) \approx x^{-\alpha}$ implies that is asymptotically scale-invariant. In general it is convenient to write from a minimum value as: $p(x) = \frac{\alpha - 1}{x_{\min}}\left(\frac{x}{x_{\min}}\right)^{-\alpha}; \alpha > 1$

Moments of power law distribution

$$\langle x^m \rangle = \int_{x_{\min}}^{\infty} x^m p(x)dx = \frac{\alpha - 1}{\alpha - 1 - m} x_{\min}^m$$

Which is well defined for $m < \alpha - 1$. That means for $m \geq \alpha - 1$ all moments diverge. Can have no defined average or variance!! It is preferential ‘rich-getting –richer’.

Discharge (Strong & Intermittent-Weak)

1) Point Discharge:

$$v(t) = V_0 \exp\left(-t/RC\right) \qquad \text{Strong}$$

Here the Capacitor discharges with constant time constant with constant farads and constant load resistance. If the *RC* circuit is point quantity then it is true as the relaxation process will be uniform throughout the discharge process.

2) Distributed Discharge:

$$v(t) = V_0 \exp\left(-t/\{R(t)\}\{C(t)\}\right)$$

Intermittent Weak

Say the capacitor is spatially distributed, the discharge event may not be uniformly distributed in time. More so the capacitance farad may be function of time or even the farads may be having relation to the charge stored. This gives imperfect weak discharge with several time-constants in the discharge period!!

Discussion Normal Distribution vis-à-vis Power Law Distribution:

. Unlike the Normal-Distribution, in the Power Law Distribution ‘no meaningful average, that can be assumed to be representing unique relaxation time-constants; have several time constants of relaxation.

. Unlike the Normal-Distribution which mostly occur in static environment the Power-Law-Distribution occur when the following conditions are possibly met:

1. Variety.
2. Inequality.
3. Dependency.
4. Finite Resources.

The weak & intermittent relaxation meaning the initial state of the disturbed parameter decays to the equilibrium value weakly with sporadic weakening intensity (rate). The strong and intense relaxation has strong and intense rate.

Relaxation through several states (Variety of time constants)

$$\Psi(t) = [\Psi] = [\psi_1(t) \quad \psi_2(t) \quad \psi_3(t) \quad * \quad * \quad \psi_\infty(t)]$$

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ * \\ \psi_N(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & * & a_{1N} \\ a_{21} & a_{22} & * & a_{2N} \\ * & * & * & * \\ a_{N1} & a_{N2} & * & a_{NN} \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} \\ e^{-\lambda_2 t} \\ * \\ e^{-\lambda_N t} \end{bmatrix}$$

$$\Psi(t) = \left\| [\Psi] \right\| = \sqrt{[\Psi]^T [\Psi]}$$

Relaxation in complex process some comments:

. Sufficiently high micro structural disorder can lead statistically to macroscopic behavior well approximated by Fractional Calculus.

. Damping (relaxation) behavior of materials if modeled by Linear Differential Equations (LDE); with constant coefficient cannot include 'long-memory, that fractional order derivatives require.

. Rubber molecules (presumably) cannot remember past here (perhaps) LDE with constant coefficient can be involved. Such systems have 'exponential-decay'-system without memory. For large times the value goes to zero-(quickly).

. Many materials with 'complex' microscopic dissipative mechanisms may macroscopically show Fractional Order Differential Equation behavior. Damping (relaxation) models may involve relatively fewer fitted parameters compared to integer order complex models.

. Fractional Order behavior may be an artifact of many complex internal dissipative mechanisms-each of them with out memory.

General Decay Law of Equilibrium System:

Observable property of condense material A defined with $\Psi(t)$ as autocorrelation function. Normalize $A(t)$ so that it averages zero, then dimensionless auto-correlation function is:

$$\Psi(t) \equiv \frac{\langle A(t)A(0) \rangle}{\langle A^2(0) \rangle}$$

$A(t)$ Observable is physical quantity say stress, strain rate, voltage current , etc…

Obeys

$$\frac{d\Psi(t)}{dt} = -\int_0^t d\tau K(t-\tau)\Psi(t)$$

Meaning the process evolution (relaxation) is wrapped up (convoluted) in the above integral expression with ‘Memory-Kernel’. That is the value at present instant (or present state) is being influenced by all the states the system has been running from initial time (space) $\tau = 0, 1, 2 \ldots t$

Specifying the Memory Kernel in Relaxation Process:

The decay equation is generally describing the condense matter relaxation process, the specification of the memory kernel is difficult.

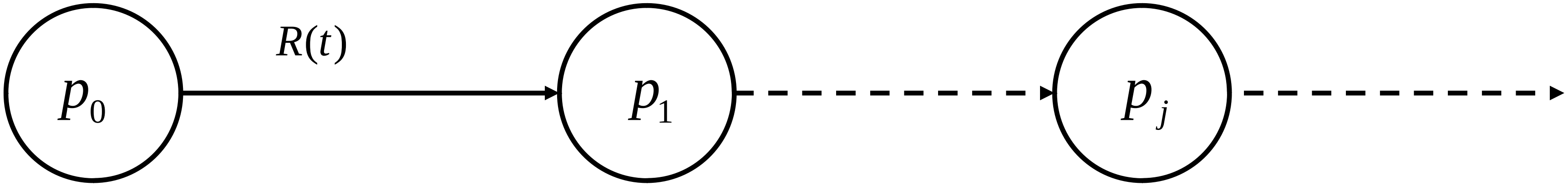
One way to ascribe this kernel as a representative particle in the material is subjected to fluctuating force exerted by its molecular environment and the time average of these fluctuating force is related to memory kernel, by ‘fluctuating dissipating theorem’ (Langevian-Brownian Motion)

Another way is ‘return to equilibrium’ in dynamic process governed by random evolution. This assumes that relaxation process occur as independent random variable in time. A heuristic physical motivation for describing the dynamical evaluation of material composed of interacting particles as random-walk in phase-space. This random walk picture gives vivid conceptual picture of relaxation in many body system and occurrence of ‘relaxation event times (tau’s) after random (independent) time intervals. The intensity of these fluctuations has implications for character of the relaxation process and general ‘universality class’ of relaxation emerge from this approach. Relation between relaxing quantity may be related thus with ‘renewal theory’-a generalization of Poisson’s process with arbitrary hold times.

Generalized Poisson's process:

The system begins at state zero at initial time 0, and change its state at time T, where the T is to be randomly drawn from $p(x) = \lambda e^{-\lambda x}$ What is the probability that system will be in state-1 at some arbitrary time: $P_1(t) = \int_0^t \lambda e^{-\lambda t} dt = 1 - e^{-\lambda t}$

$P_0(t) = e^{-\lambda t}$ is the probability the system is at state-0 at the arbitrary time t.



$$R(t) = \lambda; \frac{d}{dt} p_0(t) = -\lambda p_0; \frac{d}{dt} p_1(t) = -\lambda p_1(t) + \lambda p_0(t); \ldots$$

Probability density for the 'first passage' time between the relaxation increment events:
For a process with changing rate function is 'generalization' of first passage equation of The Poisson's event is-

$$\frac{d}{dt} p_0(t) = -R(t) p_0(t) + \frac{d}{dt} R(t)$$

$$R(t) = p(t) + p(t) * R(t) = p(t) + \int_0^t d\tau\, p(t-\tau) R(\tau)$$

<u>Feller's fluctuation theory</u>

Renewal Theory and Memory Kernel:

$$\frac{d}{dt}\Psi(t) = -\int_0^t d\tau\, K(t-\tau)\Psi(\tau)$$

$$\Psi(0) = 1$$

Integrate both sides:

$$\Psi(t) - \Psi(0) = -\int_0^t d\tau \int_0^t d\tau\, K(t-\tau)\Psi(\tau)$$

$$\Psi(t) = 1 - \int_0^t d\tau\, R(t,\tau)\Psi(\tau)$$

$$R(t,\tau) = R(t-\tau) = \int_0^t d\tau\, K(t-\tau)$$

$$R(t) = \int_0^t d\tau\, K(\tau)$$

Classes of relaxation:

$R(t)$ is the Renewal rate-Rate at which members drop out through death and where new members are added to keep total number of policy holder the same

$\Psi(t)$ is the decay in relative number of charter member of insurance group.

Here consider $\Psi(t)$ as probability that initial state of the dynamic system's property $A(0)$ persists (survives) up-to time 't'.

Continuum picture of Feller's fluctuating theory:

$$R(t) = p(t) + \int_0^t d\tau \, p(t-\tau) R(t)$$

Where $p(t)$ is the probability describing the time between relaxation increment events. The solution to this equation gives the rate kernel of the large-scale relaxation process**.**

The occurrence of 'universality' in $\Psi(t)$ and $R(t)$ in this model of condense matter relaxation from the observation that solution $R(t)$ from above equation for large 't' depends on existence of moments of $p(t)$

$$\langle t^n \rangle = \int_0^\infty d\tau \, (\tau)^n \, p(\tau)$$

1. Finite average and variance-Strong Relaxation
2. Infinite average and Infinite Variance-Strong Intermittent Relaxation
3. Finite average and Infinite Variance-More Intermittent Relaxation

Strong Relaxation (Strong Mixing):

Relaxation event occurs with well defined average period $\langle t \rangle < \infty$ with finite standard deviation (variance) $\langle t^2 \rangle < \infty$

Relaxation with Poisson's probability distribution $p(t) = \frac{1}{\tau_0} e^{-t/\tau_0}$ describing the time between relaxation increment events.

Then from: $\frac{d}{dt} R(t) = \frac{d}{dt} p(t) + p(t) R(t)$ The rate is: $R(t) = \frac{1}{\tau_0}$

Implying: $\Psi(t) = e^{-t/\tau_0}$ Strongly decaying to zero.

Meaning, generally $R(t)$ for any $p(t)$ with finite $\langle t \rangle$ and $\langle t^2 \rangle$ has the asymptotic behavior dependence $R(t) \approx \frac{1}{\tau_0} + \frac{C_1}{t}$ .That is rapidly approaching to constant rate.

Exponential decay (strong-relaxation/mixing) is commonly found in idealized medium,of condensed matter relaxation and more generally with one average time constant.
Having relaxation kernel with no memory.

$$R(t) = \int_0^t d\tau K(\tau) \qquad K(t) = \frac{1}{\tau_0} \delta(t)$$

$$\Psi(t) = 1 - (\tau_0)^{-1} I_t^1 \Psi(t)$$

$$D_t^1 \Psi(t) = -(\tau_0)^{-1} \Psi(t)$$

Weak relaxation-strongly intermittent relaxation:

Relaxation event with no average or standard deviation (power law) $\langle t \rangle ; \langle t^2 \rangle \to \infty$

$$R(t) \approx C_2 \frac{t^{\phi}}{t} = C_2 t^{\phi - 1}$$

$0 < \phi < 1$ For large times $t \to \infty$ implies highly intermittent relaxation

Indicates non-integral dimensions of time “Fractal Dimension” Hausdroff number a non-integer different from topological dimension. Gives degree of intermittency ‘fractal dimensions’ of time points at which relaxation process occurs.

‘Time is not flowing in uniform way perhaps has some power law in evolution’

A fractional integral/differential equation is required to have this relaxation:

$$\Psi(t) = 1 - (\tau_0)^{-1} I_t^1 \Psi(t)$$

$$D_t^1 \Psi(t) = -(\tau_0)^{-1} \Psi(t)$$

→

$$\Psi(t) = 1 - (\tau)^{-\phi} I_t^{\phi} \Psi(t)$$

$${}_0D_t^{\phi} \Psi(t) - \frac{t^{-\phi}}{\Gamma(1-\phi)} = -\tau^{-\phi} \Psi(t)$$

Rate increases with time

$$R(t) \approx t^{\phi - 1} \qquad 1 < \phi \leq 2$$

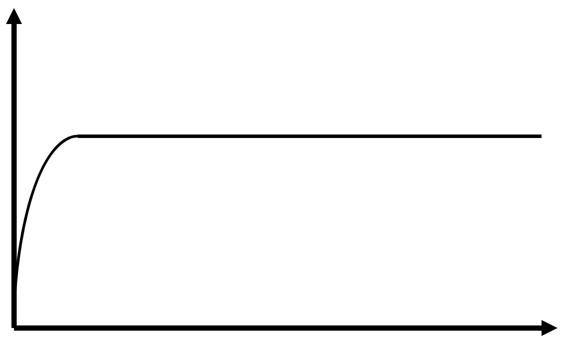
As $\phi \to 2$, the fractal dimension approaches topological dimension ,
The kernel is approaching:

$$R(t) = \int_0^t d\tau K(\tau) \to \frac{t}{\tau_0}$$ Rate is increases $$K(t) \to \frac{1}{\tau_0}$$ Tends to constant memory

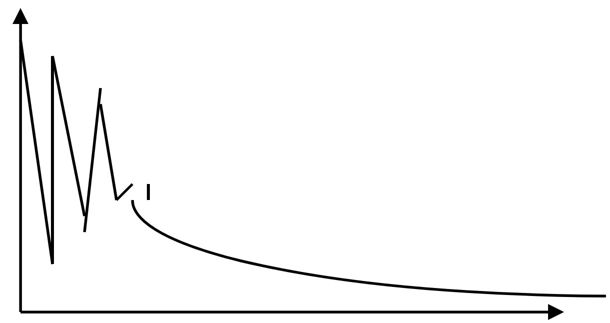
$$\Psi(t) \approx \cos\frac{t}{\tau_0}$$ Oscillatory relaxation

Relaxation process occurring at small scales in condense matter. The situation may be significant for material systems far driven from equilibrium.

$R(t)$



$\Psi(t)$



Rate Generalization and Riemann-Liouvelli (Fractional) Differentials.

The classes of mixing can be generalized by rate with degree of intermittency as:

$$R_{\phi}(t) \approx \frac{\Omega_0}{\Gamma(\phi)}(t)^{\phi-1} \qquad 0 < \phi \leq 1$$

Ω_0 Is 'coupling-constant' governing the relaxation rate intensity $\Gamma(\phi)$ is normalization factor

$$\lim_{\phi \to 1} R_{\phi}(t) \approx \Omega_0 = 1/\tau_0$$

Strong mixing with no intermittency

Insertion of ϕ, characterizing degree of intermittency or RL operator yields FDE

$$\Psi(t) = 1 - \int_0^t d\tau\, R(t,\tau)\Psi(t) = 1 - \Omega_0 I_t^{\phi}\Psi(t)$$

The relaxation process is described by Mittag-Leffler function (exact solution)

$$\Psi(t) = E_{\phi}(-\Delta_{\tau}) \quad \text{With} \quad \Delta_{\tau} = \left(t/\tau_0\right)^{\phi} \quad \& \quad \tau_0 = (\Omega_0)^{-1/\phi}$$

Asymptotically for weak relaxation (mixing)

$$\Psi(t,\phi) \approx \left(\frac{t}{\tau_0}\right)^{-\phi} ;\, t \to \infty$$

Scale Invariant Memory Kernel from Power Law relaxation rate for weak mixing

$$\Psi(t;\phi) \approx \left(t \big/ \tau_0 \right)^{-\phi}$$

$$K(t,\phi) = \frac{d}{dt} R(t) = \frac{d}{dt} \frac{\Omega_0}{\Gamma(\phi)} t^{\phi-1} = \frac{\Omega_0}{\Gamma(\phi-1)} t^{\phi-2}$$

$$0 < \phi \leq 1$$

$$K(\mu t;\phi) = \mu^{\phi-2} K(t;\phi)$$

We may take a note that rate in strong relaxation is constant and its integral from zero to infinity is infinite, and the function $\Psi(t)$ decays to zero very fast. For rate that decays fast with time (say exponentially) so that its integral from zero to infinity is ‘finite’ then the function $\Psi(t)$ no longer decays to zero at large times.

This Non-Ergodic limiting behavior corresponds to material which is not in equilibrium-common in glassy materials.

Weak class of relaxation process involving fractional operator leads to breakdown of ergocity as classically defined. This ϕ may define ‘degree of ergocity’ or ‘quasi-mixing’ or ‘quasi-relaxation’.

Sense for intermittency & fractional differential operator (comment)

The occurrence of weak mixing (relaxation) leads to non-trivial constitutive relations involving fractional operators. The Mittag-Leffler function $\Psi(t;\phi) \approx \left(t/\tau_0 \right)^{-\phi}$

Which can be good approximations over appreciable time scales (Scale-Invariance)-to evaluate these weak relaxation processes in condense matter physics. For example if we introduce this approximation for shear relaxation, into the function relating stress & strain rate, in the limit of linear responses then we get equations for hybrids as in between pure Newtonian liquid $\phi = 1$ and pure Hook's solid $\phi = 0$. $\phi \in [0,1]$

$$\sigma(t) \approx k I_t^{1-\phi} \dot{\varepsilon}(t)$$

Provides good approximation for 'polymer-gel' and is useful expression for understanding the 'internal' effect on the asymptotic frequency dependence of 'viscosity' of the small molecule liquids.

Note that usual definition of viscosity is not valid in this formulation; thus it becomes appropriate to define 'new type of transport coefficients'.

$$v(t) \approx k D_t^{\phi} \dot{q}(t) \quad \phi \in [-1,1]$$

These fractional dynamic representations are also valid for electrical devises (Fractances) amalgamation of pure resistance pure capacitors and pure inductances. More-so the observed electrochemical process gives Warburg impedances in battery dynamics as half order element!!

Spatial Clustering:

The previous discussed models for relaxations in condensed matter does not account for heterogeneity in matter which develop transiently through inter-particle interactions. Cooled liquids develops large scale heterogeneity which modifies the relaxation process This gives rise to ‘stretched exponential’ relaxation (mixing); even for systems which were locally ‘strong-relaxing’.

$$\Psi(t) \approx \exp\left[-\Omega_0 t^{1-\beta}/(1-\beta)\right]$$

$\beta = 2/3$ linear

$\beta = 1/2$ sheet

$\beta = 2/5$ clumps

This denotes measure of fluid heterogeneity is related to geometry of cluster, can occur in clusters having fractal structures!!

Memory Kernel for Stretched Exponential Relaxation:

$$\Psi(t) \approx \exp\left[-\Omega_0 t^{1-\beta}/(1-\beta)\right]$$

$$K(t,\tau) = \Omega_0 \tau^{-\beta} \delta(t-\tau)$$

$\beta = 0$ Strong relaxation without clustering heterogeneity.

$$\frac{d}{dt}\Psi(t) = -\int_0^t d\tau\, K(t-\tau)\Psi(\tau)$$

$$= -\int_0^t d\tau \Omega_0 \tau^{-\beta}\delta(t-\tau)\Psi(\tau) = -\Omega_0 \int_0^t d\tau \delta(t-\tau)\left\{\tau^{-\beta}\Psi(\tau)\right\}$$

$$= -\Omega_0 t^{-\beta}\Psi(t)$$

$$\boxed{\frac{d\Psi(t)}{dt} = -\Omega_0 t^{-\beta}\Psi(t)}$$

Relaxation with intermittency and cluster heterogeneity $\Psi(t,\phi,\beta)$

The relaxation process $\Psi(t,\phi,\beta)$ will be Hybrid of Mittag-Leffler and Stretched Exponential with new Rate and Memory Kernel describing the process:

$$R_{\phi,\beta}(t,\tau) = \Omega_0 \tau^{-\beta}(t-\tau)^{\phi-1}/\Gamma(\phi) \qquad \phi > 0 ;\ \beta \leq 1$$

$$\Psi(t;\phi,\beta) = 1 - \int_0^t d\tau\, R_{\phi,\beta}(t,\tau)\Psi(\tau;\phi,\beta)$$

$$\Psi(t;\phi,\beta) = 1 - \Omega_0 \int_0^t d\tau \left[(t-\tau)^{\phi-1}\tau^{-\beta}/\Gamma(\phi)\right]\Psi(\tau;\phi,\beta)$$

Erdelyl-Kober Fractional Operator:

$$I_t^{p,q} f(t) = t^{-(p+q)} \int_0^t d\tau \left[(t-\tau)^{q-1}\tau^{p}/\Gamma(q)\right] f(\tau)$$

Relation to Riemann-Liouvelli Operator is:

$$I_t^{p,q} f(t) = t^{-(p+q)} I_t^{q}\left[t^{p} f(t)\right]$$

Solution to Relaxation Equation with intermittency & clustering:

$$\Psi(t;\phi,\beta)=\sum_{k=0}^{\infty} a_k(\phi,\beta)\left[z_\Omega(\phi,\beta)\right]^k \; ; \; a_0(\phi,\beta)=1$$

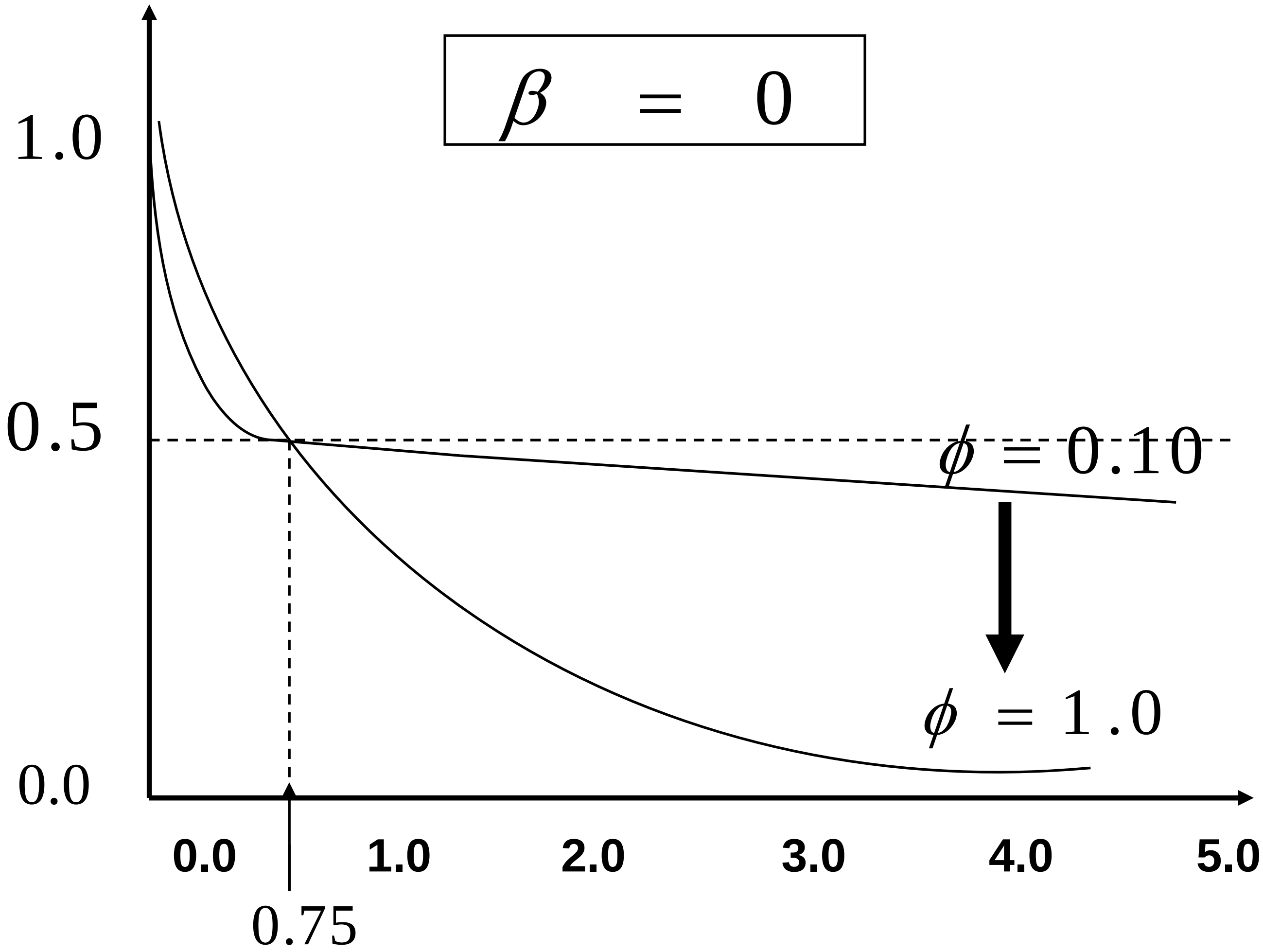
$$a_k(\phi,\beta)=\prod_{m=1}^{k}\frac{\Gamma(1+m\hat{\phi}-\phi)}{\Gamma(1+m\hat{\phi})}; \; k>0$$

$$\hat{\phi}=\phi-\beta \; ; \; 0<\hat{\phi},\phi \; ; \; \beta\le 1$$

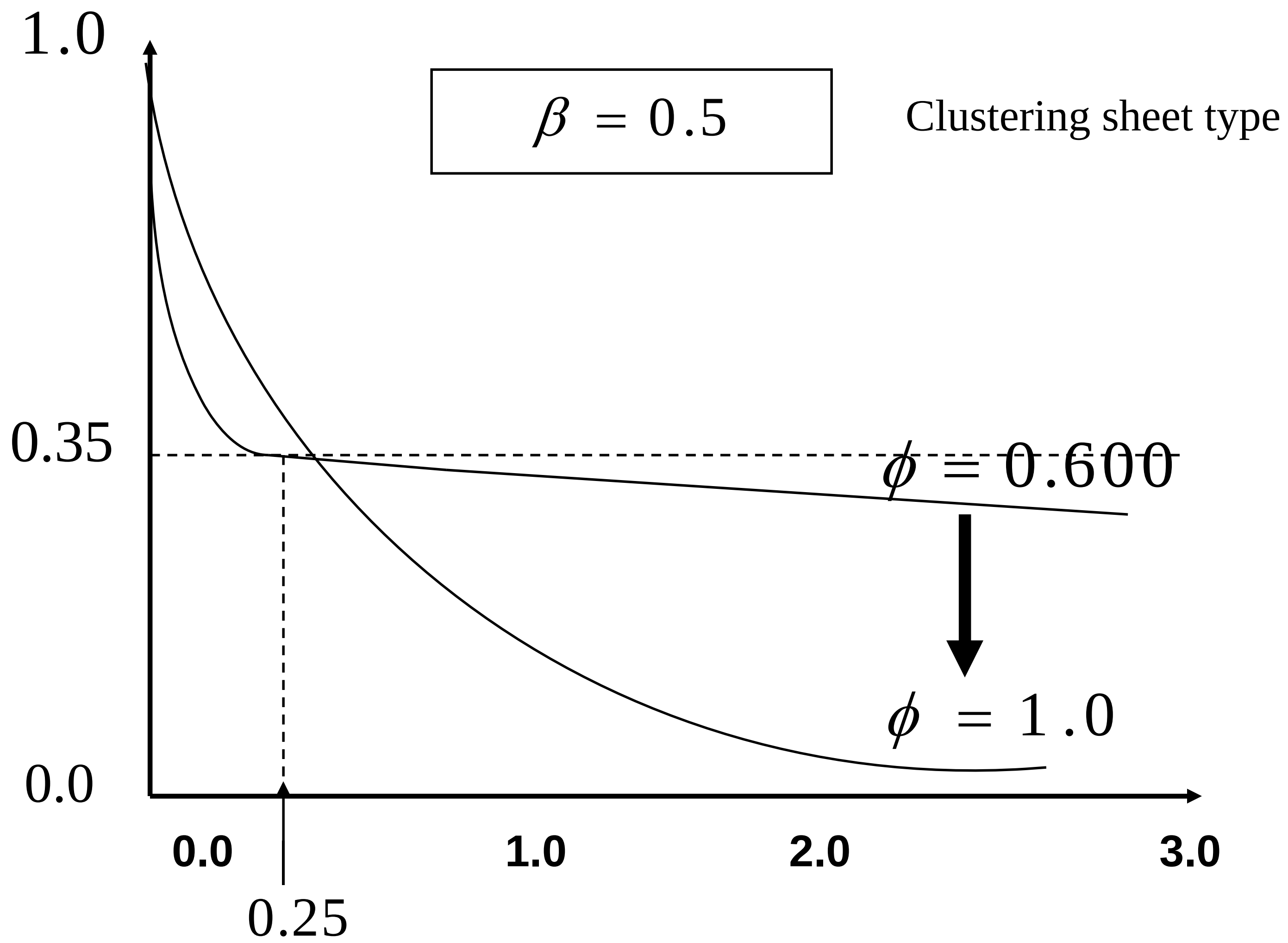
$$z_\Omega(\phi,\beta)=\Omega_0\tau^{\hat{\phi}}=\left(t/\tau_*\right)^{\hat{\phi}} \; ; \; \tau^*(\phi,\beta)=\Omega_0^{-1/\hat{\phi}}$$

Mittag-Leffler for $\beta=0$ and stretched exponential for $\phi=1$ otherwise hybrid of Mittag-Leffler & Stretched exponential.

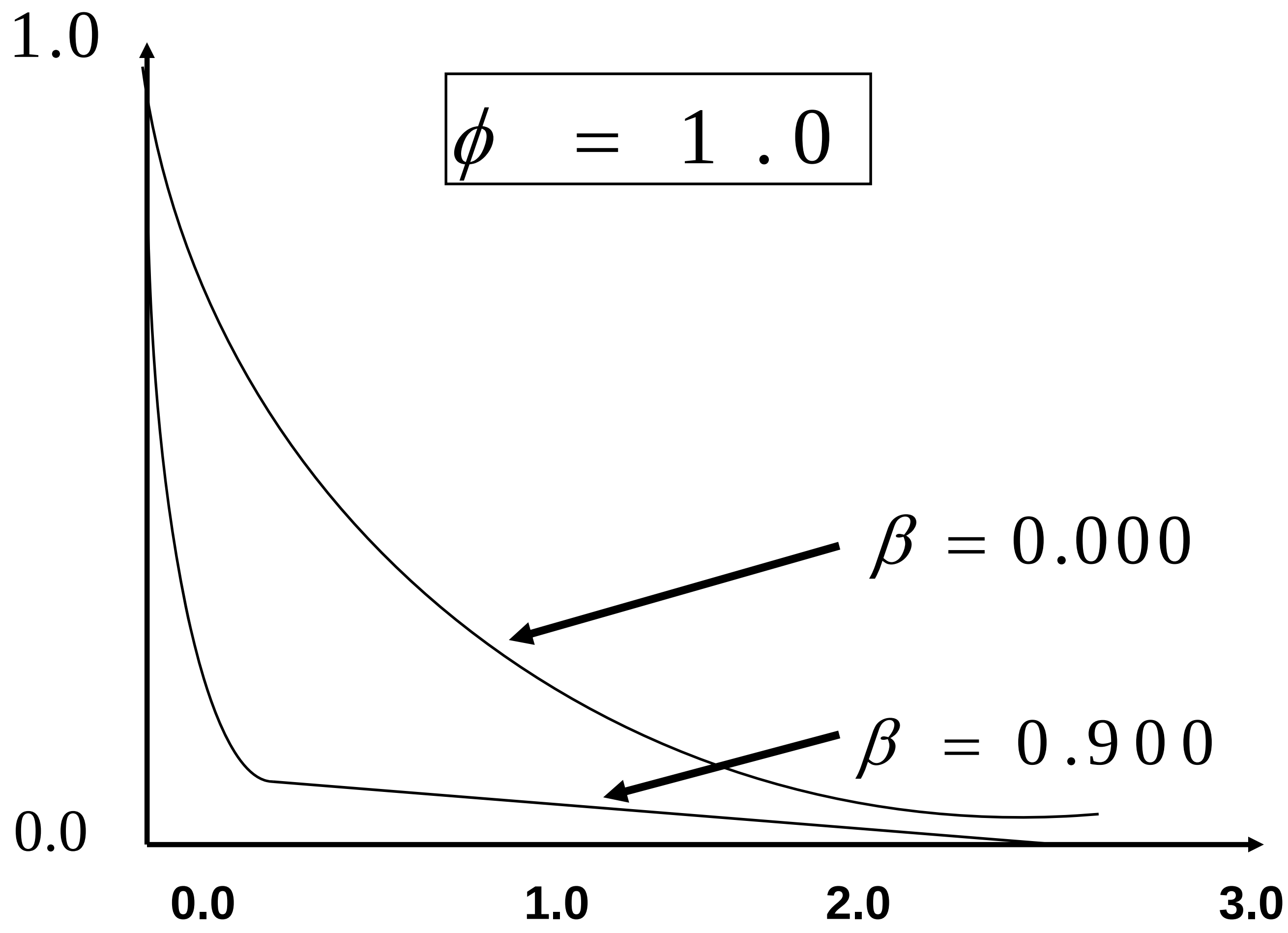
Plots of relaxation curves weak relaxation without clustering (ML)



Plots of relaxation curves Hybrids of ML & Stretched Exponential



Plots of relaxation curves Strong Relaxation with Clustering:



$\Psi(t;\phi \to 1, \beta)$ Is stretched exponential relaxation Kohrash-William's Watts law widely used to correlate relaxation data in complex liquids.

Epilogue

Perhaps the XXI century will try to think more exact and extend this exact thinking by the paradox which appeared to Leibniz, three hundred years ago by………..

………………speaking to the Nature by a language which it understands the best that is the Fractional Calculus.

There are perhaps easier ways too

and Fractional Calculus can be further generalized to Complex Order Differintegrations

..some natural laws must have this context