SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS IN FORMAL WAY

MATHEMATICO-PHYSICS OF GENERALIZED CALCULUS

Module IV

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Demonstration to Solve Fractional Differential Equation-Numerically

Obtaining the Step response

$$a\frac{D^n}{Dt^n}y(t)+by(t)=u(t) \qquad 0<n<1$$

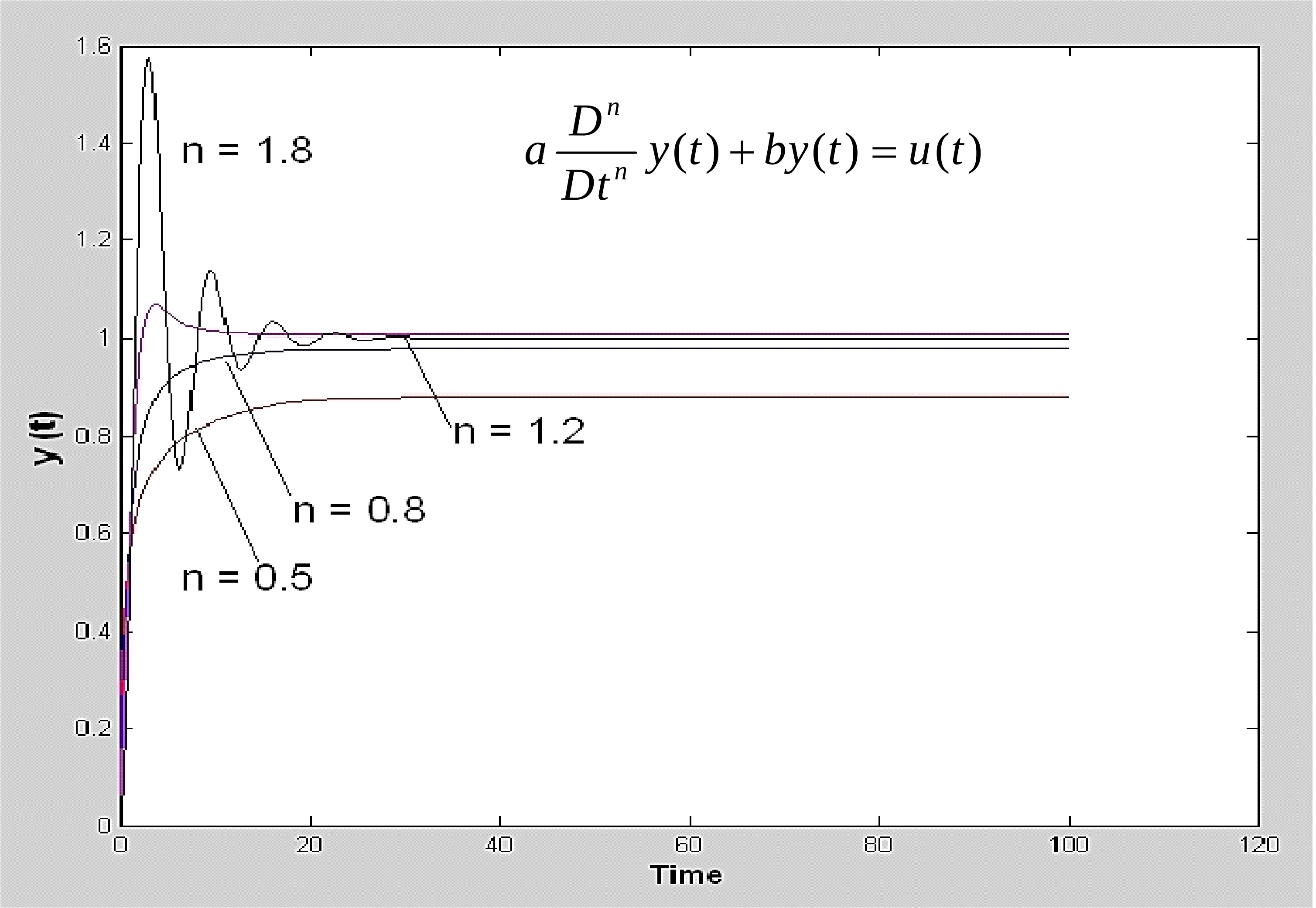
Equation above is **Grunwald-Letnikov** discretized directly gives the solution for a simple *n*-th order FO system; like **FIR Realization**

Step response of the system represented by above equation (Tracking Filter) is studied for various values of *n*.

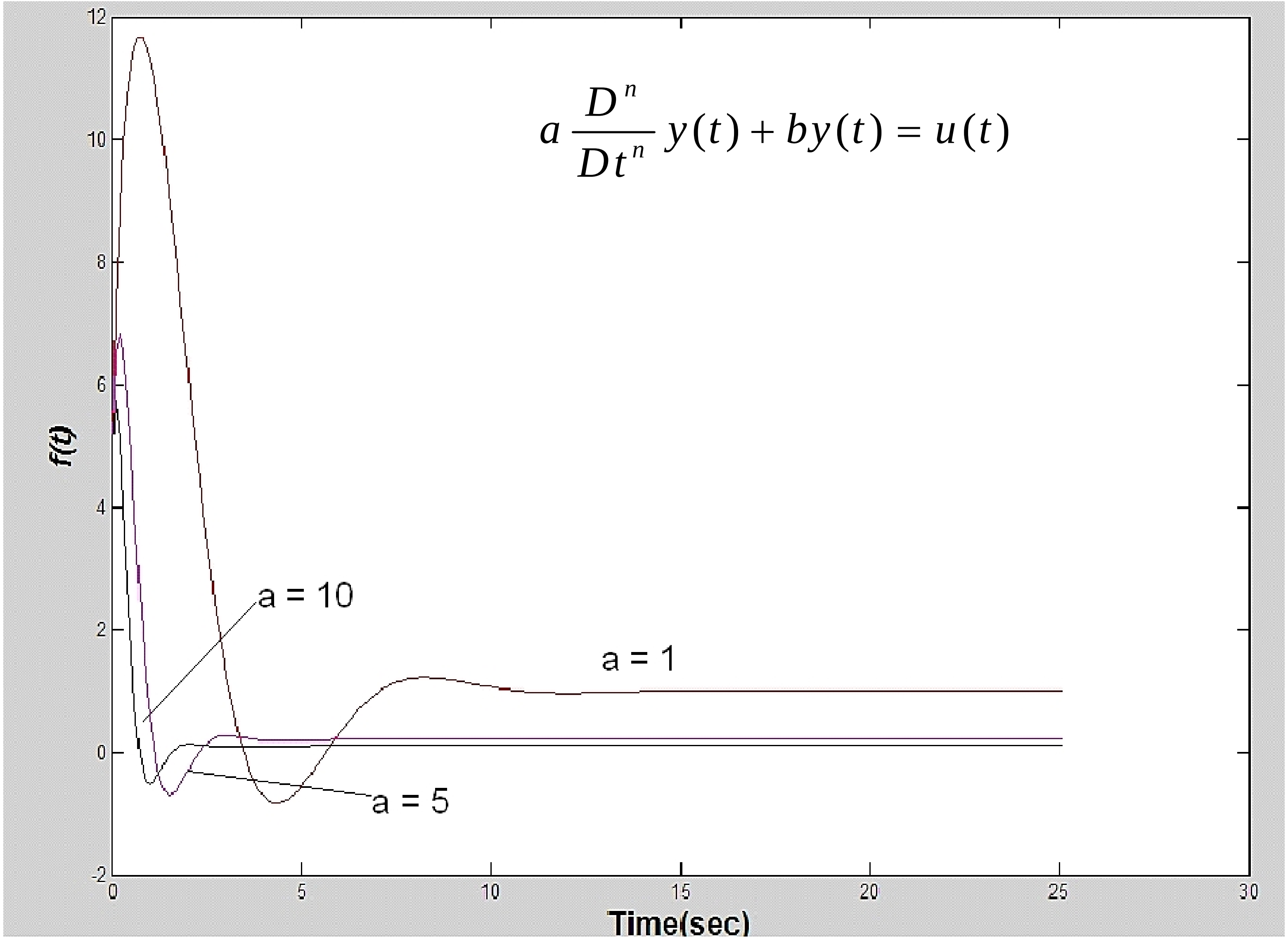
$$y(K\Delta T)=\frac{u(K\Delta T)-a\dfrac{1}{\Delta T^n}\displaystyle\sum_{j=0}^{m}(-1)^j\frac{\Gamma(n+1)}{\Gamma(n-j+1)\Gamma(j+1)}y(K\Delta T-j\Delta T)}{b}$$

For all the plots, the total number of points taken for evaluation is 1000.

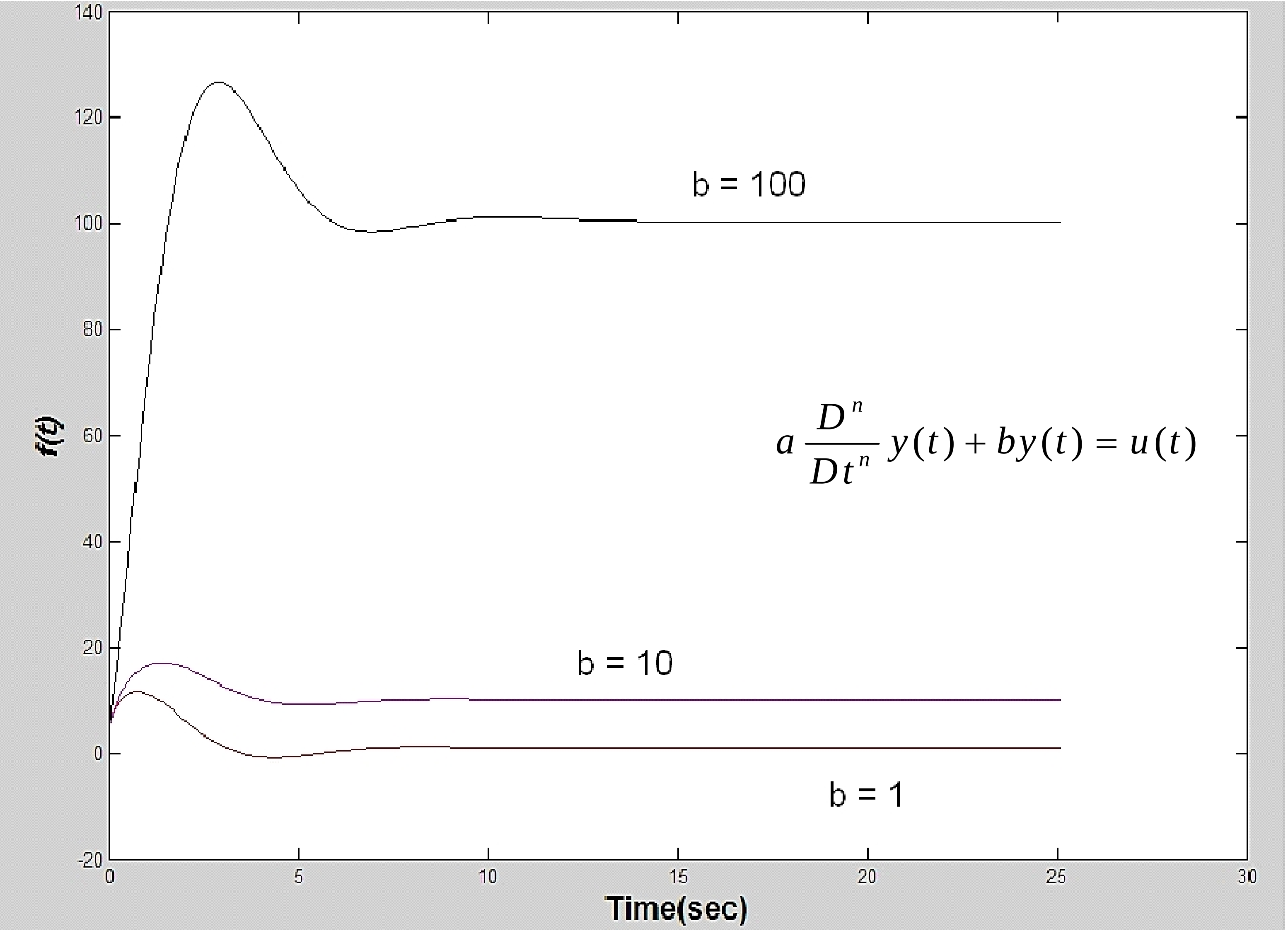
Result: Step response of the system for different values of n using a=b=1 and y (0) =0



Step response for different values of *a*, depicting increased damping for greater values of *a*.

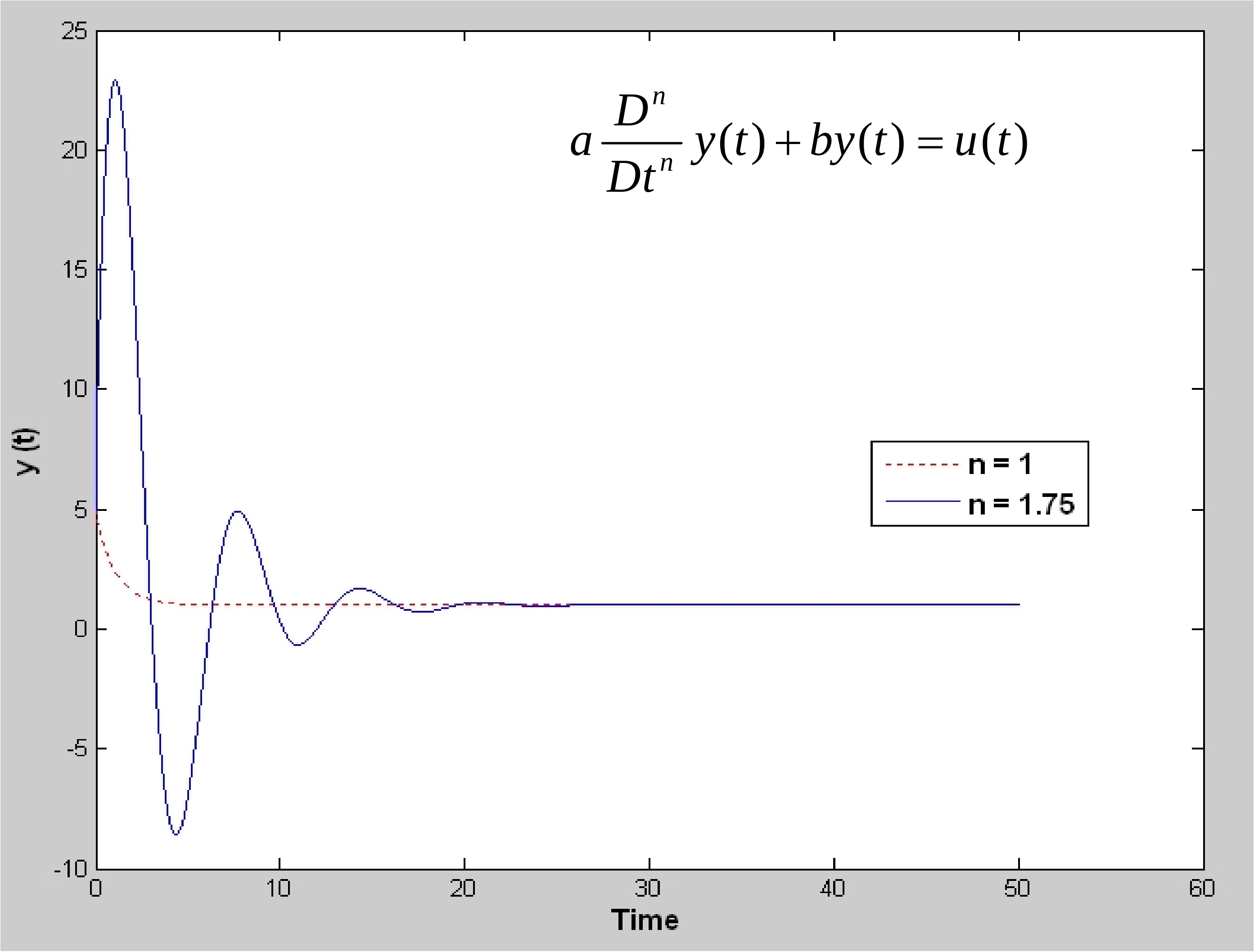


Step response for different values of parameter *b*.



$$a \frac{D^n}{Dt^n} y(t) + by(t) = u(t)$$

Effect of initial conditions on a system with
n **=1, 1.75 for the step response**



Fractional Differential Equation:

Solution finding to such an ODE in general is not an easy task. FDE are as rigorous as ODE

$$x^2 D^2 y(x) + xDy(x) + (x^2 - k^2)y(x) = 0; x \geq 0$$

Is Bessel's equation requires substantial effort to solve

In fact the only class of equations for which we can find an explicit solution is class of LDE with constant coefficients (or reducible to this form)

$$D^2 y(t) + aDy(t) + by(t) = 0$$ *a* & *b* are constants.

If α, β are distinct zeros (roots) of the Indicial Polynomial: $P(x) = x^2 + ax + b$

Then $\exp(\alpha t); \exp(\beta t)$ are fundamental solutions.

While if $\alpha = \beta$ then $\exp(\alpha t); t\exp(\alpha t)$ are fundamental solutions.

Note $\exp(\alpha t) = E_t(0, \alpha)$

$\exp(\beta t) = E_t(0, \beta)$

Formulating Fractional Order Differential Equation (Formally)

$$[D^{r_n} + b_1 D^{r_{n-1}} + b_2 D^{r_{n-2}} + \dots\dots\dots\dots + b_n D^0]\, y(t) = 0$$

$r_n, r_{n-1}, r_{n-2}, \dots\dots, r_0$ is sequence of decreasing non-negative rational number

$b_n, b_{n-1}, b_{n-2}, \dots\dots, b_0$ are constants

Let q be the least common multiplier of the of nonzero r_j

$$[D^{r_n} + b_1 D^{r_{n-1}} + b_2 D^{r_{n-2}} + \dots\dots\dots\dots + b_n D^0]\, y(t) = 0$$

$$[D^{nv} + b_1 D^{(n-1)v} + b_2 D^{(n-2)v} + \dots\dots + b_n D^0]\, y(t) = 0$$

Where $q = 1/v$ is Fractional Differential Equation of order (n , q)

(n , q)=(1,1) is First order LDE

(n , q)=(2,1) is Second order LDE

(n , q)=(4,3) is Fractional Differential Equation $D^{4v}[y(t)] = D^{4/3}[y(t)] = 0$

Indicial Polynomial is:

$$P(x) = x^n + b_1 x^{n-1} + \dots\dots + b_0$$

Fractional Differential Operator is: $P(D^v) = D^{nv} + b_1 D^{(n-1)v} + \dots\dots + b_0 D^0$

(Homogenous) Fractional Diff. Equation is:

$$P(D^v)[y(t)] = 0$$

Number of Solutions

$$D_t^{4/3}[y(t)] = 0$$

Shall it have 4/3=1.3333 number of solutions?

Try $y(t) = t^{1/3}$

$$D^{4/3}\left[t^{1/3}\right] = \frac{\Gamma\left(\frac{1}{3}+1\right)t^{\frac{1}{3}-\frac{4}{3}}}{\Gamma\left(\frac{1}{3}-\frac{4}{3}+1\right)} = \frac{\Gamma\left(\frac{4}{3}\right)t^{-1}}{\Gamma(0)} = 0$$

Therefore $y_1(t) = t^{1/3}$ is one solution

The second solution, can it be: $y_2(t) = D^{(1)}t^{1/3} \equiv t^{\left(\frac{1}{3}-1\right)} = t^{-2/3}$

Let us try this

$$y_2(t) = t^{-2/3}$$

$$D^{4/3}\left[t^{-2/3}\right] = \frac{\Gamma\left(-\frac{2}{3}+1\right)t^{\frac{2}{3}-\frac{4}{3}}}{\Gamma\left(-\frac{2}{3}-\frac{4}{3}+1\right)} = \frac{\Gamma\left(\frac{1}{3}\right)t^{-2}}{\Gamma(-1)} = 0$$

$y(t) = C_1 t^{1/3} + C_2 t^{-2/3}$ is solution of (4,3) order FDE $D^{4/3}[y(t)] = 0$

Similarity with ODE number of independent solutions:

Equation	$y_1(t)$	$y_2(t)$	$y(t)$	Number of Solutions
$D^{(1)}y(t)=0$	$Ct^0 = C$	Nil	C	one
$D^{(2)}y(t)=0$	C_1t^1	$C_2t^0 = C_2$	$C_1t + C_2$	two
$D^{(3)}y(t)=0$			$C_1t^2 + C_2t + C_3$	three
$D^{(4/3)}y(t)=0$	$C_1t^{1/3}$	$C_2t^{-2/3}$	$C_1t^{1/3} + C_2t^{-2/3}$	two

nv=(4)(1/3)=1.33, The nearest integer greater than nv is N=2. Thus (4,3) FDE is having two linearly independent solutions (not 1.333..). Observation is that : $y_1(0)=0$ but $y_2(0)=\infty$ still a solution.

For

$N = nv;\ y_1(0) = ..y_N(0) = 0(C)$

$N > nv;\ y_1(0) = y_2(0) = \ldots = y_{N-1}(0) = 0(C)$

$y_N(0) = \infty$

Finding solution of homogeneous FDE

Direct Method

Laplace Transform Method

Useful tool is 'Green's Function' (Fractional Green's Function)-to get unique solution to Non-Homogeneous Fractional Differential Equation

Ordinary Differential Equation(n,1)

$$[D^n + a_1 D^{n-1} + \ldots + a_n D^0] y(t) = 0$$

$$P(x) = x^n + a_1 x + \ldots + a_n$$

$$D \equiv [D^n + a_1 D^{n-1} + \ldots + a_n D^0]$$

Try

$$P(D) y(t) = 0$$

$$y(t) = e^{ct}$$

$$P(D) e^{ct} = (c^n + a_1 c^{n-1} + \ldots + a_n) e^{ct} = P(c) e^{ct}$$

If 'c' is root (zero) of the indicial polynomial then $y(t) = \exp ct$ is one solution

$D^n e^{ct} = c^n e^{ct}$ integer order differentiation returns the function itself, keeping the functional form same. For fractional differentiation $D^u e^{ct} = E_t(-u, c)$

The form alters thus exponential function is not suitable candidate.

$D^u E_t(w, c) = E_t(w - u, c)$ Here the form remains same thus possible one is Miller-Ross function (variant of Mittag-Leffler), a candidate for FDE solution.

Miller-Ross function a possible choice to solve FDE

$$D^u E_t(w,c) = E_t(w-u,c)$$

$$D^u tE_t(w,c) = tE_t(w-u,c) + uE_t(w-u+1,c)$$

$$D \exp ct = c \exp ct$$

$$D[t \exp ct] = ct \exp ct + \exp ct$$

$$E_t(0,c) = \exp ct$$

$$D^u E_t(0,c) = E_t(-u,c)$$

Thus we can possibly try functions of $E_t(kv,c)$ where 'k' is integer as candidate of solution of FDE

Miller-Ross
$$E_t(w,c) = t^w \sum_{n=0}^{\infty} \frac{(ct)^n}{\Gamma(1+n+w)}$$

Mittag-Leffler
$$E_w(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(1+nw)}$$

$$D^{w-1}[D E_w(ct^w)] = cE_w(ct^w)$$

$$E_w(ct^w) = \sum_{k=0}^{\frac{1}{w}-1} c^k E_t(kw, c^{1/w})$$

Mittag-Leffler as finite sum of Miller-Ross function

Direct approach

Consider

$$[D + aD^{1/2} + b]y(t) = 0; n = 2, q = 2, v = 1/2$$

$$[D^{2\times 1/2} + aD^{1\times 1/2} + b]y(t) = 0; (n, q) \equiv (2, 2)$$

$$P(x) = x^2 + ax + b$$

For ODE (2,1), with v=1 the indicial polynomial is same for FDE (2,2), and solution, of ODE (2,1) are

$e^{ct}, D^{v=1}e^{ct} = ce^{ct}$ For FDE v=1/2 we can try $e^{ct} = E_t(0,c); D^{v=1/2}e^{ct} = D^{1/2}E_t(0,c) = E_t(-1/2, c)$

Let try out

$$\psi_1(t) = AE_t(0,c) + E_t(-1/2, c)$$

$$P(D^{1/2}) \equiv D^1 + aD^{1/2} + bD^0$$

$$P(D^{1/2})\psi_1 = (cA + ac + bA)E_t(0,c) + (c + aA + b)E_t(-1/2, c) + \frac{t^{-3/2}}{\Gamma(-1/2)}$$

Let $A = \lambda; c = \lambda^2$ then

$$P(D^{1/2})\psi_1(t) = \lambda P(\lambda)E_t(0,\lambda^2) + P(\lambda)E_t(-1/2, \lambda^2) + \frac{t^{-3/2}}{\Gamma(-1/2)}$$

If λ is root of indicial polynomial $P(x)$ then $P(\lambda) = 0$

$$P(D^{1/2})\psi_1(t) = \frac{t^{-3/2}}{\Gamma(-1/2)} \neq 0$$

Therefore ψ_1 is not a solution!

What is the solution of the FDE?

If α and β are roots of the indicial polynomial $P(x) = x^2 + ax + b$

$$\psi_1 = \alpha E_t(0, \alpha^2) + E_t(-1/2, \alpha^2)$$

$$\psi_2 = \beta E_t(0, \beta^2) + E_t(-1/2, \beta^2)$$

$$P(D^{1/2})\psi_2 = \frac{t^{-3/2}}{\Gamma(-1/2)} = P(D^{1/2})\psi_1$$

Thus we let: $\psi = \psi_1 - \psi_2$ then ψ is solution

The solution of the FDE is:

$$\psi(t) = \alpha E_t(0, \alpha^2) - \beta E_t(0, \beta^2) + E_t(-1/2, \alpha^2) - E_t(-1/2, \beta^2)$$

$$\psi(t) = \alpha e^{\alpha^2 t} \mathrm{erfc}(-\alpha\sqrt{t}) - \beta e^{\beta^2 t} \mathrm{erfc}(-\beta\sqrt{t})$$

Observations

$$\psi(0) = \alpha - \beta;\ D^{-1/2}\psi(0) = 0;\ D^{1/2}\psi(0) = \infty;\ D\psi(0) = \infty$$

Here n=1,and v=1/2. The N=1=nv only one solution

Summary

The Miller-Ross Function was used for direct method described

The property of Miller-Ross derivative and its algebraic manipulations are used as:

$$DE_t(v, a) = E_t(v-1, a)$$

$$E_t(v-1, a) = aE_t(v, a) + \frac{t^{v-1}}{\Gamma(v)}$$

$$DE_t(-1/2, a) = aE_t(-1/2, a) + \frac{t^{-3/2}}{\Gamma(-1/2)}$$

The indicial polynomial for (2,1) ODE is though same as for FDE (2,2). If the root in case of ODE is λ the FDE has root for the same indicial as λ^{v}, $\sqrt[q]{\lambda}$

The solution to FDE with direct method yielded similarity with ODE, the basis function is off course higher transcendental functions (Generalized exponential Generalized Trigonometric, Generalized hyperbolic)-which shall be playing role to solve Fractional Differential Equations.

Motivation Laplace Technique:

$$[D + aD^{1/2} + bD^0]y(t) = 0$$

$$[sY(s) - D^{(1-1)}y(0)] + a[\mathcal{L}\{D^{1/2}y(t)\}] + bY(s) = 0$$

$$[sY(s) - y(0)] + a[s^{1/2} - D^{-1/2}y(0)] + bY(s) = 0$$

$$\mathcal{L}\{D^{1/2}y(t)\} = s^{1/2}Y(s) - D^{1/2-1}y(0)$$

$$[s + as^{1/2} + b]Y(s) - y(0) - aD^{-1/2}y(0) = 0$$

$$Y(s) = \frac{C}{s + as^{1/2} + b} = \frac{C}{P(s^{1/2})}$$

$$P(x) = x^2 + ax + b$$

$$C = y(0) + aD^{-1/2}[y(0)]$$

1.How do we know $y(0)\,\&\,D^{-1/2}[y(0)]$ is finite? If 'C' is not finite the problem is serious and this approach is meaningless. Thus assume 'C' is finite.

2.How do we find inverse Laplace of $Y(s)$? $y(t) = \mathcal{L}^{-1}\{P^{-1}(\sqrt{s}\}$

Generalization of Partial Fractions:

Expand $P^{-1}(x)$ with Partial Fractions, with α and β roots (zeros) of $P(x)$

$$P^{-1}(x) \equiv \frac{1}{P(x)} = \frac{1}{x^2 + ax + b} = \frac{1}{\alpha - \beta}\left(\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right)$$

Put $x = s^{1/2}$

$$P^{-1}(s^{1/2}) \equiv \frac{1}{P(s^{1/2})} = \frac{1}{s + a\sqrt{s} + b} = \frac{1}{\alpha - \beta}\left(\frac{1}{s^{1/2}-\alpha} - \frac{1}{s^{1/2}-\beta}\right)$$

Robotnov-Hartley $$\mathcal{L}^{-1}\left\{(s^q - a)^{-1}\right\} = F_q(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma(\{n+1\}q)}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{C}{(\alpha - \beta)}\left[F_{1/2}(\alpha, t) - F_{1/2}(\beta, t)\right]$$

Is this solution same as with Miller-Ross, done with direct method?

Solution with Miller-Ross function:

$$\left\{ \frac{1}{s^{1/2} - \alpha} \right\} = \frac{1}{s^{-1/2}(s - \alpha^2)} + \frac{\alpha}{(s - \alpha^2)}$$

$$\mathcal{L}^{-1}\{(s^{1/2} - \alpha)^{-1}\} = E_t(-1/2, \alpha^2) + \alpha E_t(0, \alpha^2)$$

Using this expression we get:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{C P^{-1}(s^{1/2})\}$$

$$y(t) = \frac{C}{\alpha - \beta}\left[\alpha E_t(0, \alpha^2) - \beta E_t(0, \beta^2) + E_t(-1/2, \alpha^2) - E_t(-1/2, \beta^2)\right]$$

The solution with direct method $\psi(t)$ is multiplied by a constant $C/(\alpha - \beta)$ to get the solution with this way by Laplace, for unequal roots of Indicial Polynomial.

Roots are equal $\alpha = \beta$ for indicial polynomial $P(x)$

Then
$$Y(s) = \frac{C}{P(s^{1/2})} = \frac{C}{(s^{1/2} - \alpha)^2}$$

$$\frac{1}{(s^{1/2}-\alpha)^2} = \left[\frac{1}{s^{-1/2}(s-\alpha^2)} + \frac{\alpha}{(s-\alpha^2)}\right]^2 = \left[\frac{1}{s^{-1}(s-\alpha^2)^2} + \frac{\alpha^2}{(s-\alpha^2)^2} + 2\frac{\alpha}{s^{-1/2}(s-\alpha^2)^2}\right]$$

Use the Miller-Ross property:
$$\mathcal{L}^{-1}\left\{s^{-v}(s-a)^{-2}\right\} = tE_t(v,a) - vE_t(v+1,a)$$

to get
$$\mathcal{L}^{-1}\left\{(s^{1/2}-\alpha)^{-2}\right\} = tE_t(-1,\alpha^2) + E_t(-1+1,\alpha^2) + \alpha^2 t^0 E_t(0,\alpha^2) - \alpha^2(0)E_t(0+1,\alpha^2)$$
$$+2\alpha tE_t(-1/2,\alpha^2) + 2\alpha\frac{1}{2}E_t(-1/2+1,\alpha^2)$$

Use $E_t(-1,a) = aE_t(0,a)$,simplify to get:
$$y(t) = C[(1+2\alpha^2 t)E_t(0,\alpha^2) + \alpha E_t(1/2,\alpha^2) + 2\alpha tE_t(-1/2,\alpha^2)]$$

Also be it:
$$y(t) = C[F_{1/2}(\alpha,t) * F_{1/2}(\alpha,t)]$$

Motivation to get Linearly Independent Solutions to FDE

ODE has set of linearly independent solutions, from which Green's function is obtained via Wronskian determinant. This Green's function is solution of the homogeneous ODE and is utilized to get solution of non-homogeneous ODE

ODE $[D^2 y(t) + aDy(t) + by(t)] = 0$

$$P(x) = x^2 + ax + b$$

Let a zero of this indicial polynomial be α

$y_1(t) = e^{\alpha t}$ is a solution $Dy_1(t) = \alpha e^{\alpha t}$ is also a solution, as well as other higher derivatives. But they are not linearly independent

If β is other zero (root) of the indicial polynomial then $y_2(t) = e^{\beta t}$ is other solution and its higher derivatives, again not linearly independent.

The combination $y(t) = y_1(t) + y_2(t) = e^{\alpha t} + e^{\beta t}$ is solution. Here $y(t)$ and $Dy(t) = \alpha e^{\alpha t} + \beta e^{\beta t}$ are obviously solution and too linearly independent.

Fundamental solution corresponding to roots

$$y_1(t) = \frac{\beta y(t) - Dy(t)}{\beta - \alpha}$$

$$y_2(t) = \frac{Dy(t) - \alpha y(t)}{\beta - \alpha}$$

For $\alpha = \beta$ $\quad y_2(t) = te^{\alpha t}$

Linearly Independent Solutions of FDE:

$$[D^{3/2} - 2D - D^{1/2} + 2D^{0}]y(t) = 0; n = 3, v = 1/2\ q = 2,$$

$$[s^{3/2}Y(s) - D^{1/2}y(0) - sD^{-1/2}y(0)] - 2[sY(s) - y(0)] -$$

$$-[s^{1/2}Y(s) - D^{-1/2}y(0)] + 2[Y(s)] = 0$$

$$Y(s) = \frac{A}{P(s^{1/2})} + \frac{Bs}{P(s^{1/2})}$$

$$P(x) = x^{3} - 2x^{2} - x + 2 = (x+2)(x-1)^{2}$$

$$y(t) = A\mathcal{L}^{-1}\{P^{-1}(s^{1/2})\} + B\mathcal{L}^{-1}\{sP^{-1}(s^{1/2})\}$$

$$y_1(t) = \mathcal{L}^{-1}\{P^{-1}(s^{1/2})\};\ y_2(t) = \mathcal{L}^{-1}\{sP^{-1}(s^{1/2})\} = Dy_1(t)$$

$y_1(t) = \frac{1}{3}\left[-E_t(1/2,1) + 4E_t(1/2,4) - 2E_t(0,1) + 2E_t(0,4)\right]$ is a solution & $y_1(0) = 0$

$$y_2(t) = Dy_1(t) = \frac{1}{3}\left[-E_t(1/2,1) + 16E_t(1/2,4) - 2E_t(0,1) + 8E_t(0,4) + \frac{t^{-1/2}}{\sqrt{\pi}}\right]$$

$y_2(0) = \infty$ is yet a solution. Here (N=2) for (3,2) order FDE. nv=1.5 thus N=2, both these are linearly independent one

$\psi(t) = C_1 y_1(t) + C_2 y_2(t)$ is solution of this FDE

Explicit solution for homogeneous Fractional Differential equation (FDE)

FDE of order $(n, q); q = 1/v$ be the following:

$$[D^{nv} + a_1 D^{(n-1)v} + \ldots\ldots + a_n D^0]\, y(t) = 0$$

Indicial Polynomial $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$

Fundamental solution $y_1(t) = \mathcal{L}^{-1}\left\{ \dfrac{1}{P(s^v)} \right\}$

Integer N greater or equal to nv $\quad N - 1 < nv \le N$

"N" linearly independent solutions are $y_1(t), y_2(t), \ldots\ldots, y_j(t), \ldots. y_N(t)$

$$y_{j+1}(t) = D^j y_1(t);\ j = 0, 1, 2, \ldots\ldots\ldots N - 1$$

Explicit solution example:

$$(2, q); q = 1/v$$

$$[D^{2v} + a_1 D^{v} + a_2]y(t) = 0$$

$$P(x) = x^2 + a_1 x + a_2 = (x - \alpha_1)(x - \alpha_2)$$

$$y_1(t) = \mathcal{L}^{-1}\{P^{-1}(s^{v})\}$$

$$\alpha_1 \neq \alpha_2$$

$$P^{-1}(s^{v}) = \frac{1}{(s^{v} - \alpha_1)(s^{v} - \alpha_2)} = \frac{1}{\alpha_1 - \alpha_2}\left\{\frac{1}{s^{v} - \alpha_1} - \frac{1}{s^{v} - \alpha_2}\right\}$$

$$\alpha_1 = \alpha_2$$

$$P^{-1}(s^{v}) = \frac{1}{(s^{v} - \alpha_1)^2}$$

let

$$F_v(\alpha_i, t) = e_i(t) = \sum_{k=0}^{q-1} \alpha_i^{q-k-1} E_t(-kv, \alpha_i^2); i = 1, 2$$

$$y_1(t) = A[e_1(t) - e_2(t)]; \alpha_1 \neq \alpha_2; A = (\alpha_2 - \alpha_1)^{-1}$$

$$y_1(t) = A[e_1(t) * e_1(t)]; \alpha_1 = \alpha_2$$

$$y_1(0) = y_2(0) = \ldots\ldots\ldots y_{N-1}(0) = 0, y_N(0) = 1(C); N = nv$$

$$N \geq nv; y_N(0) = \infty$$

Explicit solution

$$(3,q); q = 1/v$$

$$[D^{3v} + a_1 D^{2v} + a_2 D^{v} + a_3 D^{0}]y(t) = 0$$

$$P(x) = x^3 + a_1 x^2 + a_2 x + a_3 = (x - \alpha_1)(x - \alpha_2)^2$$

$$\frac{1}{P(s^v)} = \frac{B_1}{(s^v - \alpha_1)} + \frac{B_2}{(s^v - \alpha_2)} + \frac{C_1}{(s^v - \alpha_2)^2}$$

$$B_1 = (\alpha_2 - \alpha_1)^{-2}; B_2 = -(\alpha_2 - \alpha_1)^{-2}; C_2 = (\alpha_2 - \alpha_1)^{-1}$$

$$y_1(t) = B_1 e_1(t) + B_2 e_2(t) + C_1[e_2(t) * e_2(t)]$$

$$e_i(t) = \mathcal{L}^{-1}\{(s^v - \alpha_i)^{-1}\} = \sum_{k=0}^{q-1} \alpha_i^{q-k-1} E_t(-kv, \alpha_i^2) = F_v(\alpha_i, t)$$

$$e_i(t) * e_i(t) = \mathcal{L}^{-1}\{(s^v - \alpha_i)^{-2}\} = \sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \alpha_i^{2q-j-k-2}\{tE_t(-[j+k]v, \alpha_i^q) + (j+k)vE_t(1-[j+k]v, \alpha_i^q)\}$$

$$D^v e(t) = \alpha e(t)$$

$$D^{2v} e(t) = \alpha^2 e(t) + \frac{t^{-1-v}}{\Gamma(-v)}$$

Non Homogeneous Fractional Differential Equation Solution

$$[D^{nv} + a_1 D^{(n-1)v} + a_2 D^{(n-2)v} + \ldots\ldots + a_n D^0]\, y(t) = x(t)$$

$$D^j y(0) = 0;\; j = 0, 1, 2, \ldots\ldots\ldots\ldots (N-1)$$

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_n$$

Green's function $K(t) = \mathcal{L}\{P^{-1}(s^v)\}$

$$y(t) = \int_0^t K(t-\xi)\, x(\xi)\, d\xi$$

$$[D^{2v} - aD^v]\, y(t) = x(t)$$

$$(n, q) \equiv (2, 4);\, v = 1/4,\, q = 4;\, y(0) = 0$$

$$P(x) = x^2 - ax = x(x-a)$$

$$K(t) = \mathcal{L}^{-1}\{P^{-1}(s^v)\} = \mathcal{L}^{-1}\{s^{-v}(s^v - a)^{-1}\} = \sum_{j=0}^{3} a^j E_t[(j-2)v, a^4]$$

$$y(t) = \int_0^t K(t-\xi)\, x(\xi)\, d\xi$$

Non-Homogeneous solution-example

$$[D^{2v} - aD^{v}]y(t) = x(t)$$

$$y(0) = 0; N = 1 \geq nv = 1/2; n = 2, v = 1/4, q = 2$$

$$x(t) = \sin bt; X(s) = b(s^2 + b^2)^{-1}$$

$$Y(s) = \frac{X(s)}{P(s^v)} = \frac{b}{s^v(s^v - a)(s^2 + b^2)}$$

$$y(t) = \frac{1}{a^8 + b^2}\sum_{k=1}^{4} a^{k-1}[bE_t((k+1)v - 1, a^4) - bC_t((k+1)v - 1, b) - a^4 S_t((k+1)v - 1, b)]$$

$$y(0) = 0 = y^{N-1}(0); N = 1$$

$$[D^{6v} + D^{v}]y(t) = x(t); v = 1/6, n = 6; P(x) = x(x^5 + 1)$$

$$x(t) = A\left[55t + \frac{36}{\Gamma(5v)}t^{11v}\right]$$

$$X(s) = \frac{55A}{s^2}\left(1 - \frac{1}{s^{5v}}\right)$$

$$Y(s) = \frac{X(s)}{P(s^v)} = \frac{55A(s^{5v} - 1)}{s^{5v+2}} = \frac{55A}{s^3}$$

$$y(t) = \frac{55A}{2}t^2$$

Fractional Integral Equations and its solution:

$m, q > 0$ integers and $v = 1/q$ a positive fraction. Then

$$[D^0 + b_1 D^{-v} + b_2 D^{-2v} + \dots\dots\dots\dots + b_m D^{-mv}]\, y(t) = x(t)$$

Where the forcing function at RHS is of exponential order and the Laplace of above equation is:

$$[1 + b_1 s^{-v} + b_2 s^{-2v} + \dots\dots\dots\dots + b_m s^{-mv}]\, Y(s) = X(s)$$

$$[s^{mv} + b_1 s^{(m-1)v} + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots + b_m]\, Y(s) = s^{mv} X(s)$$

$$Y(s) = \frac{s^{mv} X(s)}{R(s^v)}$$

$$R(x) = x^m + b_1 x^{m-1} + b_2 x^{m-2} + \dots\dots\dots + b_m$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s^{mv} X(s)}{R(s^v)} \right\}$$

Fractional Integral Equation Explicit Solution:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s^{mv}X(s)}{R(s^{v})}\right\}$$

If we wish to be more explicit and express $y(t)$ in terms of $x(t)$ we should make further a assumptions $R(D^{v})$ is fractional differential operator of order (m,q) and if $K(t)$ be the Green's function associated with $R(D^{v})=0$ that is: $K(t) = \mathcal{L}^{-1}\left\{R^{-1}(s^{v})\right\}$

Let "M" be the integer just grater than "mv" i.e. $M-1 < mv \le M$

$$s^{mv}X(s) = \mathcal{L}\left\{D^{mv}x(t)\right\} + \sum_{k=0}^{M-1} s^{M-k-1}[D^{k-M+mv}x(0)]$$

$$Y(s) = \frac{\left[\mathcal{L}\left\{D^{mv}x(t)\right\}\right]}{R(s^{v})} + \sum_{k=0}^{m-1}[D^{k-M+mv}x(0)]\frac{s^{M-k-1}}{R(s^{v})}$$

$$^{***}\mathcal{L}\left\{D^{p}K(t)\right\} = s^{p}L\left\{K(t)\right\};\ p = 0,1,2,\ldots\ldots\ldots\ldots M-2$$

$$Y(s) = \frac{\mathcal{L}\left\{D^{mv}x(t)\right\}}{R(s^{v})} + \sum_{k=0}^{M-1}[D^{(k-M+mv)}x(0)]\mathcal{L}\left\{D^{M-k-1}K(t)\right\}$$

$$y(t) = \int_{0}^{t} K(t-\xi)[D^{mv}x(\xi)]d\xi + \sum_{k=0}^{M-1}[D^{k-M+mv}x(0)][D^{M-k-1}K(t)]$$

*** Final value theorem

A slight extension of final value theorem

$M-1 < mv \le M$ is an integer just greater than "*mv*"; let *p* is integer from 0 to *M* - 1

$D^p x(t)$ be piecewise continuous for *p* = 0, 1,……(*M* - 1)

We have used the following:

$$\mathcal{L}\left\{D^p K(t)\right\} = s^p \mathcal{L}\left\{K(t)\right\} - \sum_{j=0}^{p-1}\left[D^j K(0)\right]s^{p-1-j} = s^p K(s)$$

$$D^j K(0) = 0;\ j = 0,1,\ldots\ldots.M-2$$

$$\mathcal{L}\left\{D^p K(t)\right\} = s^p L\left\{K(t)\right\};\ p = 0,1,2\ldots\ldots\ldots M-2$$

$$\lim_{s\to\infty} s^{v+1}\mathcal{L}\left\{f(t)\right\} = 0 \to \lim_{t\to 0} D^v f(0) = 0$$

$$v = 0; \lim_{s\to\infty}[s\mathcal{L}\left\{f(t)\right\}] = 0 \to \lim_{t\to 0} f(0) = 0$$

$$K(s) = \frac{1}{s^{mv} + b_1 s^{(m-1)v} + b_2 s^{(m-2)v} + \ldots.b_m}$$

$$\lim_{s\to\infty} s^{P-1}K(s) = s^{p-1}[s^{mv} + b_1 s^{(m-1)v} + b_2 s^{(m-2)v} + ..+ b_m] = (\infty)^{-1} = 0$$

$$p-1 = M-2 < mv$$

$$D^j K(0) = 0;\ j = 0,1,2,\ldots\ldots M-2$$

Simplification solution of Fractional Integral Equation

$$y(t) = \int_0^t K(t-\xi)[D^{mv}x(\xi)]d\xi + \sum_{k=0}^{M-1}[D^{k-M+mv}x(0)][D^{M-k-1}K(t)]$$

$$M \geq mv$$

For “$k = 0$” and $M > mv$ $D^{-M+mv}x(t)$ is a fractional integral and

$$D^{-M+mv}x(0) = \int_0^0 (\Phi_{-M+mv}x(t-\xi)d\xi = 0$$

Thus for $M > mv$; rewriting the indexes

$$y(t) = \int_0^t K(t-\xi)[D^{mv}x(\xi)]d\xi + \sum_{j=0}^{M-2}[D^{mv-1-j}x(0)][D^{j}K(t)]$$

For “$k = 0$” and $M = mv$ $D^{-M+mv}x(0) = D^{0}x(0) = x(0)$

Thus for $M = mv$ and rearranging the indexes

$$y(t) = \int_0^t K(t-\xi)D^{M}x(\xi)d\xi + \sum_{j=0}^{M-1}[D^{j}x(0)][D^{j}K(t)]$$

In short: $M = 1; M > mv; y(t) = \int_0^t K(t-\xi)D^{mv}x(\xi)d\xi$

$$M = 1; M = mv, y(t) = \int_0^t K(t-\xi)Dx(\xi)d\xi + x(0)K(t)$$

Fractional Integral Equations examples-explicit solution

$$[D^{0} + bD^{-2v}]y(t) = x(t)$$

$$[1 + s^{-2v}b]Y(s) = X(s)$$

$$[s^{2v} + b]Y(s) = s^{2v}X(s)$$

$$Y(s) = \frac{s^{2v}X(s)}{[s^{2v} + b]}; R(x) = x^{2} + b$$

$$K_{q}(t) = \mathcal{L}^{-1}\left\{[s^{2v} + b]^{-1}\right\}$$

$$M = 1 > mv;:: y(t) = \int_{0}^{t} K_{q}(t-\xi) D^{mv} x(\xi)\, d\xi$$

$$M = 1 = mv;:: y(t) = \int_{0}^{t} K_{q}(t-\xi) D x(\xi)\, d\xi + x(0)K(t)$$

Fractional Integral Equations examples-explicit solution:

$$[D^{0} + bD^{-2v}]y(t) = x(t)$$

Say $q = 2$, hence $v = 1/q = 1/2$; and $m = 2$, $M = mv = 1$

$$K_2(t) = \mathcal{L}^{-1}\left\{[s + b]^{-1}\right\} = E_t(0, -b) = e^{-bt}$$

$$y(t) = \int_0^t e^{-b(t-\xi)} D\, x(\xi)\, d\xi + x(0)e^{-bt}$$

For $q = 3$, $v = 1/3$, $m = 2$, $M = 1 > mv$

$$K_3(t) = \mathcal{L}^{-1}\left\{\left[s^{2/3} + b\right]^{-1}\right\} = C_t(-1/3, -b^{3/2}) + b^{-1/2}S_t(-2/3, -b^{3/2}) - b^{1/2}S_t(0, -b^{3/2})$$

$$C_t(v, a) = C_t(v, -a); S_t(v, a) = -S_t(v, -a)$$

$$y(t) = \int_0^t [C_{t-\xi}(-0.33, b^{1.66}) - b^{-0.5}S_{t-\xi}(-0.66, b^{1.5}) + b^{0.5}S_{t-\xi}(0, b^{1.5})]D^{0.66}x(\xi)d\xi$$

Fractional integral equation example-explicit solution

$$[D^0 + bD^{-2v}]y(t) = x(t)$$

For $q = 4, v = 1/4, m = 2, M = 1 > mv$

$$[D^0 + bD^{-1/2}]y(t) = x(t)$$

$$K_4(t) = \mathcal{L}^{-1}\left\{[s^{1/2} + b]^{-1}\right\} = E_t(-0.5, b^2) - bE_t(0, b^2)$$

$$y(t) = \int_0^t [E_{t-\xi}(-0.5, b^2) - bE_{t-\xi}(0, b^2)]D^{1/2}x(\xi)\,d\xi$$

Simplifying with identities & properties of Miller-Ross functions

$$y(t) = x(t) - be^{b^2 t}\int_0^t [D^{1/2}x(\xi) - bx(\xi)]e^{-b^2\xi}\,d\xi$$

Fractional Integral equation-direct solution

$$[D^0 + bD^{-2v}]y(t) = x(t)$$

$$x(t) = t^\lambda; \; X(s) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1}}$$

$$Y(s) = \frac{s^{2v}}{s^{2v}+b} X(s) = \frac{\Gamma(\lambda+1)}{s^{\lambda+1-2v}(s^{2v}+b)}$$

$q = 2; v = 1/2$

$$Y(s) = \frac{\Gamma(\lambda+1)}{s^\lambda(s+b)}$$

$$y(t) = \Gamma(\lambda+1)E_t(\lambda, -b)$$

$q = 3, v = 1/3$

$$Y(s) = \Gamma(\lambda+1)\left[\frac{s}{s^\lambda(s^2+b^3)} + \frac{b^2}{s^{\lambda+v}(s^2+b^3)} - \frac{b}{s^{\lambda-v}(s^2+b^3)}\right]$$

$$y(t) = \Gamma(\lambda+1)[C_t(\lambda, b^{3/2}) + b^{1/2}S_t(\lambda+v, b^{3/2}) - b^{1/2}S_t(\lambda-v, b^{3/2})$$

$q = 4, v = 1/4$

$$Y(s) = \Gamma(\lambda+1)\left[\frac{1}{s^\lambda(s-b^2)} - \frac{b}{s^{\lambda+1/2}(s-b^2)}\right]$$

$$y(t) = \Gamma(\lambda+1)[E_t(\lambda, b^2) - bE_t(\lambda+1/2, b^2)]$$

Is there a easy way out to tackle

Extra Ordinary Differential equation Systems?