FRACTIONAL CALCULUS
for
Applied Science & Engineering

MATHEMATICO-PHYSICS OF GENERALIZED CALCULUS

<u>Module-III</u>

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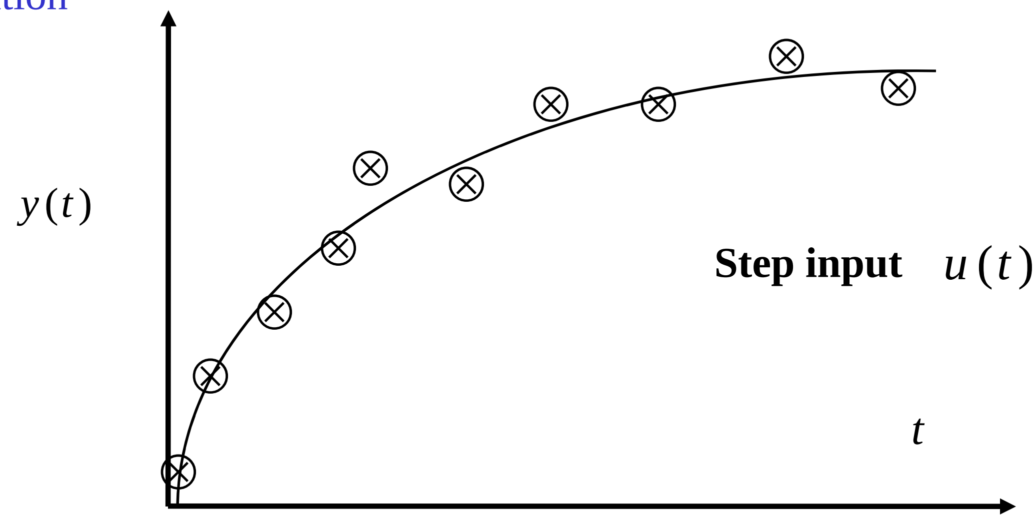
Curve fitting-A

System identification



Set of measured values $y_i^*(i = 0, M)$,average error margin $Q = \frac{\sum_{i=0}^{M} (y_i^* - y_i)}{M + 1}$

$$1.8675 \frac{d^2}{dt^2} y(t) + 5.518 \frac{d}{dt} y(t) + 0.0063 y(t) = u(t)$$

$$Q \approx 3 \times 10^{-3}$$

$$0.7943 \frac{d^{2.571}}{dt^{2.571}} y(t) + 5.2385 \frac{d^{0.83}}{dt^{0.83}} y(t) + 1.5960 y(t) = u(t)$$

$$Q \approx 10^{-4}$$

$$6.288 \frac{d^{1.0315}}{dt^{1.0315}} y(t) + 1.8508 y(t) = u(t)$$

$$Q \approx 4 \times 10^{-4}$$

Curve fitting-B

Life span estimation, Predictive Maintenance, Reliability analysis…

. During a certain period, after installation of a wire on load, an enhancement of its properties is observed. Say yield point.

. Then properties of wires become worse and worse until it breaks down.

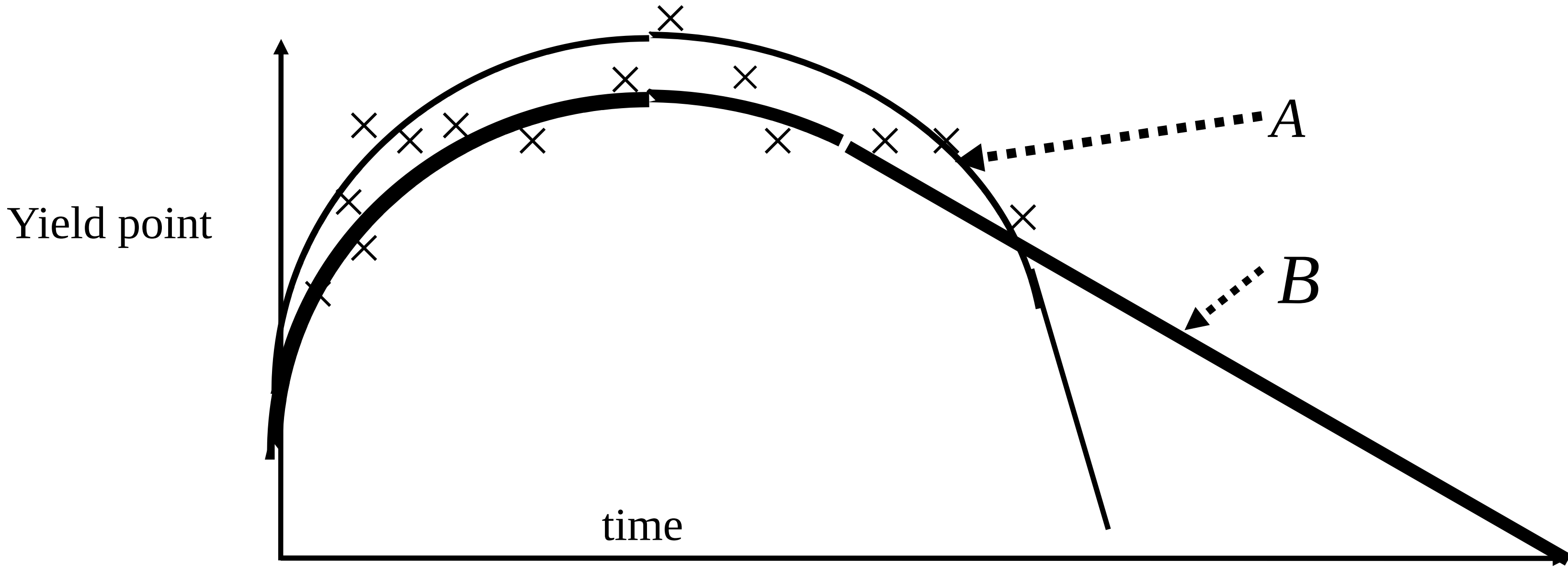
. The period of enhancement is shorter than the period of decrease of property and the general shape of the process curve is not symmetric.

Set of experimental measurements $y_1, y_2, \dots, y_n$ is fitted with fractional differential equation with $y(t) = a_0 + a_1 t + a_2 t^2 + \dots - a_{m0} D_t^{-\alpha} y(t) \quad (0 < \alpha \le m)$

$a_0, a_1, a_2, \dots a_{m-1}$ initial values of fitted function and (m - 1) derivatives.

The fractional integration and its fractional order represents the cumulative impact of the previous history loading on the present state of wire.
The order of fractional integration is related to shape of memory function of wire material.

Experimental fit quadratic and fractional order regression



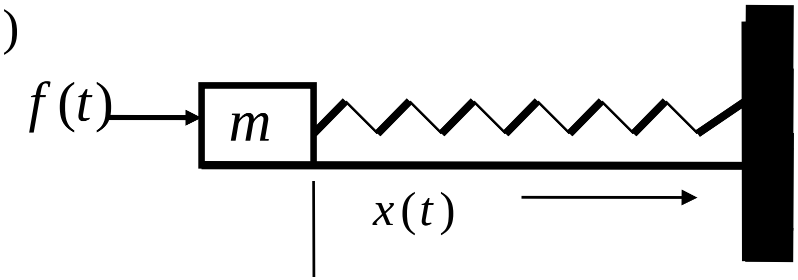
$$A \quad y(t) = 0.033t^2 + 0.562t + 10.723$$

$$B \quad y(t) = -0.046\,{}_0D_t^{-1.32}y(t) + 1.2760t + 10.1955$$

It is obvious that the order of fractional integration would be different for different wires because they work in different conditions. Thus it is necessary to apply this regression in each case separately. Main problem is that each particular wire changes its property due to certain very peculiar causes(heredity/history). The order 1.32 is for this particular wire of 2.4mm diameter at this loading, a 2.8mm diameter wire will have different order

Generalization of Newtonian mechanics and differential equations

$$m x^{//}(t) + b_0 x^{/}(t) + k x(t) = f(t)$$



$$x(0) = 0$$

$$x^{/}(0) = 0$$

Mass concentrated at point
Mass less spring
Frictionless spring
Infinite wall

$$(m s^2 + b_0 s + k) X(s) = F(s)$$

Spring with friction

$$k_q s^q X(s) = F_{sp}(s)$$

$$0 \le q \le 1$$

$$(m s^2 + b_0 s + k_q s^q + k) X(s) = F(s)$$

$$(m s^2 + b_0 s + k_{q_n} s^{q_n} + k_{q_{n-1}} s^{q_{n-1}} + \ldots + k_{q_1} s^{q_1} + k_{q_0}) X(s) = F(s)$$

$$\left(\sum_{n=0}^{N=2} k_n s^{q_n}\right) X(s) = F(s)$$

$$\left(\int_0^2 k(q) s^q dq\right) X(s) = F(s)$$

Distributed mass
Spring with mass
Spring with friction
Damping with spring action
Non conservation system
Leaky wall/termination

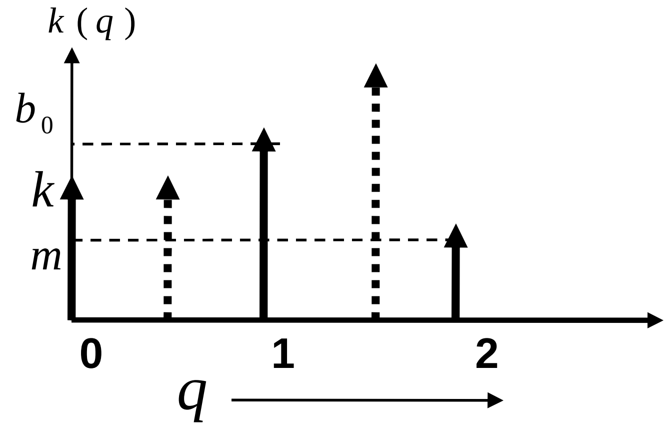
System Identification & order distribution

Integer Order:

$$mx''(t) + b_0 x'(t) + kx(t) = f(t)$$

$$(ms^2 + b_0 s + k)X(s) = F(s)$$

$$\{\int_0^\infty [m\delta(q-2) + b_0\delta(q-1) + k\delta(q)]s^q dq\} X(s) = F(s)$$



Fractional Order

$$(ms^2 + b_1 s^{3/2} + b_0 s + k_1 s^{1/2} + k_0) X(s) = F(s)$$

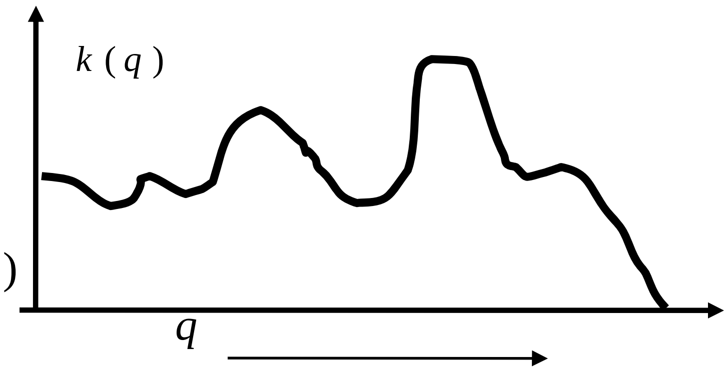
$$\{[m\delta(q-2) + b_1\delta(q-1.5) + b_0\delta(q-1) + k_1\delta(q-0.5) + k_0\delta(q)]s^q dq\} X(s) = F(s)$$

$$m\frac{d^2 x(t)}{dt^2} + b_1 \frac{d^{3/2} x(t)}{dt^{3/2}} + b_0 \frac{dx(t)}{dt} + k_1 \frac{d^{1/2} x(t)}{dt^{1/2}} + k_0 = f(t)$$

Continuous Order

$$\left(\int_0^\infty k(q) s^q dq\right) X(s) = F(s)$$

$$\{\mathcal{L}^{-1}\left(\int_0^\infty k(q) s^q dq\right)\} * x(t) = f(t)$$



Order distribution based feed back control system

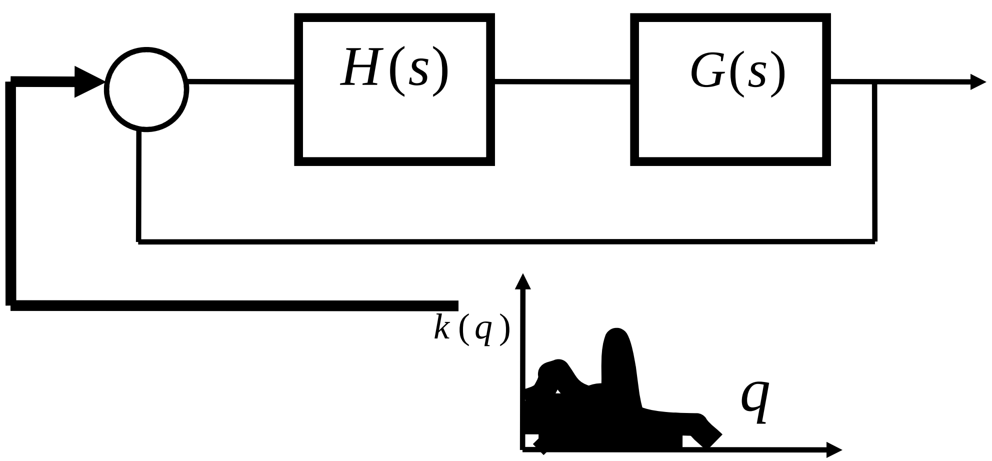
Reaction of a system depends on order value.
Reaction of a system depends on amplitude of order
A first (integer) order system cannot go into oscillations.
Presence of fractional order and its strength can give oscillations.
Why not control system order and its strength?

A futuristic automatic controller

$$G(s) = \frac{1}{s^2 + a}, \quad H(s) = \int_0^{\infty = 2} k(q) s^q \, dq$$

$$T(s) = \frac{H(s)G(s)}{1 + H(s)G(s)}$$

$$T(s) = \frac{\int_0^2 k(q) s^q \, dq}{s^2 + a + \left(\int_0^2 k(q) s^q \, dq\right)}$$

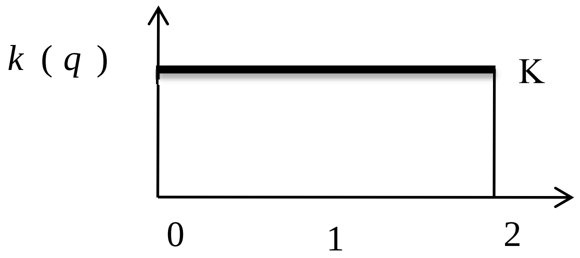


Demanded order distribution-

Solving continuous order system

Let the continuous order system be represented as:

$$X(s) = \frac{F(s)}{\int_{-\infty}^{+\infty} k(q)s^q dq} = \frac{F(s)}{P(s)} \quad \text{where} \quad P(s) = \int_{-\infty}^{+\infty} k(q)s^q dq$$



Let the order distribution be uniform from 0 to 2 with K as order strength $k(q) = \mathrm{K} \;\; ; 0 \le q \le 2$

Then:

$$P(s) = \int_{-\infty}^{\infty} k(q)s^q dq$$

$$\ldots\ldots\ldots\ldots = \int_0^2 \mathrm{K}s^q dq = \int_0^2 \left(\mathrm{K}e^{q\ln(s)}\right)dq$$

$$\ldots\ldots\ldots\ldots\ldots = \mathrm{K}\left[\frac{s^2 - 1}{\ln(s)}\right]$$

For delta excitation: $F(s) = 1$

Fundamental Response of system is: $X(s) = \dfrac{\ln(s)}{\mathrm{K}\left[s^2 - 1\right]}$

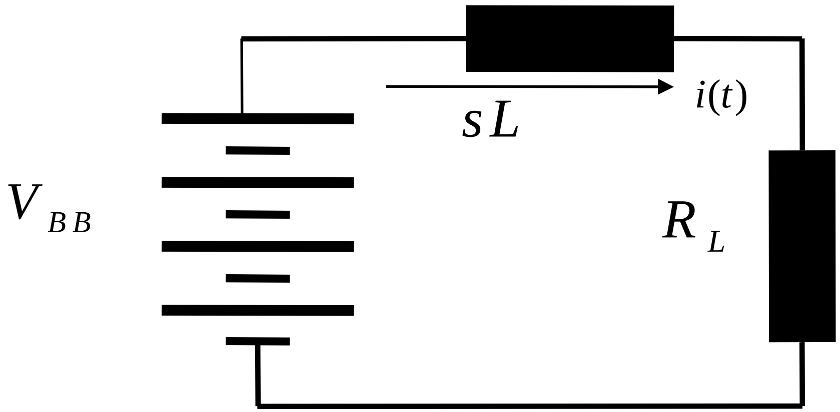
$$x(t) = \mathcal{L}^{-1}\left\{\frac{\ln(s)}{\mathrm{K}\left[s^2 - 1\right]}\right\}$$

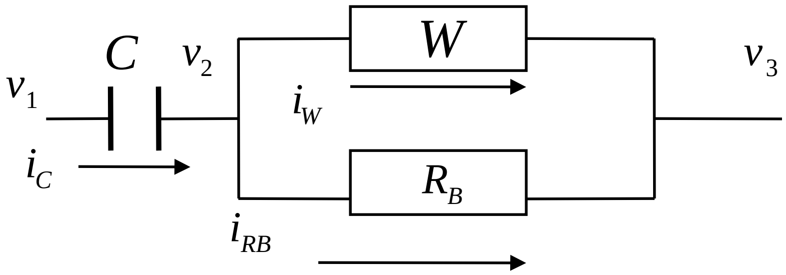
To other type inputs convolution with this will give solution

A good mathematical research topic!

Circuit theory

Fractional order source
Fractional order load
Fractional order connectivity





Circuit equation

$$L\frac{di(t)}{dt} + R_L i(t) = V_{BB}$$

Inside battery

$$i(t) = i_C(t)$$

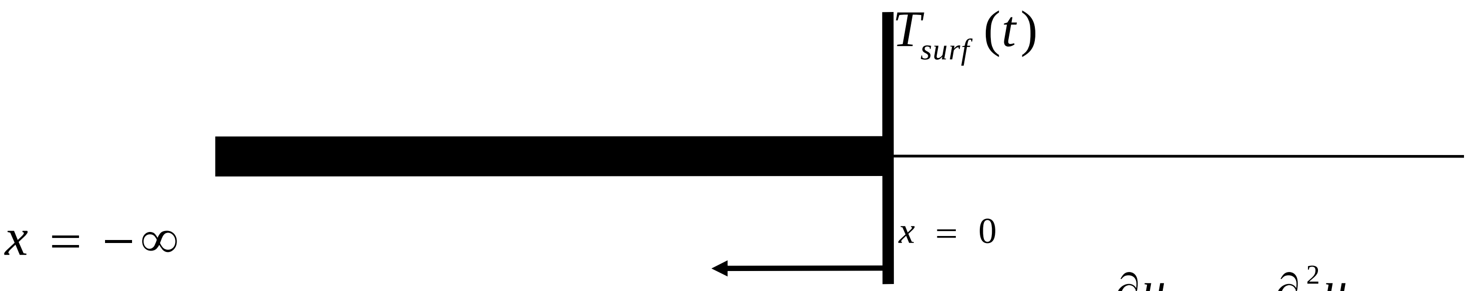
$$v_1(t) - v_2(t) = \frac{1}{C}\int_0^t i_C(t)dt + v_{1-2}(0)$$

$$i_C(t) = i_W(t) + i_{RB}(t)$$

$$i_W(t) = {}_0D_t^{1/2}[v_2(t) - v_3(t)]$$

$$i_{RB}(t) = \frac{v_2(t) - v_3(t)}{R_B}$$

Heat flux and temperature for semi infinite heat conductor



$$c\rho \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(t,x) = T(t,x) - T_0$$

$$T(0,x) = T_0, u(0,x) = 0$$

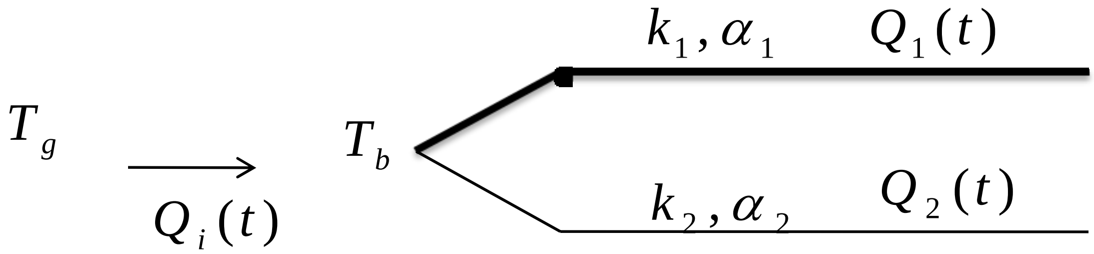
$$T(t,0) = T_{surf}(t), u(t,0) = T_{surf}(t) - T_0$$

$$Q(t) = \frac{\partial T(t,0)}{\partial x}$$

$$Q(t) = \frac{k}{\sqrt{\alpha}} D_t^{1/2} [T_{surf}(t) - T_0]$$

$$\alpha = \sqrt{\frac{k}{c\rho}}$$

Heat flux measurement with single TC



General equation heat flow relating conducted heat flux $Q_i(t)$ through semi infinite heat conductor is

$$Q_i(t) = \frac{k_i}{\sqrt{\alpha_i}} D_t^{1/2} T_b$$

, with

$$\alpha_i^2 = k_i / c\rho$$

$$Q_i(t) - Q_1(t) - Q_2(t) = mc\frac{dT_b}{dt}$$

Convective heat input is:

$$Q_i(t) = hA\left(T_g(t) - T_b(t)\right)$$

$$\frac{T_b(s)}{T_g(s)} = \frac{1}{\left(\frac{mc}{hA}\right)s + \frac{1}{hA}\left(\frac{k_1}{\sqrt{\alpha_1}} + \frac{k_2}{\sqrt{\alpha_2}}\right)\sqrt{s} + 1}$$

Impedance RC distributed semi infinite transmission line

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R$$

$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial v(x,t)}{\partial t}$$

$$\frac{\partial^2 v}{\partial x^2} = R\frac{\partial i}{\partial x} = RC\frac{\partial v}{\partial t}$$

$$v(0,t) = v_1(t), v(\infty,t) = 0$$

$$i(t) = \frac{1}{R\sqrt{\alpha}} \cdot {}_a D_t^{1/2} v(t)$$

$$\alpha = \frac{1}{RC}$$

$$Z(s) = \sqrt{\frac{R}{C}}\left(\frac{1}{s^{1/2}}\right)$$

Basic building block for fractional order immittance realization of arbitrary order to make fractional order analogue function generator and fractional order analogue PID controller.

Fractional Divergence

To define non-local flux of material flowing through anisotropic media, lossy volume and heterogeneous ambient.

Non Fickian diffusion phenomena

Anomalous diffusion

$$\left(\mathrm{div}^{\alpha} J\right) = \nabla^{\alpha} J \equiv \lim_{\Delta V \to \mathrm{REV}} \frac{1}{V} \oint_{S} J \, . \vec{n} \, dS = \Phi(x)$$

$$\frac{d^{\beta} \Phi(x)}{d x^{\beta}} + B^{2} \Phi(x) = 0$$

$$\beta = 1 + \alpha$$

$$1 \leq \beta \leq 2$$

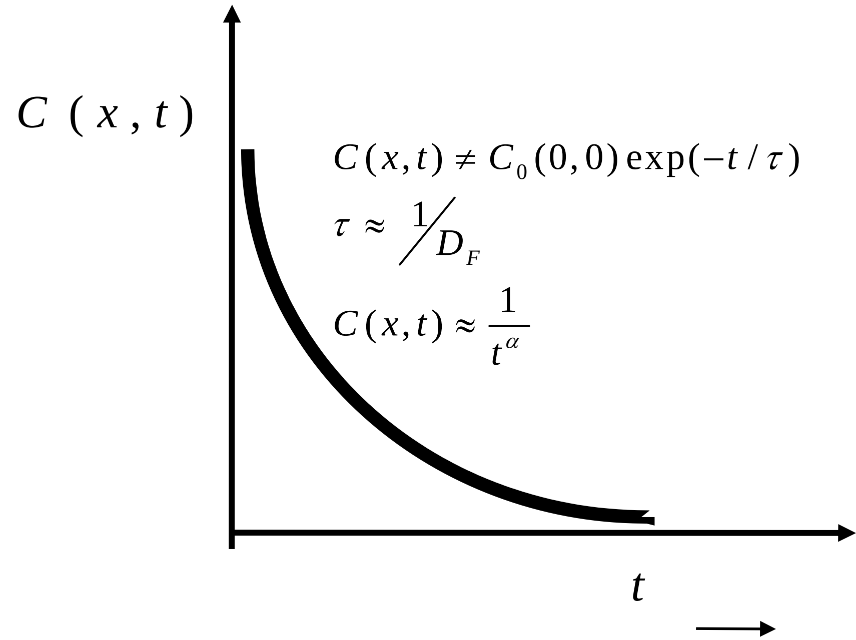
General relaxation phenomena

Electrode Electrolyte interface, derivation of Warburg law
Application in Electrochemistry.
Non-Fickian reaction kinetics.
Power law in anomalous diffusion
Time constant aberration
Magnetic flux diffusion studies in geophysics

$$ {}_0\partial_t^{(1)} C(x,t) = \mathbb{D}_F \partial_x^2 C(x,t) $$

$$ {}_0 D_t^{\alpha} C(x,t) = \mathbb{D}_{NF} \frac{\partial^2}{\partial x^2} C(x,t) $$

$$ C(x,t) \approx \frac{1}{t^{\alpha}};\ 0 < \alpha \le 1 $$



Reaction to impulse excitation
Non exponential reaction

Fractional Curl

In between dual solution in electrodynamics

$$\rho_e \leftrightarrow \rho_m$$

$$(E, H, D, B, \mu, \varepsilon) \leftrightarrow (H, -E, B, -D, \varepsilon, \mu)$$

$$(E, \eta H) \leftrightarrow (\eta H, -E)$$

$$E_{fd} = \frac{1}{(ik)^{\alpha}} (\nabla \times)^{\alpha} E$$

$$\eta H_{fd} = \frac{1}{(ik)^{\alpha}} (\nabla \times)^{\alpha} H$$

Future R&D in in-between mapping of Right Handed Maxwell systems and Left Handed Maxwell Systems (RHM)-(LHM)

Mapping/modelling in between pure Thevenin and pure Norton circuits.

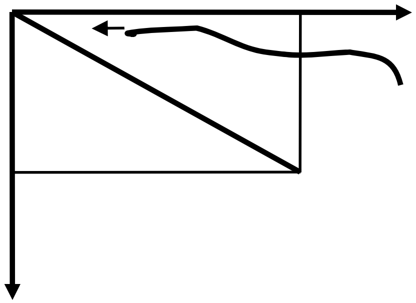
Electrodynamics

Wave propagation in media with losses.

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \varepsilon_0 \chi_0 \frac{\partial^\alpha E}{\partial t^\alpha} + \frac{\partial^2 E}{\partial x^2} = 0$$

$$1 \leq \alpha < 2$$

$$D^\alpha E \sin \omega t = E \omega^\alpha \sin\left(\omega t + \frac{\pi \alpha}{2}\right)$$



$$\frac{\pi \alpha}{2}$$

Power factor modelling in AC machines, a new field of R&D.

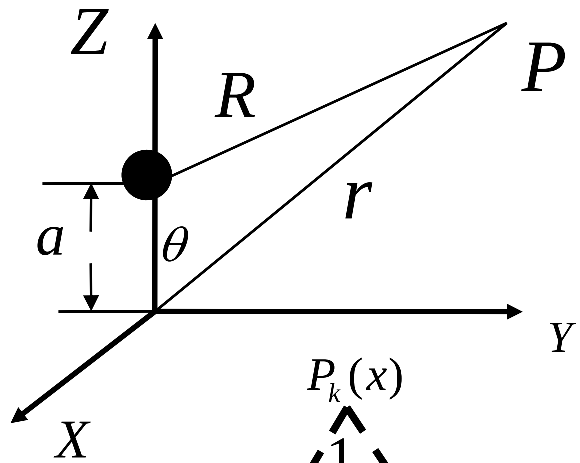
Electrodynamics

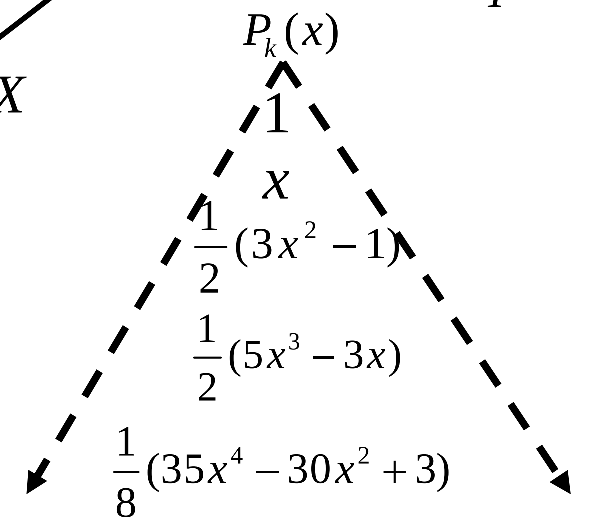
Multi-pole expansion

$$\Phi(r,\theta) \propto \frac{q}{R} = \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

$$= \frac{q}{r} + \frac{qa}{r^2}(\cos\theta) + \frac{qa^2}{2r^3}(3\cos^2\theta - 1) + \dots$$

$$= \frac{q}{r}\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^k P_k(\cos\theta)$$



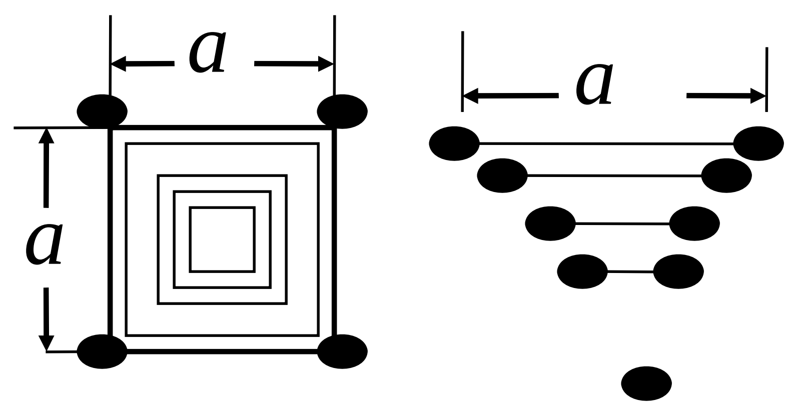


Fractional mutipole
Fractal charge distribution

$\alpha = 0 \rightarrow 2^0$ Mono
$\alpha = 1 \rightarrow 2^1$ Dipole
$\alpha = 2 \rightarrow 2^2$ Quadra

$$\Phi = \left(\frac{qa^\alpha}{4\pi\varepsilon_0}\right)\left({}_{-\infty}D^\alpha\right)\left(\frac{1}{r}\right)$$

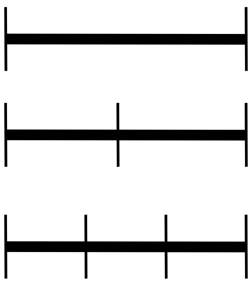
$$\Phi = \frac{qa^\alpha\Gamma(1+\alpha)}{4\pi\varepsilon_0(r)^{1+\frac{\alpha}{2}}}P_\alpha(\cos\theta)$$



Fractional Legendre polynomial, Fractional poles, dipole, monopole
self similarity-fractal distribution

Fractal Geometry & Fractional Calculus

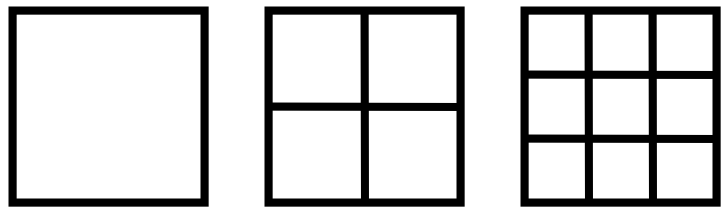
$$d_F = \lim_{\varepsilon \to 0} \frac{\log N}{\log\left(\frac{1}{\varepsilon}\right)}$$



$N = 2$ $r = 2, \varepsilon = 1/2$

$N = 3$ $r = 3, \varepsilon = 1/3$

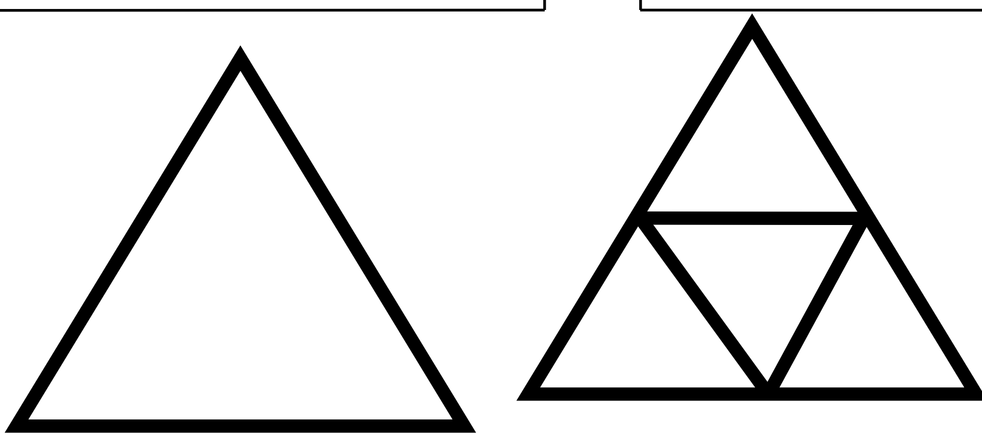
$$d_F = 1$$



$N = 4$, $r = 2, \varepsilon = 1/2$

$N = 9$, $r = 3, \varepsilon = 1/3$

$$d_F = 2$$



$N = 3$

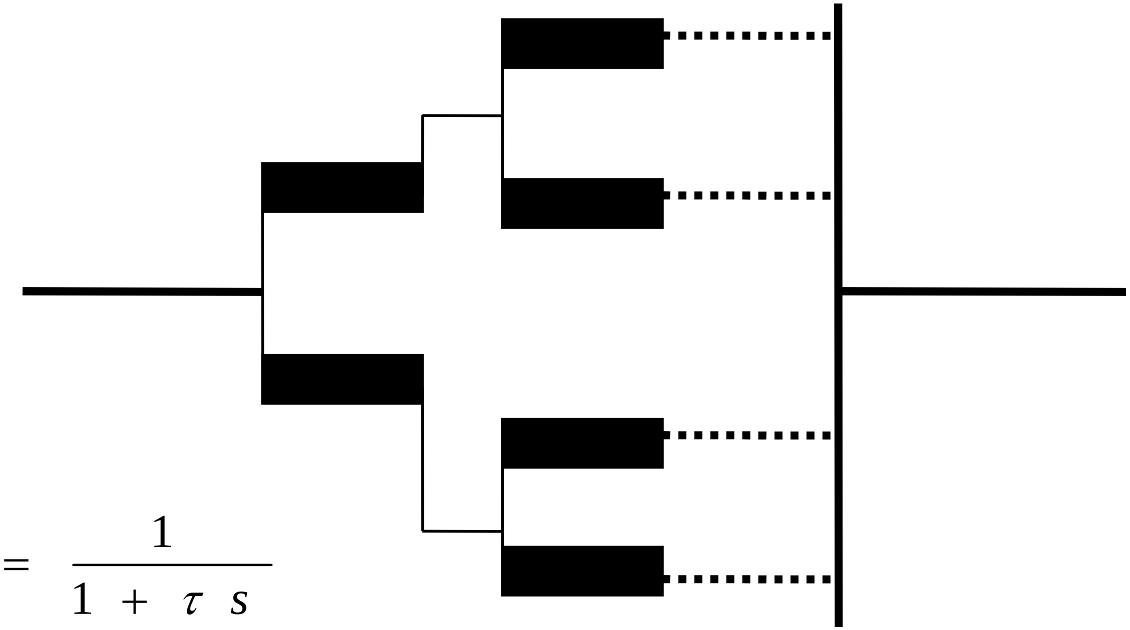
$r = 2, \varepsilon = 1/2$

$$d_F = \frac{\log 3}{\log\left(\frac{1}{1/2}\right)} = \log 3 / \log 2 = 1.585$$

Application to graph theory and reliability analysis of software, data structure, cancer cell growth as future R&D topic on use of Local Fractional Calculus.

Relation of fractal dimensions and order of fractional calculus

Time constant aberration and transfer function of flow through a
Fractal structure and relation to its fractal dimension.



$$G_1(s) = \frac{1}{1 + \tau s}$$

$$G_2(s) = \frac{1}{1 + (\tau s)^{\lambda}}$$

$$\lambda \leftrightarrow d_F$$

Relation of order to the fractal dimension

Fractional calculus and multifractal functions

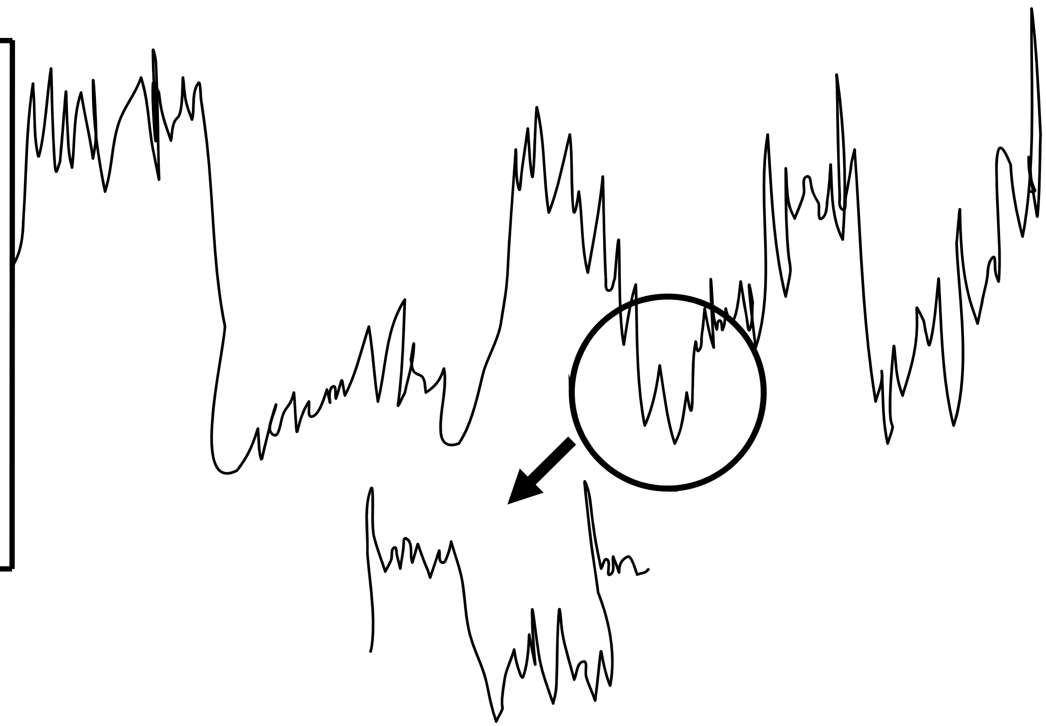
Fractals and multifractal functions and corresponding curves or surfaces are found in numerous non-linear, non-equilibrium phases like low viscous turbulent fluid motion, self similar and scale independent processes, continuous but nowhere differentiable curves.

Weistrauss

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi n)$$

$$0 < a < 1, b > 0, ab > 1 + \frac{3}{2}\pi$$

$$d = \frac{\log a}{\log b + 2}$$



Fractality implies $d > 1$ and it is scale independent, has no smaller scale

Viscoelasticity

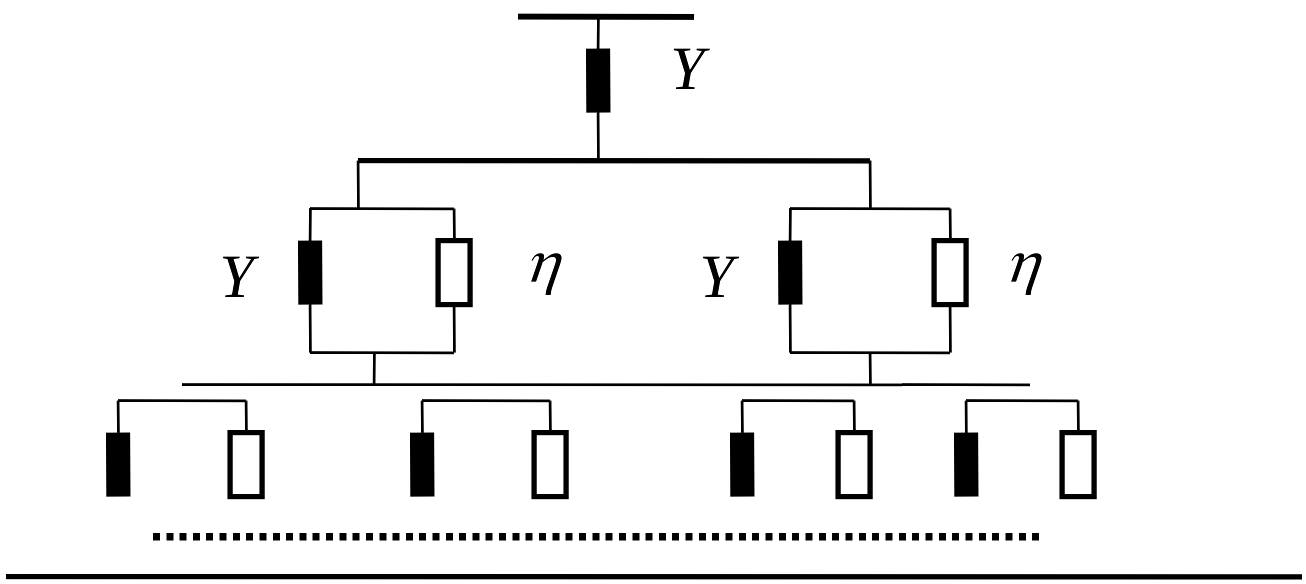
$$\sigma(t) = K\, {}_0D_t^{\alpha}\, \varepsilon(t)$$

$$0 < \alpha < 1$$

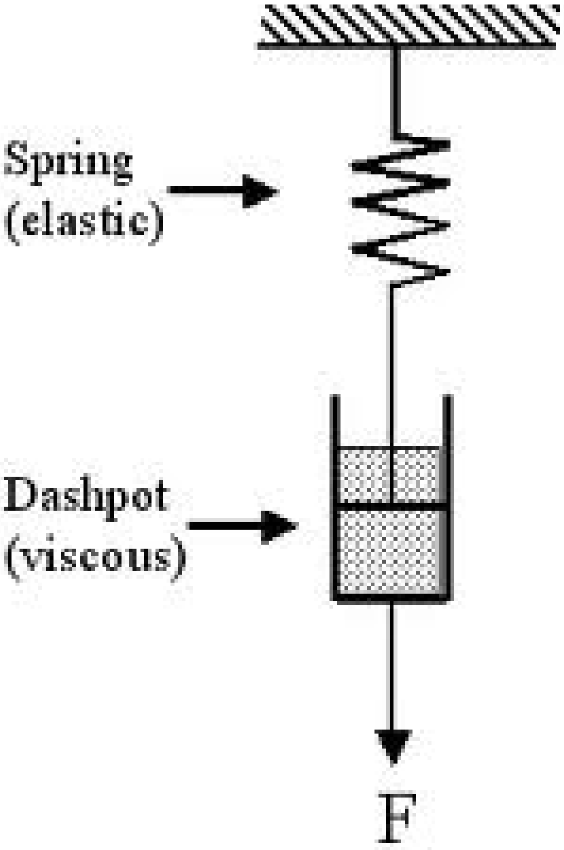
$$\sigma(t) = Y\varepsilon(t) = Y\, {}_0D_t^{0}\, \varepsilon(t)$$ Pure solid Hook's law

$$\sigma(t) = \eta \frac{d}{dt}\varepsilon(t) = \eta\, {}_0D_t^{1}\, \varepsilon(t)$$ Newtonian fluid

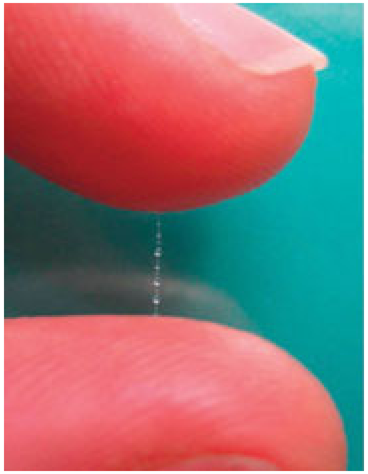
Ideally no matter is pure solid nor is pure fluid



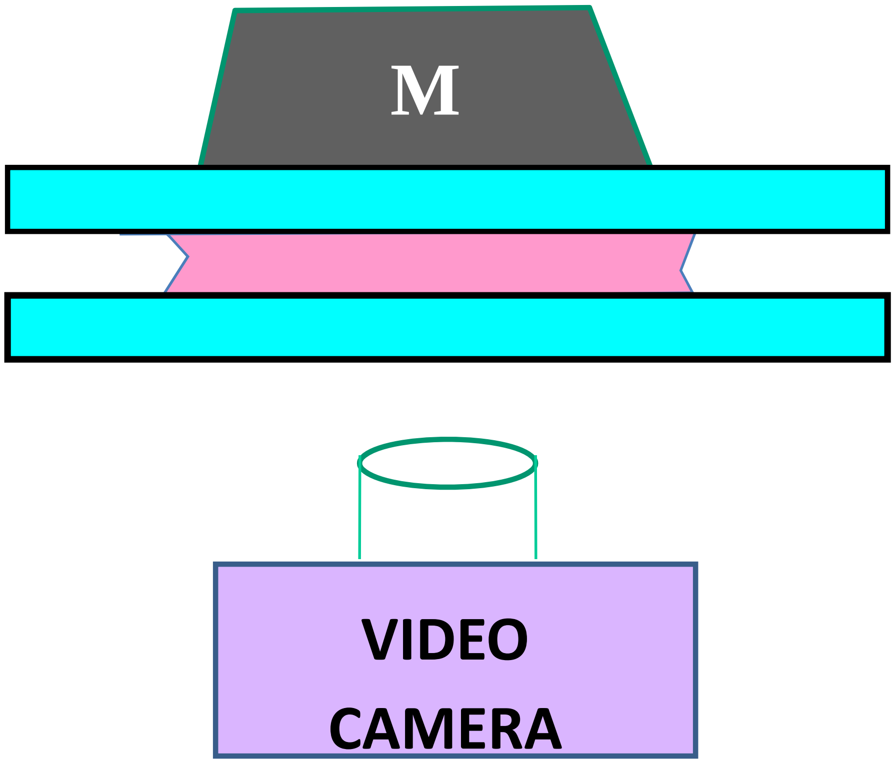
Viscoelastic Model



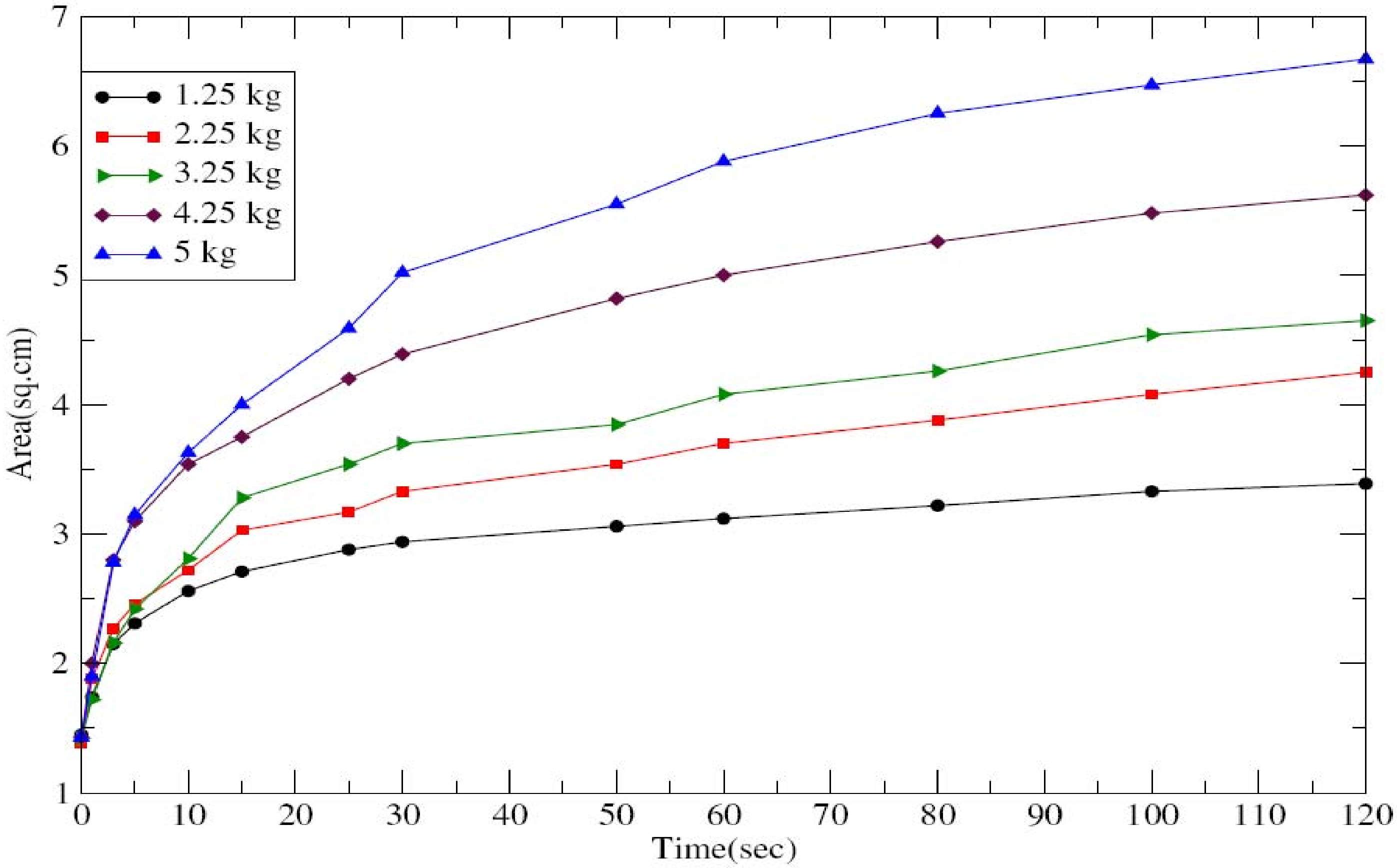
Maxwell Viscoelastic Model



Our Experiment – forcing a fluid to spread under a load

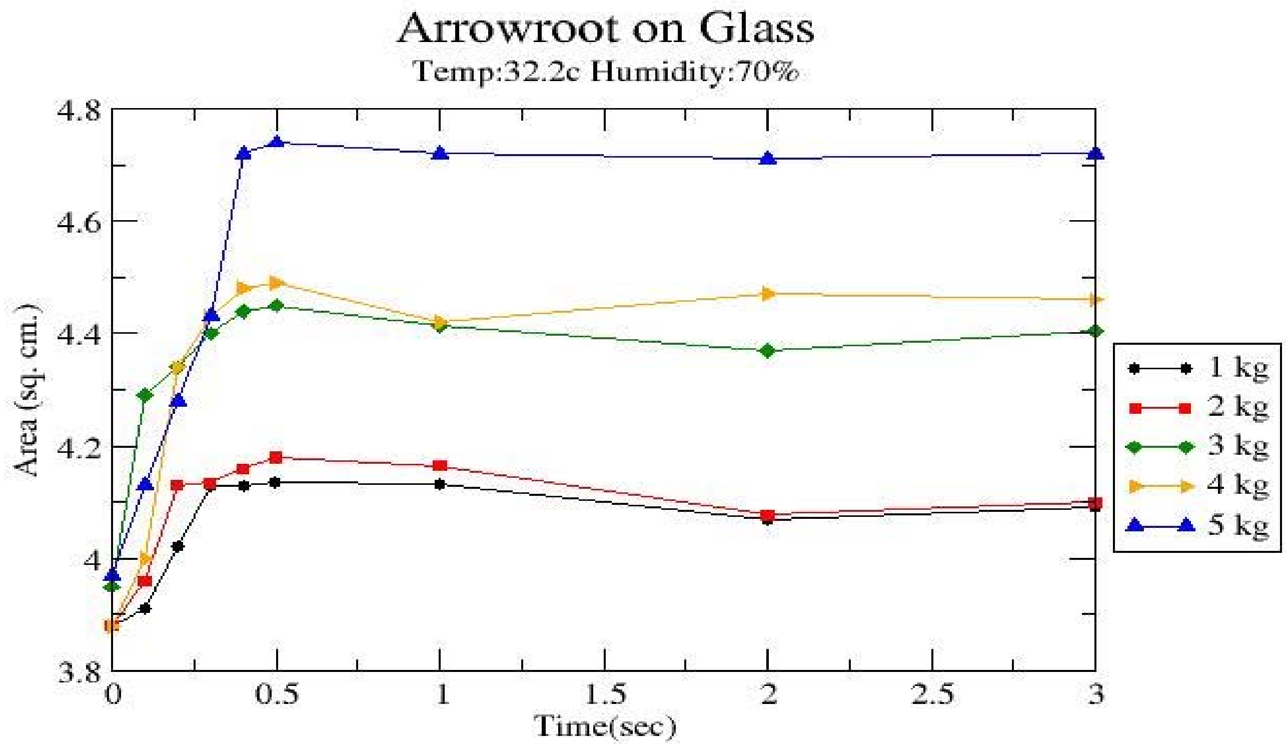


Case of Newtonian Fluid



Non-Newtonian case

Area-Time plot



Viscoelasticity with variable fractional order value

$$\beta \frac{d^q \varepsilon}{dt^q} + E\varepsilon = \sigma$$

$\beta \equiv$ Similar to viscosity, but units differ from 'viscous constant'

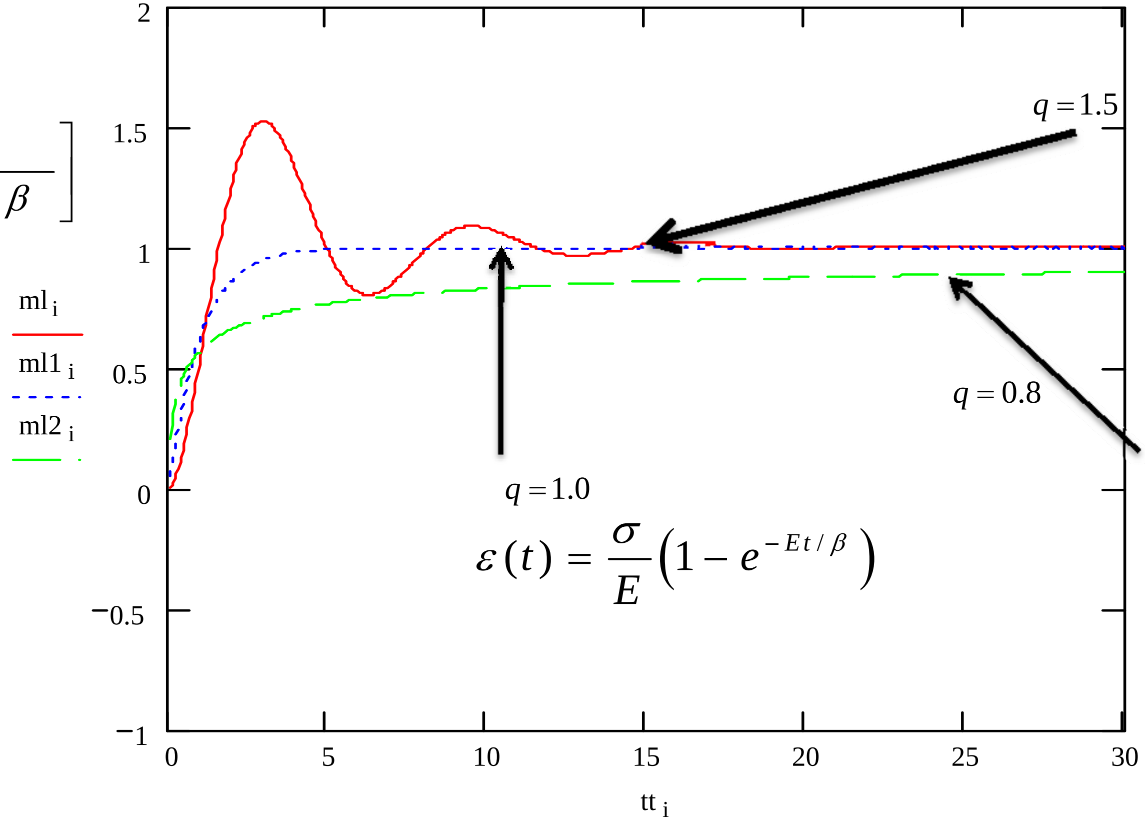
$E \equiv$ Young's Modulus

$$\sigma < \sigma_c$$

$$\varepsilon(s) = \frac{\sigma}{\beta}\left[\frac{1}{s(s^q + E/\beta)}\right]$$

$$= \frac{\sigma}{E}\left[\frac{1}{s} - \frac{s^{q-1}}{s^q + E/\beta}\right]$$

$$\varepsilon(t) = \frac{\sigma}{E}\left[1 - E_q\left(-\frac{E\,t}{\beta}\right)\right]$$



Biology

Muscles and joint tissues in musco-skeletal system seem to behave as visco-elastic material, as fractional integrator, then this could be compensated by fractional order differentiator dynamics of neurons.

Membrane reaction relation as power law to frequency of current

$$X(\omega) = X_0 \omega^{-\alpha}$$

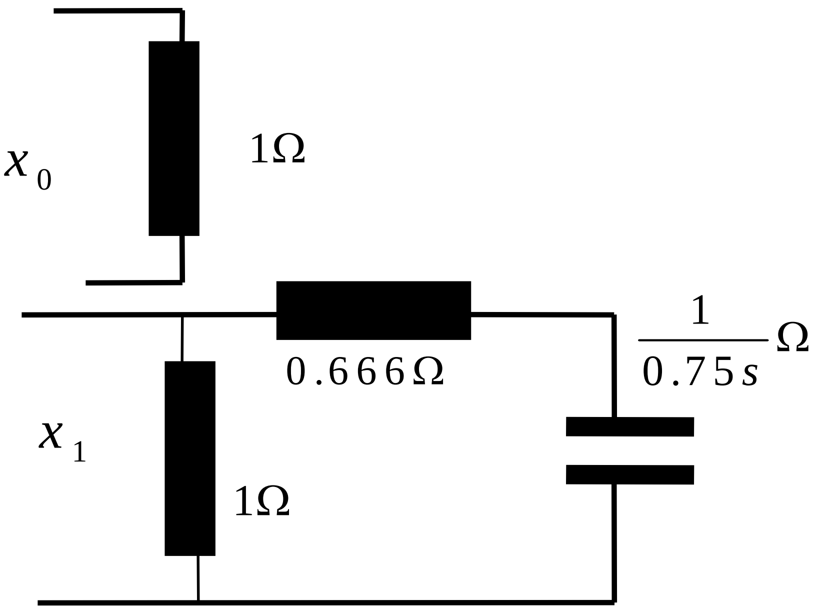
$$G(s) = X_0^{/} s^{-\alpha}$$

Motor discharge rate to rate of change of position

$$\frac{R(s)}{V(s)} = \frac{\tau_1 (s\tau_2 + 1)}{s\tau_1 + 1} s^{\beta - \alpha}$$

Circuit Synthesis

Synthesis of fractional order immittances
Newton method of root evaluation



$$a = x^n,\ x = (a)^{1/n},\ x_0 = 1$$

$$x_k = x_{k-1}\frac{(n-1)(x_{k-1})^n + (n+1)a}{(n+1)(x_{k-1})^n + (n-1)a}$$

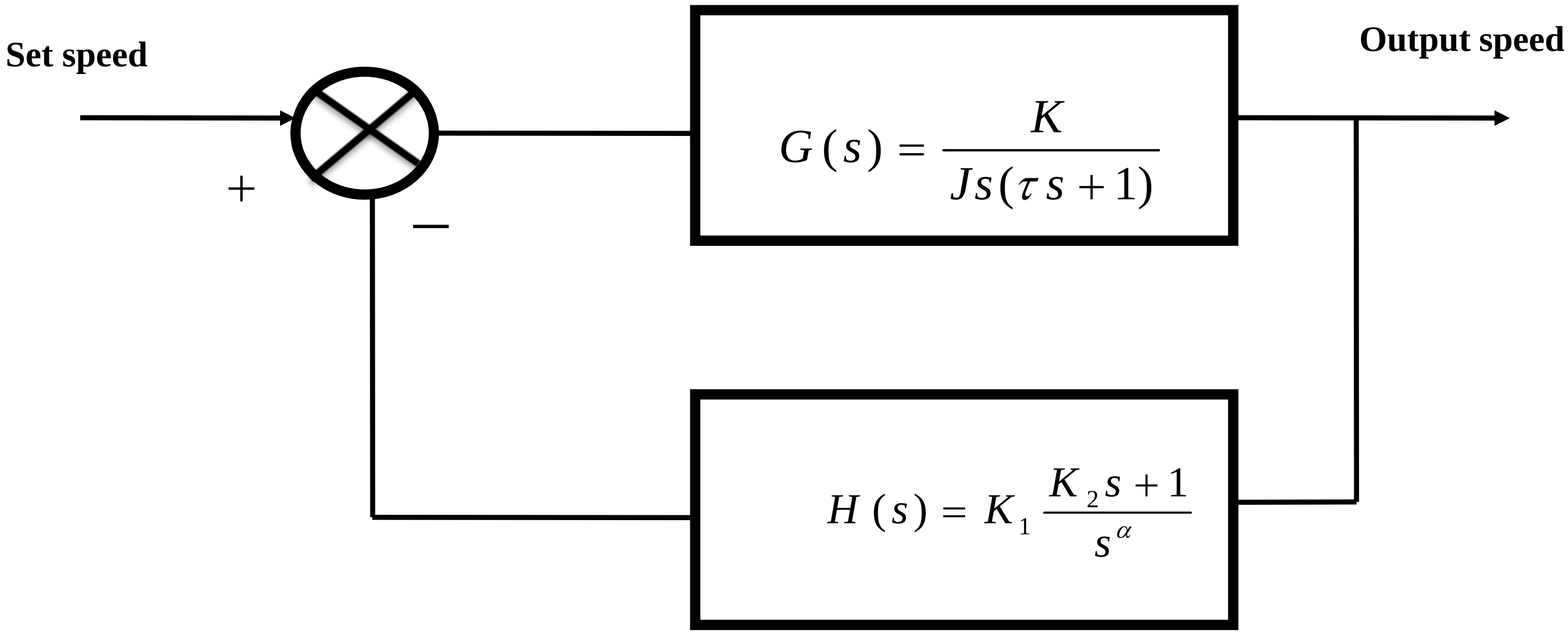
$$n = 3,\ a = \frac{1}{s},\ x_0 = 1,$$

$$x_1 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s+2}{2s+1}$$

$$x_2 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s^5 + 24s^4 + 80s^3 + 92s^2 + 42s + 4}{4s^5 + 42s^4 + 92s^3 + 80s^2 + 24s + 1}$$

$$s^{0.5} = \frac{s+3}{3s+1},\ \left(\frac{1}{s^{0.5}}\right) = \frac{3s+1}{s+3},\ \frac{1}{s^{0.25}} = \frac{3s+5}{5s+3},\ \frac{1}{s^{0.15}} = \frac{1+1.35s}{1.35s+s}$$

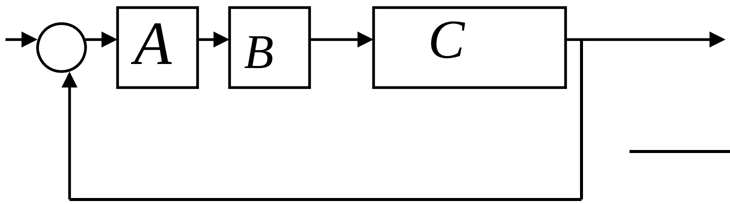
Fuel efficient control system

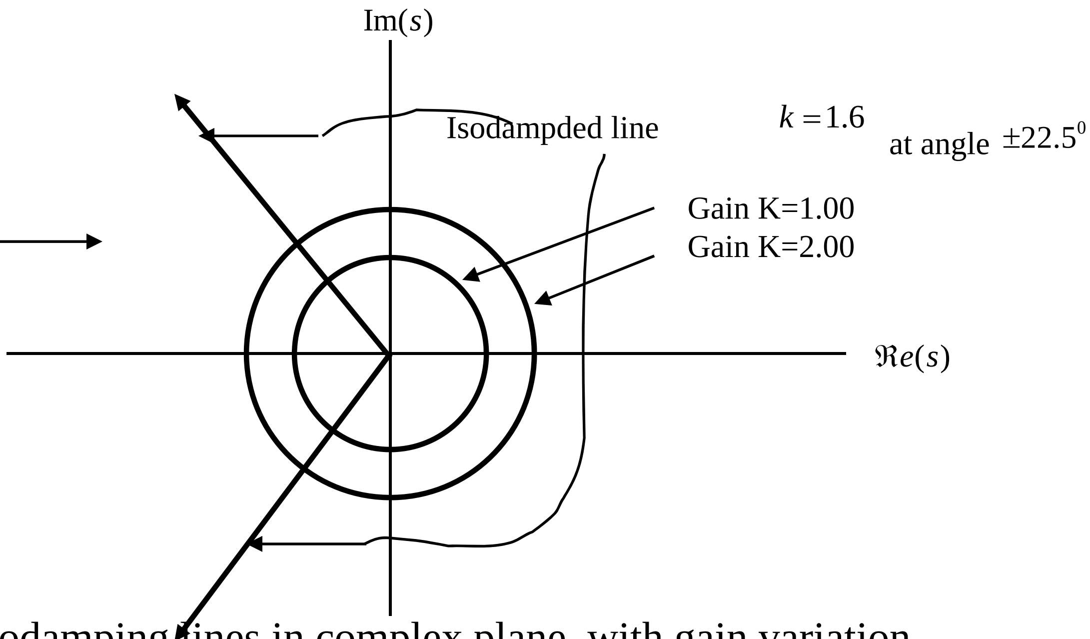


The constant close loop phase gives a feature of ISO-DAMPING where the peak overshoot is invariant on parametric spreads, giving fuel efficiency, avoidance of plant spurious excursions and trips, enhances safety and increases plant operational longevity & more robustness in control. This scheme also takes lesser controller effort as compared to tuned integer order control system.

Overshoot Independent of Gain Spread-Isodamped systems

ISODAMPING
Root-Locus





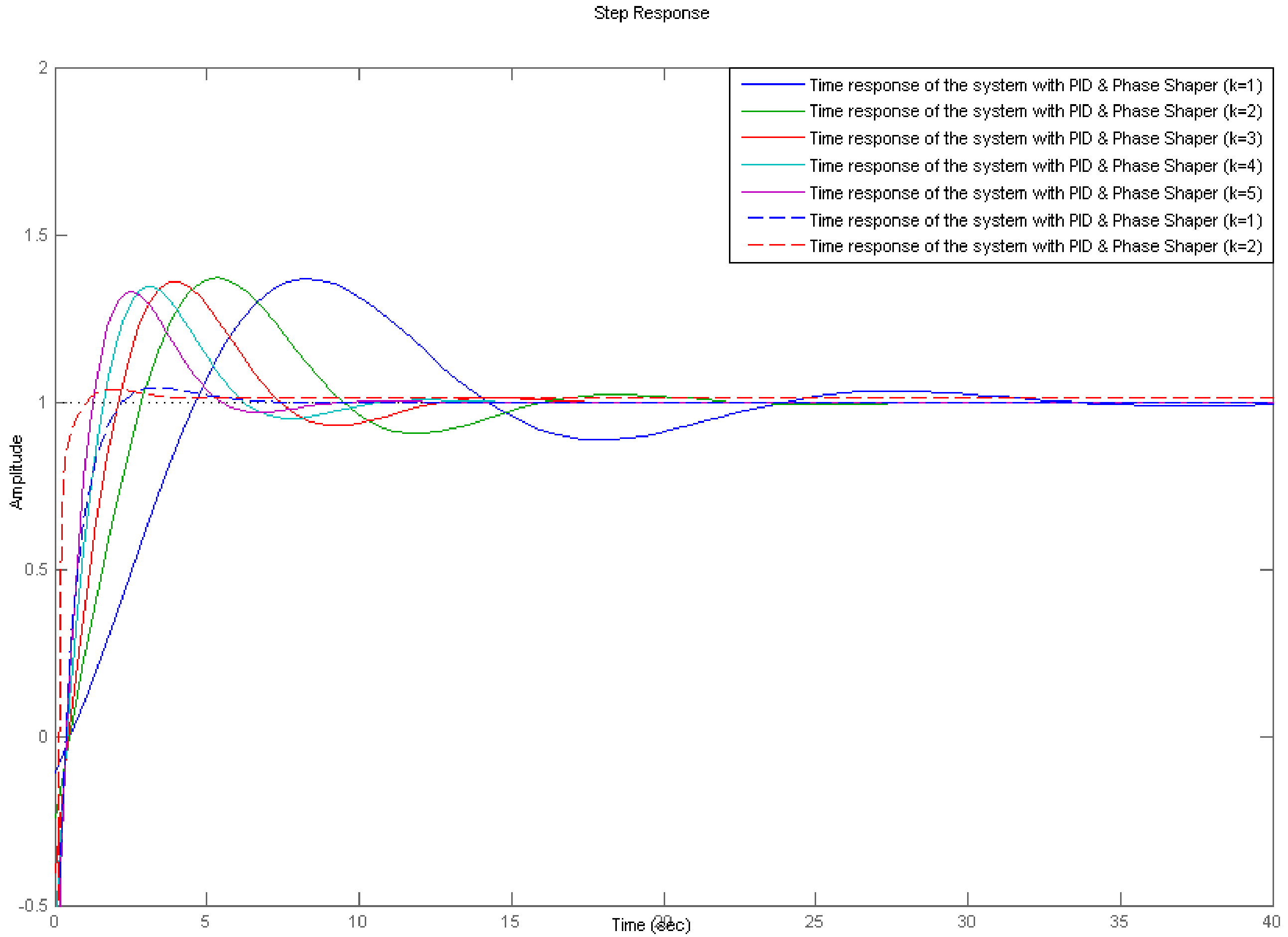
Isodamping lines in complex plane, with gain variation

Observations
As the gain is varied the peak overshoot is more or less constant.
Obtained lesser controller effort.
These are preliminary results on a DC Motor (*C*) TF tuned with PID (*B*) and forwarded by a fractional phase shaper (*A*).
OLTF of good control system shows a fractional order integral form of order between 1-2.

$$k = 1.6, PM = 36^0, \varsigma = 0.38, M_p = 40.5\%, M_r = 1.7013$$

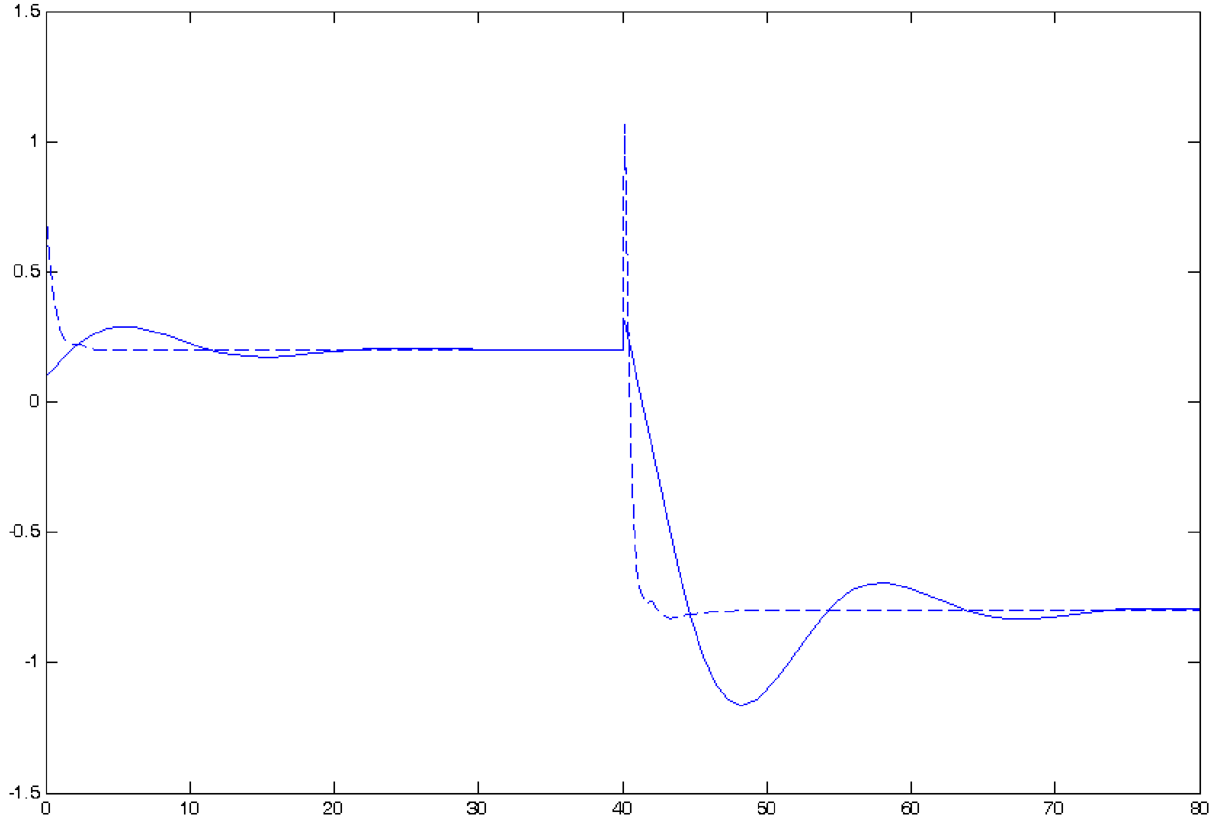
$$k = 1, PM = 90^0, \varsigma = 1, M_p = 0\%, M_r = 1.000$$

Iso-damped response with wide parametric spreads



The scalar gain, as shown in Figure can be varied by 500% keeping the overshoot constant. The advantage of the phase shaper becomes evident considering the fact that the PID controller alone cannot handle such large variation in gain. The closed loop system, with the PID controller alone becomes unstable with two fold increase in gain.

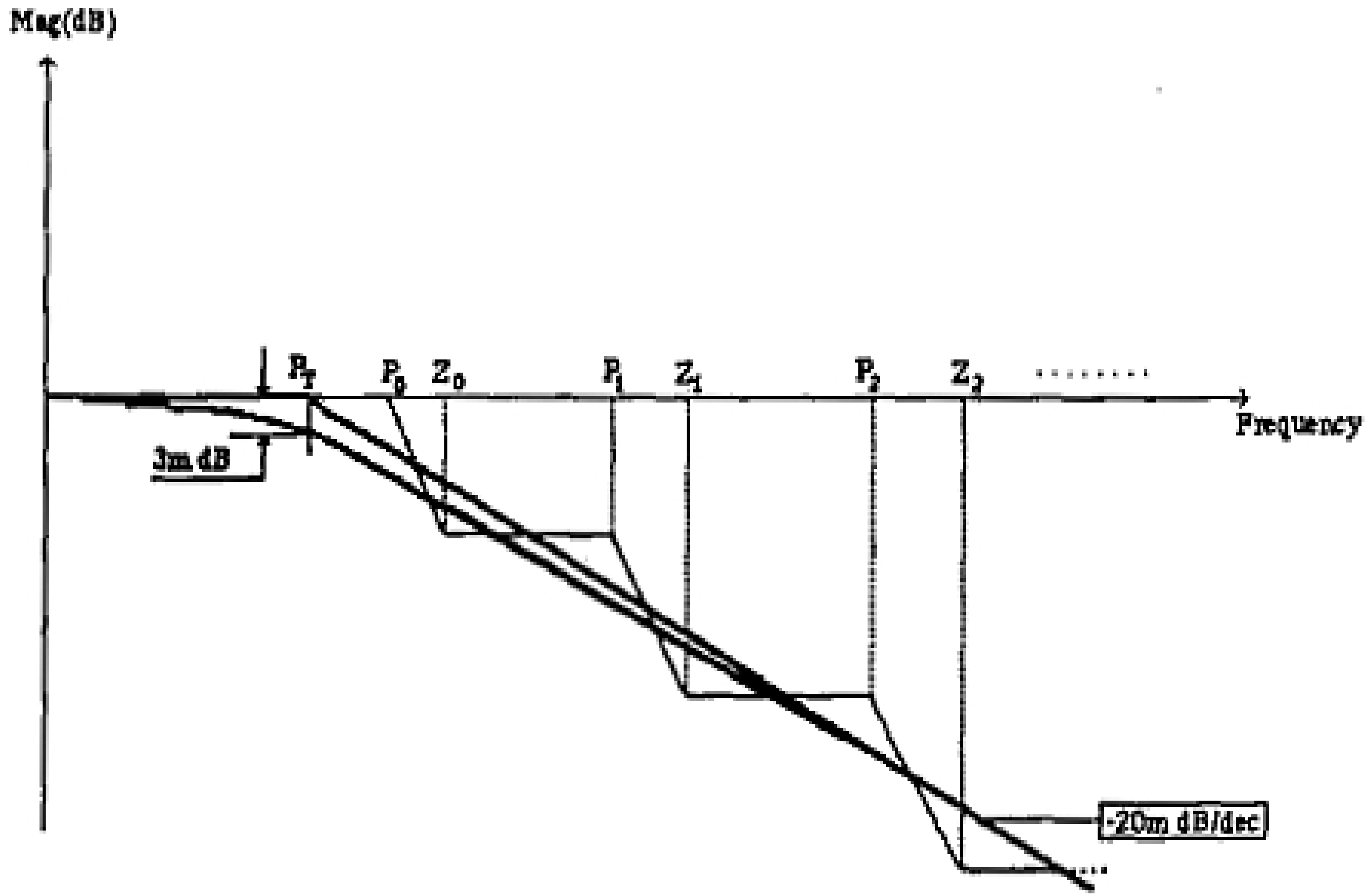
Controller effort is lesser with Fractional order PID



Controller output signal with and without the phase shaper, represented by solid line and dashed line respectively.

Fractal Pole-Zero & Constant Phase Element

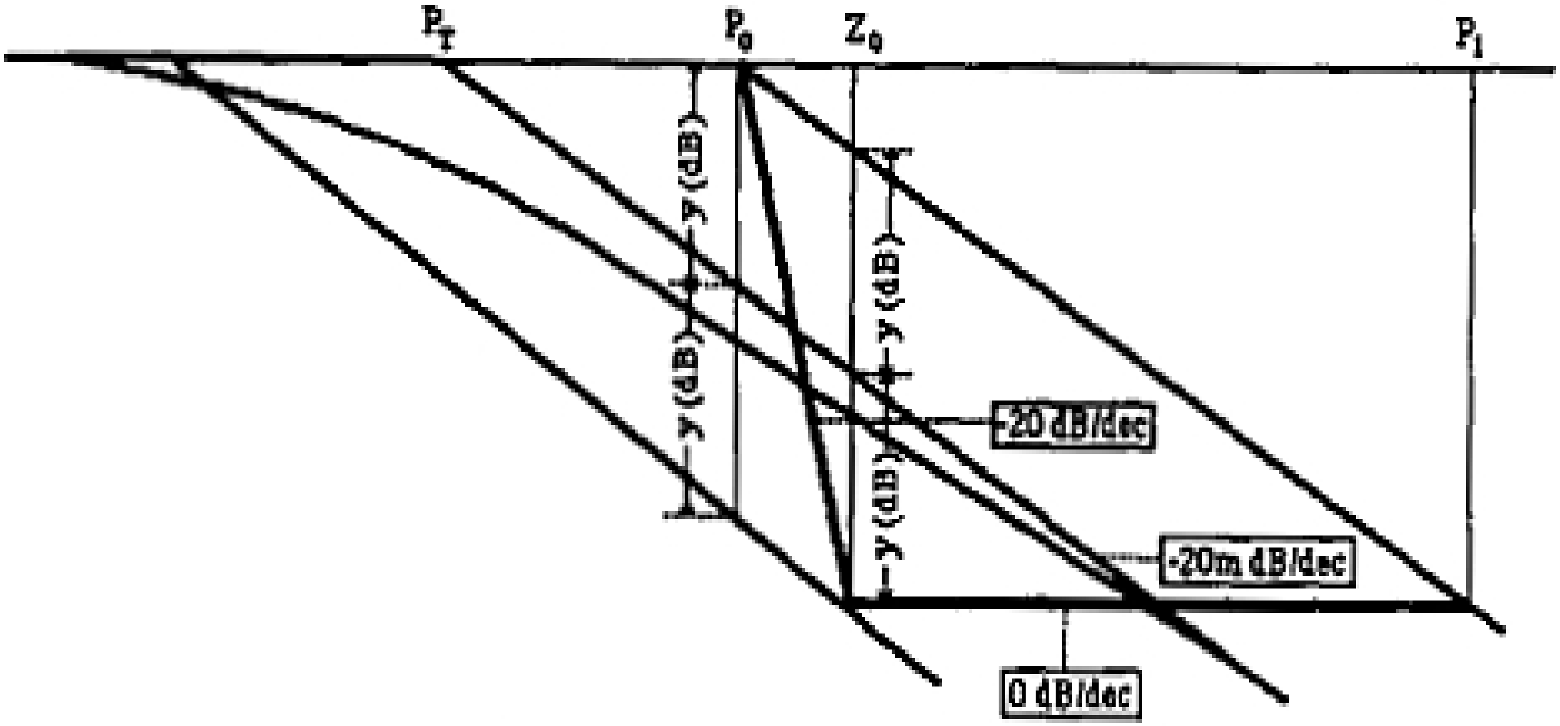
Realization of Fractional Order Integrator Differentiator and Transfer Functions by Fractal Singularity Structure-Bode's dream



Bode plot of an FPP with slope of -20mdB/dec. and its approximation as zigzag straight lines with individual slopes of -20dB/dec. and 0dB/dec.

contd…

Choosing the singularities for approximation by assuming
a constant error between the -20 dB/decade line and the zigzag lines.



contd…

Recursive Algorithm
Finite range of frequency, can be truncated
to a finite number *N*, and the approximation becomes, with interlaced poles and zeros:

$$H(s) = \frac{1}{\left(1 + \frac{s}{p_T}\right)^m} \approx \frac{\prod_{i=0}^{N-1}\left(1 + \frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1 + \frac{s}{p_i}\right)}$$

$$p_o = p_T 10^{[y/20m]}$$ the first pole,

$$z_0 = p_o 10^{[y/10(1-m)]}$$ the first zero,

$$p_1 = z_o 10^{[y/10m]}$$ the second pole,

$$z_1 = p_1 10^{[y/10(1-m)]}$$ the second zero,

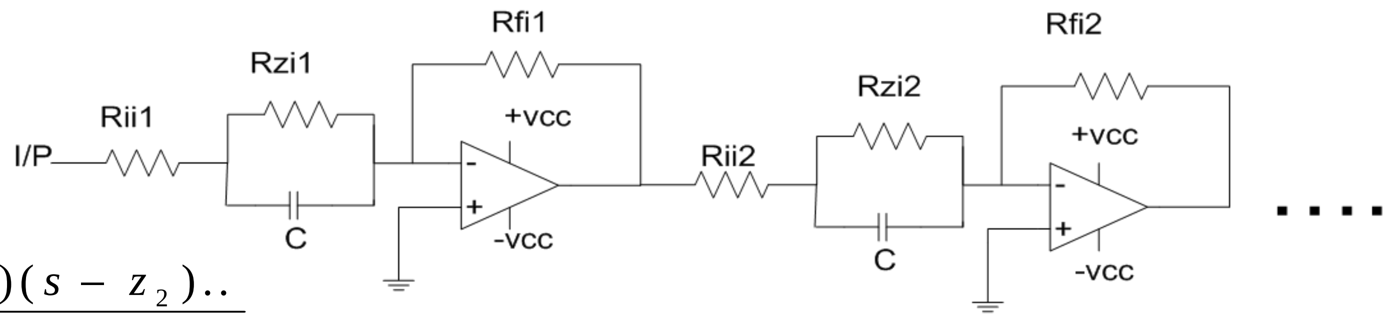
……

……

$$z_{N-1} = p_{N-1} 10^{[y/10(1-m)]}$$ the *N*th zero,

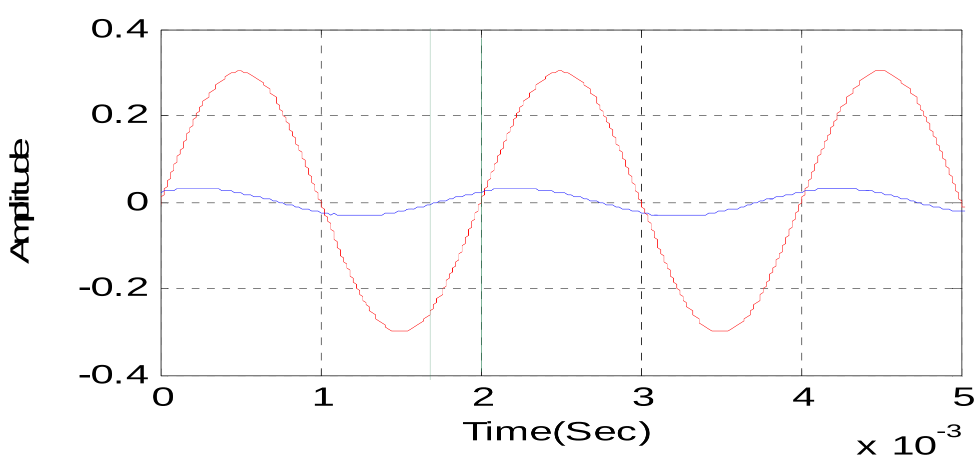
$$p_N = z_{N-1} 10^{[y/10m]}$$ the (*N*+1)th pole,

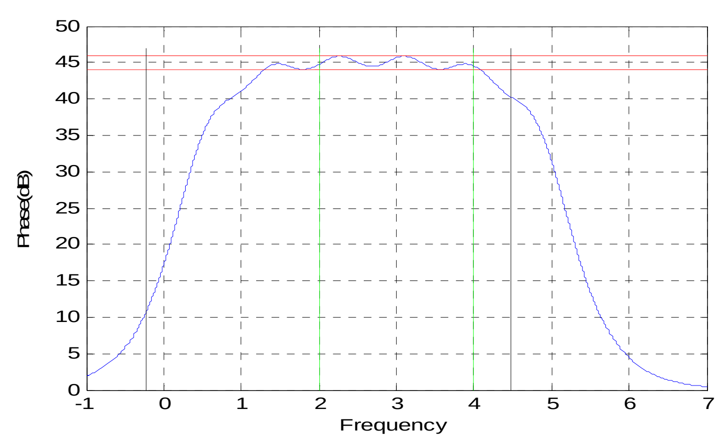
Fractal real poles and real zeros interlaced to give half order differentiator:



$$s^{1/2} \cong \frac{(s - z_1)(s - z_2)..}{(s - p_1)(s - p_2)..}$$

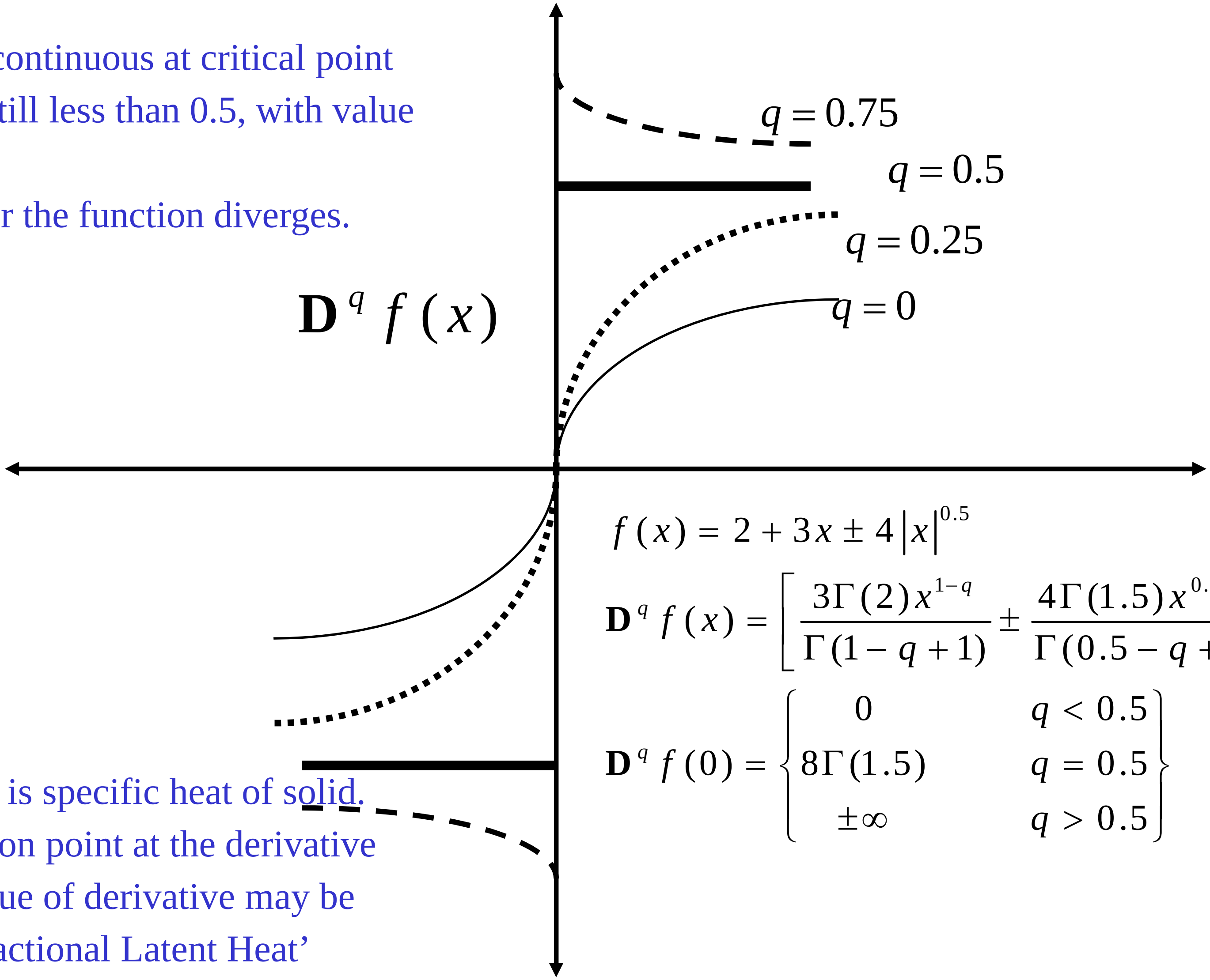
i	Zi	Pi	Ci	Rfi= Rii		Rzi	
				Ω	TP	Ω	TP
1	2.2537	6.0406	1µ	264.07k	500k	443.71k	500k
2	15.955	42.764	1µ	37.30k	50k	62.67k	100k
3	112.95	302.75	680nf	11.21k	20k	18.83k	20k
4	799.65	2143.3	68nF	10.94k	20k	18.39k	20k
5	5661.1	15173	10nF	10.51k	20k	17.64	20k
6	40078	107420	1nF	14.85k	20k	24.95k	50k





Fractional Differentiability at Critical Point:

The function is continuous at critical point from zero order till less than 0.5, with value zero.
Beyond 0.5 order the function diverges.



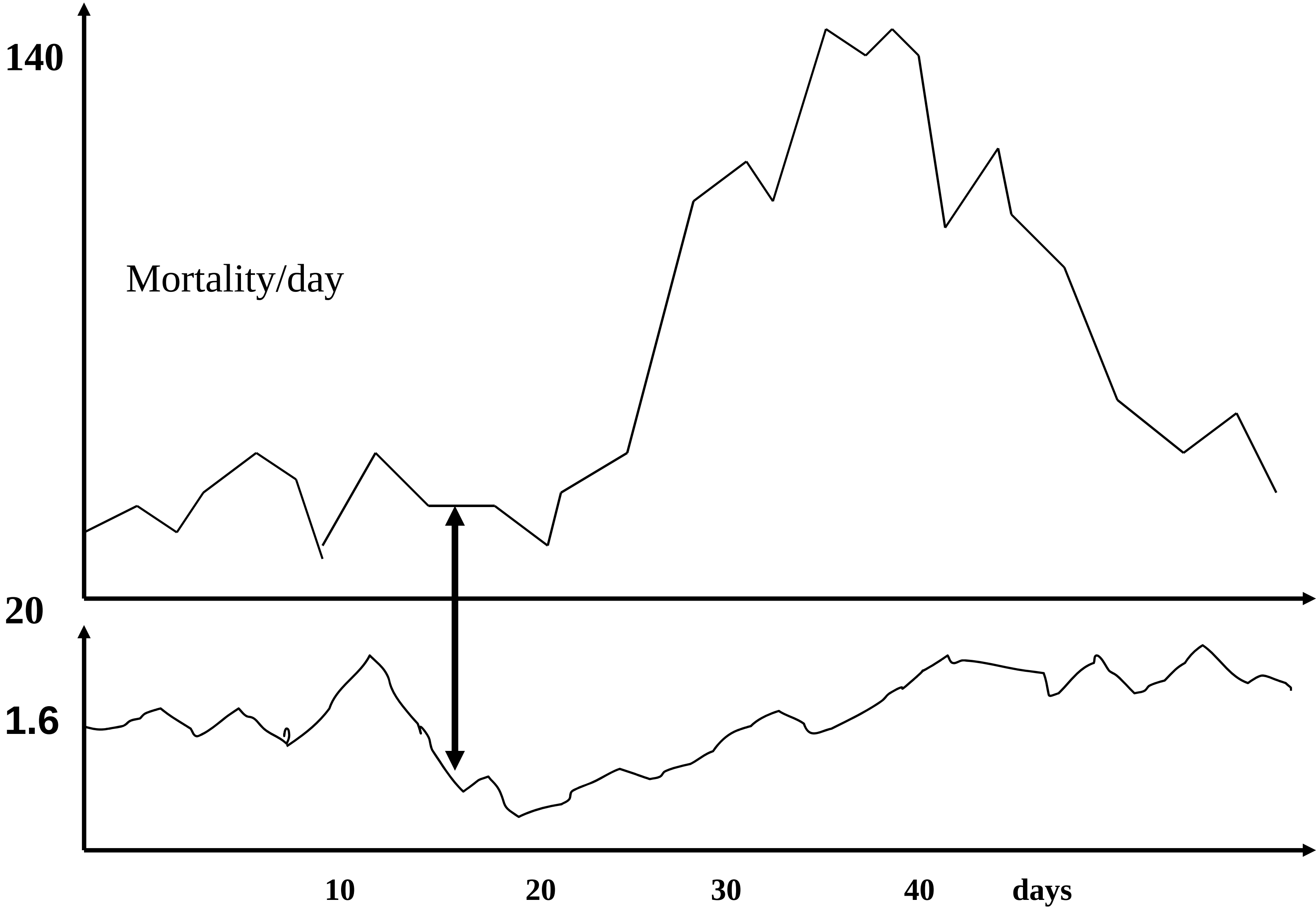
$$f(x) = 2 + 3x \pm 4|x|^{0.5}$$

$$\mathbf{D}^q f(x) = \left[\frac{3\Gamma(2)x^{1-q}}{\Gamma(1-q+1)} \pm \frac{4\Gamma(1.5)x^{0.5-q}}{\Gamma(0.5-q+1)}\right]$$

$$\mathbf{D}^q f(0) = \begin{cases} 0 & q < 0.5 \\ 8\Gamma(1.5) & q = 0.5 \\ \pm\infty & q > 0.5 \end{cases}$$

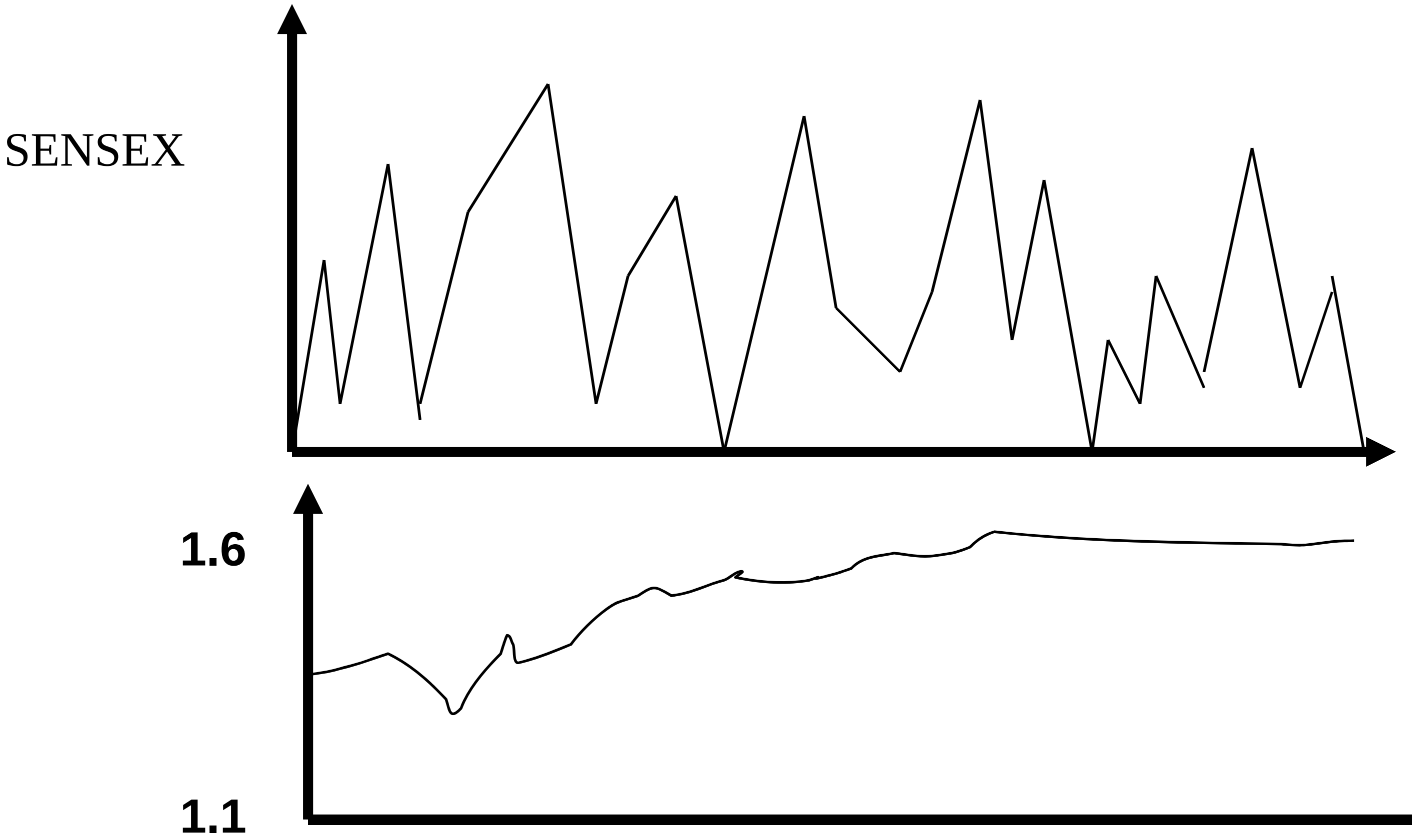
Say the function is specific heat of solid.
At phase transition point at the derivative order 0.5 the value of derivative may be Regarded as ‘Fractional Latent Heat’

Fractal Dimension indicating on-set of Epidemic



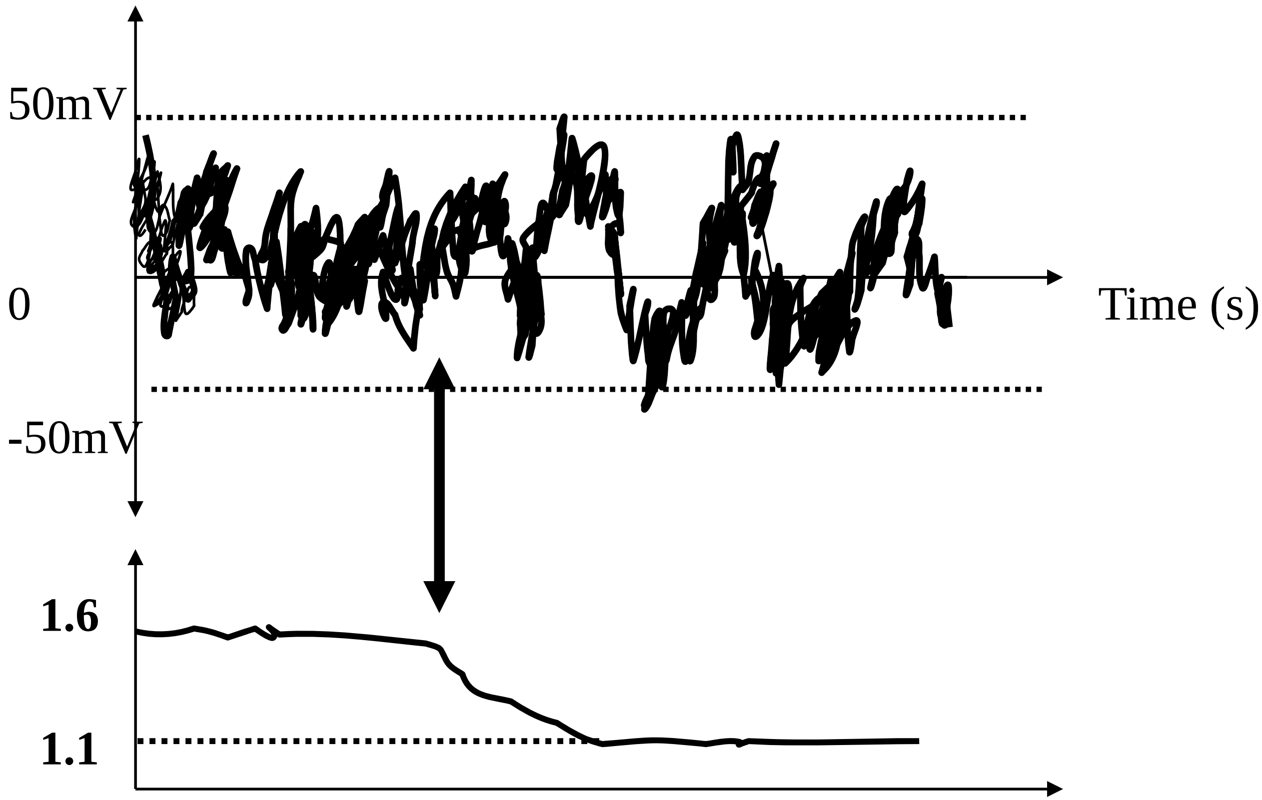
The cause of epidemics exhibited significant change in fractal dimension. Initially behaved as Brownian Motion 1.4-1.6, then dropped to 1.3-1.1 indicating on set of burst between 0-16 day (became regular from irregular) and again raised to 1.4 behaves variably.

A normal sensex of stock market



The dimension shows normal irregular ‘Brownian’ trading, with dimension slewing towards 1.5-1.6 indicating no bull bear or crash or financial irregularity! Trading is regular with normal irregularity as expected like White-Noise.

Exactly where the signal starts in high noise background



Signal buried in 85% white noise, the change in dimension indicates the first arrival time of signal.

Identification of singularity by LFD

1. Single singularity:

$$f(x) = ax^{\alpha} \qquad 0 < \alpha < 1$$

At ‘zero’ critical order gives the ‘order of singularity and Local Fractional Derivative gives strength of singularity.

$$\mathbf{D}^{\alpha} f(0) = a\Gamma(\alpha + 1)$$

2. Multiple singularity:

$$f(x) = ax^{\alpha} + bx^{\beta} \qquad 0 < \alpha < \beta < 1$$

$$\mathbf{D}^{\alpha} f(0) = a\Gamma(\alpha + 1) + b\frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} x^{\beta - \alpha} = a\Gamma(\alpha + 1)$$

Write:

$$G(x, \alpha) = f(x) - \left[f(0) + \frac{\mathbf{D}^{\alpha} f(0)}{\Gamma(\alpha + 1)} x^{\alpha} \right] = bx^{\beta}$$

$$\frac{d^q G(x, \alpha)}{dx^q} = b\frac{\Gamma(\beta + 1)}{\Gamma(\beta - q + 1)} x^{\beta - q}$$

$q = \beta$ is critical order

This way one extracts secondary singularity hidden by primary singularity

Fractional Taylor's series by LFD

Let $F(x_0, x - x_0; q) = \dfrac{d^q\left(f(x) - f(x_0)\right)}{\left[d(x - x_0)\right]^q}$ it is clear that $\mathbf{D}^q f(x_0) = F(x_0, 0, q)$

Using RL Integration, and by integration by parts we get

$$f(x) - f(x_0) = \frac{1}{\Gamma(q)} \int_0^{x-x_0} \frac{F(x_0, \xi; q)}{(x - x_0 - \xi)^{-q+1}} d\xi$$

$$= \frac{1}{\Gamma(q)} \left[F(x_0, \xi; q) \int (x - x_0 - \xi)^{q-1} d\xi \right]_0^{x-x_0} + \frac{1}{\Gamma(q)} \int_0^{x-x_0} \frac{dF(x_0, \xi; q)}{d\xi} \frac{(x - x_0 - \xi)^q}{q}$$

Provided the II term exists!

$$f(x) - f(x_0) = \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} (x - x_0)^q + \frac{1}{\Gamma(q+1)} \int_0^{x-x_0} \frac{dF(x_0, \xi; q)}{d\xi} (x - x_0 - \xi)^q d\xi$$

$$f(x) = f(x_0) + \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} (x - x_0)^q + R_q(x, x_0) \quad \text{for} \quad 0 < q < 1$$

More general:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{\Gamma(n+1)} \Delta^n + \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} \Delta^q + R_q(x, \Delta)$$

$$\Delta = x - x_0 > 0$$

Usage:

1. LFD provides the coefficient A in approximating function $f(x)$

By $f(x_0) + \frac{A}{\Gamma(q+1)}(x - x_0)^q$ in vicinity of x_0 .The terms are non trivial

For $q = \alpha$, the critical order

For $q = 1$ we get equation for tangent $f(x) = f(x_0) + \mathbf{D}^1(x_0)[(x - x_0)^q]$
This forms an equivalence class modeled by linear behavior. All curves passing through a point x_0 having same tangent.

2. Analogously all the functions (curves) with same 'critical order' α and same $\mathbf{D}^q$ will form an equivalence class modeled by power law x^α
This generalizes definition of tangents.

3 Useful to approximate irregular (non-differentiable) functions by piece-wise smooth (scaling) function; and survey of singularities.

4. Useful as Fractional curve fitting, start point of 'Fractional Differential Geometry'.
5. A useful world of mathematics for "CALCULUS ON FRACTALS"

Line/surface/volume integrals of Fractal Distribution:

Fractal Distribution represented by Fractional Continuous Medium and then we perform the integration.

The fractional Integrals are considered as an approximate integrals on fractals. This type of new approach is applicable in processes where fractal features of the process or the medium impose the necessity of using non traditional tools in regular smooth physical equations.

Smoothening the microscopic characteristics over physically infinitesimal Volume/surface/line transforms the initial fractal distribution into fractional continuum model. The order of fractional integration is of fractal dimension.

$$ {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)\,du $$

$$ {}_0D^{-d} f(r) = \int_V f(r)\,dV_d \approx \int_V \frac{r^{d-3}}{\Gamma^3(d/3)} dV_3 \qquad dV_d = K_3(r,d)\,dV_3 \quad K_3(r,d) = \frac{r^{d-3}}{\Gamma^3(d/3)} $$

$$ 2 < d < 3 $$

$$ dS_d = K_2(r,d)\,dS_2 \qquad 1 < d < 2 \qquad K_2(r,d) = \frac{r^{d-2}}{\Gamma^2(d/2)} $$

$$ dL_d = K_1(r,d)\,dL_1 \qquad 0 < d < 1 \qquad K_1(r,d) = \frac{r^{d-1}}{\Gamma(d)} $$

Some laws on Fractal Geometries

Flux through a fractal surface:

A flowing quantity trough a fractal surface be represented as:

$$\phi_{S_d} = \int_S \left(J(r,t) \bullet dS_d \right) \qquad dS_d \equiv K_2(r,d)\, dS_2 \qquad dS_d = \frac{r^{d-2}}{\Gamma^2(d/2)} dS_2$$

$$1 < d < 2$$

Gauss's law on Fractal:

$$\int_{\partial W} \left(J(r,t) \bullet dS_2 \right) = \int_W \mathbf{div}[J(r,t)]\, dV_3$$

$$dS_d = K_2(r,d)\, dS_2 \qquad dV_d = K_3(r,d)\, dV_3$$

$$\int_{\partial W} \left(J(r,t) \bullet dS_d \right) = \int_W (K_3(r,d_3))^{-1} \mathbf{div}[K_2(r,d_2) J(r,t)]\, dV_d$$

Stroke's law on Fractal:

$$\int_L \left(E \bullet dL_1 \right) = \int_S [\mathbf{curl} E]\, dS_2$$

$$dL_d = K_1(r,d)\, dL_1 \qquad dS_d = K_2(r,d)\, dS_2$$

$$\int_L \left(E \bullet dL_d \right) = \int_S \left(K_2(r,d_2) \right)^{-1} [\mathbf{curl} K_1(r,d_1) E]\, dS_d$$

Existence of Magnetic charges?

In normal cases of smooth geometry $\mathbf{div}\, B = 0$ indicating no magnetic charges at point exists . Magnetic mono-pole not possible.

Fractional generalization however gives: $\mathbf{div}[K_2(r, d_2) B] \neq 0$

$$\mathbf{div}\, B = B . \mathbf{grad}\, K_2(r, d_2)$$

For $d_2 \neq 2$; $\mathbf{grad}\, K_2(r, d_2) \neq 0$ indicating $\mathbf{div}\, B \neq 0$

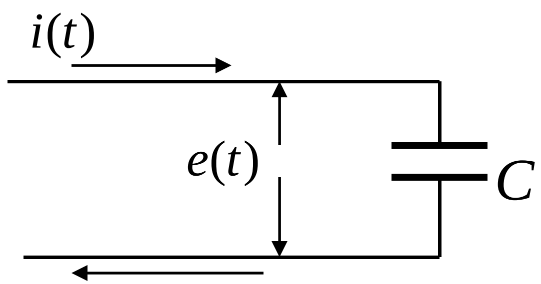
Existence of 'magnetic monopole charges' with magnitude of

$$e_m \approx B . \nabla K_2(r, d_2)$$

For fractal distribution we have thus all sets of conservation laws and set of Maxwell equations and electrodynamics do get modified.

This method perhaps is suitable for dusty plasma cases.

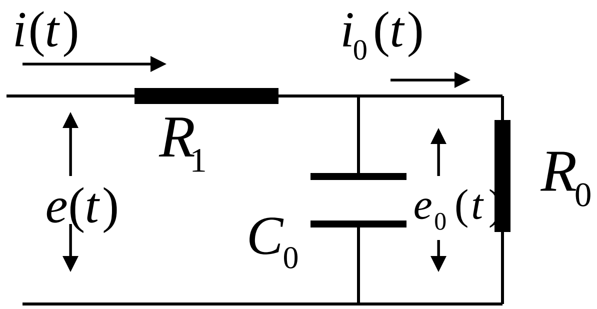
Self similar repeated prolong structure terminal relation and semi-differentiation



$$e(t) = \frac{1}{C}\int_0^t i(\xi)\,d\xi = \frac{1}{C}\frac{d^{-1}}{dt^{-1}}i(t)$$

$$i(t \le 0) = 0 = e(t \le 0)$$

$$s \leftrightarrow \frac{d}{dt};\ s^{-1} \leftrightarrow \frac{d^{-1}}{dt^{-1}};\ s^{v} \leftrightarrow \frac{d^{v}}{dt^{v}}$$



$$i_0(t) = \frac{e_0(t)}{R_0} \quad \text{.............................(1)}$$

$$i(t) - i_0(t) = C_0\frac{de_0(t)}{dt} \quad \text{...............(2)}$$

$$i(t) = \frac{e(t) - e_0(t)}{R_1} \quad \text{......................(3)}$$

Eliminating $e_0(t)$ & $i_0(t)$ from (1), (2) & (3)

$$[R_0 + R_1]i(t) + R_0R_1C_0\frac{di(t)}{dt} = e(t) + R_0C_0\frac{de(t)}{dt}$$

Continued Fraction Expansion form

$$i(0) = 0 = e(0)$$

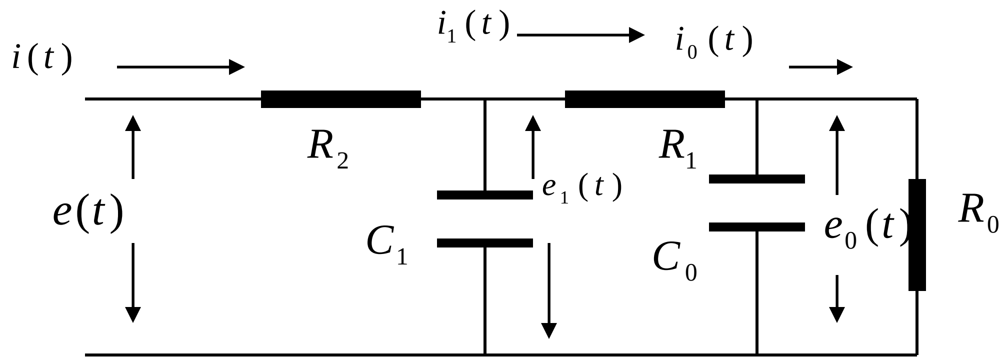
$$[R_0 + R_1]i(t) + R_0 R_1 C_0 \frac{di(t)}{dt} = e(t) + R_0 C_0 \frac{de(t)}{dt}$$

$$I(s)[R_0 + R_1 + R_0 R_1 C_0 s] - R_0 R_1 C_0 i(0) = E(s)[1 + R_0 C_0 s] - R_0 C_0 e(0)$$

$$\frac{E(s)}{I(s)} = \frac{R_0 + R_1 + R_0 R_1 C_0 s}{1 + R_0 C_0 s}$$

$$\frac{E(s)}{R_1 I(s)} = 1 + \frac{\frac{1}{R_1 C_0}}{s + \frac{1}{R_0 C_0}}$$

Expand the circuit further



$$i(t) = \frac{e(t) - e_1(t)}{R_2}$$

$$i(t) - i_1(t) = C_1 \frac{d e_1(t)}{d t}$$

$$\frac{E(s)}{R_2 I(s)} = 1 + \frac{\frac{1}{R_2 C_1}}{s + \frac{I_1(s)}{C_1 E_1(s)}}$$

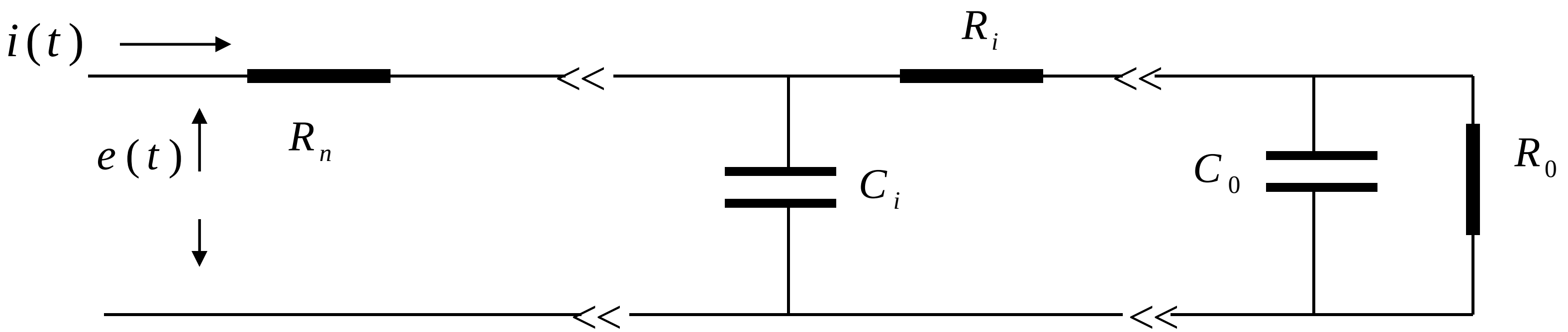
$$\frac{E_1(s)}{R_1 I_1(s)} = 1 + \frac{\frac{1}{R_1 C_0}}{s + \frac{1}{R_0 C_0}}$$

$$\omega_0 \equiv (\tau_0)^{-1} = (R_0 C_0)^{-1}, \omega_1 \equiv (R_1 C_0)^{-1}$$

$$\omega_2 \equiv (R_1 C_1)^{-1}, \omega_3 \equiv (R_2 C_1)^{-1}$$

$$\frac{E(s)}{R_2 I(s)} = 1 + \frac{\omega_3}{s + \frac{\omega_2}{1 + \frac{\omega_1}{s + \omega_0}}}$$

Generalizing and expanding to infinity



$$\frac{E(s)}{R_n I(s)} = 1 + \cfrac{\omega_{2n-1}}{s + \cfrac{\omega_{2n-2}}{1 + \cfrac{\omega_{2n-3}}{s + \ldots \cfrac{\omega_1}{s + \omega_0}}}} = 1 + \frac{\omega_{2n-1}}{s +} \frac{\omega_{2n-3}}{1 +} \frac{\omega_{2n-3}}{s +} \ldots \frac{\omega_2}{1 +} \frac{\omega_1}{s +} \frac{\omega_0}{1}$$

$$\omega_{2j} = (R_j C_j)^{-1}; \omega_{2j+1} = (R_{j+1} C_j)^{-1}$$

Simplifying CFE

$$\frac{E(s)}{R_n I(s)} = 1 + \frac{\omega_{2n-1}}{s+} \frac{\omega_{2n-2}}{1+} \frac{\omega_{2n-3}}{s+} \cdots\cdots \frac{\omega_2}{1+} \frac{\omega_1}{s+} \frac{\omega_0}{1}$$

$$v_j = \frac{\omega_j}{s}$$

$$\frac{E(s)}{R_n I(s)} 1 + \frac{v_{2n-1}}{1+} \frac{v_{2n-2}}{1+} \frac{v_{2n-3}}{1+} \cdots\cdots\cdots\cdots \frac{v_2}{1+} \frac{v_1}{1+} \frac{v_0}{1}$$

$$C_0 = C_1 = C_2 = \cdots\cdots\cdots\cdots = C_{n-1} = C$$

$$R_0 = R_1 = R_2 = \cdots\cdots\cdots\cdots = R_{n-1} = R ; R_n = \frac{1}{2} R$$

$$v_{2n-1} = \frac{2}{RCs} = 2v$$

$$\frac{2E(s)}{RI(s)} = 1 + 2 \frac{v}{1+} \frac{v}{1+} \frac{v}{1+} \cdots\cdots\cdots\cdots \frac{v}{1+} \frac{v}{1}$$

CFE in limit of very large number of stages:

By induction $$\frac{2E(s)}{RI(s)} = 1 + 2\,\frac{v}{1+}\,\frac{v}{1+}\cdots\cdots\frac{v}{1+}\,\frac{v}{1} \quad \cdots\cdots(1)$$

$$\frac{v}{1+}\,\frac{v}{1+}\cdots\cdots\frac{v}{1+}\,\frac{v}{1} = \frac{\sqrt{4v+1}}{1+\left[\frac{\sqrt{4v+1}-1}{\sqrt{4v+1}+1}\right]^{2n+1}} - \frac{\sqrt{4v+1}}{2} - \frac{1}{2} \quad \cdots\cdots(2)$$

From (1) and (2) and dividing by $2\sqrt{v}$ we obtain:

$$\frac{E(s)}{I(s)}\sqrt{\frac{Cs}{R}} = \sqrt{\frac{4v+1}{4v}}\left[\frac{[\sqrt{4v+1}+1]^{2n+1} - [\sqrt{4v+1}-1]^{2n+1}}{[\sqrt{4v+1}+1]^{2n+1} + [\sqrt{4v+1}-1]^{2n+1}}\right] \quad \cdots\cdots(3)$$

Graphically one estimate RHS of (3) to unity as for large ‘*n*’ and seemingly wide spread of “*v*”; implying, RHS is within 2% of unity for wide frequency/time range

$$\frac{E(s)}{I(s)}\sqrt{\frac{Cs}{R}} \approx 1 \qquad 6 \le v \le \frac{1}{6}n^2$$

$$E(s) \approx \sqrt{\frac{R}{Cs}}\,I(s) \qquad 6RC \le \frac{1}{s} \le \frac{1}{6}n^2RC$$

$$e(t) \approx \sqrt{\frac{R}{C}}\,\frac{d^{-1/2}}{dt^{-1/2}}\,i(t) \qquad 6RC \le t \le \frac{1}{6}n^2RC$$

Computer delay a case of Fractional Brownian motion!!

Dynamics of delay in computer based systems demonstrate the stochastic behavior. The delay of random nature has wide spikes and if a statistics be taken, it is like a power law, with pronounced tail. Effect of network delay in control system is very widely researched topic and has practical relevance to modern computer control industry.

The classical method of fluctuation dynamics is by Gaussian assumption of the random behavior, and dynamics of the same applied to fluctuations in financial assets gives integer order differential equation formulations giving Gaussian solutions. We can develop a new extension of fractality concept for dynamics of random delay. We can propose a possible fractional calculus approach to model the evolution of stochastic dynamics of random delay. We consider the fractional form of Langevin type stochastic differential equation, and replace standard 'white noise' Gaussian stochastic driving excitation force, by 'shot-noise' whose each pulse has randomized amplitude. The proposed fractional dynamic stochastic approach allows obtaining the probability distribution function (pdf) of the modeled random delay. It can be proposed to describe the dynamics of random delay of computer control system along with fractional stationary condition as below:

$$\frac{d^q}{dt^q}\tau(t) = \lambda\tau(t) + F(t) \qquad 0 < q \le 1$$

$$\left.\frac{d^{q-1}}{dt^{q-1}}\tau(t)\right|_{t=0} = \tau_0$$

For q=1 the process is standard Langevin with solution as Poison's process as:

$$\tau(t;\tau_0,F) = e^{\lambda t}\tau_0 + \int_0^t dt' F(t') e^{\lambda(t-t')}$$

$$E_{1,1}(z) = e^z$$

$$\tau(t;\tau_0,F) = t^{q-1}E_{q,q}\left(\lambda t^q\right)\tau_0 + \int_0^t dt' F(t')(t-t')^{q-1}E_{q,q}\left(\lambda(t-t')^q\right)$$

Stock Market & Pricing etc.

Mandelbrot who introduced the term ‘fractal’ observed that in addition to being non-Gaussian, the stochastic process of financial returns show interesting property of ‘self-similarity’. That is the statistical dependencies of ‘random phenomena like financial returns. Brownian motion, have similar functional form for various time increments.

The classical method of fluctuation dynamics is by Gaussian assumption of the random behavior, and dynamics of the same applied to fluctuations in financial assets is widely used in mathematical finance because of simplification it provides in analytical calculations

Similarly dynamics of stock market may too be treated as Brownian Motion & its generalization as Fractional Brownian motion, leading to ‘long ranged correlated’ power law!!

And Several More……………………………..

Prologue

EXPRESSED DIFFERENTLY WE MAY SAY THAT NATURE WORKS WITH FRACTIONAL DERIVATIVES

WE MAY EXPRESS OUR CONCEPTS IN NEWTONIAN TERMS IF WE FIND CONVENIENT, BUT IF WE DO SO, WE MUST REALIZE THAT WE HAVE MADE A TRANSLATION INTO A LANGUAGE WHICH IS FOREIGN TO THE SYSTEM WE ARE STUDYING

FRACTIONAL CALCULUS IS THE CALCULUS OF XXI CENTURY

At the end one has to solve

Fractional Differential Equations