APPRECIATION OF GENERALIZED CALCULUS

MATHEMATICO-PHYSICS OF GENERALIZED CALCULUS

Module_II

Shantanu Das
RRPS
Reactor Control Division
BARC
2009-2010

Reimann Liouvelli (RL) fractional integration-antiderivative

Repeated *n*-fold integration generalization to arbitrary order

$$d_t^{-1} f(t) = \int_0^t f(\tau) d\tau$$

$$d_t^{-2} f(t) = \int_0^t \int_0^t f(\tau) d\tau\, d\tau = \int_0^t (t-\tau) f(\tau) d\tau$$

$$d_t^{-3} f(t) = \int_0^t \int_0^t \int_0^t f(\tau) d\tau\, d\tau\, d\tau = \frac{1}{2}\int_0^t (t-\tau)^2 f(\tau) d\tau$$

$$d_t^{-n} f(t) = \underbrace{\int_0^t \int_0^t \cdots\cdots\cdots\cdots \int_0^t f(\tau) d\tau}_{n} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau) d\tau$$

$$d_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau = \int_0^t \Phi_\alpha(t-\tau) f(\tau) d\tau$$

$$\Phi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$$

Fractional integration-shape change with compressed limit

$$ {}_0D_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} f(\xi)\, d\xi $$

$$ \xi = t - x^{\frac{1}{q}}, \quad d\xi = -\frac{1}{q} x^{\frac{1}{q}-1} dx = -\frac{1}{q} x^{\frac{1-q}{q}} dx $$

$$ {}_0D_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_{t^q}^0 (t - t + x^{\frac{1}{q}})^{q-1} \left(-\frac{1}{q} x^{\frac{1-q}{q}}\right) f(t - x^{\frac{1}{q}})\, dx $$

$$ {}_0D_t^{-q} f(t) = \frac{1}{q\Gamma(q)} \int_{t^q}^0 -(x^{\frac{1}{q}})^{q-1} x^{\frac{1-q}{q}} f(t - x^{\frac{1}{q}})\, dx $$

$$ {}_0D_t^{-q} f(t) = \frac{1}{\Gamma(q+1)} \int_0^{t^q} f(t - x^{\frac{1}{q}})\, dx $$

Fractional integration changes the functional shape example

$$ {}_0D_t^{-q} f(t) = \frac{1}{\Gamma(q+1)} \int_0^{t^q} f\left(t - x^{\frac{1}{q}}\right) dx $$

$f(t) = (t+2)^2 = t^2 + 4t + 4$ to find semi integration of this i.e. $D^{-1/2} f(t)$

$f(t - x^{1/q}) = (t - x^2)^2 + 4(t - x^2) + 4$

put; $t = 0$; $f(x) = 4 - 4x^2 + x^4$, is function near $t = 0$

Integration

$$ {}_0D_{t\to 0+}^{-0.5} f(t) = \frac{1}{\Gamma(1.5)} \int_0^{\sqrt{0+}} (4 - 4x^2 + x^4)\, dx $$

put; $t = 1$; $f(x) = 9 - 6x^2 + x^4$, is function at $t = 1$

$$ {}_0D_1^{-0.5} f(t) = \frac{1}{\Gamma(1.5)} \int_0^{\sqrt{1}} (9 - 6x^2 + x^4)\, dx $$

put; $t = 3$; $f(x) = 25 - 10x^2 + x^4$, is function at $t = 1$

$$ {}_0D_3^{-0.5} f(t) = \frac{1}{\Gamma(1.5)} \int_0^{\sqrt{3}} (25 - 10x^2 + x^4)\, dx $$

The function keeps on changing its shape as time grows, area under the shape changing curve.

Geometrical Interpretation of Fractional Integral

$$I_t^{\alpha} f(t) = D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(t)(t-\tau)^{\alpha-1} d\tau$$

let $$g(\tau) = \frac{1}{\Gamma(\alpha+1)} \{ t^{\alpha} - (t-\tau)^{\alpha} \}$$

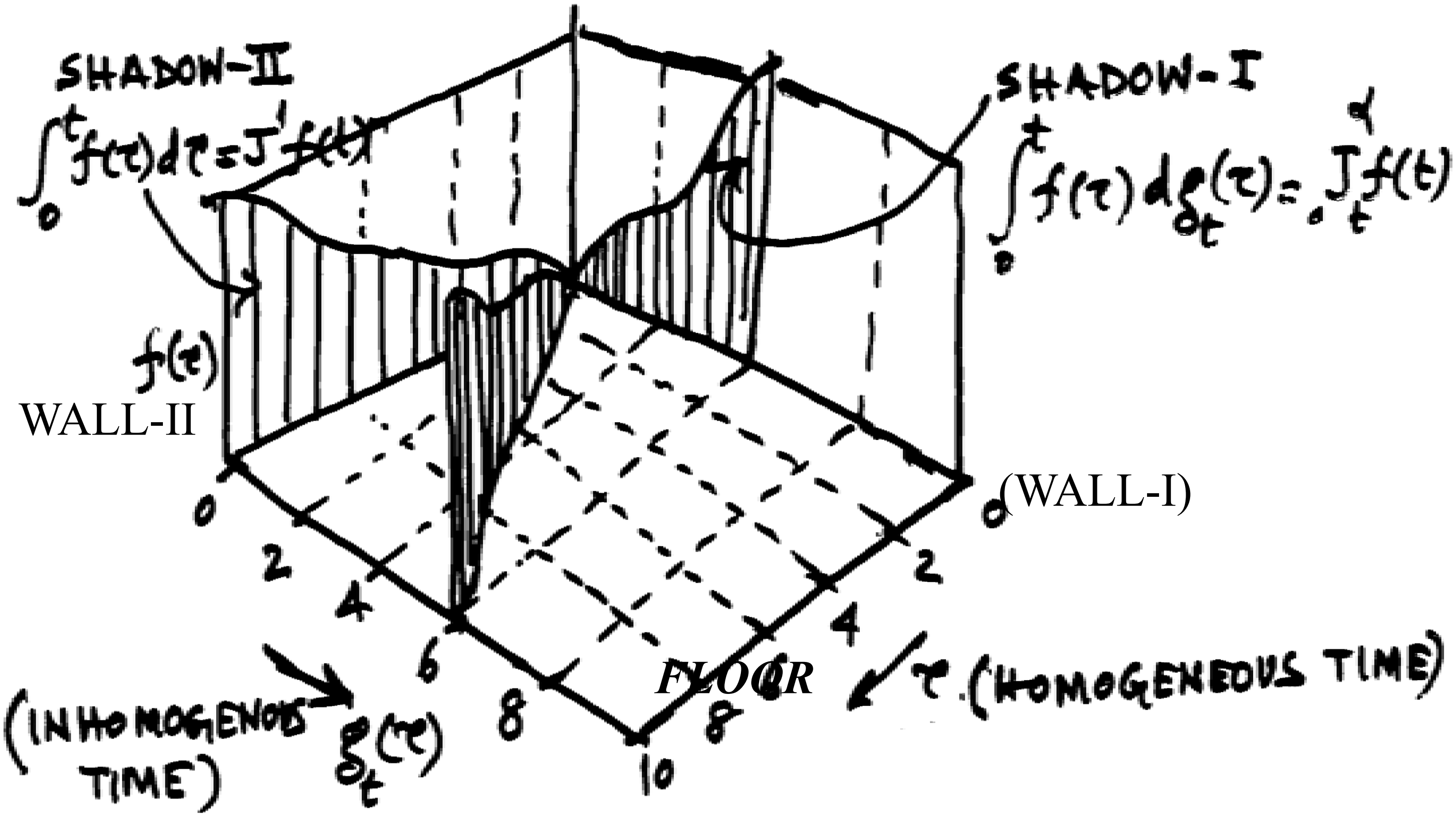
$$dg(\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau$$

$$\tau \to g(\tau)$$

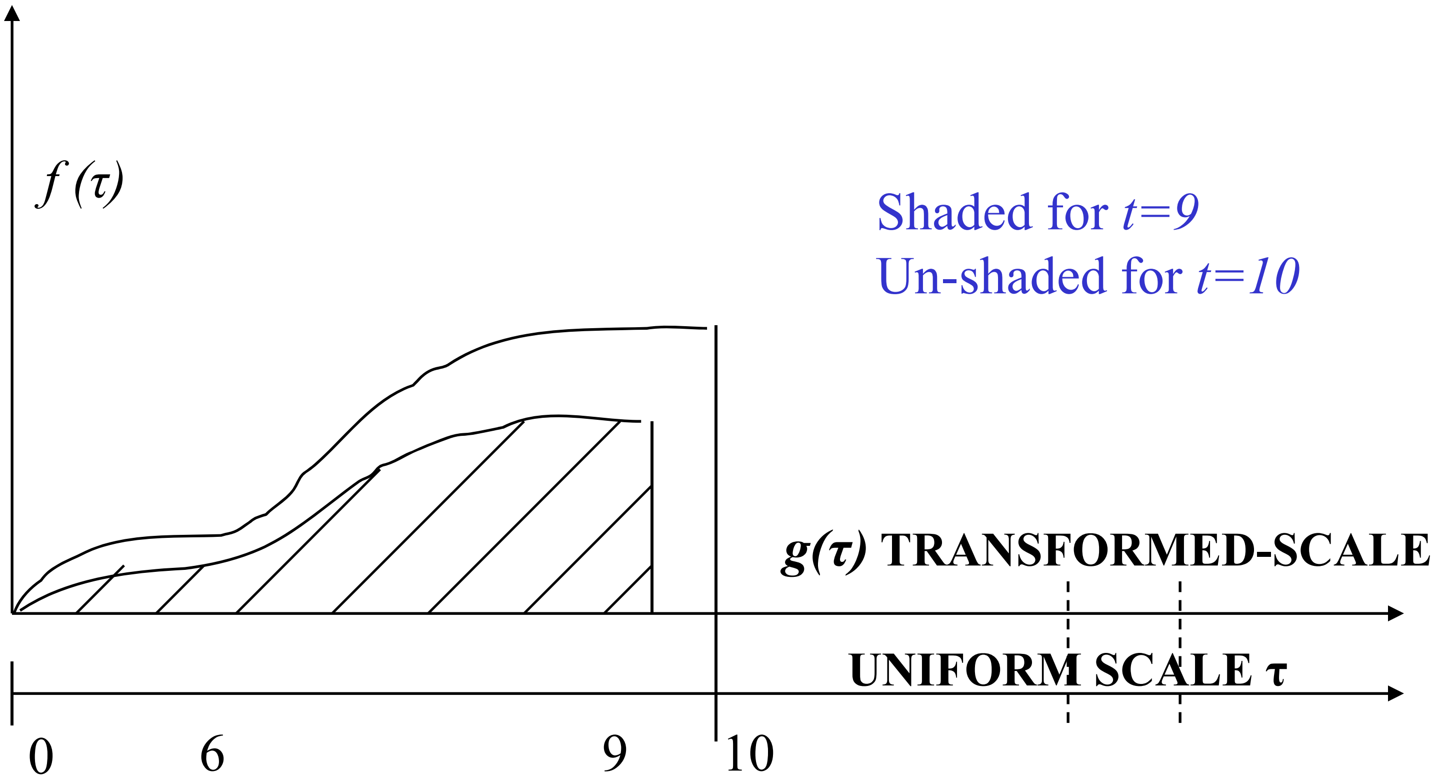
Change from Homogeneous time to Non-homogeneous time scale

$$I_t^{\alpha} f(t) = D_t^{-\alpha} f(t) = \int_0^t f(\tau) dg(\tau)$$

3-D Interpretation of Fractional Integration



Observations for ${}_0I^{\alpha}$



For t=10, τ is varied from 0-10, $g(\tau)$ is transformed & $f(\tau)$ is plotted. The representation shows the change in shape of the curve from t=9 to t=10 , and the integration under new shape. The difference with integer order integration is the new shape of f from 0-10 as compared with 0-9.

Interpretation from Geometric figure

S_N, $v(\tau)$, τ Individual Measurement
S_O, $v_O(t)$, t Observer Measurement.

$$S_N = \int_0^{\tau} v(\tau)\, d\tau = {}_0 I_t^1 v(t)$$

$$S_O = \int_0^{t} v(\tau)\, dg_t(\tau) = {}_0 I_t^{\alpha} v(t)$$

$$v(t) = {}_0 D_t^{\alpha} S_O(t)$$ ← Individual Velocity

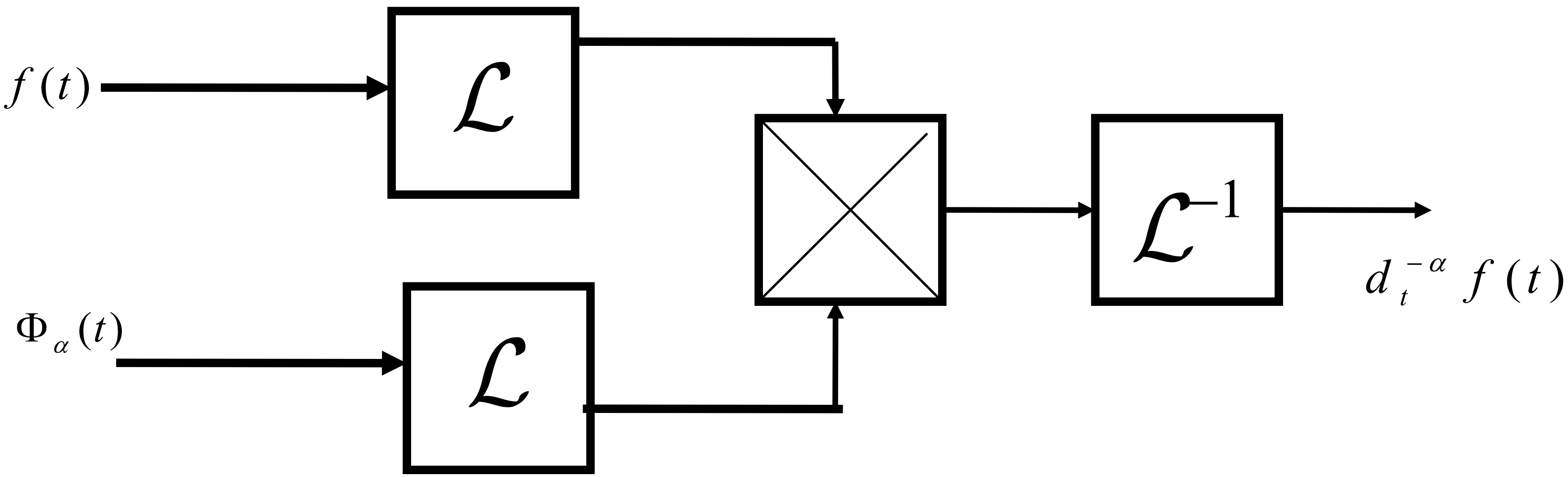
$$v_O(t) = \frac{d}{dt} S_O(t) = \frac{d\, {}_0 I_t^{\alpha} v(t)}{dt} = {}_0 D_t^{1-\alpha} v(t) = S_O^{(1)}(t)$$

Convolution with power function RL fractional integration:

$$d_t^{-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\,d\tau = [f(t)] * \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) = f(t) * \Phi_\alpha(t)$$

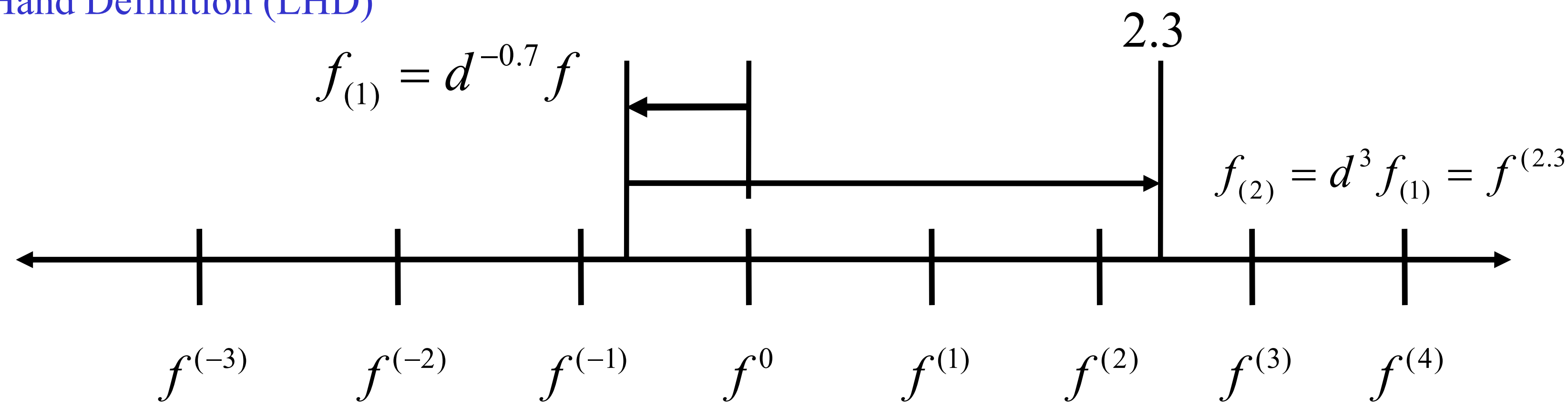
$$\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}; \mathcal{L}\left\{t^\mu\right\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}; \mathcal{L}\left\{\Phi_\alpha(t)\right\} = s^{-\alpha}$$

$$\mathcal{L}\left\{{}_0 d_t^{-q} f(t)\right\} = \frac{1}{\Gamma(q)} \mathcal{L}\left\{t^{q-1}\right\} \mathcal{L}\left\{f(t)\right\} = \frac{1}{\Gamma(q)} \left\{ \frac{\Gamma(q)}{s^q} \right\} F(s) = s^{-q} F(s)$$

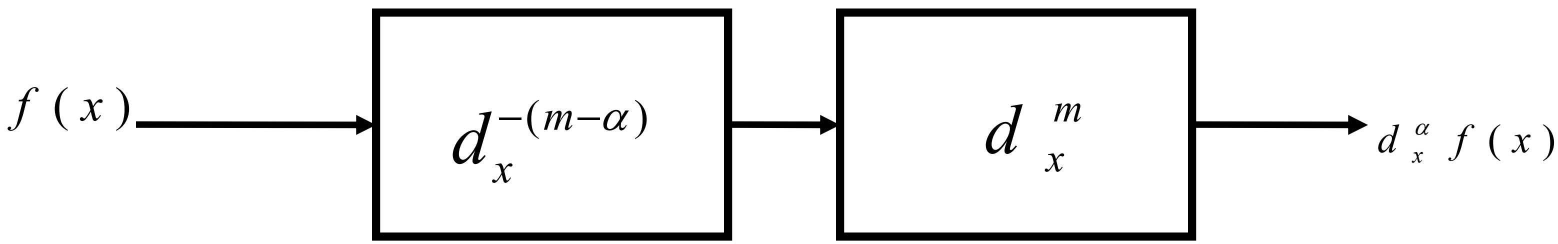


Reimann Liouvelli (RL) Fractional derivative

Left Hand Definition (LHD)



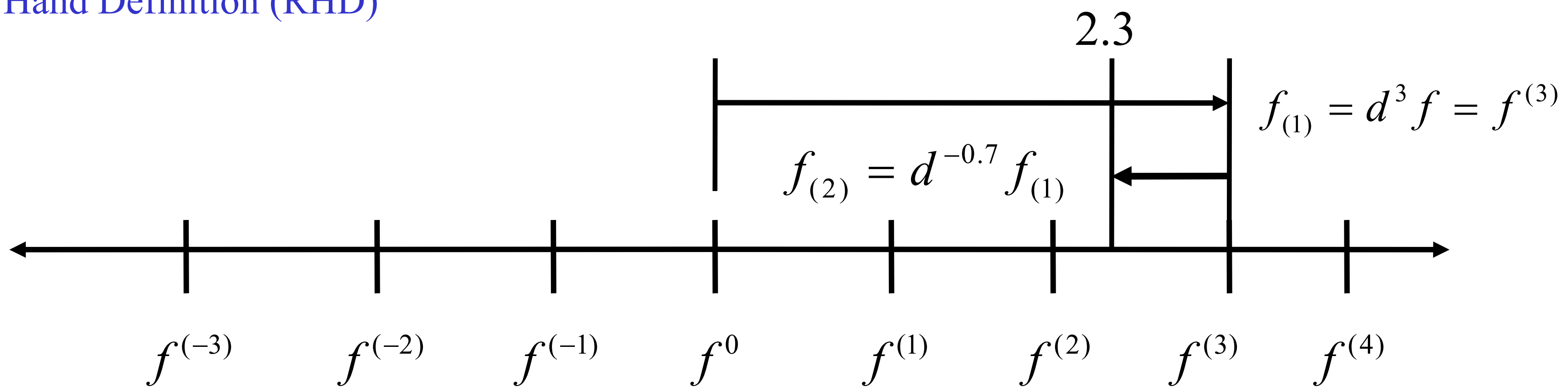
Here '*m*' is the integer just greater than fractional order of derivative



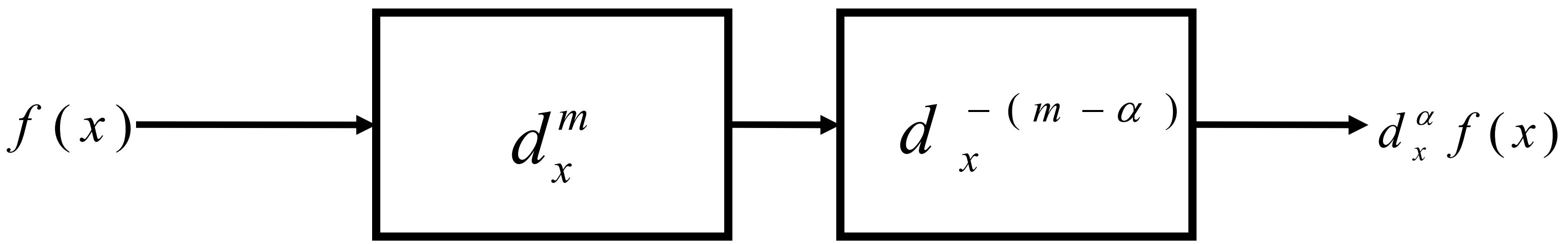
$$d_x^{\alpha} f(x) = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-\tau)^{-\alpha-1+m} f(\tau)\,d\tau\right]$$

Caputo (1967) Fractional derivative

Right Hand Definition (RHD)



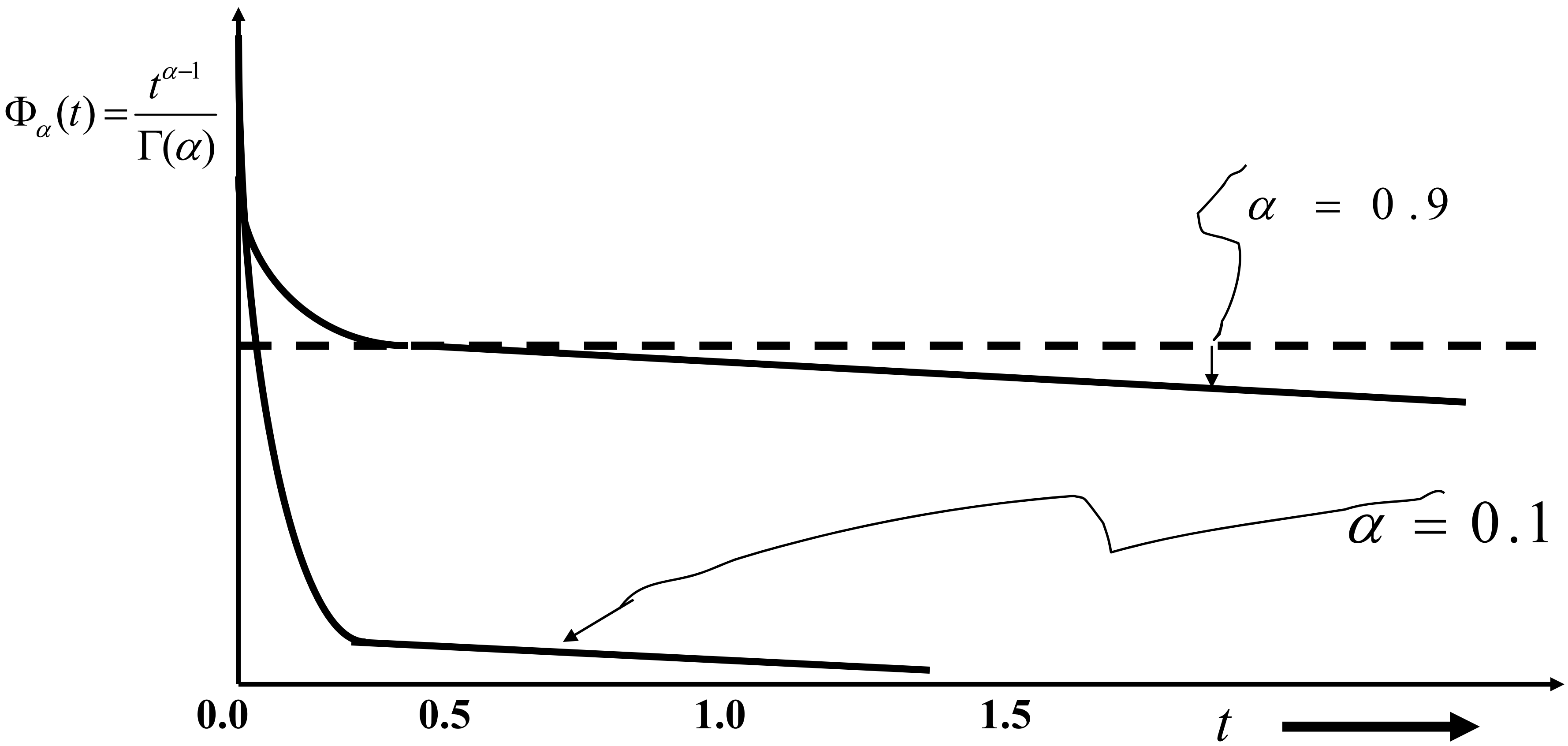
Here '*m*' is the integer just greater than the fractional order derivative



$$d_x^\alpha f(x) = \left[\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{-\alpha-1+m} \frac{d^m f(\tau)}{d\tau^m} d\tau \right]$$

Nature of Power Function in Fractional Differ-integration:

$$I_t^{\alpha} f(t) = d_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$



$\lim_{\alpha \to 1} \Phi_\alpha(t) = H(t)$ Heaviside's Unit Step Function

$\lim_{\alpha \to 0} \Phi_\alpha(t) = \delta(t)$ Unit Delta Function

Caputo-Riemann Liouvelli Fractional Derivative Equivalence

Only if the initial conditions are static to zero should these be equal.

$$\frac{1}{\Gamma(m-q)}\int_a^x \frac{f^{(m)}}{[x-y]^{q-n+1}}dy + \sum_{k=0}^{m-1}\frac{[x-a]^{k-q} f^{(k)}(a)}{\Gamma(k-q+1)} = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-q)}\int_a^x \frac{f(y)}{[x-y]^{q-m+1}}dy\right]$$

$m > q$

$$[{}_a d_t^q f(t)]_{RL} = [{}_a^C d_t^q f(t)] + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}; n > q$$

This is generalization of fundamental theorem if calculus, i.e. differentiation of integration commutes and integration of differentiation is separated by initial values of function at start.

Duality

For LHD (RL) fractional derivative of constant is not zero $d_x^{\alpha} C \neq 0 = C\left[\Gamma(1-\alpha)\right]^{-1} x^{-\alpha}$
This fact lead to RL or LHD approach to consider "limit of differentiation" (lower terminal) to minus infinity. The physical significance of this minus infinity is starting the physical processes at time immemorial!! However lower limit to minus infinity is necessary abstraction for steady state (sinusoidal) response. For LHD(RL) $d_x^{\alpha-1} f(0), d_x^{\alpha-2} f(0)$ are required. This posses physical interpretability.

For RHD the fractional derivative of the constant is zero. But this requires $f(0)=0,$ also with $f^{(1)} = f^{(2)} = \ldots f^{(m)} = 0$ in mathematical world this posses a problem.

Our mathematical tools go far beyond our physical understanding

Fractional Integration of exponential function & origin of Miller-Ross function & higher trigonometric functions:

$$f(t) = e^{at}$$

$$D^{-v}e^{at} = \frac{1}{\Gamma(v)}\int_0^t (t-\xi)^{v-1}e^{a\xi}d\xi$$

$$x = t - \xi$$

$$D^{-v}e^{at} = \frac{e^{at}}{\Gamma(v)}\int_0^t x^{v-1}e^{-ax}dx$$

Above is not elementary function, but is closely related to incomplete Gamma function

$$\gamma^*(v,t) = \frac{1}{t^v\Gamma(v)}\int_0^t \xi^{v-1}e^{-\xi}d\xi$$

$$D^{-v}e^{at} = t^v e^{at}\gamma^*(v,at) = E_t(v,a)$$

similarly

$$D^{-v}\cos at = \frac{1}{\Gamma(v)}\int_0^t \xi^{v-1}\cos a(t-\xi)d\xi = C_t(v,a)$$

$$D^{-v}\sin at = \frac{1}{\Gamma(v)}\int_0^t \xi^{v-1}\sin a(t-\xi)d\xi = S_t(v,a)$$

Fractional differentiation of exponential Miller-Ross & trigonometric functions

$$f(t) = e^{at}$$

$$D^{\mu} e^{at} = D^{m}[D^{-v} e^{at}]; \mu = m - v$$

$$D^{-v} e^{at} = E_t(v, a)$$

$$D^{m} E_t(v, a) = E_t(v - m, a) = E_t(-\mu, a)$$

$$D^{\mu} e^{at} = E_t(-\mu, a)$$

Similarly, one can have

$$D^{\mu} \cos at = C_t(-\mu, a)$$

$$D^{\mu} \sin at = S_t(-\mu, a)$$

$$f(t) = E_t(\lambda, a)$$

$$D^{\mu} E_t(\lambda, a) = D^{m}[D^{-v} E_t(\lambda, a)]; \mu = m - v$$

$$D^{-v} E_t(\lambda, a) = E_t(\lambda + v, a)$$

$$D^{\mu} E_t(\lambda, a) = E_t(\lambda - \mu, a)$$

Similarly, one can have

$$D^{\mu} C_t(\lambda, a) = C_t(\lambda - \mu, a)$$

$$D^{\mu} S_t(\lambda, a) = S_t(\lambda - \mu, a)$$

Integration & Differentiation by same formula?

$$D^{\mu} f(t) = [D^{-\nu} f(t)]_{\nu=-\mu}$$

Let $f(t) = t^{\lambda}$

$$D^{-\nu} t^{\lambda} = \frac{1}{\Gamma(\nu)} \int_0^t (t-\xi)^{\nu-1} \xi^{\lambda} d\xi ; t > 0, \lambda > -1$$

$$= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\nu+1)} t^{\lambda+\nu}$$

$$D^{\mu} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)} t^{\lambda-\mu} ; t > 0, \lambda > -1$$ Euler’s formula

But $D^{\mu} t^{\lambda} = \frac{1}{\Gamma(-\mu)} \int_0^t (t-\xi)^{-\mu-1} \xi^{\lambda} d\xi$ is absurd, as it does not converge

Paradox!

Fractional derivative and integration by sign replacement

$$D^{v}e^{t} = D^{v}\sum_{k=0}^{\infty}\frac{t^{k}}{k!} = \sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{k!\,\Gamma(k-v+1)}t^{k-v} = \sum_{k=0}^{\infty}\frac{t^{k-v}}{\Gamma(k-v+1)}$$

$$v = n$$

$$D^{n}e^{t} = \sum_{k=0}^{\infty}\frac{t^{k-n}}{\Gamma(k-n+1)} = \sum_{j=0}^{\infty}\frac{t^{j}}{j!} = e^{t}$$ Independent of order 'n'

$$n = -1$$

$$D^{-1}e^{t} = e^{t} \neq \int_{0}^{t}\exp(t)\,dt$$ Not correct integration

$$v = -1$$

$$D^{-1}e^{t} = \int_{0}^{t}\exp(t)\,dt = \sum_{k=0}^{\infty}\frac{t^{k+1}}{\Gamma(k+2)} = \exp(t) - 1$$

Correct integration

Paradox stems from fact that we compute ordinary derivative and put -1 to find integral of order one. Computing generalized fractional derivative and then putting the order equal to -1 gives the integer order integration correctly!

Standardization of symbols for fractional differintegrals

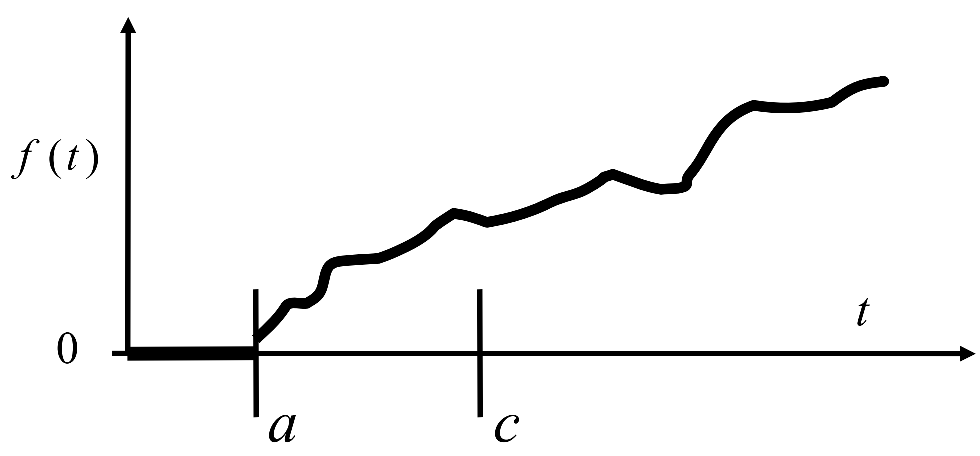
Initialized differintegration $_{c}D_{t}^{\pm q}$ $\qquad q \in \Re e^{+}$

Uninitialized differitegrations $_{c}d_{t}^{\pm q}$

Initialization function $\psi(t) = \psi(f, \pm q, a, c, t)$

$$_{c}D_{t}^{\pm q} = {}_{c}d_{t}^{\pm q} + \psi(f, \pm q, a, c, t)$$

For a function $f(t)$ born at time (space) $= a$ and the differintegration starts at time (space) $= c$

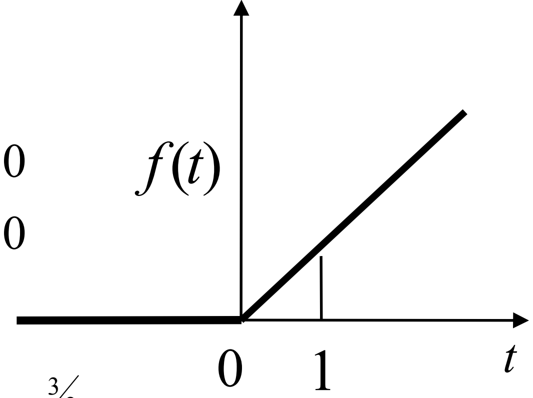


Initialized fractional integration

$$ {}_1D_t^{-1/2}\{t\} = {}_1d_t^{-1/2} + \psi(t) $$

$$ \psi(t) = \psi\{f(t) = t, -1/2, 0, 1, t\} $$

$$ f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} $$



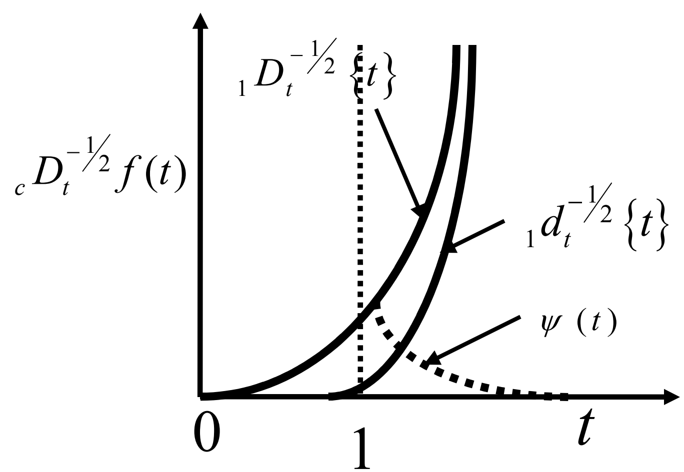
$$ {}_0D_t^{-1/2}\{t\} = {}_0d_t^{-1/2} + \{\psi(t) = 0\} = \frac{1}{\Gamma(0.5)}\int_0^t (t-\tau)^{0.5-1}\tau\, d\tau = \frac{4t^{3/2}}{3\sqrt{\pi}} $$

$$ {}_1d_t^{-1/2}\{t\} = \frac{1}{\Gamma(0.5)}\int_1^t (t-\tau)^{0.5-1}\tau\, d\tau = \frac{2(t-1)^{1/2}(2t+1)}{3\sqrt{\pi}} $$

$$ \psi(t) = \frac{1}{\Gamma(0.5)}\int_0^1 (t-\tau)^{0.5-1}\tau\, d\tau = \frac{2}{3\sqrt{\pi}}\left[2t^{3/2} - (t-1)^{1/2}(2t+1)\right] $$

$$ {}_0D_t^{-1/2}\{t\} = {}_1D_t^{-1/2}\{t\} = {}_1d_t^{-1/2}\{t\} + \psi(t) $$

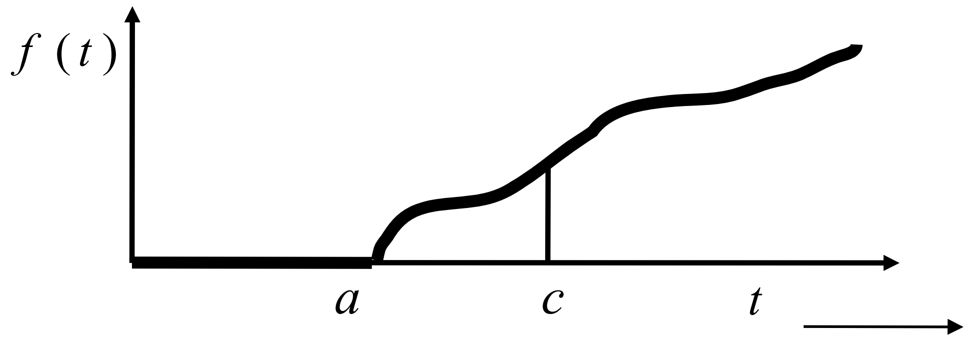
$\psi(t)$ Is the history of the functional process since birth and the history effect decays with time, memory is lost!!

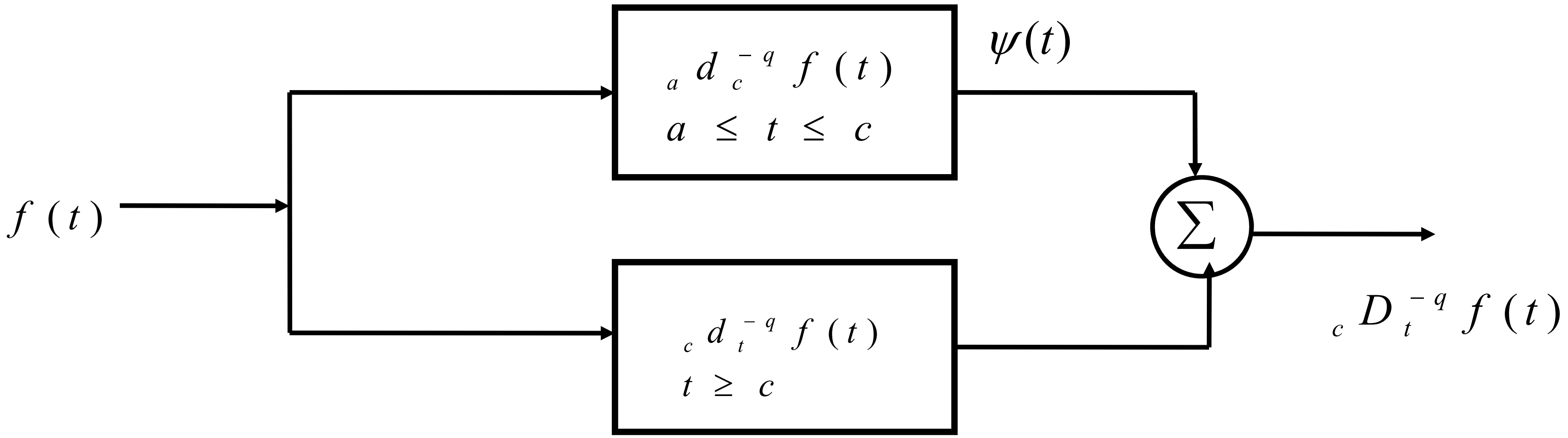


$$ {}_cD_t^{-q} f(t) = {}_aD_t^{-q} f(t) = {}_cd_t^{-q} f(t) + \psi(t) $$

$$ t \geq c \geq a $$

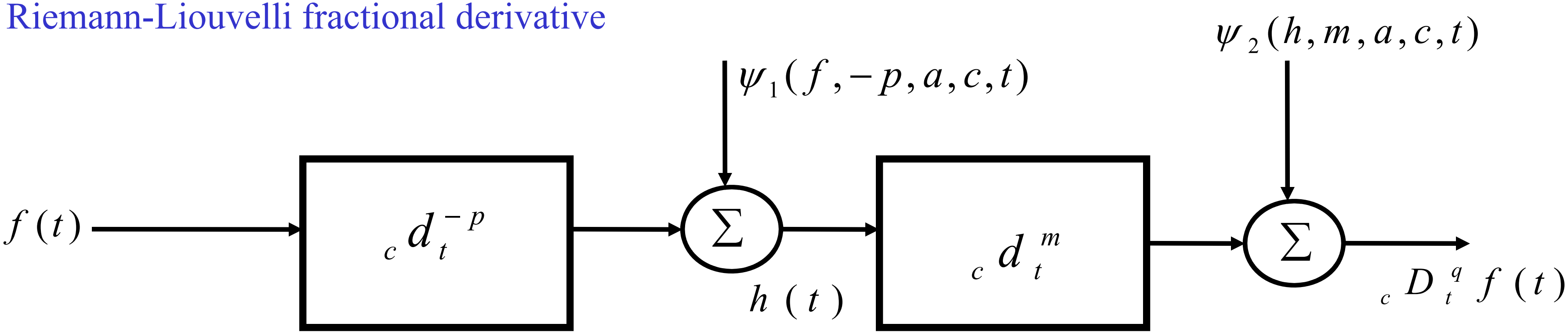
Initialization function fractional integration





Initialization of fractional derivative

Riemann-Liouvelli fractional derivative



$$q = (m - p)$$

$${}_c D_t^q f(t) = {}_a D_t^q f(t)$$

$${}_c D_t^q f(t) = {}_c d_t^m \{h(t)\} + \psi_2(h, m, a, c, t)$$

$${}_c D_t^q f(t) = {}_c d_t^m \{ {}_c d_t^{-p} f(t) + \psi_1(f, -p, a, c, t)\} + \psi_2(h, m, a, c, t)$$

$${}_c D_t^q f(t) = {}_c d_t^q f(t) + \psi_1^{(m)}(t) + \psi_2(t)$$

$${}_c D_t^q f(t) = {}_c d_t^q f(t) + \psi(f, q, a, c, t)$$

For terminal initialization $\psi_2 = 0$

For side initialization ψ_2 is arbitrary

Solution of FDE

$${}_0D_t^{1/2} f(t) + bf(t) = 0$$

$$t > 0, [{}_0D_t^{-1/2} f(t)]_{@\ t=0} = C$$

Requiring fractional initial state?

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots$$

$$\mathcal{L}\left[f^{(1/2)}(t)\right] = s^{1/2} F(s) - s^{1/2-1} f(0) = s^{1/2} F(s) - s^{-1/2} f(0)$$

$$\mathcal{L}\left[f^{(1/2)}(t)\right] = s^{1/2} F(s) - [{}_0D_t^{-1/2} f(t)]_{@\ t=0}$$

$$F(s) = \frac{C}{s^{1/2} + b}$$

$$f(t) = C t^{-1/2} E_{0.5,0.5}(-b\sqrt{t})$$

Solution of FDE with initialization function

$${}_0D_t^{1/2} f(t) + b f(t) = 0$$

$${}_0d_t^{1/2} f(t) + \psi(t) + b f(t) = 0 \quad \text{for} \quad t > 0$$

$$s^{1/2} F(s) + \psi(s) + b F(s) = 0$$

$$F(s) = -\frac{\psi(s)}{s^{1/2} + b}$$

For specific $\psi(t) = -C\delta(t)$

We get $F(s) = \dfrac{C}{s^{1/2} + b}$ $\quad f(t) = C t^{-1/2} E_{0.5,0.5}(-b\sqrt{t})$

Generally: $\mathcal{L}^{-1}\left[\dfrac{1}{s^{1/2} + b}\right] = R_{1/2,0}(-b, 0, t)$

General Solution

$$f(t) = -\int_0^t R_{1/2,0}(-b, 0, t-\tau)\psi(\tau)\, d\tau$$

Robotnov-Hartley convoluted with initialized function

Fractional Initial State:

For a Fractance device and its circuits requires type of "fractional initial conditions" $D^{\alpha-1}[i(t)]_{@\ t=0} = D^{\alpha-1} i(0)$

$$v(t) = K D^{\alpha}\{i(t)\}$$

Step Input

$$i(t) = \frac{1}{K}[D^{-\alpha} i(t)]$$

$$v(t) = V_0 ; t \geq 0$$

$$i(t) = \frac{1}{K} {}_0D_t^{-\alpha}[V_0] = \frac{V_0 t^{\alpha}}{K\Gamma(\alpha+1)}$$

But

$$ {}_0D_t^{\alpha} i(t) = \frac{V_0}{K}$$

$$ {}_0D_t^{\alpha-1}\{i(t)\} = {}_0D_t^{-1}[{}_0D_t^{\alpha}\{i(t)\}] = {}_0D_t^{-1}\left[\frac{V_0}{K}\right] = C t$$

$$\lim_{t\to 0}[{}_0D_t^{\alpha-1}\{i(0)\}] = 0$$

Here we have correlated and found that fractional initial state value.

Fractional Initial State

$$v(t)/K = D^{\alpha}\{i(t)\}$$

Impulse input

$$v(t) = B\delta(t)$$

$$i(t) = \frac{1}{K} D^{-\alpha} v(t) = \frac{1}{K}[\Phi_{\alpha}(t) * B\delta(t)] = \frac{B}{K}\Phi_{\alpha}(t)$$

$$\Phi_{\alpha}(t) = t^{\alpha-1}/\Gamma(\alpha)$$

$$i(t) = B t^{\alpha-1}/K\Gamma(\alpha)$$

$$v(t) = B\delta(t)$$

$$t > 0;\ v(t) = 0$$

Using $D^{\alpha}i(t) = v(t)/K = B\delta(t)/K$

$$[{}_0D_t^{\alpha-1} i(t)]_{@t=0} = [{}_0D_t^{-1}\left[\frac{B\delta(t)}{K}\right]]_{@t=0}$$

$$\int \delta(t)\,dt = 1$$

$$D_t^{\alpha-1} i(0) = B/K = C$$

Here also we have found that fractional initial state value a constant.

Formal methods to solve fractional differential equation

1. Laplace Transforms *

2. Indicial Polynomial & Generalized Partial Fractions *

3. Fractional Greens function *

4. Mellin Transforms

5. Power Series Method *

6. Babenko's Symbolic calculus method

7. Orthogonal Polynomial decomposition

8. Adomian Decomposition *

9. Order reduction (complementary polynomial)

10. Numerical *

11. Gelarkin Approximations and others

Caputo initialization?

Fractional initialization states are required in RL differentiation, but integer order initial states are required for Caputo differential operator-for FDE

The initialization function as described herein is for RL differential operator

The initialization function is backward continuation of function till start point as for RL-but is different initialization for Caputo formulations

The Caputo initialization is same as RL only when the static (zero) initial states else they vary.

Caputo and RL are same when the start point of function and differential process starting are same.

$$ {}_0D_t^{\alpha} f(t) = {}_0^C d_t^{\alpha} f(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{d}{dt}\psi\{f, -(1-\alpha), a, 0, t\} $$

Integer order calculus in fractional context

RL derivative

$$ {}_aD_t^1 f(t) = \frac{d}{dt} f(t) = \frac{1}{\Gamma(1)} \frac{d^2}{dt^2} \int_a^t (t-\tau)^{2-1-1} f(\tau) d\tau = \frac{d^2}{dt^2} \int_a^t f(\tau) d\tau $$

Integrate the function from *a* to *t* and then obtain second derivative.

Obtaining the differentiation in fractional context imbibes history (hereditary) of the function from start of the differentiation process.

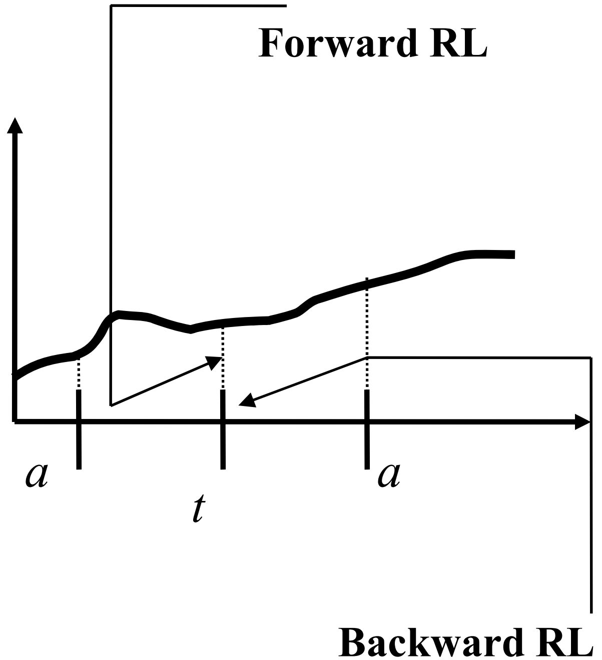
This also describes the 'non-local' behaviour in space or time.

Forward and backward differentiation integer order derivative in fractional context

$$ {}_aD_t^q\{f(t)\} = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_a^t (t-\tau)^{m-q-1} f(\tau)d\tau $$

RL derivative

$$ {}_tD_a^q f(t) = \frac{1}{\Gamma(m-q)}(-1)^m \frac{d^m}{dt^m}\int_t^a (\tau-t)^{m-q-1} f(\tau)d\tau $$



$$ {}_aD_t^1 f(t) = \frac{d}{dt}f(t) = \frac{d^2}{dt^2}\left[\int_a^t f(\tau)d\tau\right] $$

$$ = \frac{d^2}{dt^2}\left[\frac{1}{2}\{f(t)+f(a)\}(t-a)\right] $$

$$ = f'(t) + \frac{1}{2}(t-a)f''(t) $$

$$ \to f'(t) $$

$$ {}_tD_a^q f(t) = \frac{1}{\Gamma(m-q)}(-1)^m \frac{d^m}{dt^m}\int_t^a (\tau-t)^{m-q-1} f(\tau)d\tau $$

If forward and backward derivatives are equal (with sign) then fractional derivative at a POINT exist, meaning to get fractional derivative at point entire character of function be known!

$$ {}_tD_a^1 f(t) = \frac{1}{\Gamma(1)}(-1)^2 \frac{d^2}{dt^2}\int_t^a (\tau-t)^{2-1-1} f(\tau)d\tau $$

$$ = -{}_aD_t^1 f(t) $$

Local behavior at point and thus ‘local’ fractional derivative:

Note the non-local character of the Fractional Derivative defined by RL definition and the non-constant ‘fractional-derivative’ of non-zero constant. These two features makes extraction of scaling information somewhat difficult. The problem is overcome by ‘Local Fractional Derivative’ LFD. Sometimes it is desirable to have ‘Local-Character’ in wide range of applications ranging from structure of ‘differentiable’ manifolds to various physical models. Secondly, the Fractional Derivative of constant is non-zero, consequently the magnitude of Fractional Derivative changes with addition of a constant to a function. The notion of LFD must address these issues. The logic is take fractional integral of order $(1-\alpha)$ of function minus the value of function at point of interest; then Differentiate the function and put limit tending to that point of interest.

$$I(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} \left(f(t) - f(x_0) \right) dt$$

$$\mathbf{D}^{\alpha} f(x_0) = \lim_{x \to x_0} I^{/}(x)$$

$$D^{\alpha} f(x)_{@\, x = x_0} = \lim_{x \to x_0} \frac{d^{\alpha} \left(f(x) - f(x_0) \right)}{d(x - x_0)^{\alpha}}$$

Local Fractional Derivative KG-Definition

Kolwankar-Gangal definition:

$$D^{\alpha} f(x)_{@\, x=x_0} = \lim_{x\to x_0} \frac{d^{\alpha}\big(f(x)-f(x_0)\big)}{d(x-x_0)^{\alpha}}$$

$$\frac{d^{\alpha}}{dx^{\alpha}} f(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{x\to y} \frac{d}{dx} \int_y^x (x-t)^{-\alpha}\big(f(t)-f(y)\big)dt$$

In the limit the first derivative of fractional integral $I(x)$ of order $(1-\alpha)$

$$I(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x (x-t)^{-\alpha}\big(f(t)-f(y)\big)dt$$

$$D^{\alpha} f(x)_{@\, x=y} = \lim_{x\to y} I'(x)$$

Local Fractional Derivative (LFD) Kolwankar-Gangal (KG)

For a function $f : [0,1] \rightarrow \mathbf{R}$, the limit

$$\mathbf{D}^{q} f(x_0) = \lim_{x \rightarrow x_0} \frac{d^{q}\left(f(x) - f(x_0)\right)}{\left[d(x - x_0)\right]^{q}}$$

exists and is finite; then we say LFD of order q where $0 < q < 1$ at $x = x_0$ exits

In this definition the lower limit x_0 is treated as a constant. The subtraction of $f(x_0)$ corrects for the fact that fractional derivative of constant (in RL) is not zero. Where the limit $x \rightarrow x_0$ is taken to remove non-local contents. This LFD (removing the non-local contents) allows the study of point wise behavior of $f(x)$.

$$\mathbf{D}^{1} f(0) = \lim_{x \rightarrow 0} \frac{d}{dx} f(x)$$ Slope at origin!

KG Local Fractional Derivative for fractional order more than one:

$$\mathbf{D}^{q} f(x_0) = \lim_{x \to x_0} \frac{d^{q}\left(f(x) - \left[f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}}{2!}(x - x_0)^2 + \ldots \frac{f^{(N)}}{N!}(x - x_0)^N \right] \right)}{\left[d(x - x_0) \right]^q}$$

$$= \lim_{x \to x_0} \frac{d^{q}\left(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}}{\Gamma(n+1)}(x - x_0)^n \right)}{\left[d(x - x_0) \right]^q}$$

If the limit exists and is finite, where *N* is the largest integer for which *N*-th derivative of function $f(x)$ at point x_0 exists and is finite, then we say LFD of order q $N < q \leq N + 1$, at x_0 exists.

When q is positive integer, then integer order derivative is recovered.

For $q = 1, N = 0$

$$\mathbf{D}^{1} f(x_0) = \lim_{x \to x_0} \frac{d}{d[(x - x_0)]}[f(x) - f(x_0)]$$

Critical Order:

Critical order $\alpha(x_0)$

Supremum of q all Local Fractional Derivative of order less than q exists at x_0.

$$f(x) = a + bx + c|x|^{\beta} \quad 1 < \beta < 2$$

$$f(0) = a \qquad f^{(1)}(0) = b \qquad f^{(2)}(0) = \infty$$ Nearest integer $N = 1$

$$\mathbf{D}^q f(0) = \lim_{x \to 0} \frac{d^q \left(f(x) - \left[f(0) + f^{(1)}(0)(x-0) \right] \right)}{\left[d(x-0) \right]^q} = \lim_{x \to 0} \frac{d^q}{dx^q} \left[a + bx + c|x|^{\beta} - (a + bx) \right]$$

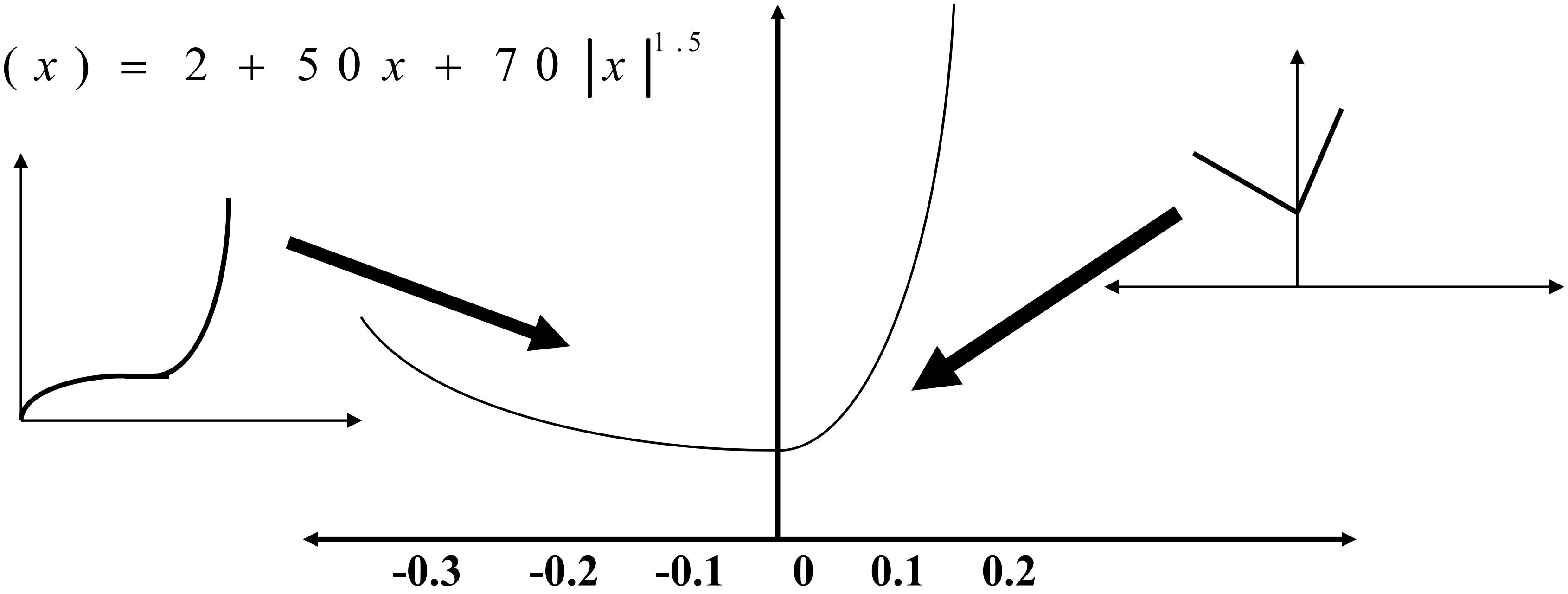
$$= \lim_{x \to 0} c \frac{\Gamma(\beta + 1)}{\Gamma(\beta - q + 1)} |x|^{\beta - q} = \begin{cases} \infty & ; q > \beta \\ 0 & ; q < \beta \end{cases}$$

Critical Order of the function $f(x)$ at $x = 0$ is $\alpha(0) = \beta$

LFD of critical order has value at origin for this function i.e. $\mathbf{D}^{\beta} f(0) = c\Gamma(\beta + 1)$

Abrupt phase transition to continuous phase transition:

$$f(x) = 2 + 50x + 70|x|^{1.5}$$



This is notion to extend Ehrenfest's classification of thermodynamic phase transition, magnetic property at critical point, or yield point (stain) beyond critical stress to continuous transition. In simplified terms magnify the critical point which takes place abruptly and approximate by polynomial to get Fractional Differentiability at critical point. Non-differentiability can be magnified and studied near critical points.

Fractional Differentiation of continuous but non-differentiable graph & its relation with 'Fractal-Dimension'

$$W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k x \qquad \lambda > 1 \qquad d_B = s \qquad 1 < s < 2$$

$$W_\lambda(0) = 0$$

Use scaling law $\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q}$ we get $\frac{d^q}{dx^q} \sin \lambda^k x = (\lambda^k)^q \frac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$

and the fractional derivative of Wierstrauss's function for $0 < q < 1$

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \frac{d^q \sin \lambda^k x}{dx^q} = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$$

Using $\frac{d^q}{dx^q} \sin x = \frac{d^q}{dx^q} \int_0^x \cos t\, dt = \frac{d^{q-1}}{dx^{q-1}} \cos x$, the FD is

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

The critical order of FD of Wierstrauss's function:

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

The fractional integral $I_x^p \cos \lambda^k x$ of order $p = 1 - q$ is bounded uniformly for all values of $\lambda^k x$. This implies that the RHS will converge for $s - 2 + q < 0$ or $q < 2 - s$ and diverge for $q > 2 - s$ at the point Zero

The value of fractional integral is zero hence FD at zero of Wierstrauss's is ZERO

This Wierstrauss's function is continuously Fractionally Differentiable locally For orders $q < (2 - s)$ and not between orders $(2 - s)$ to one .
This implies that this Wierstrauss's function has Critical Order $\alpha = (2 - s)$ at all points, which is equal roughness exponent and thereby box dimension of the graph!.

LFD is perhaps a tool to extract local dimension of the irregular rough function

Critical Order LFD and Fractal (Box) dimension

$f : [0,1] \to \mathbf{R}$, be a continuous (real) function

If $\lim_{x \to x_0} \frac{d^q \left(f(x) - f(x_0) \right)}{\left[d(x - x_0) \right]^q} = 0$ for $q < \alpha$

Then $\dim_B f(x_0) \leq 2 - \alpha$

Holder Exponent $\alpha(x_0)$ of a function $f(x)$ defined by this is the largest exponent such that there exists a polynomial $P_n(x)$ that satisfies

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^{\alpha}$$

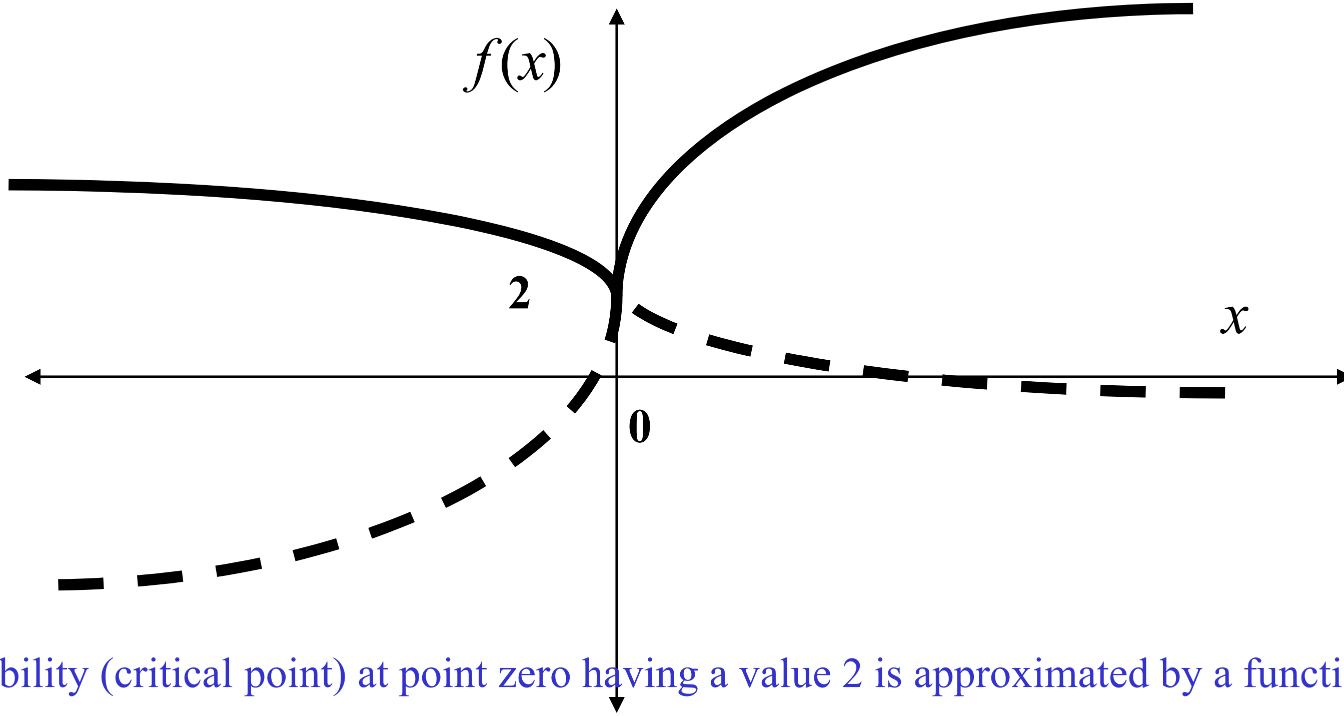
There is clear change in behavior when q crosses the Critical Order. $\alpha(x_0)$

Phase Transition (Non-Differentiability) at critical point:

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^h$$

$x_0 = 0;\ P(x) = 2 + 3x$ Critical Point at zero and the polynomial is linear.

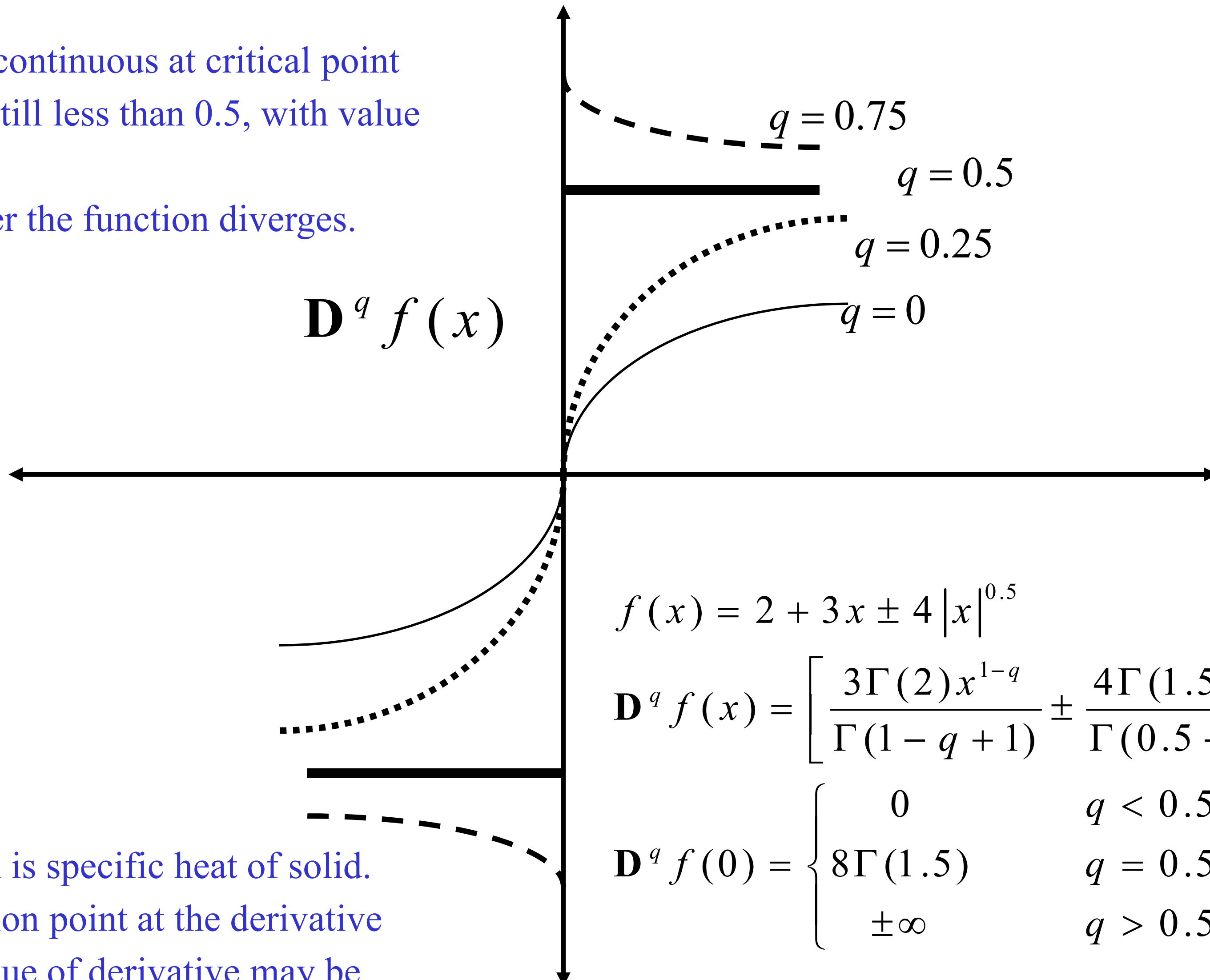
$$f(x) = (2 + 3x) \pm 4\left| x \right|^{1/2}$$



Non-Differentiability (critical point) at point zero having a value 2 is approximated by a function.
Response function of several processes diverge algebraically near critical point.
Example Vander wall's equation at critical point.

Fractional Differentiability at Critical Point:

The function is continuous at critical point from zero order till less than 0.5, with value zero.
Beyond 0.5 order the function diverges.



$$f(x) = 2 + 3x \pm 4|x|^{0.5}$$

$$\mathbf{D}^q f(x) = \left[\frac{3\Gamma(2)x^{1-q}}{\Gamma(1-q+1)} \pm \frac{4\Gamma(1.5)x^{0.5-q}}{\Gamma(0.5-q+1)} \right]$$

$$\mathbf{D}^q f(0) = \begin{Bmatrix} 0 & q < 0.5 \\ 8\Gamma(1.5) & q = 0.5 \\ \pm\infty & q > 0.5 \end{Bmatrix}$$

Say the function is specific heat of solid.
At phase transition point at the derivative order 0.5 the value of derivative may be regarded as 'Fractional Latent Heat'

Grunwald-Letnikov(GL) fractional differintegration

$$f^{/}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f^{//}(x) = \lim_{h_1 \to 0} \frac{\lim\limits_{h_2 \to 0} \frac{f(x+h_1+h_2) - f(x+h_1)}{h_2} - \lim\limits_{h_2 \to 0} \frac{f(x+h_1) - f(x)}{h_2}}{h_1}$$

$$f^{//}(x) = \lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) - f(x)}{h^2}$$

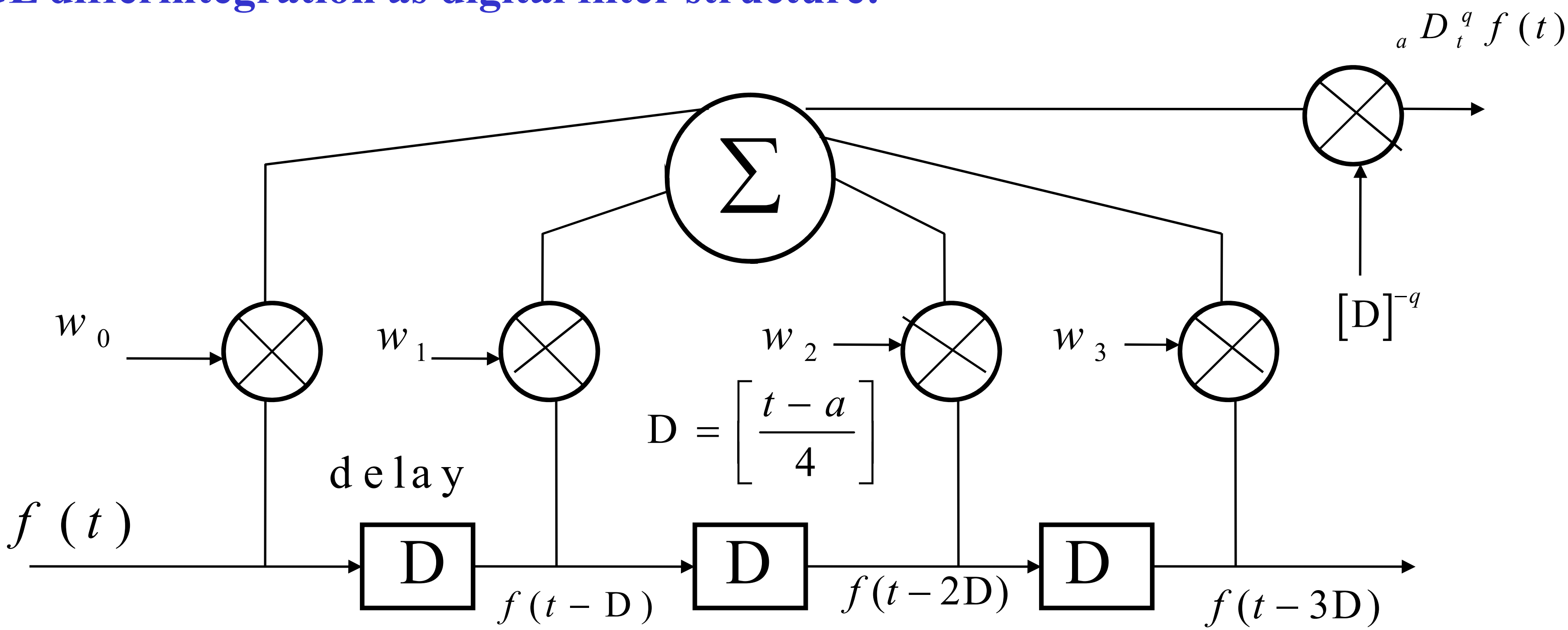
$${}_a D_x^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{m=0}^{\left[\frac{x-a}{h}\right]} (-1)^m \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)} f(x-mh)$$

$$\binom{\alpha}{m} = \frac{\alpha!}{m!(\alpha-m)!} \leftrightarrow \frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}$$

$${}_a D_x^{-\alpha} f(x) = \lim_{h \to 0} h^{\alpha} \sum_{m=0}^{\left[\frac{x-a}{h}\right]} (-1)^m \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)} f(x-mh)$$

$$\begin{bmatrix} -\alpha \\ m \end{bmatrix} = \begin{pmatrix} -\alpha \\ m \end{pmatrix} = \frac{-\alpha(-\alpha-1)..(-\alpha-m+1)}{m!} = (-1)^m \frac{(\alpha+m-1)!}{m!(\alpha-1)!} \leftrightarrow (-1)^m \frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}$$

GL differintegration as digital filter structure:



$$ {}_aD_t^q f(t) = \lim_{\Delta T \to 0} \frac{(\Delta T)^{-q}}{\Gamma(-q)} \sum_{k=0}^{N-1} \frac{\Gamma(k-q)}{\Gamma(k+1)} f(t - k\Delta T) = \lim_{D \to 0} [D]^{-q} \sum_{k=0}^{N-1} w_k f(t - kD) $$

Digital filter FIR/IIR
Tustin Discretization with Generating Function
Matrix approach FFT for weights
Short Memory Principle

Approximation from GL

$$ {}_aD_t^\alpha f(t) = \lim_{h \to 0} \frac{{}_a\Delta_h^\alpha f(t)}{h^\alpha} $$

$$ {}_a\Delta_h^\alpha f(t) = \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t - jh) $$

For a wide class of functions, important for applications, both the definitions of RL and GL are equivalent. This allows one to use RL for problem formulation and then apply GL approximations through definitions of fractional difference.

$$ {}_aD_t^\alpha f(t) \approx \frac{1}{h^\alpha} {}_a\Delta_h^\alpha f(t) $$

$$ {}_{t-L}D_t^\alpha f(t) \approx h^{-\alpha} \sum_{j=0}^{N(t)} \omega_j^{(\alpha)} f(t - jh) $$

$$ N(t) = \min\left\{\left[\frac{t}{h}\right], \left[\frac{L}{h}\right]\right\}, \omega_0^{(\alpha)} = 1; \omega_j^{(\alpha)} = \left(1 - \frac{1+\alpha}{j}\right)\omega_{j-1}^{(\alpha)} $$

ACCURACY AND ERROR

OFF COURSE FOR THIS SIMPLIFICATION PENALTY IS PAID IN TERMS OF ACCURACY. FOLLOWING RULE FOR INTERVAL *a* TO *b* IS APPLIED FOR ERROR ESTIMATION:

$$\Delta(t) = \left| {}_{a}D_{t}^{\alpha} f(t) - {}_{t-L}D_{t}^{\alpha} f(t) \right| \leq \frac{M L^{-1}}{\left|\Gamma(1-\alpha)\right|}$$

$$(a + L \leq t \leq b)$$

$$f(t) \leq M$$

$$a \leq t \leq b$$

THIS INEQUALITY CAN BE USED FOR DETERMINING THE MEMORY LENGTH PROVIDED THE REQUIRED ACCURACY IS DEFINED AS ε

$$\Delta(t) \leq \varepsilon$$

$$(a + L \leq t \leq b)$$

IF

$$L \geq \left(\frac{M}{\varepsilon \left|\Gamma(1-\alpha)\right|} \right)$$

SHORT MEMORY PRINCIPLE

FOR $t >> a$, THE NUMBER OF ADDEND IN FRACTIONAL DERIVATIVE APPROXIMATION BECOMES ENORMOUSLY LARGE. HOWEVER IT FOLLOWS FROM THE EXPRESSION FOR LARGE COEEICIENTS IN GL APPROXIMATION ,THAT FOR LARGE t THE ROLE OF "HISTORY" OF THE BEHAVIOR OF THE FUNCTION NEAR THE 'LOWER TERMINAL' $t = a$ CAN BE NEGLECTED UNDER CERTAIN ASSUMPTIONS (OF ACCURACY). THOSE OBSERVATION LEAD US TO FORMULATION OF "SHORT-MEMORY-PRINCIPLE", WHICH MEANS TAKING INTO ACCOUNT THE BEHAVIOR OF FUNCTION IN "RECENT PAST".

$${}_{a}D_{t}^{\alpha} f(t) \approx {}_{t-L}D_{t}^{\alpha} f(t) \qquad (t > a + L)$$

IN OTHER WORDS THE FRACTIONAL DIFFERINTEGRATION WITH LOWER LIMIT a IS APPROXIMATED BY FRACTIONAL DIFFERINTEGRATION WITH MOVING LOWER LIMIT (t - a). DUE TO THIS APPROXIMATION THE NUMBER OF ADDENDS IS ALWAYS NOT GREATER THAN $[L / T]$. AND $N(t)$=min $\{[t / h],[L / h]\}$

Fourier series representation of Stochastic Finite Difference & Derivative

Functions series is:

$$X(t) = \sum_{\omega=-\infty}^{\omega=+\infty} X(\omega) e^{i\omega t} \quad \text{and} \quad \Im\left[X(t)\right] = X(\omega) = \frac{1}{2\pi}\int_0^{2\pi} X(t) e^{-i\omega t} dt$$

Also the stochastic fractional difference is:

$$\Delta_{(-)}^{\alpha} X(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k X(t - k\tau)$$

$$D_{(-)}^{\alpha} X(t) = \lim_{\tau \to 0} \frac{\Delta_{(-)}^{\alpha}}{\tau^{\alpha}} \qquad \alpha > 0$$

$$\Im\left[D_{(-)}^{\alpha} X(t)\right] = \lim_{\tau\to 0} \frac{1}{2\pi\tau^{\alpha}} \int_0^{2\pi} e^{-i\omega t} \Delta_{(-)}^{\alpha} X(t) dt$$

$$\Im\left[D_{(-)}^{\alpha} X(t)\right] = \lim_{\tau\to 0} \frac{1}{\tau^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(-1)^k}{2\pi} \int_0^{2\pi} e^{-i\omega t} \sum_{\omega'=-\infty}^{\omega'=+\infty} X(\omega') e^{i\omega'(t-k\tau)} dt$$

Integral is zero for $\omega \neq \omega'$

And is 2π for $\omega = \omega'$

$$\Im\left[D_{(-)}^{\alpha} X(t)\right] = \lim_{\tau\to 0} \frac{1}{\tau^{\alpha}} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k e^{-i\omega k\tau} X(\omega)$$

$$\Im\left[D_{(-)}^{\alpha} X(t)\right] = \lim_{\tau\to 0} \frac{1}{\tau^{\alpha}} \left(1 - e^{-i\omega\tau}\right)^{\alpha} X(\omega)$$

Result of Fourier series for fractional derivative forward & backward

$$\Im\left[D^{\alpha}_{(\mp)}X(t)\right] = \lim_{\tau\to 0}\frac{1}{\tau^{\alpha}}\left(1-e^{\mp i\omega\tau}\right)^{\alpha} X(\omega)$$

$$\Im\left[D^{\alpha}_{(\mp)}X(t)\right] \approx \left(\pm i\omega\right)^{\alpha} X(\omega)$$

Thus we see that effect of Fractional Derivative operating on function is to multiply the Fourier amplitude by $(i\omega)^{\alpha}$ or by its conjugate.

Therefore:

$$D^{\alpha}_{\mp}X(t) = \sum_{\omega=-\infty}^{\omega=+\infty}\left(\pm i\omega\right)^{\alpha} X(\omega)e^{i\omega t}$$

Where it is clear that for non-integer index (order) the fractional derivative operator are non-local in time.

MARIX APPROACH TO APPROXIMATION OF GL DIFFERINTEGRATION

DIFFERENTIATION:

$$ {}_{a}D_{t_k}^{\alpha} f(t) \approx \frac{\Delta^{\alpha} f(t_k)}{h^{\alpha}} = h^{-\alpha} \sum_{j=0}^{k} (-1)^{j} \binom{\alpha}{j} f_{k-j} $$

$$ k = 0, 1, 2, 3, \ldots N $$

$$ \begin{bmatrix} h^{-\alpha}\Delta^{\alpha} f(t_0) \\ h^{-\alpha}\Delta^{\alpha} f(t_1) \\ h^{-\alpha}\Delta^{\alpha} f(t_2) \\ * \\ h^{-\alpha}\Delta^{\alpha} f(t_N) \end{bmatrix} = \frac{1}{h^{\alpha}} \begin{bmatrix} w_0^{(\alpha)} & 0 & 0 & 0 & 0 \\ w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 & 0 \\ w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 \\ * & * & * & * & * \\ w_N^{(\alpha)} & w_{N-1}^{(\alpha)} & * & w_1^{(\alpha)} & w_0^{(\alpha)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ * \\ f_N \end{bmatrix} $$

$$ w_j^{(\alpha)} = (-1)^{j} \binom{\alpha}{j}, \; j = 0, 1, 2, 3 \ldots N $$

BINOMIAL COEFFICIENT OF SERIES

GL MATRIX COEFFICIENT THROUGH GENERATING FUNCTION:

$$Q(z) = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k \leftrightarrow \mathrm{trunc}_N(Q(z)) \stackrel{\mathrm{def}}{=} \sum_{k=0}^{N} w_k^{(\alpha)} z^k = Q_N$$

$$\mathrm{B}_N^{(\alpha)} = \frac{1}{h^\alpha} \begin{bmatrix} w_0^{(\alpha)} & 0 & 0 & 0 & 0 \\ w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 & 0 \\ w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & 0 & 0 \\ * & * & * & * & * \\ w_N^{(\alpha)} & w_{N-1}^{(\alpha)} & * & w_1^{(\alpha)} & w_0^{(\alpha)} \end{bmatrix} = \beta_\alpha(z) = h^{-\alpha}(1-z)^\alpha$$

FOR FRACTIONAL INTEGRATION:

$$\mathrm{I}_N^\alpha = \left(\mathrm{B}_N^\alpha\right)^{-1}$$

$$\mathrm{I}_N^\alpha \leftrightarrow \varphi_N(z) = \mathrm{trunc}_N\left(\beta_\alpha^{-1}(z)\right) = \mathrm{trunc}_N = \left(h^\alpha (1-z)^{-\alpha}\right)$$

COEFFICIENT MATRIX FOR FRACTIONAL INTEGRATION GL

$$\mathrm{I}_N^{\alpha} = h^{\alpha} \begin{bmatrix} w_0^{(-\alpha)} & 0 & 0 & 0 & 0 \\ w_1^{(-\alpha)} & w_0^{(-\alpha)} & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ w_{N-1}^{(-\alpha)} & w_{N-1}^{(-\alpha)} & w_1^{(-\alpha)} & w_0^{(-\alpha)} & 0 \\ w_N^{(-\alpha)} & w_{N-1}^{(-\alpha)} & * & w_1^{(-\alpha)} & w_0^{(-\alpha)} \end{bmatrix}$$

$$w_j^{(-\alpha)} = (-1)^j \binom{-\alpha}{j} = \begin{bmatrix} \alpha \\ j \end{bmatrix}$$

$$j = 0, 1, 2 \ldots, N$$

$$w_0^{(-\alpha)} = 1, \; w_j^{(-\alpha)} = \left(1 - \frac{1-\alpha}{j} \right) w_{j-1}^{(-\alpha)}$$

COEFFICIENTS FOR FRACTIONAL DIFFERENTIATION AS WEIGHTS OF FOURIER SERIES IN FFT FORM

$$(1-z)^{\alpha} = \sum_{k=0}^{N} (-1)^{k} \binom{\alpha}{k} z^{k} = \mathrm{trunc}_{N} \sum_{k=0}^{\infty} w_{k}^{(\alpha)} z^{k}$$

SUBSTITUTE

$$z = e^{-j\varphi}$$

THEN

$$\left(1 - e^{-j\varphi}\right)^{\alpha} = \sum_{k=0}^{\infty} w_{k}^{(\alpha)} e^{-jk\varphi}$$

THE COEFFICIETS OF FRACTIONAL DIFFERENTIATION THEN AS EXPRESSED IN FFT AS:

$$w_{k}^{(\alpha)} = \frac{1}{2\pi j} \int_{0}^{2\pi} f_{\alpha}(\varphi) e^{jk\varphi} d\varphi$$

$$f_{\alpha}(\varphi) = \left(1 - e^{-j\varphi}\right)^{\alpha}$$

ARE

REIMANN-LIOUVILLE (RL)
GRUNWALD-LETNIKOV (GL)

EQUIVALENT ?

YES

RL (CAPUTO) FOR ANALYTICAL
GL FOR NUMERICAL COMPUTATIONS

About weights of GL in fractional differintegration

$$D_{+}^{q} f(x) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{q}} \sum_{k=0}^{\infty} w_k f(x - k\Delta x)$$

$$w_k = \frac{\Gamma(k-q)}{\Gamma(-q)\Gamma(k+1)}$$

Is apparent that fractional derivative is limit of a weighted average of the values over the function from minus infinity to point of interest (x), these weights corresponds (in limit) to a power function defined by the order of the fractional derivative (q). This averaging is for forward derivative. For backward derivative, this is limit of a average of values over the function from point of interest (x) to plus infinity. Therefore the forward fractional derivative operator has memory of the function from minus infinity to x, and backward derivative has memory of the function from x to plus infinity.

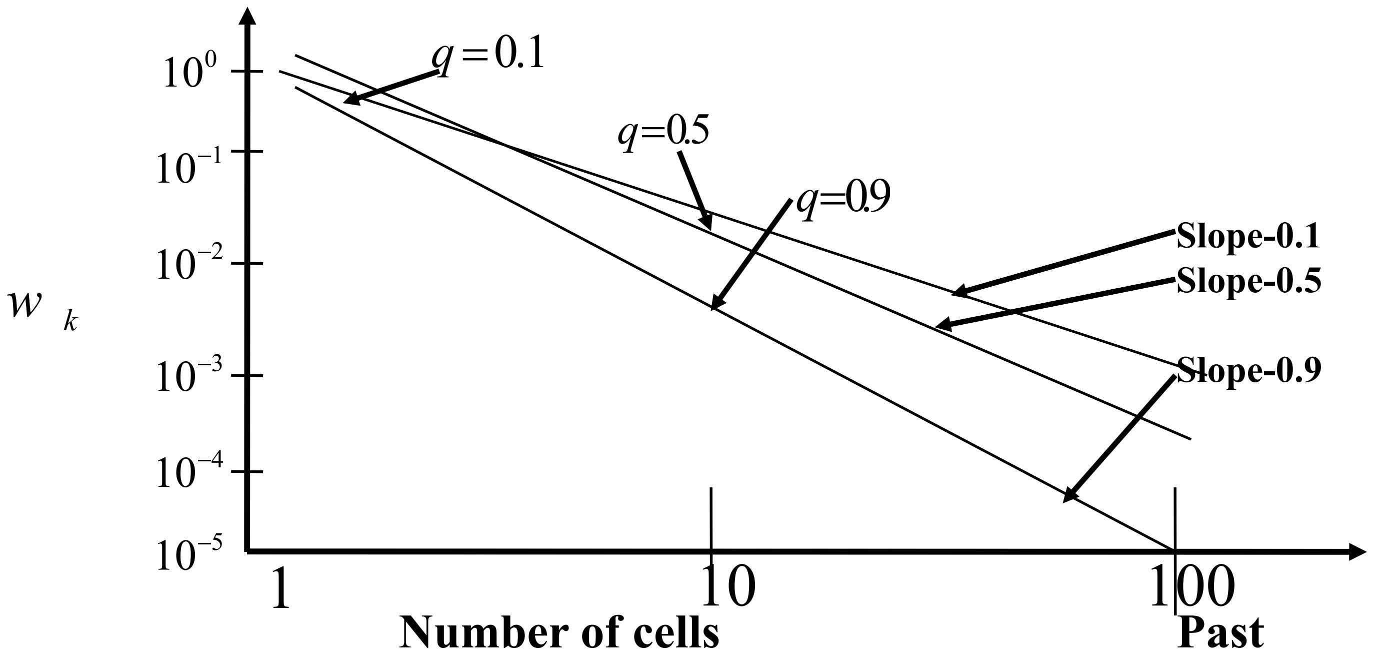
Thus point fractional derivative at a point x has a unique power law 'memory' both forward and backward on function

Local fractional Derivative $D^{q} = \frac{1}{2} D_{+}^{q} + \frac{1}{2} D_{-}^{q}$ at a point depends on the character of entire function.

Integer order derivative depends only on local behaviour meaning slope of function at point.

Fractional derivative is non-local phenomena

Strength of weights and power law exponents of fractional derivative.



Log-log plot demonstrating power law decay in weights placed on the 100 closest cells in calculating q-th derivative. Weights depending on fractional derivative for 0.1, 0.5, 0.9.
The larger order derivative place more weights on proximal cells and dependence on distal cells decrease very quickly as distance x (or t) increases.
The lower order derivatives place relatively less weight on proximal cell and dependence on distal cell decrease very slowly as x (or t) increases.
When derivative order approached unity (integer) then the locality of the definition is achieved.
The integer order derivative is local property whereas the fractional order derivative is non-local point property

USE OF FRACTIONAL DIFFERENTIATION

LET $f(x)=x^{1.5}$, $f'(x)=1.5x^{0.5}$, $f''(x)=0.75x^{-0.5}$, HERE $f''(x)$, $f'''(x)$, FIRST & SECOND DERIVATIVE, DO NOT REDUCE THE AMOUNT OF INFORMATION NEEDED, SINCE EACH OF THESE DERIVATIVES DEPEND ON x.

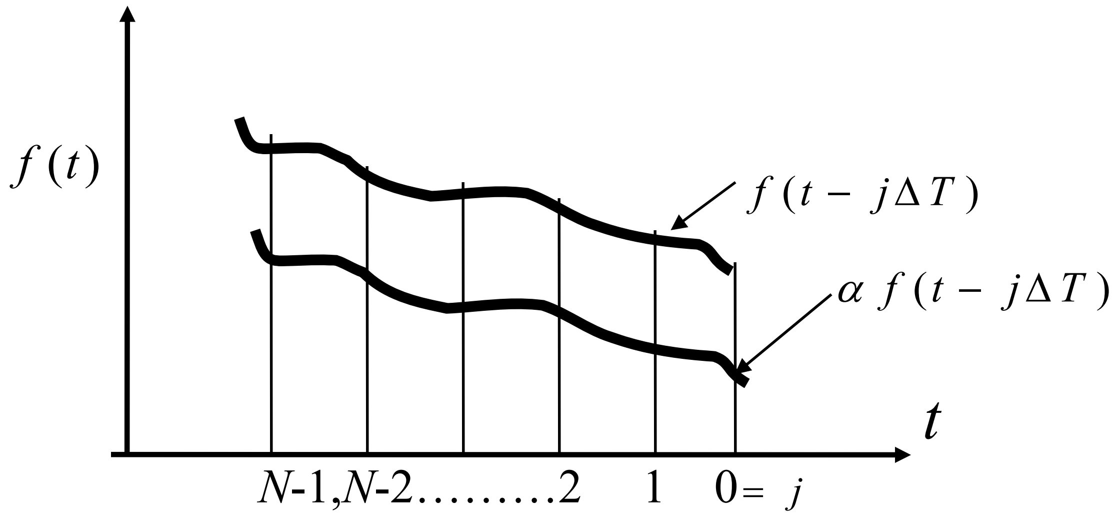
IF A FRACTIONAL DIFFERENTIAL OPERATOR IS CHOSEN IN WHICH THE FRACTIONAL ORDER DIFFERENTIATION MATCHES POWER LAW SCALING OF THE FUNCTION, THEN THE CURVATURE IS REDUCED TO A CONSTANT & ALL THE SCALING IS CONTAINED IN ORDER OF THE DERIVATIVE & THAT CONSTANT.

$$ {}_0D_+^{\alpha} x^{u} = \frac{\Gamma(u+1)}{\Gamma(u-\alpha+1)} x^{u-\alpha} $$

$$ {}_0D_+^{1.5} x^{1.5} = \frac{\Gamma(1.5+1)}{\Gamma(1.5-1.5+1)} x^{1.5-1.5} = \Gamma(2.5) \quad = 1.33 $$

THIS HAS RELEVENCE IN NON-LOCAL DIVERGENCE OPERATION WHICH GIVES FRACTIONAL DIVERGENCE INCLUDING DISPERSION.

Infinitesimal element fractional integration



$$ {}_aD_t^{-1} f(t) = \lim_{\Delta T \to 0} \left\{ \ldots\ldots\ldots + \Delta T \left[f(t - j\Delta T) + f(t-(j+1)\Delta T) \ldots\ldots\ldots \right] \right\} $$

$$ {}_aD_t^{-q} f(t) = \lim_{\Delta T \to 0} \left\{ \ldots\ldots\ldots + \Delta T^{q} \left[\alpha f(t - j\Delta T) + \beta f(t-(j+1)\Delta T) \ldots\ldots\ldots \right] \right\} $$

$$ 0 < q < 1, \alpha = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}, \beta = \frac{\Gamma(j+1-q)\Gamma(j+1)}{\Gamma(j-q)\Gamma(j+2)} $$

Fractional integration can be viewed a area under the curve $\alpha f(t - j\Delta T)$
Multiplied by ΔT^{q-1}
In between volume $[\alpha f(t - j\Delta T).\Delta T].\Delta T$ and area $\{\alpha f(t - j\Delta T).\Delta T\}$

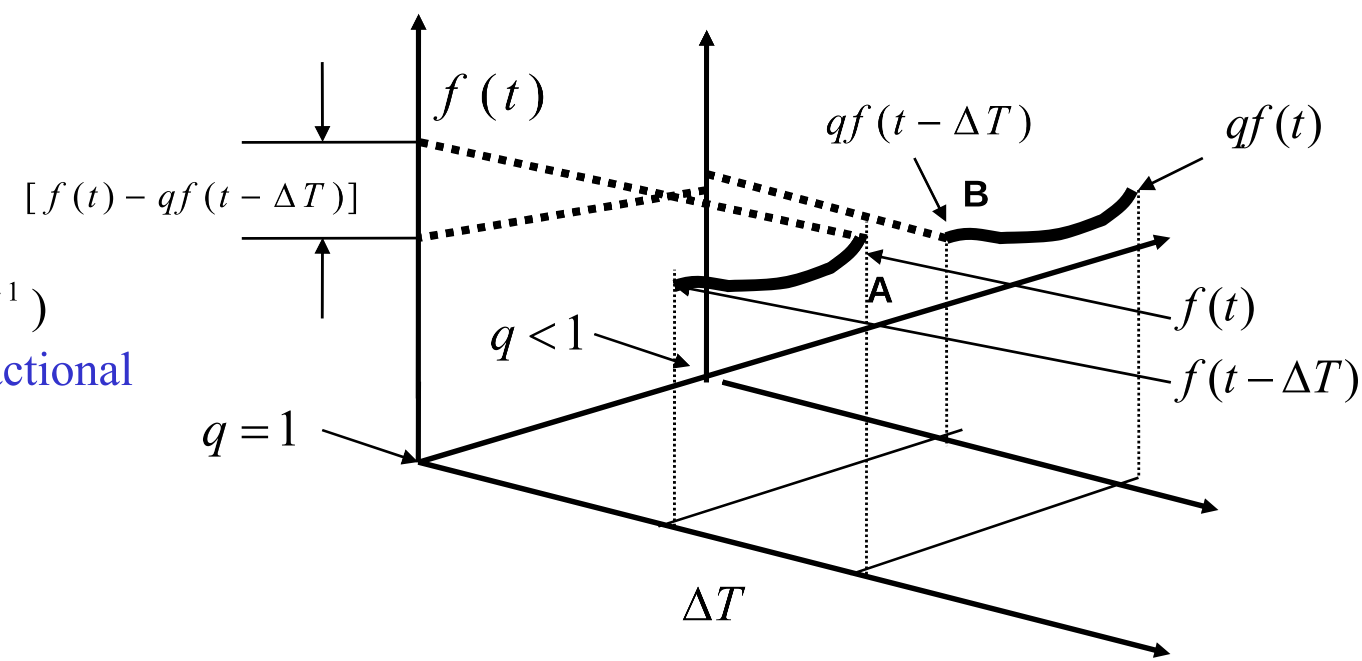
Infinitesimal element fractional differentiation

$$ {}_aD_t^q f(t) = \lim_{\Delta T \to 0} \frac{f(t) - qf(t-\Delta T)}{\Delta T^q} + \ldots\ldots $$

$$ {}_aD_t^1 f(t) = \lim_{\Delta T \to 0} \frac{f(t) - f(t-\Delta T)}{\Delta T} $$

Fractional derivative can be viewed as fractional slope, fractional rate of change. Fractional derivative is slope between $f(t)$ and $qf(t-\Delta T)$
i.e. equal to $\dfrac{f(t) - qf(t-\Delta T)}{\Delta T}$ multiplied by $\left(\dfrac{1}{\Delta T^{q-1}}\right)$

Slope between A & B multiplied by (ΔT^{-q+1}) is fractional slope of fractional differentiation



Salient points observed in the discussion:

The distributed effect of parameters distributed over large space gives half order of derivative or integration.

Can this be taken as general rule that semi infinite distributed self similar structures behave with half order of calculus?

If the distribution in space gives order of derivative as fractional order suggesting non-local behaviour, can we say event distributed in time (historical behaviour hereditary character temporal memory behaviour) be represented with fractional differ-integration of time?

The solution seems to have self similar pattern, time/space power series with fractional power, real order power.

Reality of systems are naturally not point quantity thus fractional calculus is the language what nature understands better.

Fractal dimension and LFD are related and can be Critical Point magnifier.

Fractional Calculus in Real Life