# GENERALIZATION
# OF
# NEWTONIAN-LEIBNIZ
# INTEGER ORDER CALCULUS

## MATHEMATICO-PHYSICS OF GENERALIZED CALCULUS

**Module-I**

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# Generalization of repeated differentiation & integration

Repeated differentiation

$$\frac{d^n f(x)}{dx^n}$$

Anti-differentiation

Indefinite integral

$$\frac{d^{-1} f(x)}{[dx]^{-1}} \equiv {}_0 d_x^{-1} f \equiv \int_0^x f(y)dy$$

$$\frac{d^{-1}}{d[x-a]^{-1}} \equiv {}_a d_x^{-1} f \equiv \int_a^x f(y)dy$$

$$\frac{d^{-2} f}{[dx]^{-2}} \equiv {}_0 d_x^{-2} f \equiv \int_0^x dx_1 \int_0^{x_1} f(x_0)dx_0$$

$$\frac{d^{-n} f}{[dx]^{-n}} \equiv {}_0 d_x^{-n} f \equiv \int_0^x dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \ldots\ldots\ldots \int_0^{x_2} dx_1 \int_0^{x_1} f(x_0)dx_0$$

## Generalization of repeated differentiation

$$\frac{d^1 f}{dx^1} \equiv \frac{d}{dx} f(x) = \lim_{\Delta x \to 0} \left\{ (\Delta x)^{-1} [ f(x) - f(x - \Delta x) ] \right\}$$

$$\frac{d^2 f}{dx^2} \equiv \lim_{\Delta x \to 0} \left\{ (\Delta x)^{-2} [ f(x) - 2 f(x - \Delta x) + f(x - 2\Delta x) ] \right\}$$

$$\frac{d^3 f}{dx^3} \equiv \lim_{\Delta x \to 0} \left\{ (\Delta x)^{-3} [ f(x) - 3 f(x - \Delta x) + 3 f(x - 2\Delta x) - f(x - 3\Delta x) ] \right\}$$

$$\Delta x = \left[ \frac{x - a}{N} \right]$$

$$\frac{d^n f}{dx^n} \equiv \lim_{N \to \infty} \left\{ \left[ \frac{x - a}{N} \right]^{-n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} f\left( x - j \left[ \frac{x - a}{N} \right] \right) \right\}$$

$$n \in Z +$$

# Generalization of repeated integration (Riemann Sum)

$$\frac{d^{-1}f}{[d(x-a)]^{-1}} \equiv \int_a^x f(y)dy \equiv \lim_{\Delta x\to 0}\{[\Delta x][f(x)+f(x-\Delta x)+f(x-2\Delta x)+\ldots f(a+\Delta x)\} =$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \lim_{\Delta x\to 0}\left\{(\Delta x)\sum_{j=0}^{N-1} f(x-j\Delta x)\right\}$$

$$\frac{d^{-2}f}{[d(x-a)]^{-2}} \equiv \int_a^x dx_1 \int_a^{x_1} f(x_0)dx_0 \equiv \lim_{\Delta x\to 0}\{[\Delta x]^2[f(x)+2f(x-\Delta x)+3f(x-2\Delta x)+\ldots Nf(a+\Delta x)\} =$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \lim_{\Delta x\to 0}\left\{(\Delta x)^2\sum_{j=0}^{N-1}[j+1]f(x-j\Delta x)\right\}$$

$$\frac{d^{-3}f}{[d(x-a)]^{-3}} \equiv \int_a^x dx_2 \int_a^{x_2} dx_1 \int_a^{x_1} f(x_0)dx_0 \equiv \lim_{\Delta x\to 0}\left\{[\Delta x]^3\sum_{j=0}^{N-1}\frac{[j+1][j+2]}{2} f(x-j\Delta x)\right\}$$

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} \equiv \lim_{\Delta x\to 0}\left[(\Delta x)^n\sum_{j=0}^{N-1}\binom{j+n-1}{j} f(x-j\Delta x)\right]$$

$$\frac{d^{-n}f}{[d(x-a)]^{-n}} \equiv \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^n\sum_{j=0}^{N-1}\binom{j+n-1}{j} f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\}$$

$n \in Z+$

# Generalization of factorial & Gamma Function

Complete Gamma function defined by Euler limit

$$\Gamma(x) = \lim_{N \to \infty}\left[\frac{N!N^{x}}{x(x+1)(x+2)\ldots(x+N)}\right]$$

Integral transform definition

$$\Gamma(x) \equiv \int_{0}^{\infty} y^{x-1}\exp(-y)\,dy \qquad x > 0$$

Recursive formulation

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(n+1) = n\Gamma(n) = n!$$

$$\Gamma(x-1) = \frac{\Gamma(x)}{(x-1)}$$

For negative integers

$$\Gamma(0) = \infty = \Gamma(-1) = \Gamma(-2) = \ldots.\Gamma(-N)$$

Ratio of Gamma for negative integers

$$\frac{\Gamma(-n)}{\Gamma(-N)} = [-N][-N+1]\ldots..[-n-2][-n-1] = (-1)^{N-n}\frac{N!}{n!}$$

Reciprocal Gamma function is single valued for all arguments

$$\frac{1}{\Gamma(x)} \approx \frac{x^{\frac{1}{2}-x}}{\sqrt{2\pi}}\exp(x);\; x \to \infty$$

Incomplete Gamma

$$\gamma^{*}(c,x) = \frac{1}{c^{x}\Gamma(x)}\int_{0}^{c} y^{x-1}\exp(-y)dy = \exp(-x)\sum_{j=0}^{\infty}\frac{x^{j}}{\Gamma(j+c+1)}$$

# Generalization of binomial coefficients-General differeintegration

The repeated differentiation of a function is compactly represented as follows for positive integer:

$$\frac{d^n f}{dx^n} \equiv \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^{-n}\sum_{j=0}^{N-1}(-1)^j\binom{n}{j}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\}$$

And the repeated integration is:

$$\frac{d^{-n} f}{[d(x-a)]^{-n}} \equiv \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^{n}\sum_{j=0}^{N-1}\binom{j+n-1}{j}f\left(x-j\left[\frac{x-a}{N}\right]\right)\right\}$$

The binomial generalization for positive $n$ (real) is:

$$(-1)^j\binom{n}{j}=\binom{j-n-1}{j}=\binom{-n}{j}=\frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)}$$

Using the above generalization of binomial coefficients we get a following formula for ‘fractional’ order differintegral:

$$\frac{d^q f}{[d(x-a)]^q} \equiv \lim_{N\to\infty}\left\{\left[\frac{x-a}{N}\right]^{-q}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}f(x-j\Delta x)\right\}$$

$$q\in\Re e \qquad \Delta x=\left[\frac{x-a}{N}\right]$$

The index order $q$ is real number of either sign

Positive representing ‘fractional’ differentiation and negative representing fractional integration

# Generalizing the limit of finite differences:

$$\Delta_{(+)} = 1 - E_{\tau}^{-1} = \left(1 - e^{-\tau D}\right) = \left(1 - e^{-\tau \frac{d}{dt}}\right)$$ a finite difference operator

Generalize this $$\Delta_{(+)}^{\alpha} = \left(1 - e^{-\tau \frac{d}{dt}}\right)^{\alpha}$$ for non integer $\alpha$

Fractional forward and backward derivatives are thus:

$$\lim_{\tau \to 0} \frac{\Delta_{(+)}^{\alpha}}{\tau^{\alpha}} X(t) = \lim_{\tau \to 0} \frac{\left(1 - e^{-\tau \frac{d}{dt}}\right)^{\alpha}}{\tau^{\alpha}} X(t) = \frac{d^{\alpha}}{dt^{\alpha}} X(t)$$

$$\lim_{\tau \to 0} \frac{\Delta_{(-1)}^{\alpha}}{\tau^{\alpha}} X(t) = \lim_{\tau \to 0} \frac{\left(1 - e^{\tau \frac{d}{dt}}\right)^{\alpha}}{\tau^{\alpha}} X(t) = \left(-\frac{d}{dt}\right)^{\alpha} X(t)$$

$$D_{(\pm)}^{\alpha} X(t) = \lim_{\tau \to 0} \frac{\Delta_{(\pm)}^{\alpha}}{\tau^{\alpha}}; \alpha > 0$$

# Expansion of Fractional Difference:

$$\Delta^{\alpha}_{(-)} X(t) = \left(1 - E_{\tau}^{-1}\right)^{\alpha} X(t)$$

Expanding as power series

$$\left(1 - E_{\tau}^{-1}\right)^{\alpha} = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (-1)^{k} E_{\tau}^{-k}$$

Using generalization of binomial & factorial and then substitution in above

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha - k)!} = \frac{\Gamma(\alpha + 1)}{\Gamma(k+1)\Gamma(\alpha + 1 - k)}; \alpha > 0$$

and rearranging as series

$$\Delta^{\alpha}_{(-)} X(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^{k} X(t - k\tau)$$

Fractional difference which rather depends on the values of the function in the vicinity of '$t$', depends on the entire history!!

$$\binom{\alpha}{k} = 0$$

$$k > \alpha$$

# Generalization of product rule for multiple integration & differentiation

Integration by parts:

$$\int fg = g\int f - \int\left[\int f\right]d^{(1)}g$$

$$d^{-1}[fg] = g\,d^{-1}f - d^{-1}\left\{g^{(1)}d^{-1}f\right\}$$

$$d^{-1}[fg] = g\,d^{-1}f - g^{(1)}d^{-2}f + d^{-1}\left\{g^{(2)}d^{-2}f\right\}$$

$$d^{-1}[fg] = \sum_{j=0}^{\infty}(-1)^j g^{(j)}d^{-1-j}(f) = \sum_{j=0}^{\infty}\binom{-1}{j}g^{(j)}d^{-1-j}(f)$$

$$d^{-2}[fg] = \sum_{j=0}^{\infty}\binom{-2}{j}g^{(j)}d^{-2-j}(f)$$

$$d^{-n}[fg] = \sum_{j=0}^{\infty}\binom{-n}{j}g^{(j)}d^{-n-j}f$$

Product rule for differentiation:

$$d^{(1)}[fg] = gf^{(1)} + fg^{(1)}$$

$$d^{(n)}[fg] = \sum_{j=0}^{n}\binom{n}{j}g^{(j)}d^{(n-j)}f$$

# Leibniz's Rule for arbitrary order-a generalization

The finite sum in repeated differentiation of product of functions could equally well extend to infinite series as $\binom{n}{j} = 0\,;\; j > n$

$$d^{-n}[fg] = \sum_{j=0}^{\infty} \binom{-n}{j} g^{(j)} d^{-n-j} f \qquad d^{(n)}[fg] = \sum_{j=0}^{n} \binom{n}{j} g^{(j)} d^{(n-j)} f$$

The generalized product rule is:

$$d^{q}[fg] = \sum_{j=0}^{\infty} \binom{q}{j} d^{(q-j)}\{f\}\, d^{(j)}\{g\}$$

$$q \in \Re e$$

# Generalized repeated differ-integration of monomial

Euler formulation (1730)

$$f(x) = x^m$$

$$\frac{d}{dx}(x^m) = m x^{m-1}$$

$$\frac{d^2}{dx^2}(x^m) = m(m-1)x^{m-2}$$

$$\int x^m dx = \frac{1}{m+1} x^{m+1}$$

$$\iint x^m dx dx = \frac{1}{(m+1)(m+2)} x^{m+2}$$

$$\frac{d^n}{dx^n}(x^m) = m(m-1)(m-2)\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\ldots.(m-n+1)\Gamma(m-n+1)$$

$$\frac{\Gamma(m+1)}{\Gamma(m-n+1)} = m(m-1)\ldots(m-n+1)$$

For any arbitrary index $m, n \in \Re e$ Differ-integration is:

$$\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}; m+1 > 0; (m-n+1) > 0$$

Examples of Euler formula:

$$\frac{d^{0.5}}{dx^{0.5}}(x) = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)} x^{1-0.5} = \frac{\Gamma(2)x^{0.5}}{\Gamma(1+0.5)} = \frac{1}{0.5\Gamma(0.5)}\sqrt{x} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$\frac{d^{0.5}}{dx^{0.5}}(x^0) = \frac{1}{\sqrt{\pi x}} \neq 0$$

$$\frac{d^{-0.5}}{dx^{-0.5}}\sqrt{x} = \frac{\Gamma(0.5+1)x^{0.5-(-0.5)}}{\Gamma(0.5-\{-0.5\}+1)} = \frac{0.5\Gamma(0.5)}{\Gamma(2)} x = \frac{\sqrt{\pi}}{2} x$$

$$\frac{d}{dx} x = \frac{\Gamma(1+1)x^{1-1}}{\Gamma(1-1+1)} = \frac{\Gamma(2)}{\Gamma(1)} = 1, \quad \frac{d^{-1}}{dx^{-1}} x = \frac{\Gamma(1+1)x^{1-(-1)}}{\Gamma(1-\{-1\}+1)} = \frac{x^2}{2}$$

# Composition Rules Generalized:

1. Composition & Commutation is valid for fractional integration

$$D^{-\mu}[D^{-\nu} f(t)] = D^{-(\mu+\nu)}[f(t)] = D^{-\nu}[D^{-\mu} f(t)]$$

2. Derivative of Fractional Integral

$$D[D^{-\nu} f(t)] = D^{-\nu}[D f(t)] + [t^{\nu-1} f(0)]/\Gamma(\nu)$$

3. Fractional Integral of Derivative

$$D^{-\nu-1}[D f(t)] = D^{-\nu}[f(t)] - [t^{\nu} f(0)]/\Gamma(\nu+1)$$

4. Composition of arbitrary order

$$D^{u}[D^{v} f(t)] \neq D^{u+v}[f(t)]$$

$$D^{u}[D^{v} f(t)] \neq D^{v}[D^{u} f(t)]$$

$$f(x) = x^{1/2}, u = 1/2, v = 3/2$$

$$D_x^{u}[x^{1/2}] = D_x^{1/2}[x^{1/2}] = \frac{1}{2}\sqrt{\pi}$$

$$D_x^{v}[x^{1/2}] = D_x^{3/2}[x^{1/2}] = 0$$

$$D_x^{u}[D_x^{v}(x^{1/2})] = 0$$

$$D_x^{v}[D_x^{u}(x^{1/2})] = D_x^{3/2}(\frac{1}{2}\sqrt{\pi}) = -\frac{1}{4}x^{-3/2}$$

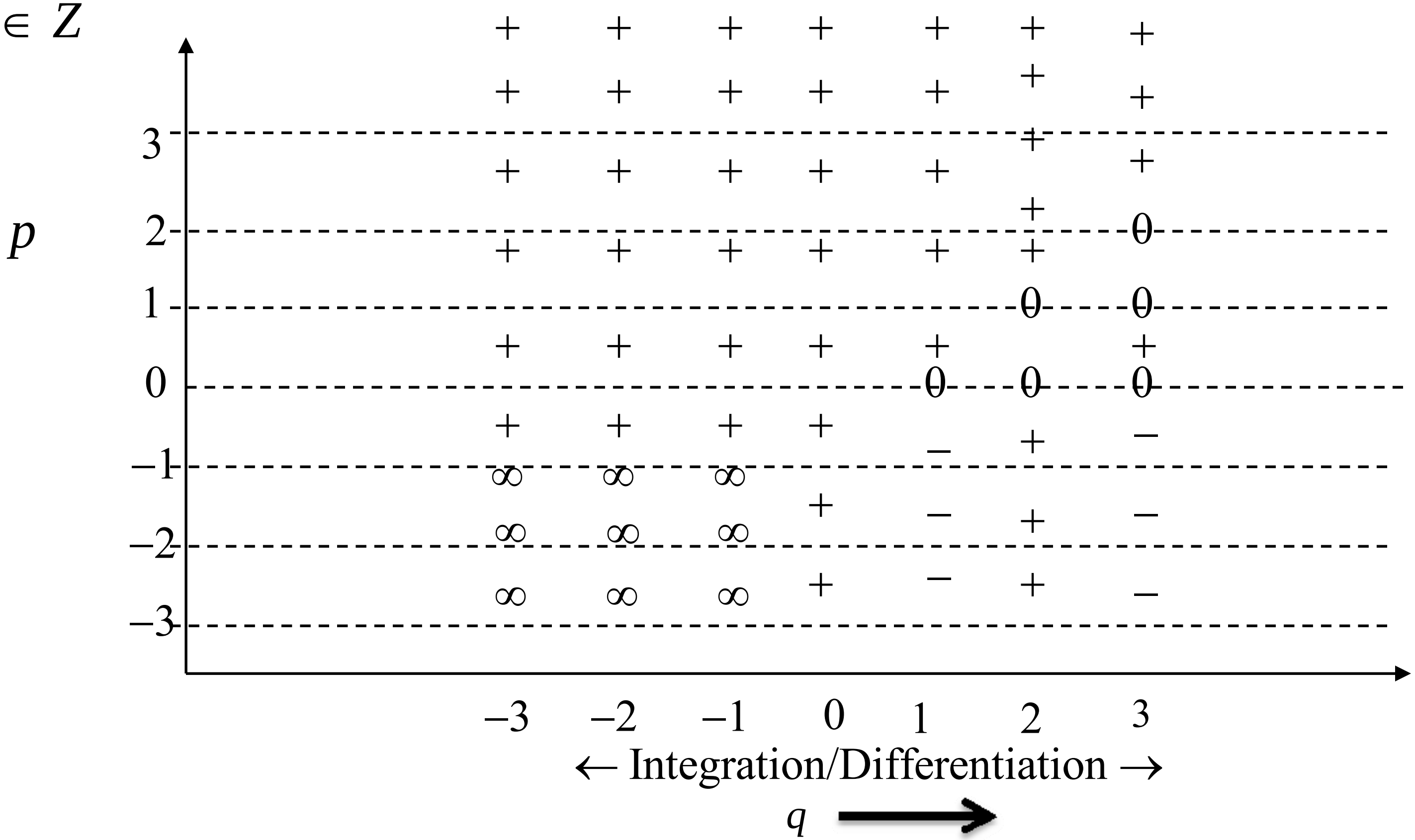
$$D_x^{u+v}[x^{1/2}] = D_x^{2}[x^{1/2}] = -\frac{1}{4}x^{-3/2}$$

# Existence of differ integration-integer order for power function (monomial)

$$\frac{d^{q}[x-a]^{p}}{[d(x-a)]} = \begin{cases} \dfrac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)} & \begin{cases} q = 0,1,2\ldots\ldots\forall p \\ q = -1,-2,-3\ldots;\ p > -1 \end{cases} \\ \infty & q = -1,-2,-3\ldots;\ p \le -1 \end{cases}$$

$p \in \Re e$

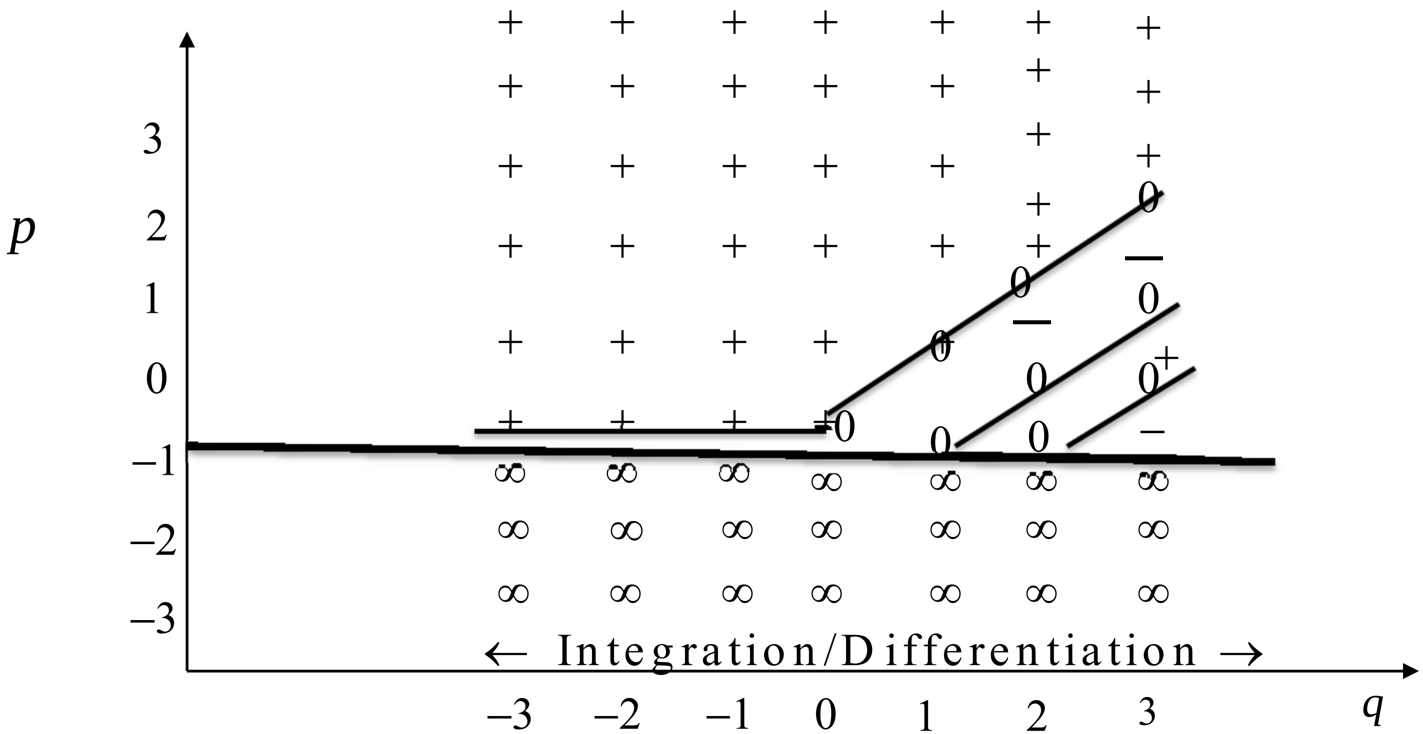
$q \in Z$



# Existence of fractional differ integration of power function (monomial)

$$\frac{d^q [x-a]^p}{[d(x-a)]^q} = \begin{cases} \dfrac{\Gamma(p+1)[x-a]^{p-q}}{\Gamma(p-q+1)} & ; p > -1 \\ \infty & ; p \le -1 \end{cases}$$

$p \in \Re e \qquad q \in \Re e$




Breakdown of differeintegration formula for $p \le -1$ is associated with "pole" of order one or greater at start point of differintegration. Functions for which pole occurs anywhere in the open interval ($a$,$x$) leads to similar difficulties thus excluded.

## Existence of differ-integral:

Indeed most of the "special-functions" of mathematical physics are differintegrable.

Condition is: $\lim_{x \to a} (x-a) f(x) = 0$

$$\lim_{(a+) \to a} \int_a^{(a+)} f(x)\,dx = 0$$

The function be represented as series expansion:

$$f(x) = [x-a]^p \sum_{j=0}^{\infty} a_j [x-a]^{j/n} ;\ a_0 \neq 0 \ \&\ p > -1$$

as product of a power of $[x-a]$ and analytic function of $[x-a]^{1/n}$ where '*n*' is integer

**Candidate function be defined on a closed interval** $a \le x \le b$ **, that they be bounded in half open interval** $a < x \le b$ **, and be "better-behaved" than** $[x-a]^{-1}$

Non-Differentigrable: $(x-a)^{-1}, (x-a)^{-2}, \sqrt{\frac{9}{4} - (x-a)^2}$

Differintegrable $\ln(x-a),\ 1/\sqrt{x-a},\ [x-a],\ \frac{1}{2}\sin[\pi(x-a)] - 1,\ \frac{\sin\sqrt{x-a}}{(x-a)^{3/4}}$

## Series representation for differential equation solution

$$m\,x''(t) + k\,x(t) = f_0(t);\ m = 1,\ k = 1$$

$$x(0) = x'(0) = 0;\ f_0(t) = \delta(t)$$

In the absence of spring (no opposing force) the displacement will be

$$x_0(t) = d^{-2} f_0(t) = d^{-2}\delta(t) = t$$

In springs presence this displacement manifests as opposing force to the external force and the first mode of displacement is

$$x_1(t) = -d^{-2} x_0(t) = -d^{-2}\{t\} = -t^3/3!$$

This displacement generates force as $f_1 = -k x_1(t) = -x_1(t) = t^3/3!$

Again this force manifests as displacement as

$$x_2(t) = d^{-2} f_1 = d^{-2}\{t^3/3!\} = t^5/5!$$

Again this displacement generates force as

$$f_2 = -k x_2 = -x_2 = -t^5/5!$$

# Series representation for differential equation solution

| **Mode** | **Force** $f_i(t)$ | **Displacement** $x_i(t)$ |
|---|---|---|
| $i = 0$ | $f_0 = \delta(t)$ | $x_0(t) = t$ |
| $i = 1$ | $f_1(t) = -t$ | $x_1(t) = -t^3/3!$ |
| $i = 2$ | $f_2(t) = t^3/3!$ | $x_2(t) = t^5/5!$ |
| $i = 3$ | $f_3(t) = -t^5/5!$ | $x_3(t) = -t^7/7!$ |

$$x(t) = x_0 + x_1 + x_2 + \ldots = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \approx \sin(t)$$

# Using monomial integration in solving differential equation

$x^{//}(t) + x(t) = f(t)$

$f(t) = \delta(t)$

$x(0) = 0, x^{/}(0) = 0$

$x(t) = \sin t$

We can rearrange and double integrate both sides and iterate as follows:

$$x^{//}(t) = f(t) - x(t)$$

$$\iint_{0\to t} x^{//}(t)dt = \iint_{0\to 0+} f(t)dt - \iint_{0+\to t} x(t)dt$$

$$x(t) - x(0) - tx^{/}(0) = \iint_{0\to 0+} f(t)dt - \iint_{0+\to t} x(t)dt$$

$$x(t) = x(0) + t[d_t^1 x(t)]_{@t=0} + \left(d_t^{-2} f(t)\right) - d_t^{-2}(d_t^{-2} f(t)) + d_t^{-2}(d_t^{-2}(d_t^{-2} f(t))) - ..$$

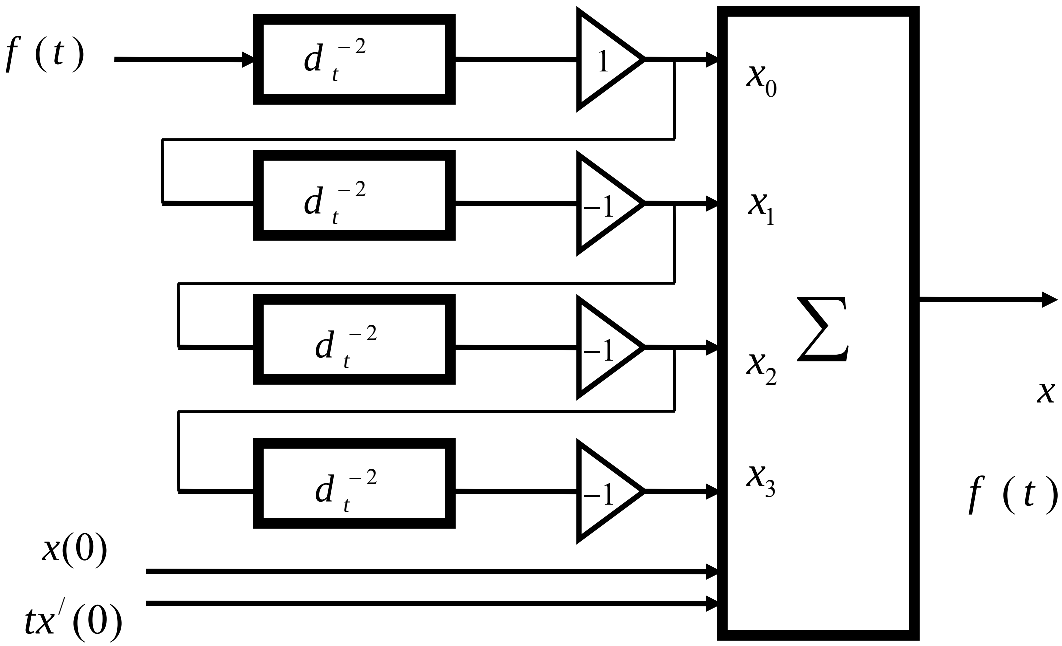
Using these we get

$$d^{-1}\delta(t) = 1$$

$$d^{-2}\delta(t) = t$$

$$d^{-3}\delta(t) = \frac{t^2}{2}$$

$$d^{-4}\delta(t) = \frac{t^3}{3\times 2}$$




$$x(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + .. \approx \sin t$$

$$f(t) = \delta(t), x(0) = 0, x^{/}(0) = 0$$

Solution is infinite series (convergent) with integer power monomial

# Using monomial differ-integration to solve fractional Differential equation:

Example oscillator with fractional loss component

$$x''(t) + d^{1/2}x(t) + x(t) = f(t)$$

$$x(t) = x(0) + tx'(0) + d^{-2}f(t) - d^{-2}x(t) - d^{-3/2}x(t)$$

$$x(0) = 0,\ x'(0) = 0,\ f(t) = \delta(t)$$

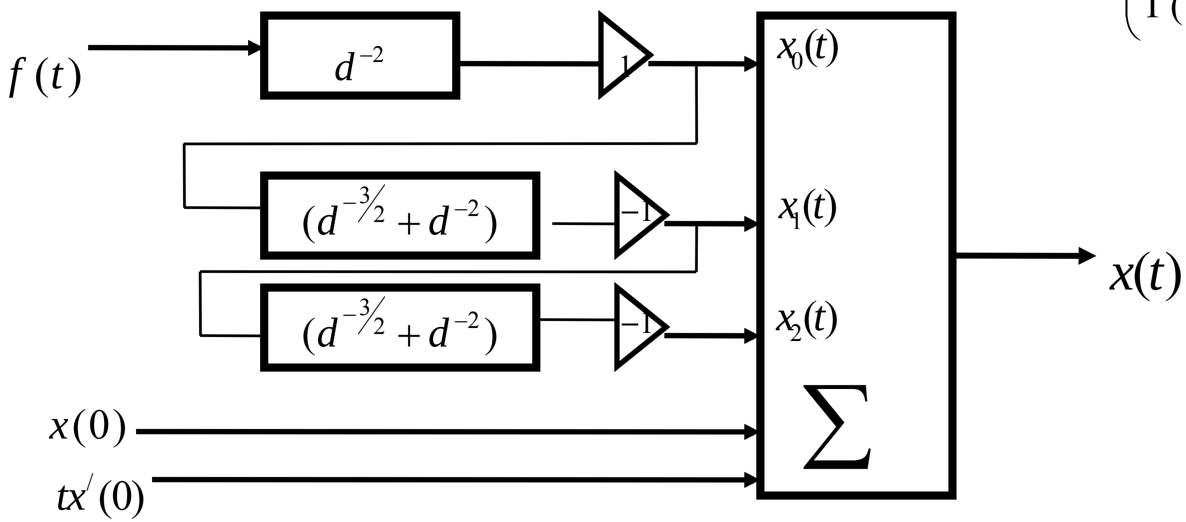
Euler's generalization $d^n x^m = \dfrac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)}$

Iterate the same and get series

$$x_0(t) = d^{-2}\delta(t) = t$$

$$x_1(t) = -(d^{-3/2} + d^{-2})(t) = -\left(\frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^3}{\Gamma(4)}\right)$$

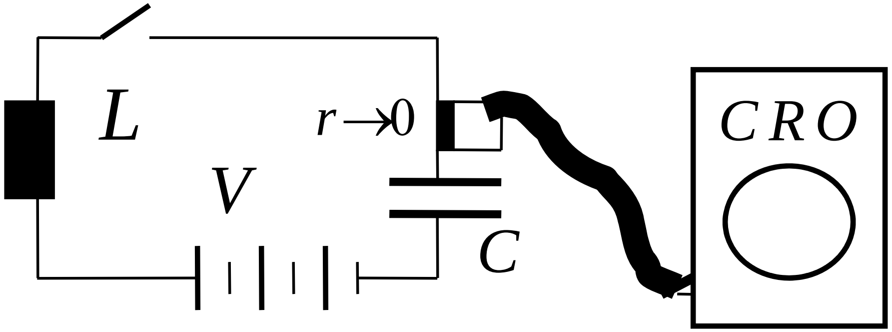
$$x_2(t) = (d^{-3/2} + d^{-2})\left(\frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^3}{\Gamma(4)}\right) = \frac{t^4}{\Gamma(5)} + 2\frac{t^{9/2}}{\Gamma(11/2)} + \frac{t^5}{\Gamma(6)}$$




$$x(t) = t - \frac{t^{2.5}}{\Gamma(3.5)} - \frac{t^3}{\Gamma(4)} + \frac{t^4}{\Gamma(5)} + 2\frac{t^{4.5}}{\Gamma(4.5)} + \frac{t^5}{\Gamma(6)} + ..$$

Solution is infinite series convergent with integer & fractional power terms

# Fractional oscillator an example:

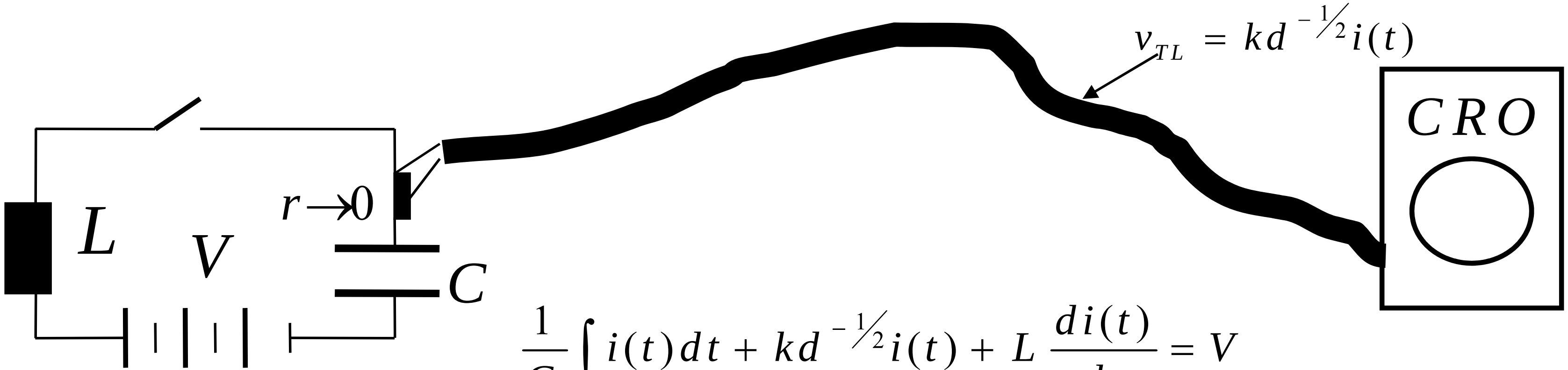



Short CRO cable circuit as oscillator

$$\frac{1}{C}\int i(t)\,dt + \lim_{r\to 0} r\,i(t) + L\frac{di(t)}{dt} = V$$

$$\frac{1}{C}i(t) + L\frac{d^2 i(t)}{dt^2} = \frac{dV}{dt} = V\delta(t)$$

Long lossy CRO cable as Semi infinite TL half derivative




$$v_{TL} = kd^{-1/2} i(t)$$

$$\frac{1}{C}\int i(t)\,dt + kd^{-1/2} i(t) + L\frac{di(t)}{dt} = V$$

$$\frac{1}{C}i(t) + k\frac{d^{1/2} i(t)}{dt^{1/2}} + L\frac{d^2 i(t)}{dt^2} = \frac{dV}{dt} = V\delta(t)$$

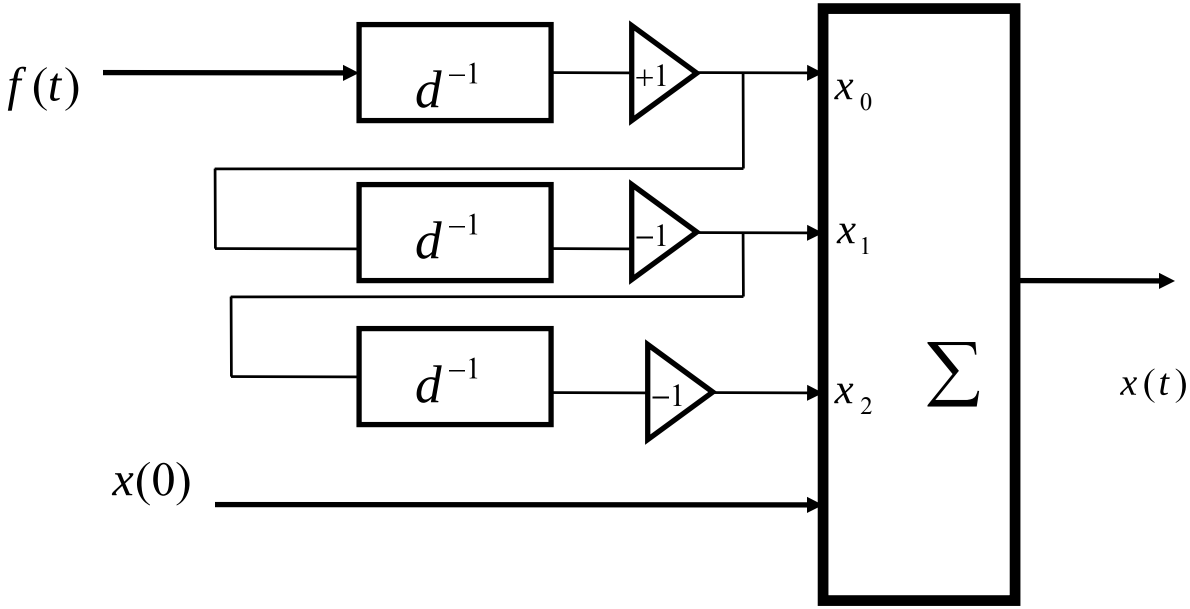
# First order system and monomial integration

$$x^{/}(t) + x(t) = f(t)$$

$$d^{-1}d^{1}x(t) + d^{-1}x(t) = d^{-1}f(t)$$

$$x(t) - x(0) = d^{-1}f(t) - d^{-1}d^{-1}f(t) + d^{-1}d^{-1}d^{-1}f(t) - ..$$

$$x(0) = 0,\ f(t) = \delta(t)$$




$$x(t) = 1 - t - \frac{t^2}{2!} + \frac{t^3}{3!} - \ldots \approx \exp(-t)$$

Solution is infinite series convergent with integer power terms.

# First order system with fractional loss term monomial solution

$$x^{/}(t) + d^{1/2}x(t) + x(t) = f(t)$$

$$f(t) = \delta(t),\ x(0) = 0$$

Euler relation $$\frac{d^n}{dx^n}x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

$$x(t) = x(0) + d^{-1}f(t) + \sum_{n=1}^{\infty}(-1)^n(d^{-1} + d^{-1/2})^n f(t)$$

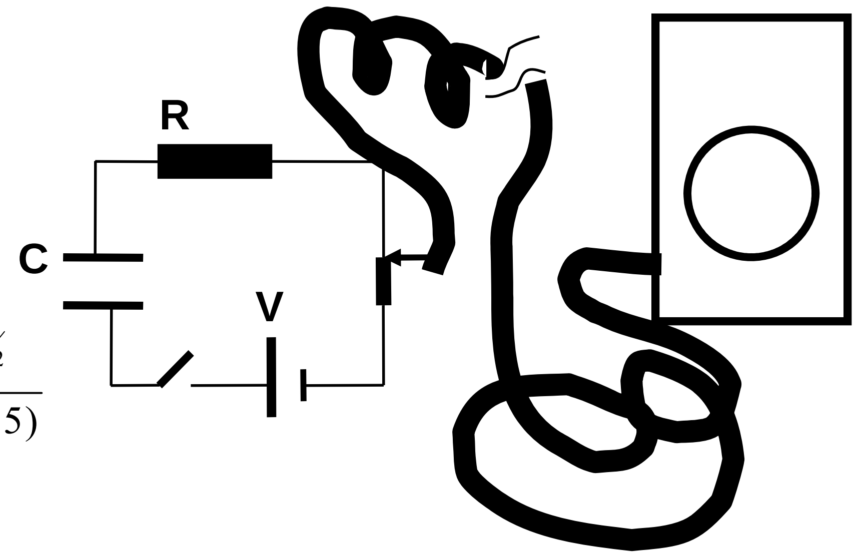
$$x_0 = d^{-1}\delta(t) = 1$$

$$x_1 = (d^{-1} + d^{-1/2})x_0 = d^{-1}(1) + d^{-1/2}(1) = t + \frac{t^{1/2}}{\Gamma(1.5)}$$

$$x_2 = (d^{-1} + d^{-1/2})x_1 = d^{-1/2}t + d^{-1/2}\frac{t^{1/2}}{\Gamma(1.5)} + d^{-1}t + d\frac{t^{1/2}}{\Gamma(1.5)}$$

$$= \frac{t^{3/2}}{\Gamma(2.5)} + \frac{t}{\Gamma(2)} + \frac{t^2}{2} + \frac{t^{3/2}}{\Gamma(2.5)}$$

$$x(t) = 1 - t - \frac{t^{1/2}}{\Gamma(1.5)} + \frac{t}{\Gamma(2)} + \frac{2t^{3/2}}{\Gamma(2.5)} + \frac{t^2}{2} - \ldots.$$




$$\frac{1}{C}\int i(t)dt + Ri(t) + kd^{-1/2}i(t) = V$$

$$R\frac{di(t)}{dt} + kd^{1/2}i(t) + \frac{1}{C}i(t) = \frac{dV}{dt} = V\delta(t)$$

Distributed effect of long TL comes as fractional derivative/integral term.
behaves as half order element , will it give II order response for I order system?

# Observations

Solution of integer order differential equation yields a series which is convergent-assuming existence of solution.

Solution of fractional order differential equation yields a series which is convergent-assuming existence of solution.

Integer order differential equations yields a series solution with integer power monomials.

Fractional order differential equations yields a series solution with integer as well as fractional power monomials.

Integer order differential equation yields convergent series to transcendental function.

Fractional order differential equations thus should yield some series –convergent to similar "higher fractional transcendental functions"

# Power series functions used in fractional calculus Higher Transcendental Functions

Exponential function forms basis in the integer order calculus so is MITTAG LEFFLER function for the fractional calculus

**Mittag-Leffler**

$$E_q(at^q) = \sum_{n=0}^{\infty} \frac{a^n t^{nq}}{\Gamma(nq+1)} \leftrightarrow \frac{s^q}{s(s^q - a)} = \frac{s^{q-1}}{s^q - a}$$

**Agarwal**

$$E_{a,b}(t^q) = \sum_{n=0}^{\infty} \frac{t^{\left(n+\frac{b-1}{a}\right)}}{\Gamma(na+b)} \leftrightarrow \frac{s^a}{s^b(s^a-1)} = \frac{s^{a-b}}{s^a-1}$$

**Erdelyi**

$$E_{a,b}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(am+b)} \leftrightarrow \sum_{m=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(ma+b)s^{m+1}}$$

**Robotnov-Hartley**

$$F_q(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)q-1}}{\Gamma(\{n+1\}q)} \leftrightarrow \frac{1}{(s^q - a)}$$

**Miller-Ross**

$$E_t(q,a) = t^q \exp(at)\gamma^*(q,at) = \sum_{k=0}^{\infty} \frac{a^k t^{k+q}}{\Gamma(q+k+1)} \leftrightarrow \frac{s^{-q}}{s-a}$$

# *R*-Function

*R*-Function

$$R_{q,v}[a,c,t] = \sum_{n=0}^{\infty} \frac{(a)^n (t-c)^{(n+1)q-1-v}}{\Gamma\{(n+1)q-v\}}$$

$$R_{q,v}[a,(t-c)] \leftrightarrow \exp(-cs)\frac{s^v}{s^q-a}$$

$$R_{q,v}[-a,0,t] \leftrightarrow \frac{s^v}{s^q-a}$$

Relation to elementary function:

$$R_{1,0}(a,0,t) = \exp(at)$$

$$aR_{2,0}(-a^2,0,t) = \sin(at)$$

$$R_{2,1}(-a^2,0,t) = \cos(at)$$

$$aR_{2,0}(a^2,0,t) = \sinh(at)$$

$$R_{2,1}(a^2,0,t) = \cosh(at)$$

Relation to other higher functions

$$E_q(-at^q) = R_{q,q-1}(-a,0,t)$$ Mittag-Leffler

$$E_{q,p}(t^q) = R_{q,q-p}(1,0,t)$$ Agarwal

$$t^{1-\beta}E_{q,\beta}(t^q) = R_{q,q-\beta}(1,0,t)$$ Erdely's

$$F_q(-a,t) = R_{q,0}(-a,0,t)$$ Rob-Hartley

$$E_t(v,a) = R_{1,-v}(a,0,t)$$ Miller-Ross

# Miller-Ross function-and higher trigonometric functions with properties

$$E_t(w,c) = t^w \sum_{n=0}^{\infty} \frac{(ct)^n}{\Gamma(1+n+w)}$$

$$E_t(v,ia) = t^v \left[ \sum_{k(even)}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma(v+k+1)} + i \sum_{k(odd)}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma(v+k+1)} \right]$$

$$C_t(v,a) = t^v \sum_{j=0}^{\infty} \frac{(-1)^j (at)^{2j}}{\Gamma(v+2j+1)}; S_t(v,a) = t^v \sum_{k(odd)}^{\infty} \frac{(-1)^{\frac{k-1}{2}}(at)^k}{\Gamma(v+k+1)}$$

$$E_t(v,ia) = C_t(v,a) + iS_t(v,a)$$

$$C_t(0,a) = \cos at, S_t(0,a) = \sin at, E_t(0,a) = e^{at}$$

$$DE_t(v,a) = E_t(v-1,a); DC_t(v,a) = C_t(v-1,a); DS_t(v,a) = S_t(v-1,a)$$

$$\int_0^t E_\xi(v,a)d\xi = E_t(v+1,a); \int_0^t C_\xi(v,a)d\xi = C_t(v+1,a); \int_0^t S_\xi(v,a)d\xi = S_t(v+1,a)$$

…………………..and several more interesting relations

# Poles in first order system with fractional loss

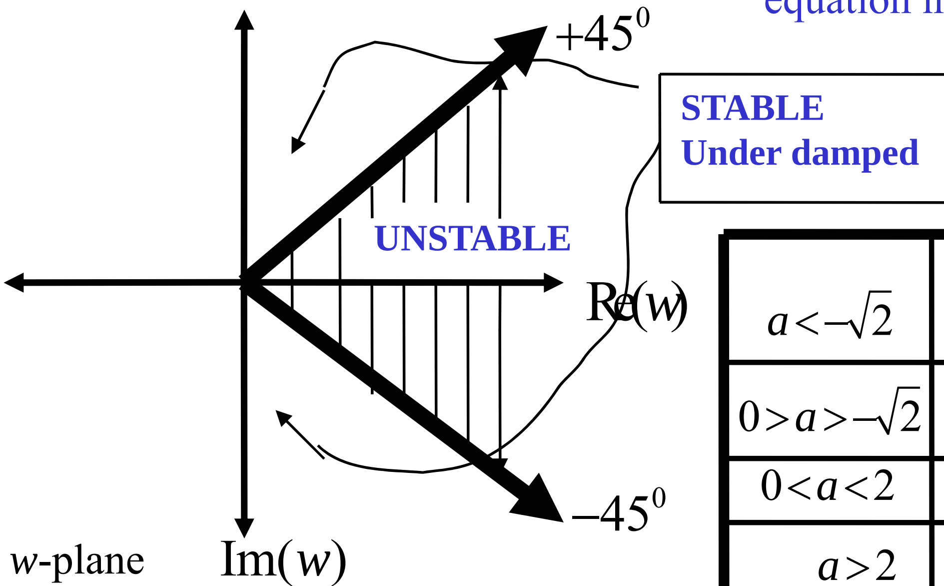
## Concept of w-plane conformal mapping

$$dx(t) + a\,d^{1/2}x(t) + x(t) = f(t)$$

$$sX(s) + a\,s^{1/2}X(s) + X(s) = F(s)$$

Characteristic equation is: $s + a\sqrt{s} + 1$ in $s$-plane

let $s^{1/2} = w$ then $w^2 + aw + 1$ is characteristic equation in $w$-plane. $\arg w = \frac{1}{2}\arg s, \bmod(w) = \sqrt{\bmod(s)}$




| | | | |
|---|---|---|---|
| $a < -\sqrt{2}$ | $\arg(w) < \pm 45^0$ | $\arg(s) < \pm 90^0$ | Unstable |
| $0 > a > -\sqrt{2}$ | $\pm 45^0 - \pm 90^0$ | $\pm 90^0 - \pm 180^0$ | Stable |
| $0 < a < 2$ | $\pm 90^0 - \pm 180^0$ | $\pm 180^0 - \pm 360^0$ | Hyperdamped |
| $a > 2$ | $\arg(w) > \pm 180^0$ | $\arg(s) > \pm 360^0$ | Ultradamped |

A first order system with fractional term may become unstable can have oscillatory behaviour and can behave as stable second order stable under damped systems. Classical order definition with number of energy storage element and or number of initial condition can give misleading information about the response in presence of fractional order terms.

# Solution of fractional differential equation (in ML function)

Fractional differential equation of a tracking filter

$$\frac{d^{0.25}}{dt^{0.25}}y(t) + y(t) = x(t)$$

$$y(0) = 0$$

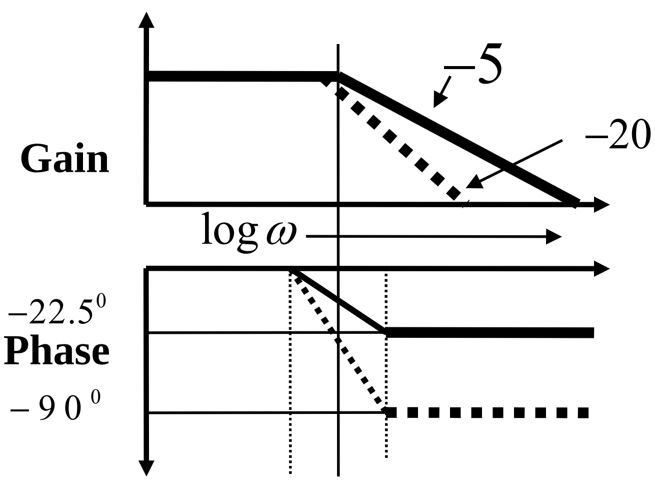
$$s^{0.25}Y(s) + Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^{0.25}+1}X(s)$$ For step excitation $X(s) = 1/s$

$$Y(s) = \frac{1}{s}\left(\frac{1}{s^{0.25}+1}\right) = \frac{1}{s}\left(1-\frac{s^{0.25}}{s^{0.25}+1}\right) = \frac{1}{s} - \frac{s^{0.25}}{s(s^{0.25}+1)} = \frac{1}{s} - \frac{s^{0.25-1}}{s^{0.25}+1}$$

$$y(t) = \mathcal{L}^{-1}Y(s) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s^{0.25-1}}{s^{0.25}+1}\right)$$

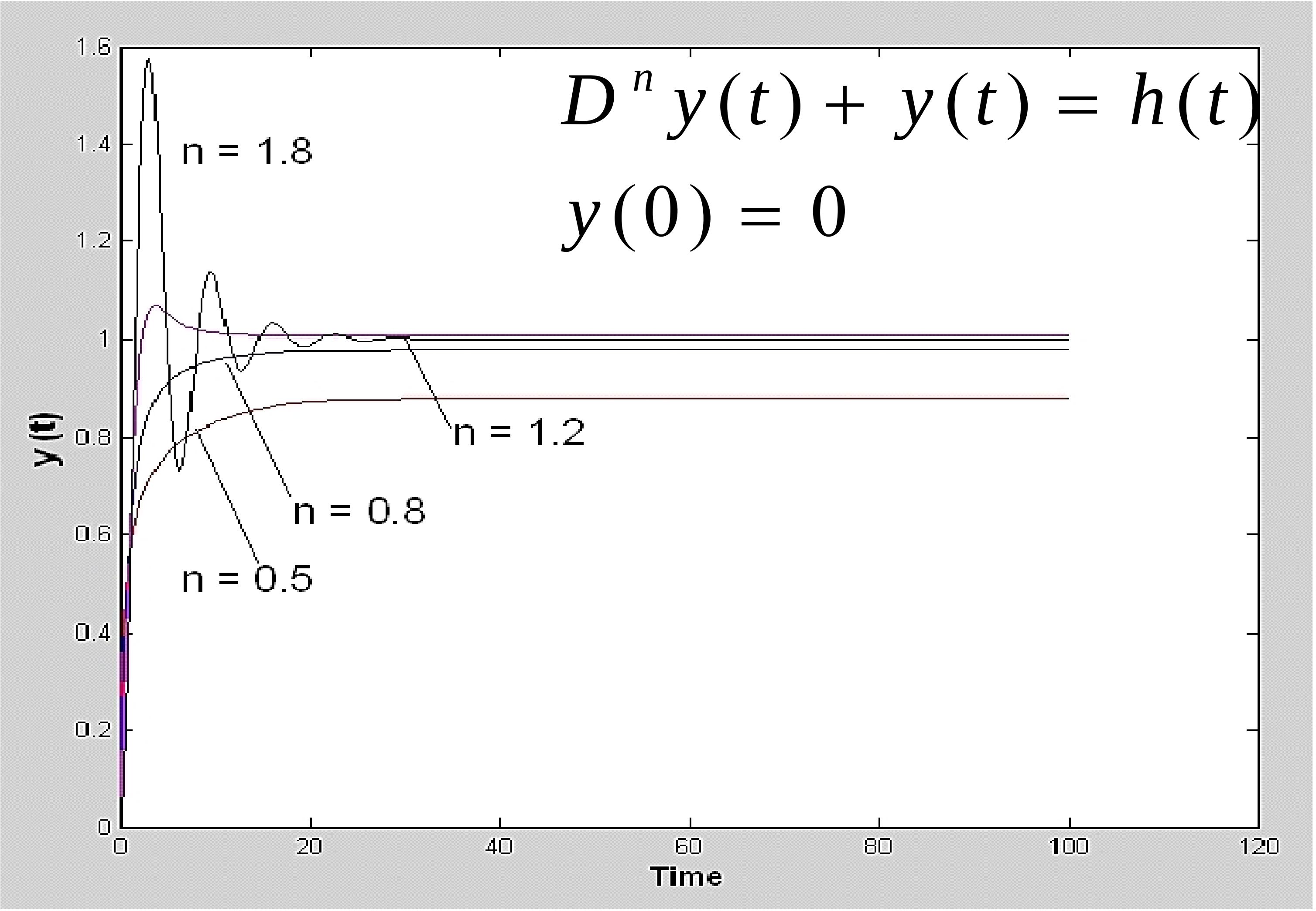
$$y(t) = 1 - E_{0.25}(-t^{0.25})$$




For first order solution is:

$$y(t) = 1 - E_1(-t) = 1 - \exp(-t)$$

## Result: Step response of the system for different values of differential order



# Series solution of FDE:

Relaxation equation, in mechanics, where FO damping or slow loading is force; the inertia plays no role (example creep test). This is similar to tracking filter

$$D^{1/3} x(t) + x(t) = y(t)$$

$$x(0) = 0,\ y(t) = h(t)$$

$$X(s) = \frac{1}{s(1+s^{1/3})} = \frac{[1-(-s^{1/3})]^{-1}}{s^{4/3}}$$

$$|s| << 1$$

Expanding numerator by Binomial Theorem

$$X(s) = \sum_{n=4}^{\infty} \frac{(-1)^n}{s^{n/3}}$$

Using Laplace $t^n \leftrightarrow \frac{n!}{s^{n+1}}$ & generalizing:

$$x(t) = \sum_{n=4}^{\infty} (-1)^n \frac{t^{(n/3)-1}}{\Gamma(n/3)}$$

# Series solution of FDE

Velocity profile of falling ball in fluid with FO damping:

$$Dv(t) + D^{\alpha}v(t) + v(t) = 1$$

$$v(0) = 0$$

$$V(s) = \frac{1}{s(1 + s + s^{\alpha})} = \frac{[1 - (-s^{-1} - s^{\alpha-1})]^{-1}}{s^2}$$

$$[(s^{-1} + s^{\alpha-1})] < 1; \alpha < 1$$

$$s >> 1$$

$$V(s) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \binom{n}{r} \frac{1}{s^{n+2-r\alpha}}$$

$$v(t) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \binom{n}{r} \frac{t^{n+1-r\alpha}}{\Gamma(n+2-r\alpha)}$$

# Comment regarding system order

On contrary to widely accepted opinion in integer order theory, the first order system cannot go into instability or oscillations, the presence of fractional order elements in the first order system can give a counterintuitive result.

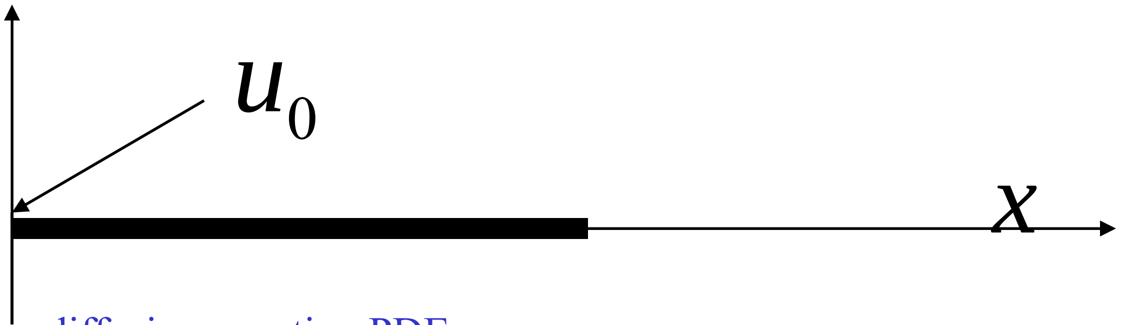
On contrary to widely accepted opinion that chaos cannot occur in continuous-time system of order less than three (in presence of non-linearity as feed back), fractional order system of order less than three can display chaotic behaviour, with non linear feed back.

Order definition in classical theory saying the order is number of energy storage elements, or number of initialization constants required or the nature of output of damped nature, is not therefore valid in the presence of fractional order element.

Order of fractional value in the differential equation system can be solved by some tricks similar to Ordinary Differential Equations.

# Partial Differential Equation & Operational calculus

Oliver Heaviside




Semi-infinite system diffusion equation PDE

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$$a^2 = \frac{c_p \rho}{k}, \quad a^2 = RC$$

Initial condition $\quad u(x,0) = 0, \; x > 0$

Boundary condition $\quad u(0,t) = u_0$

Operator $\quad s \equiv \partial / \partial t$

$$\frac{\partial^2 u}{\partial x^2} - a^2 s u = 0$$

## Solution

$$\frac{\partial^2 u}{\partial x^2} - a^2 s u = 0$$

$$m = \pm a\sqrt{s}$$

$$u(x,s) = A\exp(-a\sqrt{s}\,x) + B\exp(+a\sqrt{s}\,x)$$

$$x \to \infty,\ u(x,0) = 0,\ B = 0$$

$$x \to 0,\ u(0,t) = u_0 = A$$

$$u(x,s) = u_0 \exp(-a\sqrt{s}\,x)$$

Expanding exponential as power series:

$$u(x,s) = u_0 + u_0\sum_{n=1}^{\infty}\frac{(-ax\sqrt{s})^n}{n!} = u_0 + \sum_{n=1}^{\infty}\frac{(-ax)^n (s)^{\frac{n}{2}}}{n!}u_0$$

segregating odd & even terms and then rearranging:

$$u(x,s) = u_0 - \sum_{m=0}^{\infty}\frac{(ax)^{2m+1}}{(2m+1)!}s^m\left(s^{1/2}u_0\right) + \sum_{n=0}^{\infty}\frac{(ax)^{2n}}{(2n)!}s^n u_0$$

putting

$$d^{1/2}u_0 \to s^{1/2}u_0 \equiv \frac{u_0}{\sqrt{\pi t}};\ d^n u_0 \to s^n u_0 = 0$$

$$s^m \to d^m / dt^m$$

Putting this we get the even terms as *n*-th integer order derivative components of a constant a zero.

## Solution (contd.)

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \left[ \frac{d^m}{dt^m} t^{-1/2} \right] = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma(-\frac{1}{2}+1)}{\Gamma(-\frac{1}{2}-m+1)} t^{-\frac{1}{2}-m}$$

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma\left(1/2\right)}{\Gamma\left(\frac{1}{2}-m\right)} \frac{1}{t^{m+1/2}}$$

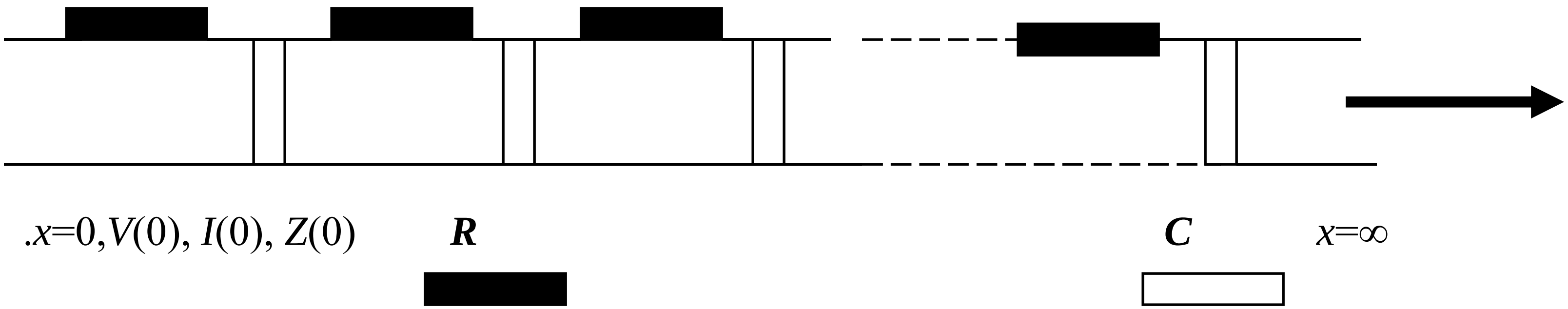
using $\Gamma(-m + 1/2) = \dfrac{[-4]^m m!\sqrt{\pi}}{(2m)!}$ We obtain the following:

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+\frac{1}{2}}}$$

writing $\dfrac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+\frac{1}{2}}} \equiv 2 \displaystyle\int_0^{y=\frac{ax}{2\sqrt{t}}} y^{2m} dy$ and using $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (x^2)^m \equiv \exp(-x^2)$

$$u(x,t) = u_0 - \frac{2u_0}{\sqrt{\pi}} \int_0^{ax/2\sqrt{t}} \exp(-y^2)\, dy$$

# Semi Infinite Lossy Transmission Line




Assuming lossy *RC* line the Boundary Value Problem can be defined in terms of voltage or current variables . Since a semi infinite line is considered , the measureable inputs and outputs are at the left, while at the right end the values are finite. The constitutive and continuity equations are:

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R \qquad \frac{\partial i(x,t)}{\partial x} = -C\,\frac{\partial v(x,t)}{\partial t}$$

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R$$

# Semi Infinite Lossy Transmission line (contd.)

$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial v(x,t)}{\partial t}$$

Differentiation first w.r.t. $x$ and then substituting into the second one we get:

$$\frac{\partial^2 v}{\partial x^2} = -R\frac{\partial i}{\partial x} = RC\frac{\partial v}{\partial t}$$

Choose $\alpha = \frac{1}{RC}$ to get diffusion equation

$$\frac{\partial v(x,t)}{\partial t} = \alpha \frac{\partial^2 v(x,t)}{\partial x^2}$$

Conditions are $v(0,t) = v_I(t), v(\infty,t) = 0$

$v(x,0)$, given with $i(x,t) = -\frac{1}{R}\frac{\partial v(x,t)}{\partial x}$

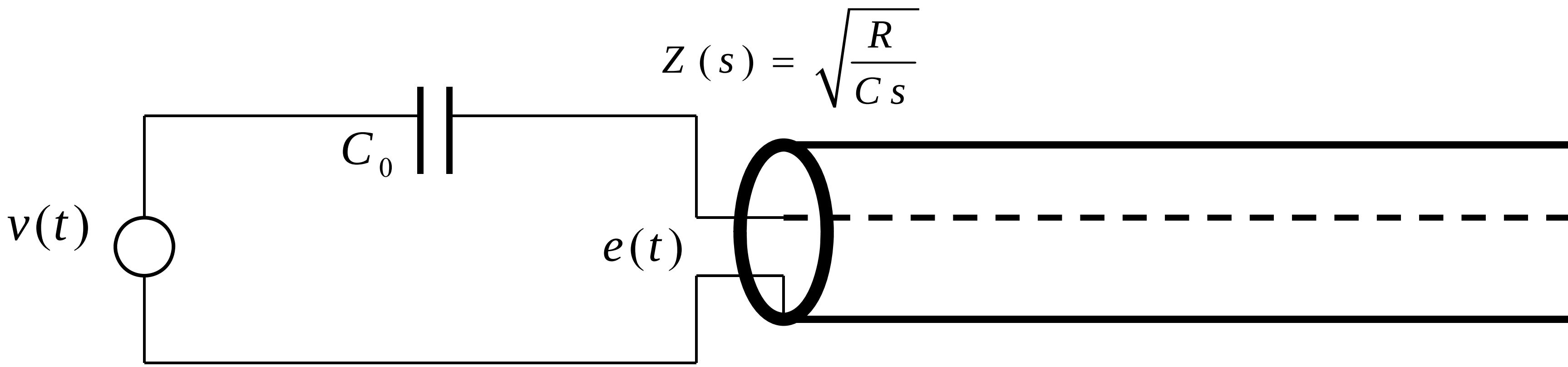
In this formulation ($v$) is the voltage ($i$) is the current ($v_I$) is time-dependent input variable. A classical solution using iterated Laplace used to solve this problem and the driving point impedance is

$$Z(0,s) = \frac{V(0,s)}{I(0,s)} = \sqrt{\frac{R}{C}}\frac{1}{\sqrt{s}}$$

Semi-Derivative $i(x,t) = \frac{1}{R\sqrt{\alpha}}\frac{d^{1/2} v(x,t)}{dt^{1/2}}$

Semi Derivative is natural part of normal integer order diffusion equation.

# Fractional Integral Equation-charging of long cable:




$$\frac{E(s)}{V(s)} = \frac{\sqrt{\frac{R}{C\ s}}}{\sqrt{\frac{R}{C\ s}} + \frac{1}{C_0 s}} = \frac{C_0}{C_0 + \sqrt{\frac{C}{R\ s}}};\left[\sqrt{\frac{C}{R\ s}} + C_0\right] E(s) = C_0 V(s)$$

Fractional integral equation:

$$e(t) + b\,D^{-1/2} e(t) = v(t)$$

$$b = \frac{1}{C_0}\sqrt{\frac{C}{R}}$$

Solution

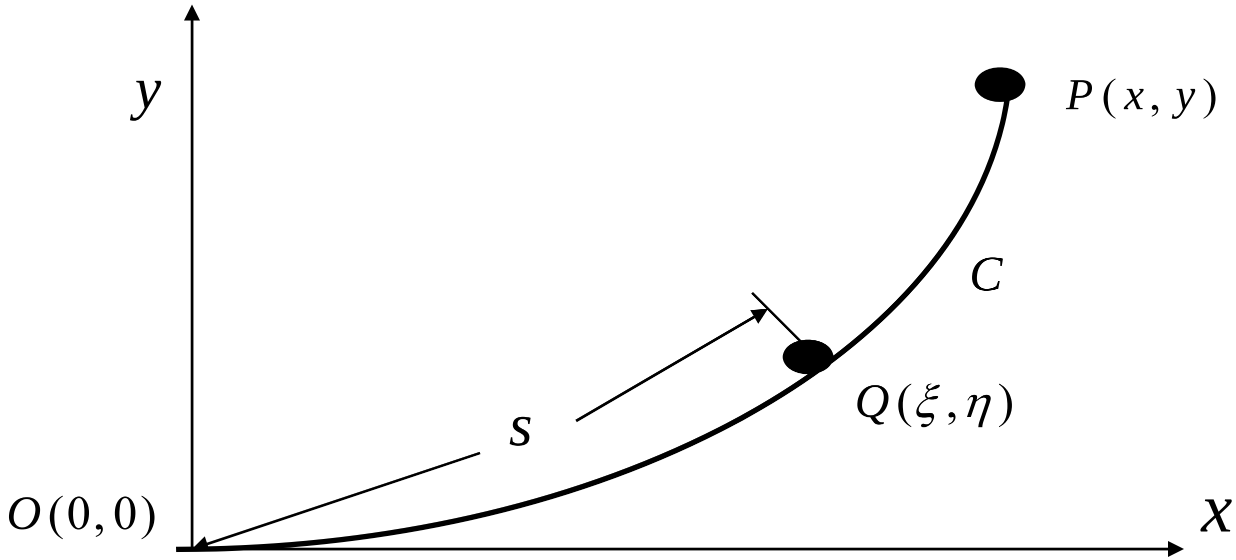
$$v(t) = kt^{\lambda}, e(t) = k\Gamma(\lambda+1)[E_t(\lambda, b^2) - bE_t(\lambda + 1/2, b^2)]$$

$$\lambda = 0, e(t) = k[E_t(0, b^2) - bE_t(1/2, b^2) = ke^{b^2 t}\mathrm{erfc}(b\sqrt{t})$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt$$

# Tautochrone-Abel's integral equation




Problem of finding the shape of curve "*C*" for which the time of decent "*T*", from point "*P*" to the origin is independent of starting (initial) point-or initial placement of ball, in frictionless path

Gain in KE=Loss in PE

$$\frac{1}{2} m \left( \frac{ds}{dt} \right)^2 = mg[y - \eta]$$

$$ds = -\sqrt{2g(y-\eta)}\,dt$$

Negative square-root implies "*s*" decreases as "*t*" increases while sliding

**Solution**

$$dt = -\frac{1}{\sqrt{2g(y-\eta)}}ds$$

Thus time of decent from "*P*" to "*O*" is "*T*" $T = -\frac{1}{\sqrt{2g}}\int_P^O \frac{1}{\sqrt{y-\eta}}ds$

Now arc length is function $s = h(\eta)$ , where "*h*" depends on shape of "*C*"

$$ds = h^{/}(\eta)d\eta$$

$$T = -\frac{1}{\sqrt{2g}}\int_y^0 (y-\eta)^{-1/2}[h^{/}(\eta)d\eta]$$

$$(\sqrt{2g})T = \int_0^y (y-\eta)^{-1/2} h^{/}(\eta)d\eta$$

$k \equiv \int_0^x (x-t)^{-1/2} f(t)dt$ This is Abel's integral equation with $\frac{d^{1/2}}{dx^{1/2}}k = \sqrt{\pi}\, f(x)$

Thus when half derivative of constant is computed the function *f(x)* is found.

It is important that fractional derivative of constant is not always zero.

Is one of the contribution of Abel.

Riemann-Liouvelli $D^{-1/2} f(x) = \frac{1}{\Gamma(1/2)}\int_0^x (x-t)^{-1/2} f(t)dt$

$$\sqrt{\pi}D^{-1/2} f(x) = \int_0^x (x-t)^{-1/2} f(t)dt$$

**Solution**

$$(\sqrt{2g})T = \int_0^y (y-\eta)^{-1/2} h'(\eta)\,d\eta$$

Where $h'(\eta) = ds/d\eta$ and let $f(y) \equiv h'(y)$, then we can write above by dividing both sides by Gamma of half as:

$$\frac{\sqrt{2g}}{\Gamma\left(1/2\right)} T = D^{-1/2} f(y)$$

$$D^{1/2}\sqrt{\frac{2g}{\pi}}T = f(y)$$

Put half derivative of constant as $D^{1/2}T = T/\sqrt{\pi y}$ to get $f(y) = \dfrac{\sqrt{2g}}{\pi}\dfrac{T}{\sqrt{y}}$

$$f(y) \equiv h'(y) = \frac{ds}{dy} = \frac{\sqrt{dx^2 + dy^2}}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$$

$$\frac{dx}{dy} = \sqrt{[f(y)]^2 - 1}$$

$$x = \int_0^y \sqrt{\frac{2gT^2}{\pi^2 \eta} - 1}\,d\eta + c$$

## Solution

$$c = 0,\ x = 0 = y$$

$$x = \int_0^y \sqrt{\frac{2gT^2}{\pi^2\eta} - 1}\, d\eta$$

Let $a = gT^2 / \pi^2$, and change of variable as: $\eta = 2a\sin^2\xi$ gives

$$x = 4a\int_0^\beta \cos^2\xi\, d\xi$$ with $$\beta = \sin^{-1}\left(\sqrt{\frac{y}{2a}}\right)$$

Parametric equation for "*C*" a cycloid

$$x = 2a\left(\beta + \frac{1}{2}\sin 2\beta\right)$$

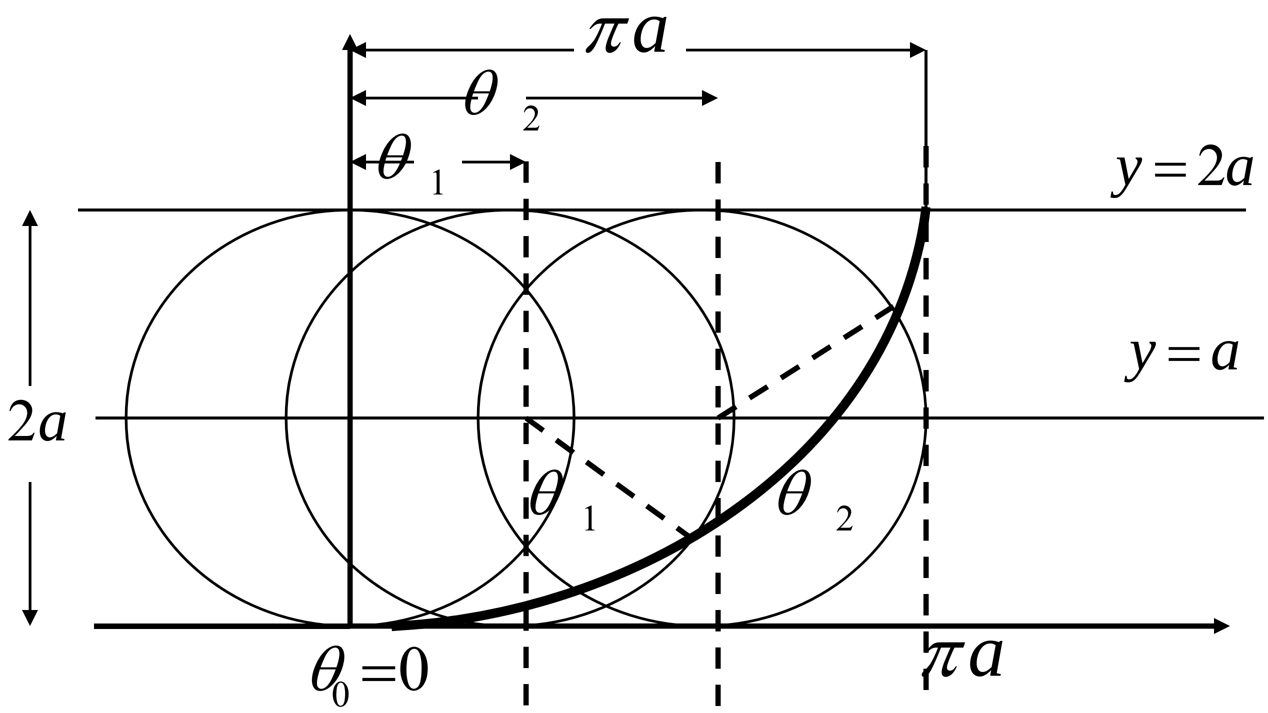
$$y = 2a\sin^2\beta$$

$$\theta = 2\beta,\ a = \frac{gT^2}{\pi^2}$$

$$x = a(\theta + \sin\theta)$$

$$y = a(1 - \cos\theta)$$

# Solution to Tautochrone is cycloid




$$x = a(\theta + \sin\theta)$$

$$y = a(1 - \cos\theta)$$

$$a = \frac{gT^2}{\pi^2}; \theta = 2\arcsin\sqrt{\frac{y}{2a}}$$

# Bernoulli's Brachistochrone-is also a cycloid

The curve *C* minimizes the time of decent from *P* to *O*, is also a cycloid. This is the problem of calculus of variation to minimize the time of decent, the solution to which also a cycloid.

In the Brachistochrone the formulation is:

$$\min I \equiv \int_{x_1}^{x_2} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \int_{x_1}^{x_2} \frac{\sqrt{1 + [f'(x)]^2}}{\sqrt{2gy}} dx$$

$$f'(x) = \frac{dy}{dx}$$

Gives curve as: $x = a(\theta - \sin\theta); y = a(1 - \cos\theta)$

For a cycloid with the above parametric equations the minimum time of decent is *T*; which is also a constant time to decent from any point of start to the bottom of cycloid. Interesting property of cycloid.
with $T = \pi\sqrt{\frac{a}{g}}$, *a* is the radius of circle forming the cycloid

# Fractional Calculus Generalizing Ohms Law

Electrical Circuit Description

Resistoductance, Resistance Inductance embedded together-Fractance

$$v(t) = K \cdot {}_0D_t^{\alpha} i(t)$$

$$\alpha = 0, K = R(\text{ohms}), \alpha = 1, K = L(\text{henry})$$

$$\text{at}; t = 0, \text{step}. v(t) = V_0$$

$$i(t) = \frac{V_0}{K.\Gamma(1+\alpha)} t^{\alpha}$$

$$\text{at} : t = 0, \text{impulse}, v(t) = B\delta(t)$$

$$i(t) = \frac{B}{K.\Gamma(\alpha)} t^{\alpha-1}$$

Fractance Device-Generalizing *R/C/L*-mixed

## *Let us appreciate Fractional Calculus*