INTRODUCTION

Mathematico-Physics of Generalized Calculus

Module-0

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Exact thinking?

Sir Isaac Newton & Gottfried Wilhelm Leibniz independently discovered Calculus in the middle of 17th century. In recognition to this remarkable discovery, J. Von Neumann remarked ,

"……the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more equivocally than anything else the inception of modern mathematical analysis, which is logical development, still constitutes greatest technical advance in <u>exact thinking</u>"

Is that all in exact thinking?

Why this subject?

Myself neither “ a mathematician nor a physicist” yet I picked up this subject about more than a decade ago, to have perhaps exact thinking, for making:

Fuel Efficient Nuclear Power Plant Controls

I discovered that this subject as:

“…..the language what our nature understands the best; and could be a efficient way to have better communication with nature.”

A language of efficient communication with the process makes myself to have efficient controls

The Greatest Discovery

One of the greatest discovery that Newton made was that:

“The nature follows Mathematics”

The intriguing question for all of us even today is

Which Mathematics?

That is why we are here discussing this Mathematico-Physics of Generalized Calculus

A passing mention to Non-Linear-Irregular Dynamics

Representing real life systems as “Non-Linear” dynamics systems, “chaotic” systems, “irregular” systems is of increasingly of interest in interdisciplinary subjects; engineering, physical-chemical science, biological science, economics, medicine, geo-physics and so on.

The ‘non-integer’ representation of dimensions or ‘fractal’ dimensions is one possible parameter to characterize a chaotic, non-linear, continuous but non-differentiable systems.

A non-differential system, irregular system, erratic system, chaotic system, rough system, noisy system, has measure parameter as fractal dimension and can be ‘fractionally differentiated’ (otherwise difficult).

It is about fractional integration fractional differentiation and fractal dimensions all are generalizations of normal integer order calculus and normal Euclidian dimensions, which enable us to possibly extract information out of these irregular behavior.

Well nature is non-linear complex fractal irregular not well behaved, far too complex to have our exact thinking to understand, well Fractional Calculus a fractional step to understand our nature.

Salute to Indian Mathematicians of Fractional Calculus

V. Balakrishnan
Anil.D. Gangal
Kiran .M. Kolwankar
H.M.Srivastava
O.P.Agarwal
S.C Dutta Ray
L.Debnath
R.K.Saxena
R.K.Raina
Rasajit Kumar Bera

…………………………

and to all exponents around the globe to have given this wonderful subject to us applied scientists and engineers, a language what nature understands the best, and thus to communicate with the nature in better and efficient way.

Implementation of fractional calculus is…….

………….in understanding nature better

………….in making effort to have this subject as “Popular Science”

………….in simple teaching and evolving the simple methods in mathematics

…………..in making working systems

……………in realizing that our physical understanding is limited and mathematical tools go far beyond our understanding

……………in appreciating the wonderful world of mathematics that lays between integer order differentiation and integration

…………….in appreciating & finding new physical meanings

Birth of Fractional Calculus

The concept of differential operator $D=d/dx$ is familiar to all who studied elementary calculus (Newtonian-Leibniz integer order calculus). For a suitable function f, the n-th derivative namely $D^n f(x) \equiv d^n f(x) / dx^n$ is well defined.

In 1695, September 30th L' Hopital inquired Leibniz, what could be ascribed to $D^n f$
If 'n' be were fraction (say ½).
Leibniz responded as apparent paradox today, will result in useful consequences tomorrow.

On that day Fractional Calculus was born

Since that time the 'fractional calculus' has drawn attention of many.
Euler, Laplace, Fourier, Abel, Riemann, Liouvelli, Laurent,……

This is three hundred year old mathematical tool, perhaps will find its application for applied science & engineering in XXI century.

What is not FRACTIONAL CALCULUS

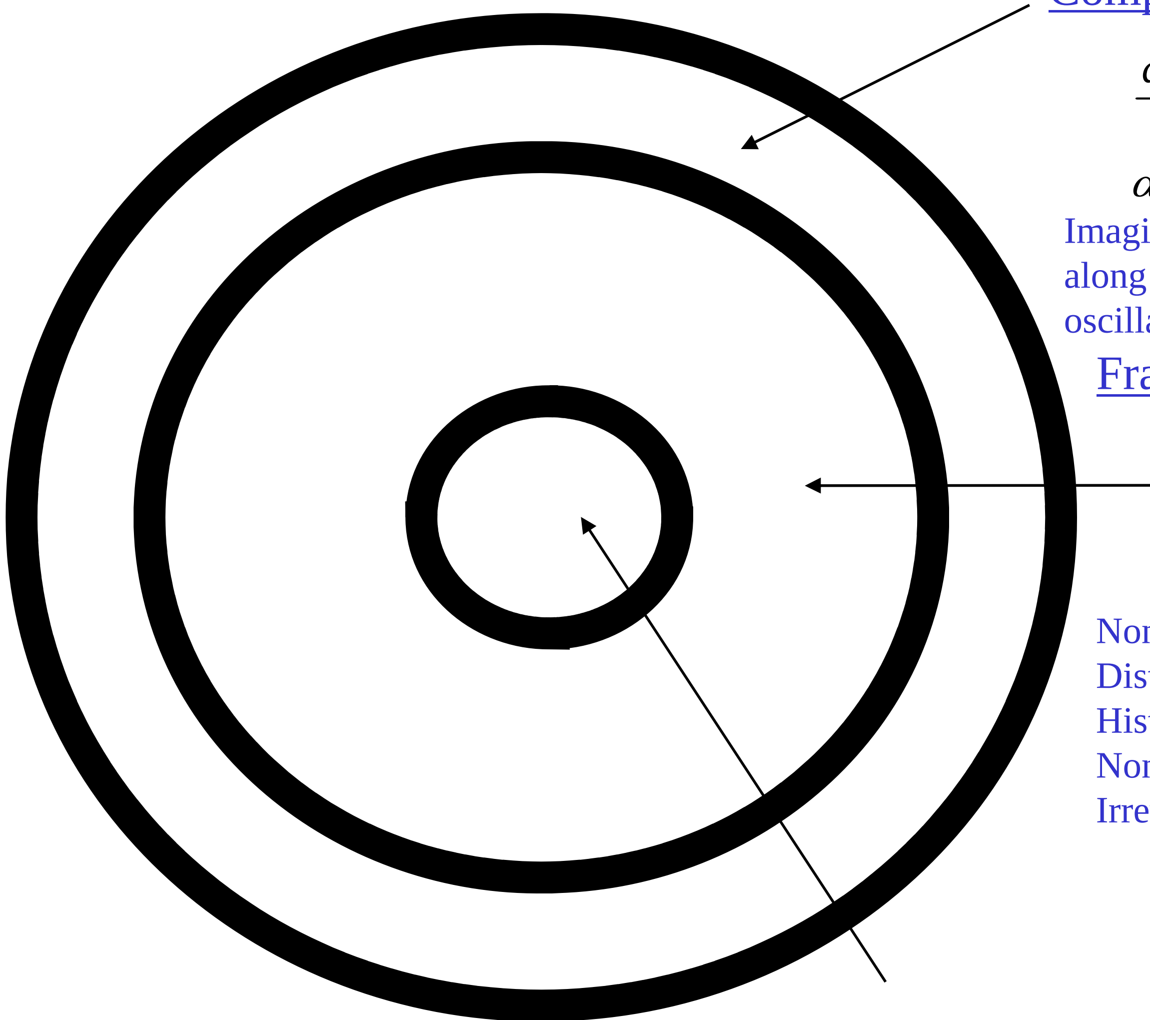
Fractional Calculus does not mean the calculus of fractions, nor does it mean a fraction of any calculus, differentiation, integration or calculus of variations.

The FRACTIONAL CALCULUS is a name of theory of integration and derivatives of arbitrary order, which unify and generalize the notion of integer order *n*-fold repeated differentiation and *n*-fold repeated integration.

FRACTIONAL CALCULUS is GENERALIZED differentiation and integration.

GENERALIZED DIFFERINTEGRATIONS

The Generalized Calculus



Complex-Order

$$\frac{d^{\alpha+i\beta} f(z)}{dz^{\alpha+i\beta}}$$

$$\alpha, \beta \in \Re e$$

Imaginary time response along with real time, fractal oscillatory frequency response

Fractional order

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}}$$

$$\alpha \in \Re e$$

Non-local
Distributed
History/heredity
Non-Markovian
Irreversibly

Integer Order

Newtonian
Point property
Symmetry

$$\frac{d^{n} f(x)}{dx^{n}}$$

$$n \in Z$$

Generalization of theory of numbers and calculations

$2^3 = 2 \times 2 \times 2 = 8$ Can be visualized

$2^{0.5} = \exp\{(0.5)\ln 2\} = 1.414$ Number exists but hard to visualize how.

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ Is a visualized quantity, but what about $(5.5)!$

Generalized factorial as GAMMA FUNCTION $(5.5)! = \Gamma(1+5.5) = \Gamma(6.5) = 287.88$

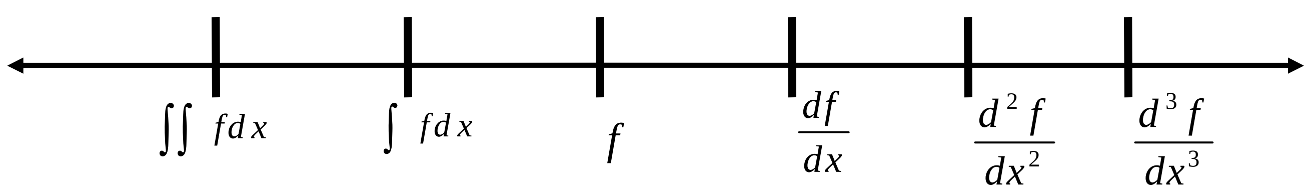
$$x^r = e^{r \ln x}, r \in \Re e$$

$$x! = \Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

$$\Gamma(x) = \lim_{n \to \infty} \frac{(n!)n^x}{x(x+1)(x+2)\ldots(x+n)}$$

Wonderful universe of mathematics lays in between one full integration and one full differentiation

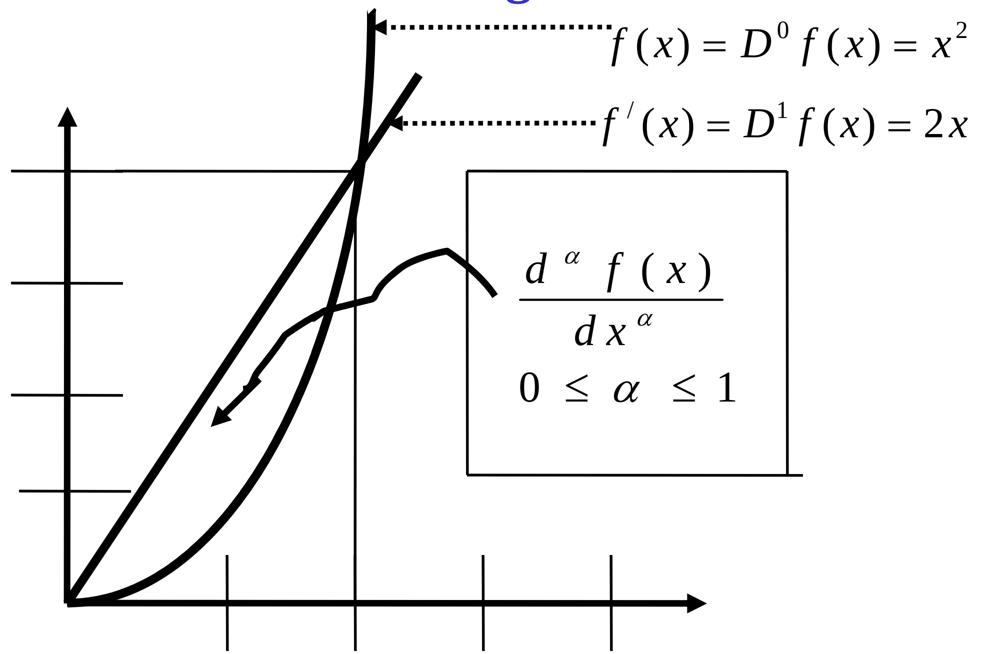


Fractional calculus gives continuum between full differ-integration

$$f(x) = x^2$$

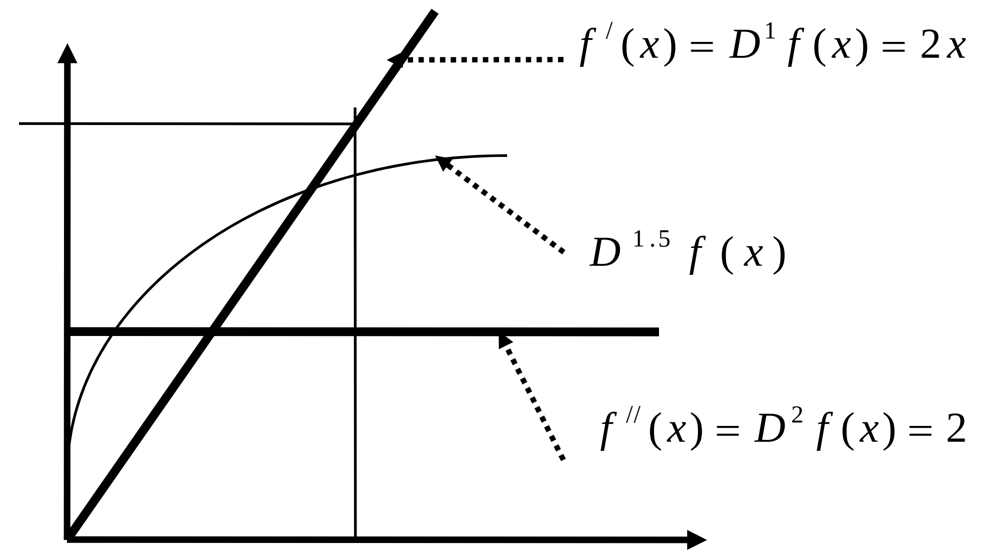
$$f'(x) = d\, f(x)/dx = 2x$$

$$f''(x) = d^2 f(x)/dx^2 = 2$$



Curve fitting will be effective by use of fractional differential equation, as compared with polynomial regression and integer order differential equation. The reason is extra freedom to closely track the curvature in continuum.

Could be a magnifier tool to observe the formation of discontinuity.



Fractional derivative the Euler (1730) formula for monomial

$$\frac{d^n f(x)}{dx^n} = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\cdots\cdots\cdots\frac{d}{dx}}_{n} f(x)$$

$$\frac{d^n}{dx^n}\left\{x^m\right\} = m(m-1)(m-2)\cdots\cdots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\cdots\cdots(m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n}{dx^n}\left\{x^m\right\} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

$$\frac{d^{0.5}}{dx^{0.5}}\left\{x\right\} = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)} x^{1-0.5} = \frac{\sqrt{x}}{\Gamma(1+0.5)} = \frac{\sqrt{x}}{0.5\Gamma(0.5)} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

For positive index the process is differentiation
For negative index the process is integration

Discrete difference to continuum limit

Backward difference (construction of up-shift operator)

Backward difference of Stochastic process & Taylor's expansion

$$X(t+\tau) = X(t) + \tau D X(t) + \frac{\tau^2}{2!} D^2 X(t) + \ldots + \frac{\tau^n}{n!} D^n X(t) + \ldots = [e^{\tau D}] X(t)$$

Derivative operator D can define 'shift operator as $E_\tau \equiv E_\tau\left(X\{t\}\right) = X(t+\tau)$

From above Taylor's expansion we can write: $E_\tau X(t) = \left\{e^{\tau D}\right\} X(t)$

Also we can formulate: $E_\tau \equiv 1 - \Delta_{(-)} = e^{\tau D}$

Backward difference operator or Up-shift operator is:

$$\Delta_{(-)} = 1 - E_\tau = 1 - e^{\tau D} = 1 - \left(1 + \tau D + \frac{(\tau D)^2}{2} + \ldots\right) \approx -\tau D$$

In a way we have defined a backward derivative operator as: $\lim_{\tau \to 0}\left(\frac{\Delta_{(-)}}{\tau}\right) = -D$

It is suggestive that D is time derivative operator and when applied to a continuous function yields: $\frac{d}{dt} X(t) = -\lim_{\tau \to 0} \frac{\Delta_{(-)} X(t)}{\tau}$

Forward Difference (Construction of down-shift operator)

With similar explanation as previous slide $E_{\tau}^{-1} X(t) \equiv X(t-\tau)$

From corresponding Taylor's expansion we can get: $E_{\tau}^{-1} \equiv 1 - \Delta_{(+)} = e^{-\tau D}$

Therefore $\Delta_{(+)} = 1 - E_{\tau}^{-1} = 1 - e^{-\tau D}$

and $\lim_{\tau \to 0} \left(\frac{\Delta_{(+)}}{\tau} \right) = D$ a forward derivative operator.

Backward derivative operator and Forward derivative operator are equivalent mathematical expression of the Derivative.

To be on safe side a physicist or engineer often writes the derivative operator in symmetric way as:

$$\lim_{\tau \to 0} \frac{\Delta_{(+)} - \Delta_{(-)}}{2\tau} = D = \frac{d}{dt}$$

Stochastic difference equation

Random time series are traditionally modeled in physical science using ‘random walks’ . If ξ_j is ‘random variable’ intending to represent a step taken at discrete time j then the random walk variable that denotes the total distance traveled in time $t = N\tau$ or after N such steps is: $X(t) = \sum_{j=1}^{N} \xi_j$

Alternatively $X(t) - X(t-\tau) = \xi_N$

We can simplify the notation by setting $\tau = 1$ and with down shift operator in unit interval without subscript as: $(1 - E^{-1})X_j = \xi_j$

Above is discrete analog of Brownian Motion and is Physicist’s guide into the world of Stochastic Process

Can we generalize and obtain a Fractional Stochastic Process, making Fractional Brownian Motion, with non-integer α like the following?

$$(1 - E^{-1})^{\alpha} X_j = \xi_j$$

And why not?

Mean Squared Displacement- a revisit:

$$\left\langle X_j(t)^2 \right\rangle = \left\langle \Delta X_j(t) \right\rangle^2 = \left\langle \frac{1}{N} \sum_{j=0}^{N} (X_j(t) - X_j(0))^2 \right\rangle$$

MSD $X_j(t) - X_j(0)$ is vector distance traveled by 'walker' over some interval of length time t ; its magnitude squared is averaged over many such intervals t . If no other 'walker' is encountered the walker traveling balastically then the distance traveled would be proportional to the time $\left\langle X_j(t) \right\rangle = vt$ (times with velocity) and MSD would increase quadratic ally as

$$\left\langle X_j^2 \right\rangle \approx t^2$$

In denser phases the time square t^2 behavior holds only for short time of the order of collision time. Beyond that time motion be better described as 'random -walk', for which MSD is proportion to time , that is $\left\langle X_j^2 \right\rangle \approx t$

The rate of growth of MSD depends on how often the walker suffers collision. At higher density it will take longer to diffuse a given distance as other walkers will impede its progress. The limiting slope of MSD considered for sufficiently long time interval for it to be in linear region is related as (by Einstein) as Diffusion constant and space dimension *d*.

$$\lim_{t \to \infty} \frac{d}{dt} \left\langle X^2 \right\rangle = 2 d \mathbb{D}$$

Well constant is function; dimension *d* can be non-Euclidian a fraction!!

Non-Linear MSD cases-Anomalous diffusion

$$\langle X^2(t) \rangle \approx t^{2H}$$

$H = \frac{1}{2}$ Normal Random walk and normal diffusion
Brownian Motion, with no memory

$$\langle X^2(t) \rangle \approx t$$

$0 \leq H < \frac{1}{2}$ Anti Persistent Random walk and sub (slow) diffusion
With short term memory
The process decays monotonically to zero hyperbolically

$$\langle X^2(t) \rangle \approx t^{1/2}$$
$$H = 1/4$$

$1 > H > \frac{1}{2}$ Persistent Random walk and super (fast) diffusion
Long-term- 'lingering' memory (Long Ranged Dependence) LRD, Autocorrelation decays as power law, Fractional Brownian Motion

$$\langle X^2(t) \rangle \approx t^{3/2}$$
$$H = 3/4$$

H **Hurst exponent**

MSD from Fick's diffusion and Brownian Motion of Perin:

The Fick's law of diffusion is:

$$\frac{\partial N(x,t)}{\partial t} = A\frac{\partial^2 N(x,t)}{\partial x^2}$$

Taking Fourier-Laplace transform of this we get:

$$\partial_t \to s$$
$$\partial_x \to ik$$

$$\frac{\partial N(k,t)}{\partial t} = A(ik)^2 N(k,t)$$

$$sN(k,s) + N_0(k,0) = -Ak^2 N(k,s)$$

$$N(k,s) = \frac{N_0(k)}{s + Ak^2}$$

The MSD is calculated from $N(k,s)$ as (moment calculations):

$$\langle \Delta x \rangle^2 = \mathcal{L}^{-1}\left\{\lim_{k\to 0}\left[-\frac{d^2}{dk^2}N(k,s)\right]\right\} = \mathcal{L}^{-1}\left\{\frac{2A}{s^2}\right\} = 2At$$

The above takes $N_0(k) = 1$ for $N(x,0) = \delta(x)$

Maxwell-Debye exponential relaxation process:

Standard Maxwell Debye relaxation is

$$\tau \frac{d}{dt}\Phi(t) = -\Phi(t)$$

$$t > 0 ; \Phi(0) = \Phi_0$$

$$\Phi(t) = \Phi_0 e^{-t/\tau}$$

Gives pure exponential solution with single relaxation time constant

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}}\Phi(t)$$

Also the Integral representation of Maxwell-Debye relaxation is:

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}}\Phi(t)$$

The integral equation can be formally extended to Fractional Integral equation by replacing $\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}} \rightarrow \frac{1}{\tau^{\beta}}\frac{d^{-\beta}}{dt^{-\beta}}$ which leads to

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau^{\beta}}\, {}_0D_t^{-\beta}\Phi(t)$$

Non Debye non-exponential relaxation process:

Kohlraush Williams Watts (KWW) relaxation law:

$$\Phi(t) = \Phi_0 e^{\left\{\left(-t/\tau\right)^{\alpha}\right\}}$$

Nutting Power Law relaxation:

$$\Phi(t) = \Phi_0 \left(1 + t/\tau\right)^{-n} ; 0 < n < 1$$

Observed in:

Dielectric relaxation, Stress Relaxation, Strain relaxation, NMR relaxation
Diffusion controlled relaxation, electrical circuits,…….; unlike normal relaxation $\Phi(t) = \Phi_0 e^{-t/\tau}$

Memory and Fractional Calculus:

1. Non-exponential relaxation implies MEMORY i.e. the underlying fundamental relaxation process are NON-MARKOVIAN

2. Natural way to incorporate memory effect is fractional calculus via the involved convolution integral in time (space). The present state is being influenced by all the states, the system has been running through at the times $t' = 0, 1, \ldots\ldots\ldots\ldots t$

3. The power-law Kernel defines the fractional expression represents a particular long memory.

$$ {}_cD_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_c^t (t-\tau)^{\beta-1} f(\tau)\, d\tau = \mathrm{P}_\beta(t) * f(t) $$

$$ {}_cD_x^{-n} f(x) = \frac{1}{(n-1)!} \int_c^x (x-\xi)^{n-1} f(\xi)\, d\xi $$

Memory Integrals:

$$\frac{d\Phi(t)}{dt} = -\int_0^t K(t-\tau)\Phi(\tau)\,d\tau = -K(t) * \Phi(t)$$

Represents Memory Integral i.e. all instances for $\tau = 0$ to $\tau = t$ contribute to situation at $\tau = t$

1. Memory breaks down i.e. Markovian Case:

$$K(t) = K_0\delta(t)$$

$$\frac{d}{dt}\Phi(t) = -\int_0^t K_0\delta(t-\tau)\Phi(\tau)\,d\tau = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \exp\{-K_0 t\}$$

$$\frac{d\Phi(t)}{dt} = -\frac{\Phi(t)}{\tau} = -K(t) * \Phi(t)$$

How the process time constant is related to Kernel of Memory integral, good research case?

2. The opposite case Constant Memory i.e. leading to oscillatory case

$$K(t) = K_0$$

$$\frac{d^2}{dt^2}\Phi(t) = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \cos(\sqrt{K_0}\,t)$$

Memory Integral (Contd.)

3. Slowly varying Kernel which for small time behaves as power law gives KWW relaxation process

$$K(t) \approx K_0 t^{\gamma}$$

$$\Phi(t) = \Phi_0 \exp\left\{-K_0 t^{\gamma+2}\right\}$$

4. Relaxation for Fractional Differential/Integral equation & its Memory Kernel

$$K(t) = K_0 t^{q-2}; \; 0 < q \leq 2$$

$$\frac{d}{dt}\Phi(t) = -\frac{1}{\tau^{q}}\left[{}_0D_t^{1-q}\Phi(t)\right]$$

$$\tau^{q} = \left[K_0 \Gamma(q-1)\right]^{-1}$$

Apply ${}_0D_t^{-1}$ on both sides to get: $\Phi(t) - \Phi_0 = -\tau^{-q}\,{}_0D_t^{-q}\Phi(t)$

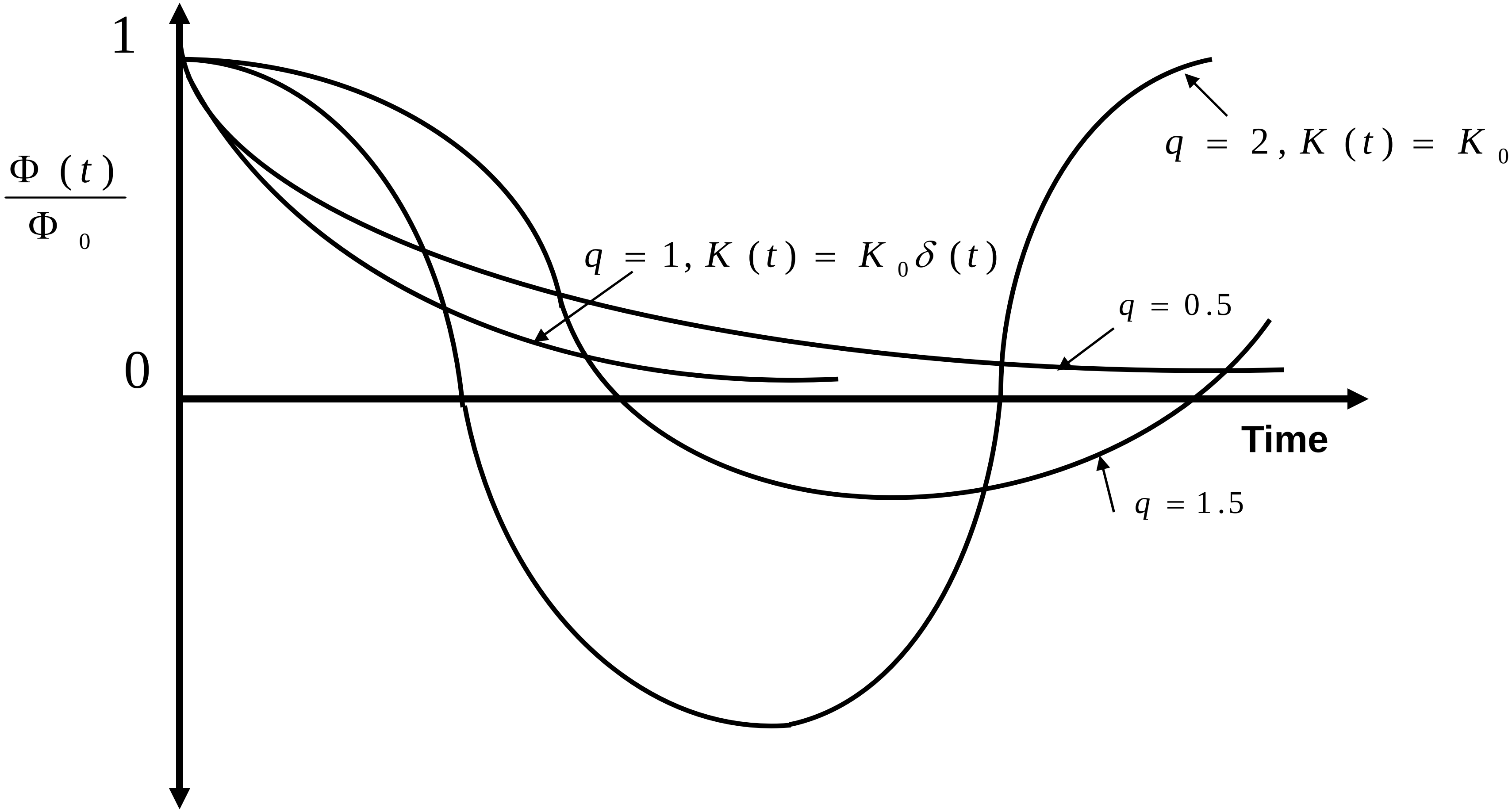
Apply ${}_0D_t^{q}$ on both sides to get FDE

$${}_0D_t^{q}\Phi(t) - \Phi_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\Phi(t)$$

Using Fractional Derivative of constant **C** as , non zero, that is $\mathrm{C}\, t^{-q} / \Gamma(1-q)$

Memory Kernel & Fractional Differential Equation for Relaxation kinetics

$$ {}_0D_t^q \Phi(t) - \Phi_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\Phi(t) $$



MARKOVIAN PROCESS:

Modeling by Markovian way, assumes that the process during its functional activity assumes a number of conformal states. Transition between the states are usually formulated as kinetic or relaxation rate constants (time constants) $k_n = \tau_n^{-1}; (n = 1, 2 \ldots N)$ - indicate how many conformational states of energy landscapes are taking part in relaxation process. Markovian models assume that probability distribution function for some events may be described by sum of exponential functions.

$$f_N(t) = \frac{a_0}{\tau_0} e^{-t/\tau_0} + \frac{a_1}{\tau_1} e^{-t/\tau_1} + \ldots\ldots + \frac{a_N}{\tau_N} e^{-t/\tau_N} = \sum_{n=0}^{N} \frac{a_n}{\tau_n} e^{-t/\tau_N}$$

The amplitude a_n and τ_N are presumed to be independent and not related. If they are correlated the Markovian is lost.

$$\tau_n = \tau \lambda^n ; \lambda > 1$$

$$a_n = a_N p^n ; 0 < p < 1$$

$$f_N(t) = \frac{a_N}{\tau} \left[e^{-\left(\frac{t}{\tau}\right)} + \frac{p}{\lambda} e^{-\left(\frac{t}{\tau}\right)\lambda^{-1}} + \frac{p^2}{\lambda^2} e^{-\left(\frac{t}{\tau}\right)\lambda^{-2}} + \ldots\ldots \right]$$

Fokker Plank Kolmogorov Equation

Basis from random walker theory and with MARKOVIAN model we get, the evolution of probability density as:

$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}\left[A(x) P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[B(x) P(x,t)\right]$$

$A(x)$ is the drift (Advection) $B(x)$ is the diffusion function

For a case of no advection, for force/potential free case and with constant $B(x)$=B we have for initial delta function $P(x,0) = \delta(x)$

A Gaussian evolution $P(x,t) = \frac{1}{\sqrt{4\pi \mathrm{B} t}} e^{-\left(\frac{x^2}{4 \mathrm{B} t}\right)}$

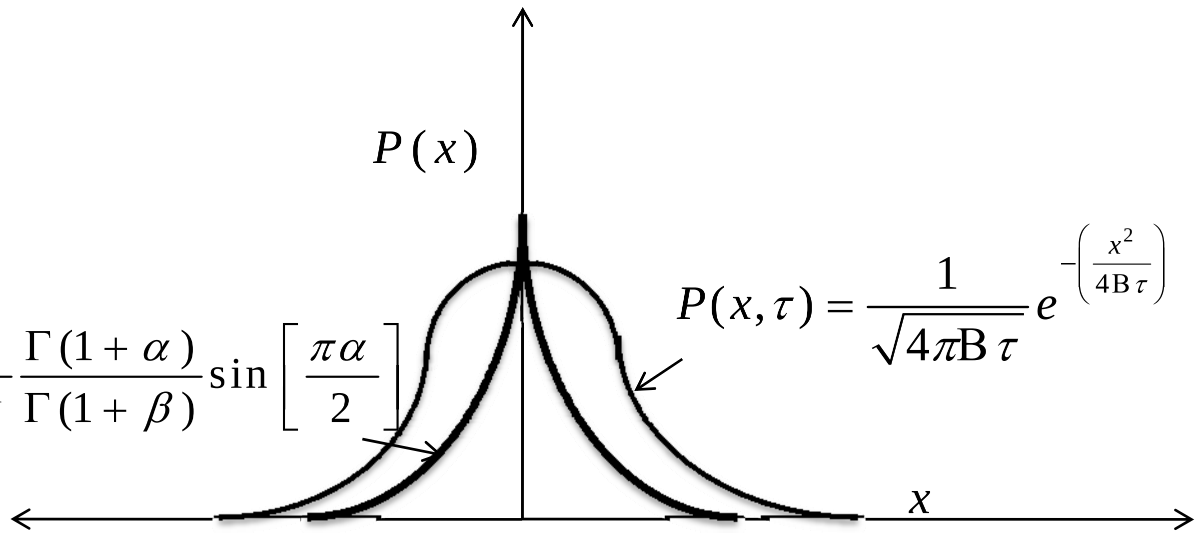
A Fractional Fokker Plank Kolmogorov Equation (without Advection)

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial x^\alpha}\left[\mathrm{B}\, P(x,t)\right]$$

Has Levy stable asymptotic power-law evolution

$$P(x,t) \approx \frac{1}{\pi} \frac{t^{-\beta}}{x^{\alpha+1}} \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} \sin\left[\frac{\pi\alpha}{2}\right] \quad \text{with} \quad \mathrm{B} = 1$$

Fokker Plank Kolmogorov Equation & Fractional FPKE



$$P(x,\tau) \approx \frac{1}{\pi}\frac{\tau^{-\beta}}{x^{\alpha+1}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin\left[\frac{\pi\alpha}{2}\right]$$

$$P(x,\tau) = \frac{1}{\sqrt{4\pi B\tau}}e^{-\left(\frac{x^2}{4B\tau}\right)}$$

The Dirac's delta at time zero plumes out at any time τ and evolves. In case of integer order FPKE the shape is dome shaped, where as for fractional order FPKE the cusp shape evolves. Well at very initial stage say at time $\tau = 0.001s$ in integer order cases there exists a large quantity at fairly large distance from origin-stating that the phase velocity requirement is very very large-infinite.

This is due to ideal Fickian case, was modified in 1948 by Catteneo, by having relaxation time constant in the Fick's second law, and thus obtaining telegrapher's equation giving finite phase velocity.

Power law relation & Asymptotic Fractals and Scale invariance:

Often one encounters functional relation in the frequency domain of the form: $\omega^{-\alpha} F(\omega)$ for example $\omega^{-\alpha}\exp(-\omega\tau)$. Such relations back transformed to time-space by generalized differentiation $\omega^{-\alpha} F(\omega) \leftrightarrow D^{-\alpha} f(t)$

Fractional expressions can arise quite naturally from the functional relations observed in so called “Spectral Domain”.

Typical asymptotic fractal is of the form. $G(t) \approx \left(\frac{t}{\tau}\right)^{-\beta} \gamma^{*}(\beta, t/\tau)$

Scale invariance in power law states that there is no small scale. Let $y(t) = t^{\beta}$ then $y(\lambda t) = (\lambda t)^{\beta} = \lambda^{\beta} y(t)$. A scaling in time axis results in simple scaling of ordinate. The law $y(t) = t^{\beta}$ is not altered by scale change. In log-log plot this means shift $\log \lambda$ on $\log t$ axis and shift $\beta \log \lambda$ in the ordinate.

This scale invariance however is the cause of stability of the systems following power laws.

Fractional Kinetics FFPK with Fractal Support (in Chaotic Dynamics)

It is recently observed that numerous physical processes do not satisfy principle of short time memory randomness-and that long power like tails in distributions of different time-scales can frequently occur in physical systems.

For the cases when Fractional Kinetics can be applied there is ‘fractal’ support in ‘phase-space’ description of system. The origin of fractal support comes from strong non-uniformity of the ‘phase-space’ where special zones of ‘particle trapping’ exists.

The (quasi) traps can have fractal structure and so can the set of ‘time-intervals’ that trajectory spends in the trap. This is how we loose universality of the kinetics description of a system (with chaotic dynamics); because it depends on the ‘local space-time’ properties of the support (or more generally speaking on the phase-space topology).

Meaning in the Fractal Support system does time flow uniformly, with unit speed? As the aberration can be in space that may evolve as temporal non-uniformity!!

Law of irreversibility and Generalizing the Classical Calculus & Dynamics

Reversibility is to idealize. Its validity or applicability in physical experiments and processes depends on the degree to which system can be isolated (or decoupled) from past history and its environment.

Classical dynamic evolution problem is $\frac{d}{dt} f(t) = \mathrm{B} f(t)$ we generalize the same to:

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) = \mathrm{B} f(t)$$

B is operator on 'phase space', Banach space .Banach spaces, including spaces of continuous functions spaces etc.

Even $\frac{\partial^2 g}{\partial t^2} = c^2 \frac{\partial^2 g}{\partial x^2}$ can be recast to $\frac{d}{dt} f(t) = \mathrm{B} f(t)$ with $f = \begin{pmatrix} g \\ h \end{pmatrix}$

and $\mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c \frac{\partial}{\partial x}$.

Where $\frac{d}{dt}$ represents basic symmetry & basic principle of locality. The fractional derivative d^{α}/dt^{α} whereas says about the history memory, heredity and principle of non-locality. Time evolution is irreversible. Processes are having history (Non-Markovian). Let us postulate that time evolution is irreversible and reversibility is idealism!!

Time evolution:

$$\frac{d}{ds} f(s) = \lim_{t\to 0} \frac{f(s) - f(s-t)}{t} = -\lim_{t\to 0} \frac{E_t^{-1} f(s) - f(s)}{t}$$

Gives $\frac{d}{dt}$ as infinitesimal generator of time translation defined by $E_t^{-1}(t) f(s) = f(s-t)$

Whereas $\frac{d^\alpha}{dt^\alpha} f(t) = \mathrm{B} f(t)$ abandons $E_t^{-1}(t)$ as generator of time evolution, represented by convolution with 'memory-kernel' and thus is replaced by:

$$E_t^{-\alpha}(t) f(t_0) = \int_0^\infty E_t^{-1}(s) f(t_0) K_\alpha\left(\frac{s}{t}\right)\frac{ds}{t}$$

While with memory Kernel $K_\alpha\left(\frac{s}{t}\right) = \delta(x-1)$ then the translation is $E_t^{-\alpha} = E_t^{-1}$

A simple translation with unit 'speed' reflects the idea of time 'flowing' uniformly with constant velocity. The idea is embodied in measuring time by comparison with periodic clocks (processes). A competing idea related to 'flow of time' represented by the convolution $E_t^{-\alpha}(t) f(t_0) = \int_0^\infty E_t^{-1}(s) f(t_0) K_\alpha(u)\, du$

is to measure time with non-periodic clocks such as decay or aging process!!

Irregularity & its identification

Normal Condition Beats: 90/70/90/70/90/70/90/70/90/70/90/70/90/70/90/70

Abnormal Condition Beats:90/70/70/90/90/90/70/70/90/90/70/90/70/70/90/70

These two series have same mean, median, and variance and the two values (90 or 70) have the same probability of occurring ½ . Statistics fail to distinguish this; yet they are different. In first, one finds the next outcome with absolute certainty. In the second series we only know that next outcome will be either 90 or 70, but our guess will be wrong in 50% cases.

Fractals and multi-fractal functions and corresponding curves or surfaces are found in numerous places in non-linear and non-equilibrium phenomena. Like turbulence, Brownian paths, attractors of some dynamics, economics, seismology records are for examples of occurrence of continuous but highly irregular (non-differentiable) curves and surfaces.

Random functions have defined mean and variance (standard deviation). The non-random (power-law) functions have non definite mean or variance. Graphs are fractal set (self-similar) with no smallest scale.

Connection between ‘local’ scaling behavior (fractal dimension) and ‘Order of Fractional Derivative’ is interesting to ‘instrument’ the irregularity of non-linear dynamic systems.

Irregular Rough Functions with fractal dimension to identify

Uni-fractals have same fractal dimensions whereas the multi-fractals have several fractal dimensions.

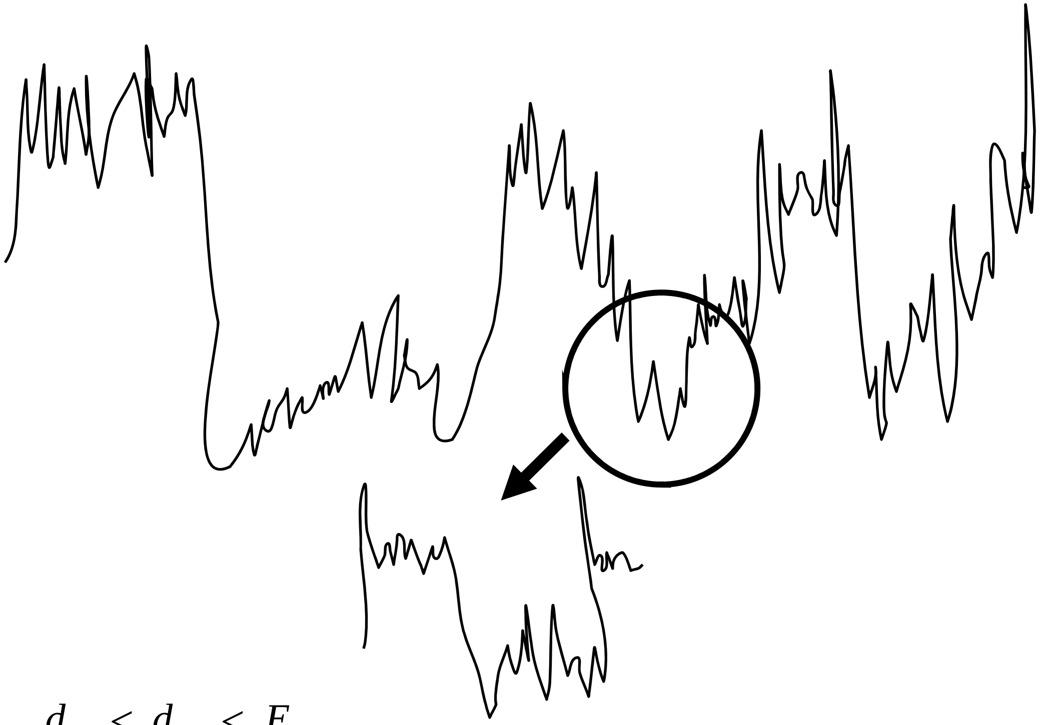
Fractal dimension is measure of roughness or irregularity or chaotic curves, irregular curves non-differentiable curves.

Weistrauss

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi n)$$

$$0 < a < 1, b > 0, ab > 1 + \frac{3}{2}\pi$$

$$d_F = \frac{\log a}{\log b + 2}$$



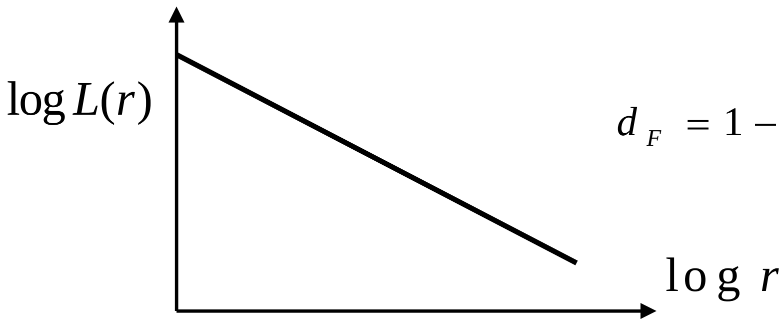
$$d_T < d_F < E$$

$$C \in R^E$$

$$1 < d_F < 2$$

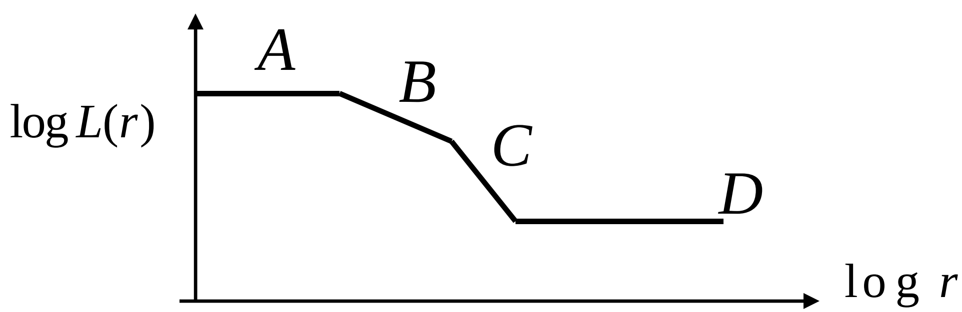
Mandelbrot-Richardson Plot:

Plot of Logarithm of Length and Log of step size is Mandelbrot-Richardson Plot. The slope of that curve is *S* then fractal dimension is (1-*S*) . The *S* of the curve is equal to or less than zero.



$$d_F = 1 - \lim_{r \to 0} \frac{\log L(r)}{\log r}$$

If the calculations of the length of the curve is performed with a step size that is too long, the main structure of the line described which gives a flat section (*D*), when the step size is too small, much less than the sampling interval, we are not able to recognize any new structure in the curve again results in flat section (*A*). The region (*B*) and (*C*) indicate two uncorrelated parts in the (irregular rough) signal.



Roughness Exponent:

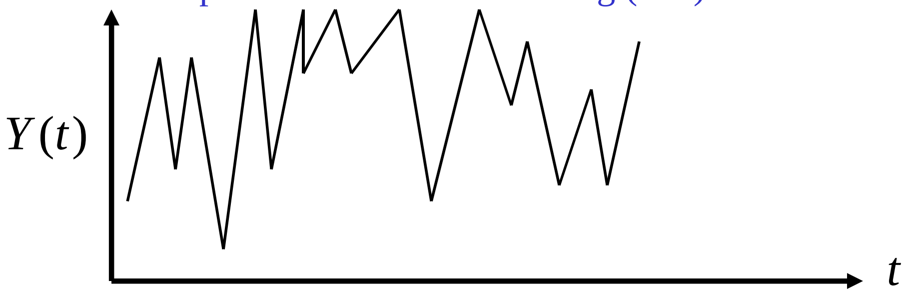
For a function $f(x)$ if there exists a polynomial $P_n(x)$ of degree $n < h$ and a constant C such that

$$|f(x) - P_n(x - x_0)| \leq C|x - x_0|^h$$

The supremum of all exponents $h(x_0) \in (n, n+1)$ is Holder Exponent, which characterizes singularity strength. Holder Exponent describes the 'local' regularity (roughness) of function $f(x)$ at $x = x_0$

Higher the exponent, more regular is local behavior. Holder exponent is local and is related to Hurst Exponent as $\overline{h} = 1 - H$ Where $\overline{h}$ is average holder exponent in the defined interval. To find Local Holder exponent one needs Wave-Let Transforms to characterize singularity

$|\Delta Y_i| = (\Delta t_i)^{1/2}$ is 'square-root' scaling of Brownian Motion, with $H = 0.5$ as Hurst exponent . To find Hurst exponent one needs Rescaling (R/S) method of time series

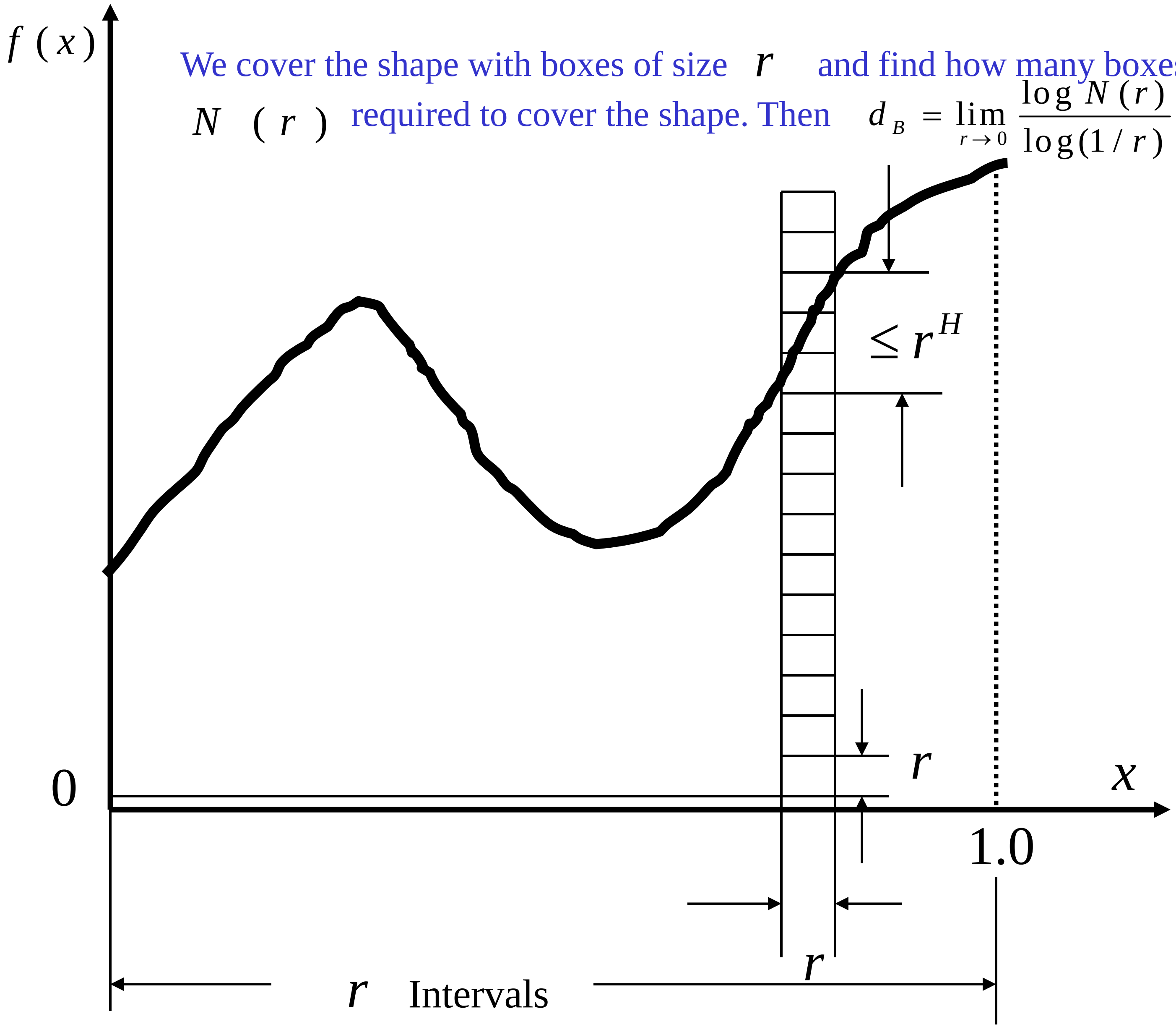


Roughness of Graph is: $H = \log|\Delta Y_i| / \log|\Delta t_i|$

Holder Exponent & (box) dimension of graph (function)

We cover the shape with boxes of size r and find how many boxes $N(r)$ required to cover the shape. Then

$$d_B = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)}$$



Box-dimension of irregular graph and Holder's exponent

1. The function $f(x)$ defined in $[0,1]$
2. Divide $0 \le x \le 1$ into r equal intervals or almost equal intervals.
3. Above each interval make column of width r .
4. In the situation of scaling condition with Holder Exponent as: $\Delta y = (\Delta x)^H$ means in each of the column of graph of $f(x)$ passes through a height r^H
5. So the number of boxes needed to cover the part of the graph in that column is about (height of graph) $\div$ (height of box)= $r^H \div r = r^{H-1}$
6. The number of columns is $1/r$ in the length 1.00.
7. The number of these boxes of side r needed to cover the entire graph is

$$(r^{H-1}) \times (1/r) = r^{H-2} = N(r)$$

8.

$$d_B = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)} = \frac{\log r^{H-2}}{\log(1/r)} = 2 - H$$

The Holder exponent of the Brownian path is ½ , the scaling; hence the box, or the fractal dimension of Brownian motion is 1.5. The white noise is also scaled by ½ and has box or fractal dimensions 1.5.

Equivalence of Mandelbrot-Richardson fractal dimension and the Box-dimension

$$d_F = 1 - \frac{\log L(r)}{\log r} = 1 + \frac{\log L(r)}{\log(1/r)}$$

$$d_F - 1 = \frac{\log L(r)}{\log(1/r)}$$

$$L(r) = (1/r)^{d_F - 1} = r^{1-d_F}$$

$$d_B = \frac{\log N(r)}{\log(1/r)}$$

$$N(r) = r^{-d_B}$$

$$N(r) = L(r) \times (1/r)$$

$$L(r) = r \times r^{-d_B} = r^{1-d_B}$$

$$d_B = d_F$$

Both the methods give fractal dimension of irregular graphs called fractal dimension or Hausdroff's dimension.

The efficient method of estimation of irregularity is good R&D.

Several dimensions an overview:

1. Embedding Euclidian dimension: The regular dimension which holds the structure. d

1. Fractal dimension : Describes irregularity and roughness, locally or globally. For self similar the dilation symmetry is uniform, but the self affine has anisotropic dilation symmetry. d_F

2. Spectral dimension: Described by density of states or the relaxation of stimulus and order of it at asymptotic late times. d_S

$$d \geq d_F \geq d_S$$

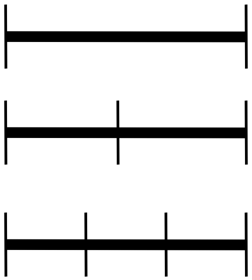
Scaling:

$$\frac{d^q f(\lambda x)}{dx^q} = \lambda^q \frac{d^q f(\lambda x)}{d(\lambda x)^q}$$

This property makes suitable for study of scaling. Local scaling behavior is it related to order of fractional derivative? However, this scaling property of fractional derivative implies study of 'self-similar' objects, distributions etc. Let $y(t) = t^\beta$ then $y(\lambda t) = (\lambda t)^\beta = \lambda^\beta y(t)$. A scaling in the time axis results in simple scaling of ordinate. The law $y(t) = t^\beta$ is not altered. In log-log plot this means a shift of $\log \lambda$ on the $\log t$ axis and shift $\beta \log \lambda$ in the ordinate. This scale-invariance is the cause of stability of system following power-law. Power-law relaxation is observed in various physical systems. Thus one often encounters algebraic relaxation $\phi(t) \approx t^{-\alpha}$ with $0 < \alpha < 1$ Therefore one often encounters fractional relations in frequency domain $\omega^{-\alpha} F(\omega)$ for example $\omega^{-\alpha} \exp(-\omega\tau)$. Fractional relations of this self-similar (fractal) form can arise quite naturally in spectral domain. in time domain fractal form is $g(t) \approx (t/\tau)^{-\alpha} \gamma^*(\beta, t/\tau)$; this form has late time power decay law.

Fractal Dimensions of self-similar figures

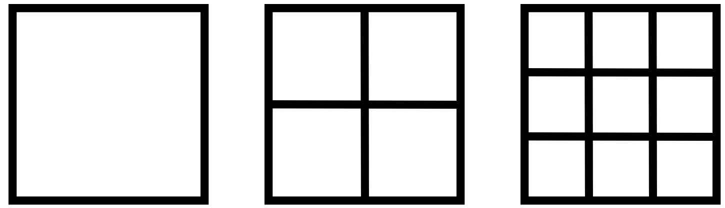
$$d_F = \lim_{\varepsilon \to 0} \frac{\log N}{\log\left(\frac{1}{\varepsilon}\right)}$$



$N = 2$ $r = 2, \varepsilon = 1/2$

$N = 3$ $r = 3, \varepsilon = 1/3$

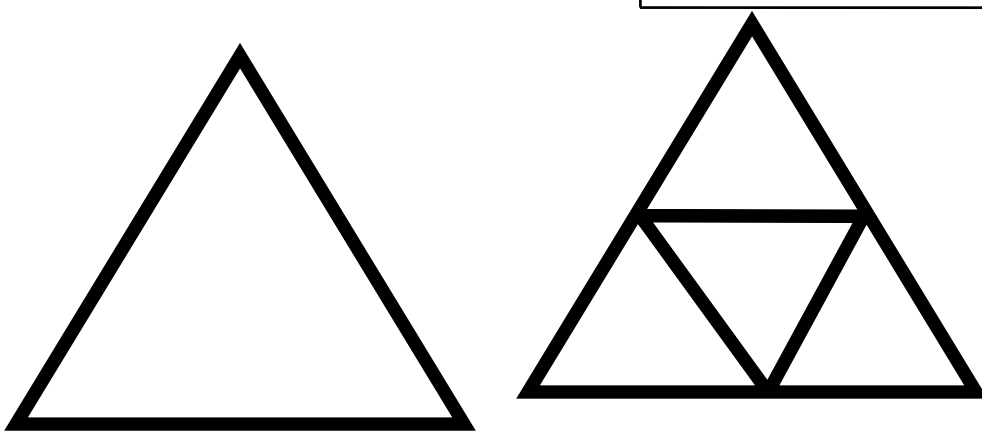
$$d_F = 1$$



$N = 4$
$r = 2, \varepsilon = 1/2$

$N = 9$
$r = 3, \varepsilon = 1/3$

$$d_F = 2$$



$N = 3$

$r = 2, \varepsilon = 1/2$

$$d_F = \frac{\log 3}{\log\left(\frac{1}{1/2}\right)} = \log 3 / \log 2 = 1.585$$

'Box' could be a circle of radius '*r*' or square of side '*r*' a sphere of radius '*r*' or triangle of side '*r*' or segment of side '*r*'. Infinitesimally all are same!!

Fractal Dimensions Fractality & Fractional calculus-a passing mention

There is relation of fractality and fractional calculus, thus we have introduced the concept of fractals and fractal dimensions.

A non-differentiable curve is Fractionally differentiable (locally).

Thus a physical process which makes a Phase change where differentiability is lost, fractional derivative (locally) gives the idea of the enlargement/manifestation of the physics of phase change.

Let us enter the World of Fractional Calculus