A General Exposition

to the course contents and for the seven lecture series for the subject

Mathematico-Physics of Generalized Calculus

for

Department of Applied Mathematics Calcutta University,
Department of Physics Jadavpur University
M.Phill. & PhD course work interdisciplinary subject

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Prelude:

For the first time (globally) formally any University is proposing to start the course on fractional calculus, i.e. at Department of Applied Mathematics of Calcutta University , & Department of Physics of Jadavpur University namely:

Mathematico-Physics of Generalized Calculus

I (by profession an Electrical and Electronics Engineer) can state that **"Mathematics goes far beyond our physical understanding"**.

Mathematics is what nature understands, but which one? We Scientists & Engineers try to find and keep on finding- **"that particular mathematics".**

As an engineer my past one and half decade of work is mainly to ascribe **physical/engineering/geometrical** sense to wonderful **three hundred years old topic** of **fractional calculus**-make **product and science out of this subject**. Why? Because it was for search to make "Fuel Efficient Control System" for Nuclear Reactors!! My small contribution is also to relate real life processes and explanation with physical behavior to the wonderful tool of mathematics that is fractional calculus; also to give physical sense to solution of solvable extraordinary differential equations, a way gives merger of two definitions of Fractional Derivative (RL) and Caputo, and this way only integer order states are required to solve FDE.

Still continuing……… and is unfinished.

Still learning Fractional Calculus by fractions!!

Book references:

1. **Functional Fractional Calculus for System Identifications & Controls (2007).** Springer Verlag (280 pages specially for Engineers)
2. **Functional Fractional Calculus Edition-II (2011)** Springer-Verlag (625 pages for science, mathematics and engineering)
3. **Mathematico-Physics of Generalized Calculus**. Department of Applied Mathematics Calcutta University. (March-2010) a draft course work book (will be modified as when required)

Jadavpur University Library , Department of Physics Jadavpur Univ. BARC library,
Power Engineering Department Jadavpur Univ. IRPE Library Calcutta University.

This work is due to blessings of **Prof. Michelle Caputo**, founder of Caputo's RHD derivative

"…..I have gone through your book, it is a nice treatment specially on modern aspects... The page 207 and beyond where you described as continuous order, I tried similar calling it order distribution. I congratulate you on your efforts, I attach initial papers of mine what was reprinted by my colleagues as my 80th anniversary for you………, Riemann sheets still puzzles me. Do join me at lunch in Rome, when you come……………." April/March 2010.

The greatest honor P.R.I.S award or promotion for me!!

and encouragement & blessings from **Prof. Rasajit Bera**, pioneer EDE solutions by ADM

In 2006 he took my lecture notes prepared lectures for a forum (where that year's Bhatnagar awardees were present). He has motivated me to carry out research in solution of FDE and told making products will be boon for this subject.

Also blessings are from **readers of the book** used by them for pure and applied science/ control science research (cited by them).

For my **late blind father's** blessings who has never seen my creations

Few journal papers references:

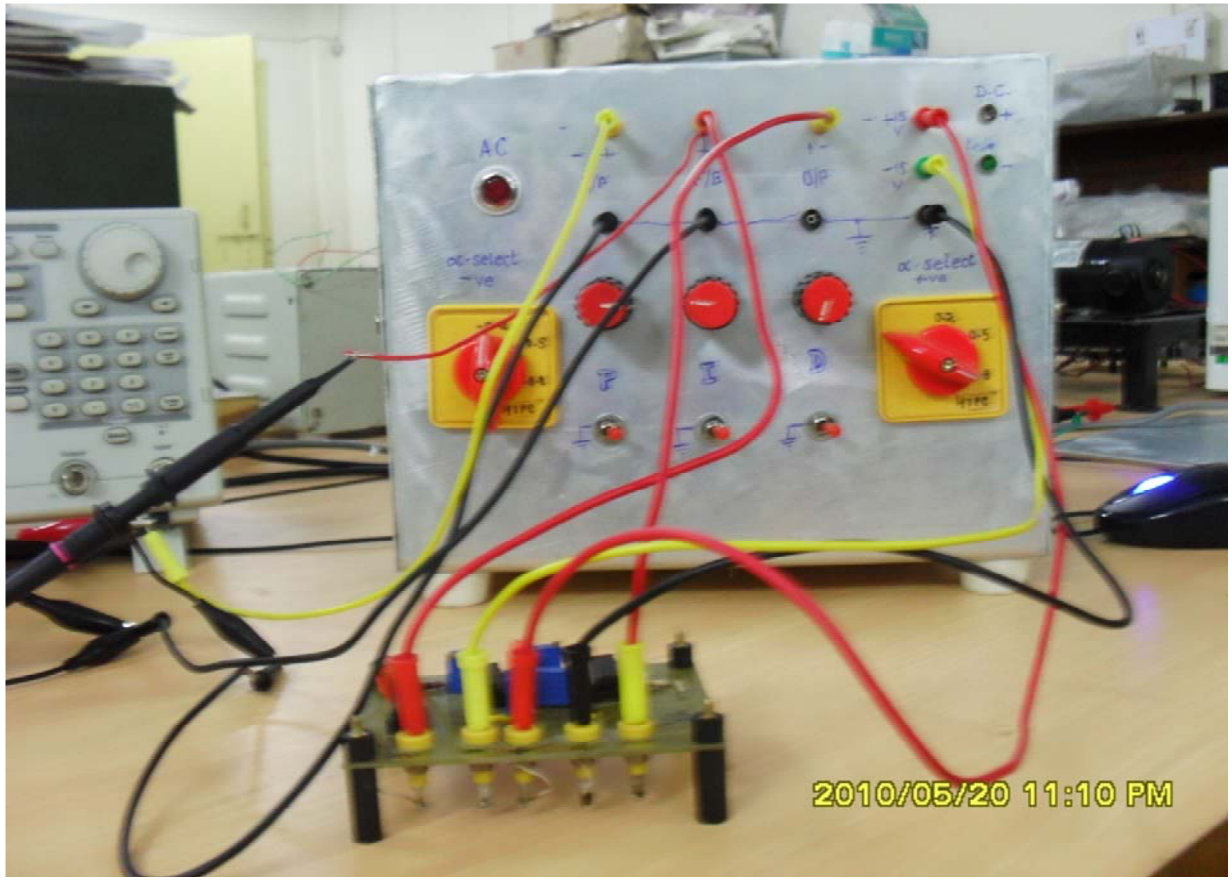
1. International Journal of Applied Mathematics & Statistics. Vol. 17; No. J10, June 2010, pp. 44-76***"Generalized Dynamic System Solution by Decomposed Physical Reactions"***.
2. International Journal Modeling and Simulation in Engineering. Volume 2010, ID739675, pp. 1-19. ***"Solution of Extraordinary Differential Equations with Physical Reasoning by Obtaining Modal Reaction Series".***
3. International Journal of Applied Mathematics & Statistics. Vol. 21; No. J11, 2011, pp131-140. ***"Fractional Stochastic Modeling for Random Dynamic Delays in Computer Control System".***
4. Elsevier ISA-Transactions 49(2010) pp.196-206 December 2009, "***Fractional Order Phase Shaper Design with Bode Integral for Iso-damped control system".***
5. Int. J. of Nuclear Energy Science & Technology, Vol.5, No.2, pp105-113, **The "*Solution of Coupled Fractional Neutron Diffusion Equation with Delayed Neutron*".**
6. Geophysical Journal No.2 T 31,2009 pp147-159. ***"Fractional Calculus to describe Half-Space Geophysical Analysis for Transient Electro-Magnetic Method".***
7. Int. J. Nuclear Energy Science & Technology Vol.3 No.2, 2006, pp139-159, ***"Fractional divergence for neutron flux profile in nuclear reactor".***
8. IEEE Transactions on Nuclear Science, Vol.57, No.3, pp1602-1612, June 2010, ***"Design of Fractional Order Phase Shaper for Iso-Damped Control of PHWR under Step-Back".***
9. Int. J of Applied Mathematics & Statistics, Vol. 23, D11, pp64-74, ***"Convergence of Riemann-Liouvelli and Caputo Derivative for Practical Solution of Fractional Order Differential Equations."***
10. Condensed Matter Archives arxiv:1012.08V2 dated 10-12-2010, "***Oscillatory Spreading and Surface Instability of a Non-Newtonian Fluid under compression."***

<u>And several more .</u>

It is reality now once considered stupidity

A great gift from mathematicians got translated into a product (circuits) for Fractional Order Controls-for “***Efficient Control System with Robustness***”. Here the Fractional Differential Equations are working in electronics circuit. This device is not existing anywhere in world This was demonstrated on 30 July 2010, and was formally inaugurated for industrialization

Under three Patents.



$$u(t) = K_P e(t) + K_I\{{}_0D_t^{-\alpha} e(t)\} + K_D\{{}_0D_t^{\beta} e(t)\}$$

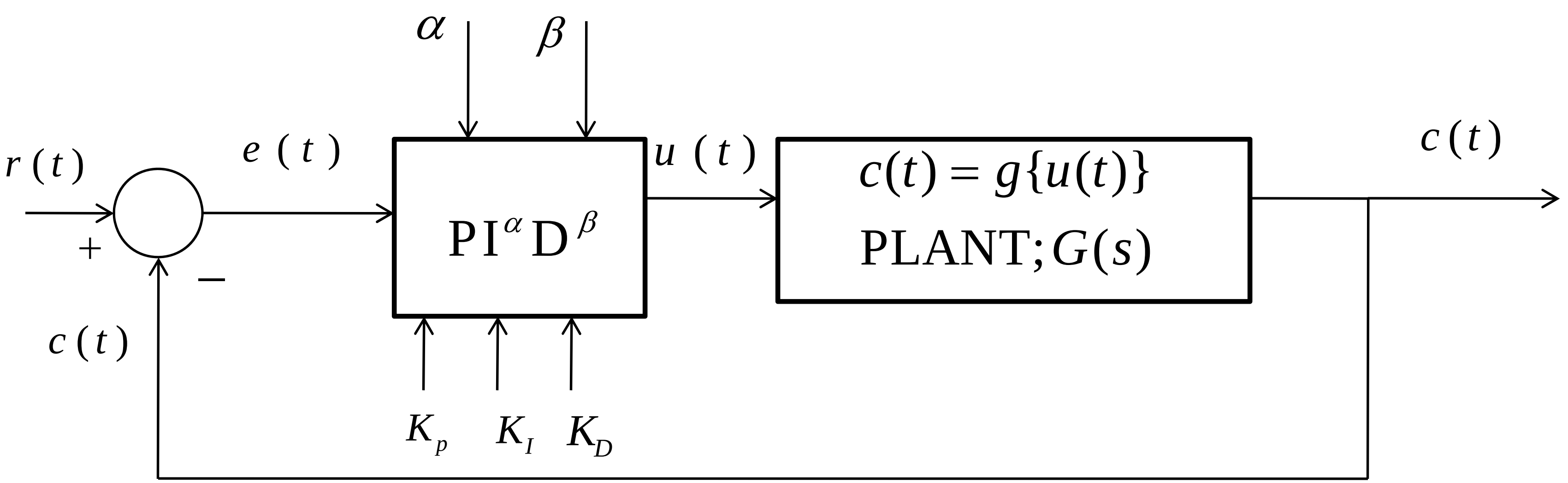
$$\alpha\, ;\, \beta \in (0,1)$$

$$e(t) = r(t) - c(t)$$

$$\mathrm{PI}^{\alpha}\mathrm{D}^{\beta}$$

A feed back control system in general

A plant is controlled by a controller manipulating desirably the plant output or performance.



$$u(t) = K_P e(t) + K_I \{ {}_0D_t^{-\alpha} e(t) \} + K_D \{ {}_0D_t^{\beta} e(t) \}$$

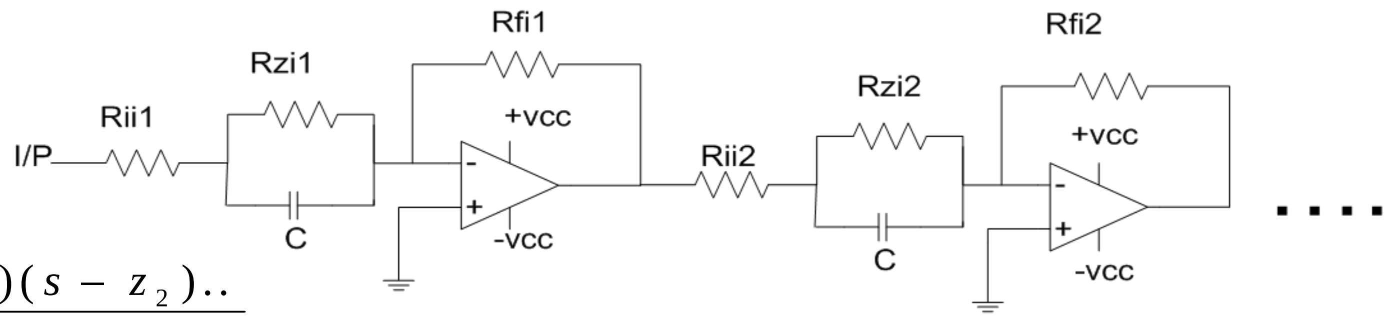
$$\frac{U(s)}{E(s)} = K_p + \frac{K_I}{s^{\alpha}} + K_D s^{\beta}$$

$$e(t) = r(t) - c(t)$$

$$E(s) = R(s) - C(s)$$

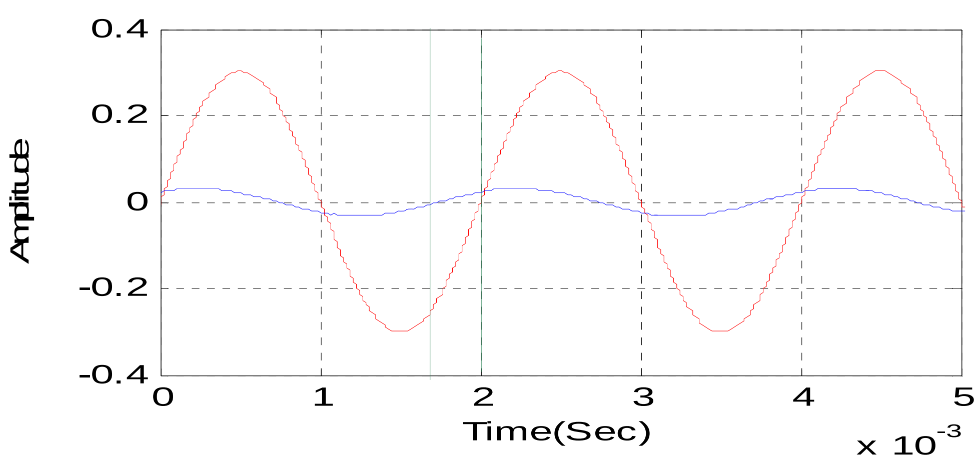
$$\frac{C(s)}{U(s)} = G(s) = \frac{(s^p + a_1 s^{p-1} + ... + a_0)}{(s^q + b_1 s^{q-1} + ... + b_0)}$$

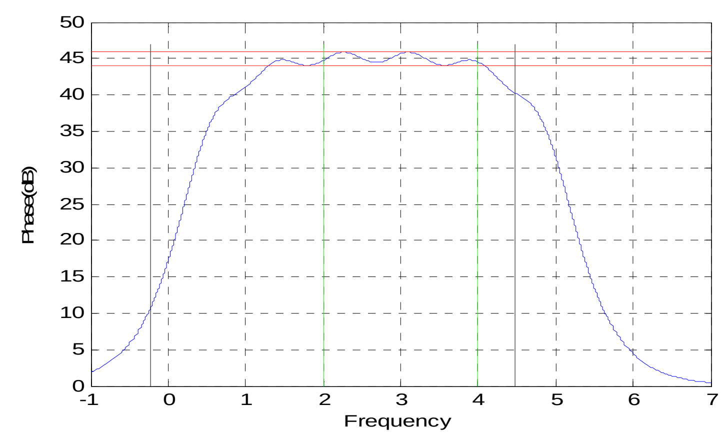
Fractal real poles and real zeros interlaced to give half order differentiator:



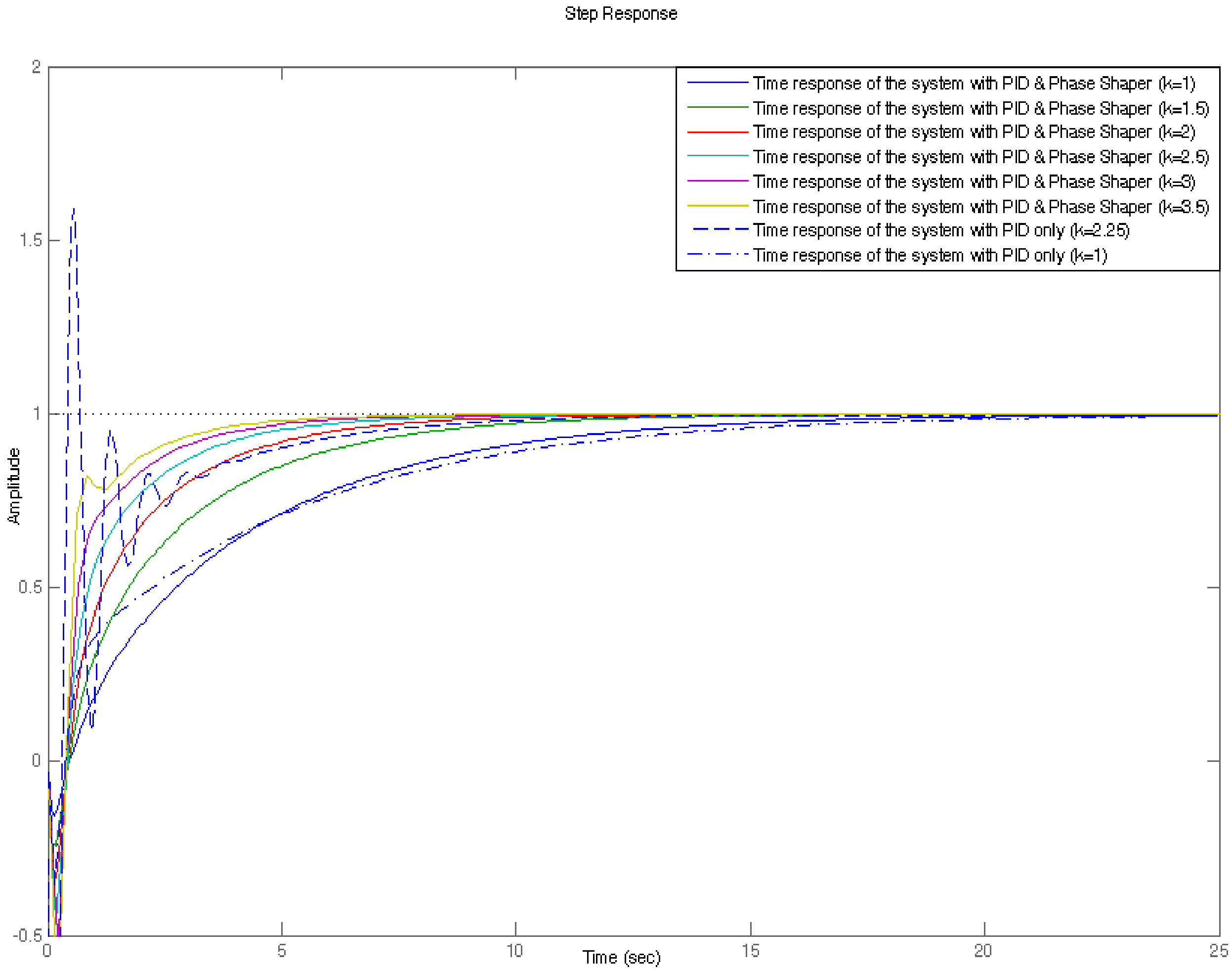
$$s^{1/2} \cong \frac{(s - z_1)(s - z_2)..}{(s - p_1)(s - p_2)..}$$

i	Zi	Pi	Ci	Rfi= Rii		Rzi	
				Ω	TP	Ω	TP
1	2.2537	6.0406	1µ	264.07k	500k	443.71k	500k
2	15.955	42.764	1µ	37.30k	50k	62.67k	100k
3	112.95	302.75	680nf	11.21k	20k	18.83k	20k
4	799.65	2143.3	68nF	10.94k	20k	18.39k	20k
5	5661.1	15173	10nF	10.51k	20k	17.64	20k
6	40078	107420	1nF	14.85k	20k	24.95k	50k



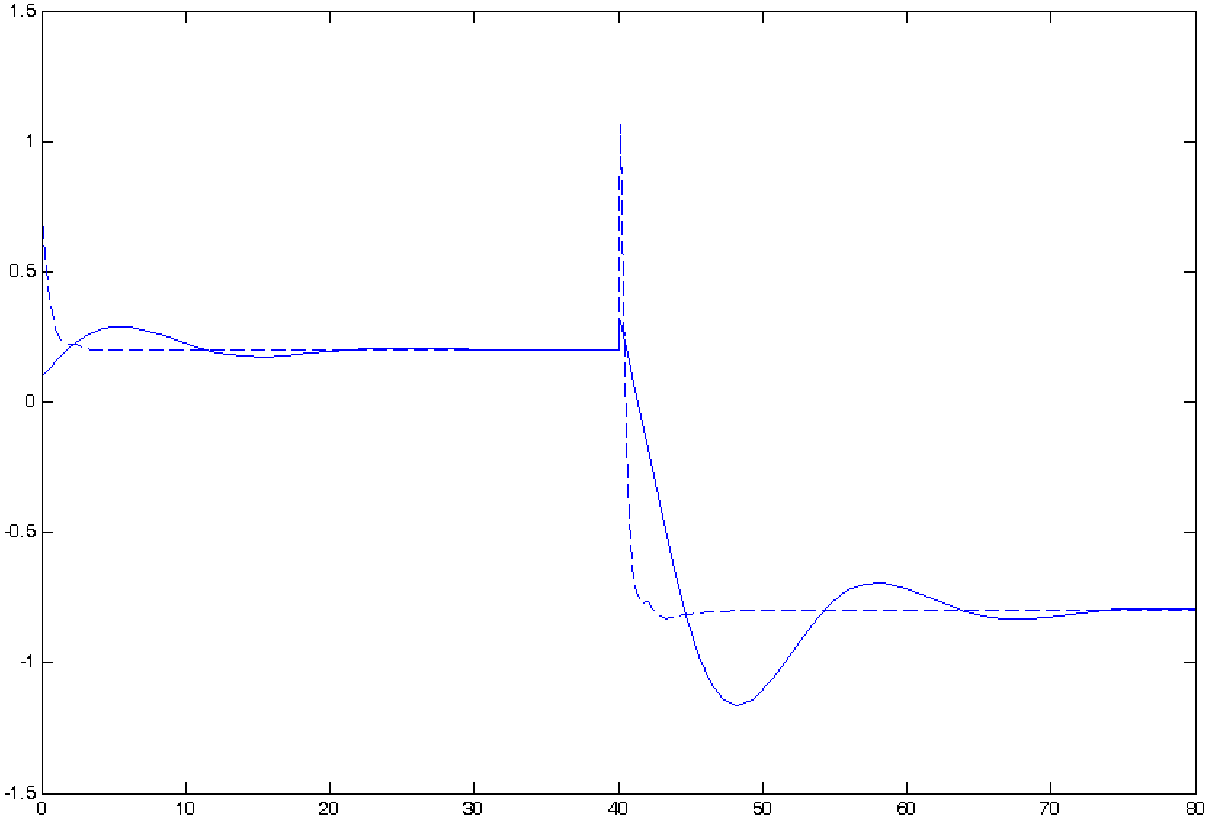


Overshoot unchanged while parameter spreads five folds with Fractional order PID



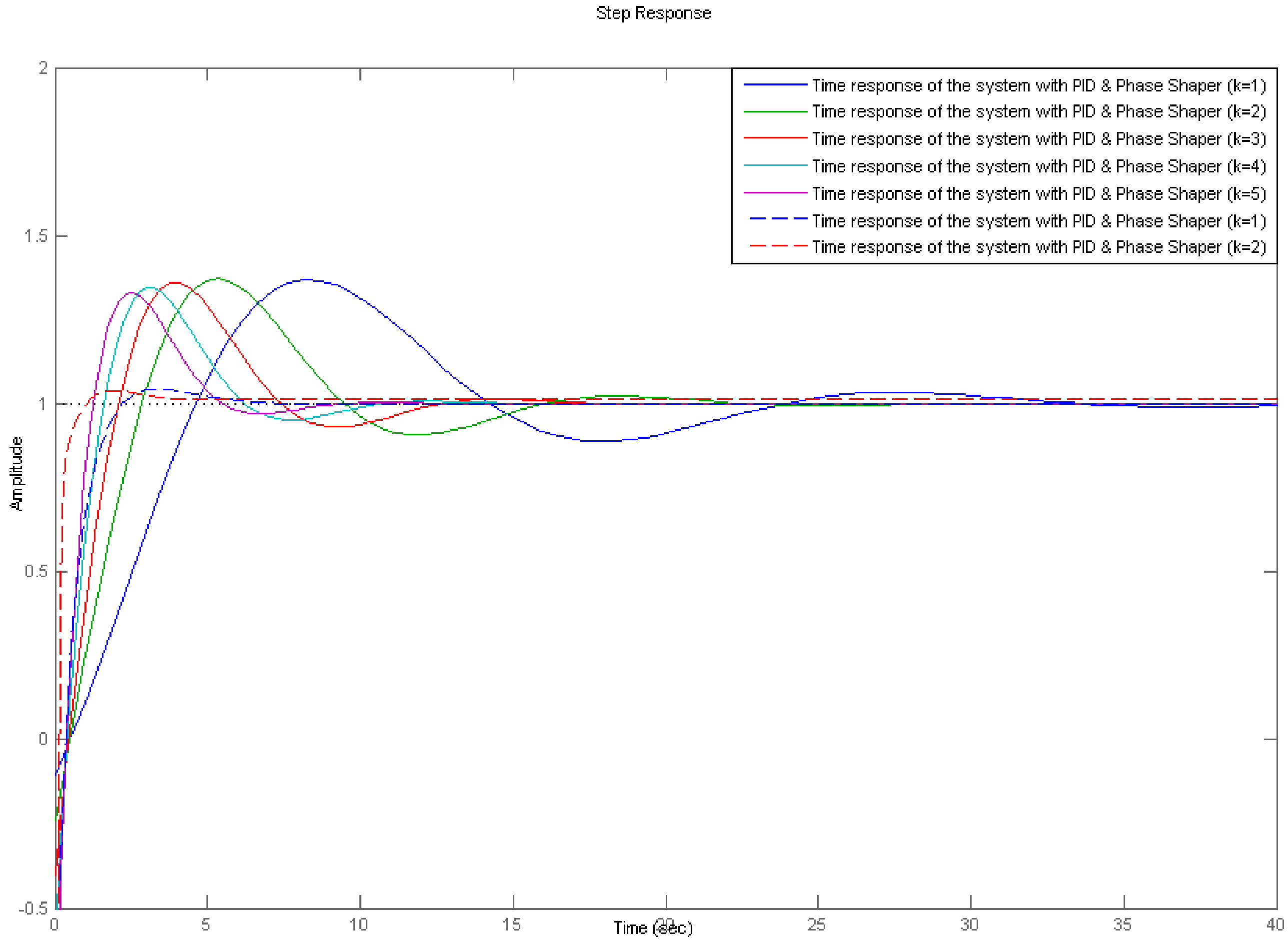
Step responses with the variation of loop gain, dashed line represents original system controlled by PI controller & continuous lines represent responses along with PI & phase shaper (Fractional order PID) for varying gains.

Controller effort is lesser with Fractional order PID



Controller output signal with and without the phase shaper, represented by solid line and dashed line respectively.

Iso-damped response with wide parametric spreads



The scalar gain, as shown in Figure can be varied by 500% keeping the overshoot constant. The advantage of the phase shaper becomes evident considering the fact that the PID controller alone cannot handle such large variation in gain. The closed loop system, with the PID controller alone becomes unstable with two fold increase in gain.

Hardwire set up to control DC Motor servo position system with FO-PID circuits

Contents of seven lecture series of

Mathemaico-Physics of Generalized Calculus

Module-0: Introduction

Module-1: Generalization of Newton-Leibniz Integer Order Calculus

Module-2 : Appreciation of Generalized Calculus

Module-3: Fractional Calculus for Applied Science & Engineering

Module-4: Solving Fractional Differential equations (FDE) in formal way

Module-5: Discussion on Natural laws of Physics

Module-6: Complex order calculus & Solution of extra ordinary differential equations

<u>The modules are not linearly independent and the lecture series is non-Markovian process!!</u>

MODULE-0
INTRODUCTION:

How this idea of utilizing fractional calculus triggered,
How Fractional Calculus can be implemented,
Idea of exact thinking,
Birth of Fractional Calculus-and seemingly paradox,
What is not fractional calculus (a misnomer),
Hierarchy of calculus,
Generalization of number theory vis-à-vis concept of fractional calculus,
Concept of continuum between one full differentiation and one full integration,
Discrete difference to continuum limit and Stochastic Difference-giving random walk (Brownian Motion), and could be extended to have Fractional Brownian Motion,
Revisiting concept of Mean Squared Displacement-and concept of anomalous (fractional) random walk,
Maxwell-Debye exponential relaxation process, Non-Debye non-exponential relaxation process,
Memory & Fractional Calculus, Concept of Memory Integrals and Convolution, Markovian Process,
Fokker-Plank-Kolmogrov Equation and its Gaussian Solution with its limitation,
Concept of Long Tail power law distribution,
Fractional Kinetics in Fractal support,
Law of Irreversibility and Generalizing Classical Dynamics,
Evolution of time,
Irregularity measure, roughness exponent, Mandelbrot’s-Richardson’s fractal dimensions,

MODULE-1:

GENERALIZATION OF NEWTON-LEIBNIZ’S INTEGER ORDER CALCULUS:

Generalization of repeated integration and differentiation,
Generalization of Factorial, Binomial Coefficients by Gamma Function,
Generalization the limit of finite differences,
Expansion of Fractional Difference,
Generalization of product rule for integration and differentiation and its unification by Leibniz’s rule,
Fractional Differentiation and Integration of monomial, and demonstrating fractional differentiation of constant is not zero,
Existence of fractional differentiation and integration,
Series representation of differential equation solution by monomial integration (by physical process of action reaction chain)-and extension of the same principle to get solution of Fractional Differential equation,
Comparison of solution of Ordinary Differential Equation vis-à-vis Fractional Differential Equation,
Introducing Higher Transcendental Functions used in Fractional Calculus,
Discussing a paradoxical condition of having instability in First Order System in presence of fractional order elements and definitions revisited regarding system order,
Examples of simple Fractional Differential Equation and its solution in series form,
Partial Differential Equation and Operational Calculus by Heaviside,
Abel’s Tautochcrone problem and Abel’s integral,
Generalization of Ohm’s law by Fractance Devices.

MODULE-2:

APPRECIATION OF GENERALIZED CALCULUS:

Riemann-Liouvelli (RL) of Fractional Integration,
Area under the ‘shape-changing’ curve to evaluate fractional integration as the time grows,
Physical interpretation of fractional integration with uniform flow of time and non-uniform flow of time,
Convolution with power function in RL integration,
Euler’s Fractional Derivative formulation for monomial,
Riemann-Liouvelli fractional derivative (Left Hand Definition) described by number line,
Caputo definition of fractional derivative (Right Hand Definition) described by number line,
Duality of two definitions and its usage,
Fractional Differentiation and Fractional Integration of exponential and trigonometric functions and generation of higher transcendental functions,
Differentiation and Integration using same formula and paradox,
Initialization function for RL Fractional Integration,
Initialization for RL fractional Derivative and symbol standardization,
Caputo initialization,
Solution of Fractional Differential Equation by Initialization Function,
Concept of Fractional Initial state and its physical interpretability,
Integer order calculus in fractional context,
Concept of Local fractional derivative,
Fractal dimension and LFD and critical order in phase transition
Grunwald-Letnikov (GL) method for fractional differeintegration, its realization as digital filter, with physical interpretation of weights of Fractional Differentiation,
Approximations and Errors to get finite dimensional GL method with short-memory principle,
Matrix approach to have GL approximation and concept of Generating function,
Fourier representation of Stochastic Finite fractional difference and fractional derivative and its physical meaning,
Use of Fractional differentiation for curve fitting,
Geometrically representing infinitesimal element for fractional integration and fractional differentiation.

MODULE-3:

FRACTIONAL CALCULUS FOR APPLIED SCIENCE & ENGINEERING:

Generalization of Newtonian Mechanics,
Concept of Order Distribution & Continuous Order Distribution in system identification,
Solving for system with continuous order distribution,
Futuristic feedback controller by continuous order distribution,
Fractional Calculus in circuit theory,
Heat flow in semi-infinite system, Impedance or RC distributed transmission line,
Fuel Efficient Control System by Fractional Controller PID or Feed Back Element,
Concept of Fractional Divergence,
Fractional Kinetics,
Concept of Fractional Curl,
Wave Propagation in media with losses,
Multi-pole expansion in electrodynamics,
Fractal Geometry & Fractional Calculus,
Fractional Order & Fractal Dimensions,
Application in material science (Visco-Elasticity) ,
Application in biology,
Circuit Synthesis,
Concept of Iso-Damping, Fractal Pole-Zero and realization of Constant Phase Element,
Application in Geophysics for Electromagnetic Flux Diffusion,
Application in Collision and Cohesion Dynamics,
Change in fractal dimension is indicator of start of process, LFD and Fractional Taylor series usage in approximating non-differentiable functions, specially at Phase Transition points,
Line surface volume integrals in fractal charge distribution and existence of magnetic monopole,
Application to realize FO by Self-Similar Repeated Structure and Continued Fraction Expansion (CFE).
A computer system delay a case of Fractional Brownian Motion

MODULE-4:

SOLVING FRACTIONAL DIFFERENTIAL EQUATION (FDE) IN FORMAL WAY:

Demonstration of solving FDE with Numerical Technique, with various orders and effect of initial condition,
Formulating FDE Formally,
Number of Solutions, Similarity with Ordinary Differential Equation,
Formal Solution of FDE,
Direct Approach,
Miller-Ross function as possible candidate solution to FDE,
Concept of Indicial Polynomial,
Motivation Laplace Transform Technique,
Generalization of Partial Fractions,
Solution with Miller-Ross and Robotnov Hartley function,
Motivation to get Linearly Independent Solutions,
Explicit solution for Homogeneous FDE,
Examples, Non-Homogeneous Solution FDE,
Examples, Fractional Integral Equation Solutions and examples.

MODULE-5:

DISCUSSION ON NATURAL LAWS OF PHYSICS:

Flow-rate vis-à-vis notch shape for water flow through dam weir,
Flux flow through heterogeneous porous medium (with fractal support),
Diffusion in Porous medium,
Classical Fick’s law of diffusion and its idealism to reality,
Modified Catteneo’s law of diffusion,
Diffusion with Memory,
Boltzmann’s superposition law, in material science of stress-strain vis-à-vis Memory Integral,
Convolution and Evolution of Process Dynamics,
Fractional Brownian Motion, Fractional Stochastic Difference,
Persistence and Anti-persistence random walks,
Non-Markovain process with memory.
Disordered relaxation, strong ,weak, intermittent ,oscillatory relaxation/discharge with memory kernel
Origination of FDE in disordered relaxation
Generalized Poisson’s law
Relaxation in clustering heterogeneity
Relaxation through several internal states
Classification of relaxation
Why a normal distribution with definite average and variance should change to power law distribution with diverging average and variance, possible physical reasoning.
Long wait time and long jump lengths will give fractional diffusion equation in time as well as space.

MODULE-6:

COMPLEX ORDER CALCULUS & SOLUTION OF EXTRAORDINARY DIFFERENTIAL EQUATIONS:

Complex order differentiation and integration,
Conjugated differintegrals,
Real Time Response and Imaginary Time Response ,
Frequency Response of Conjugated pole (zero)-Bode Plot,

FDE with non-constant coefficients and its solution by examples,

Generalized Dynamic System Solution by Decomposition method of Action-Reaction-
-A Modern Approach to Solve Extra Ordinary Differential Equation to obtain analytical approximate solution and Mathematics of Adomian Decomposition Method & Adomian Polynomial applied in Physical Reasoning,
Several examples of solution by this technique applied to FDE, ODE, Non-Linear FDE, Non-Linear ODE, Fractional Partial Differential Equations.

Acknowledgement to students and teachers working on this subject:

Stupidity is the name of innovation, and I have lived with this stupidity for long and shall continue to live.

I salute all my students who have continued the stupid thinking and worked on projects on Fractional Calculus 2006-2010.

Saptarishi Das Suman Saha, Abhishekh Bhowmik, Indranil Pan, Basudeb Mazumder (JU-PE), Jitesh Khanna, Vamsi, (IIT-KGP, EE) , Ms. Rutuja Dive (VNIT-EE), Tridip Sardar (HIT-Mathematics), Subrata Chandra, Ms Moutushi Ms Sanjukta (JU Physics), Ashrita Patra (NIT-R Mathematics)

The professors, who took this subject for projects & investigation, I salute are:

Prof. M.V. Aware, Prof. A.S. Dhabale (VNIT-EE), Prof. Susmita Sarkar, Prof. Uma Basu (CU-Mathematics), Prof. P.N. Ghosh (VC-JU), Prof. Sujata Tarfadar (HOD Physics JU), Prof. A. Gupta (JU, D-SNSE), Prof S. Sen (IIT-KGP-EE), Prof. S. Saharay (NIT-R Mathematics)

Does ‘*d* / *dt*’ represent accumulation or loss always

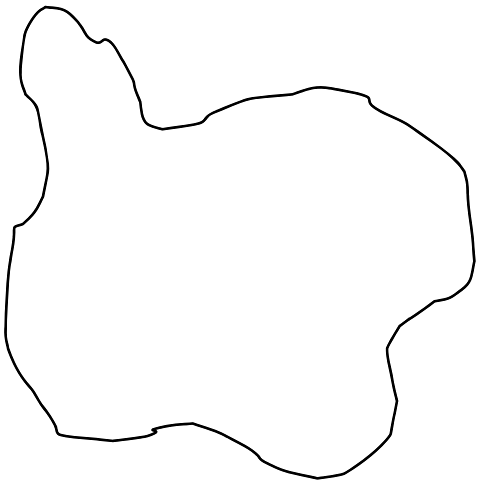
Well if there are temporary traps then?

Well if the boundary is partly reflecting?

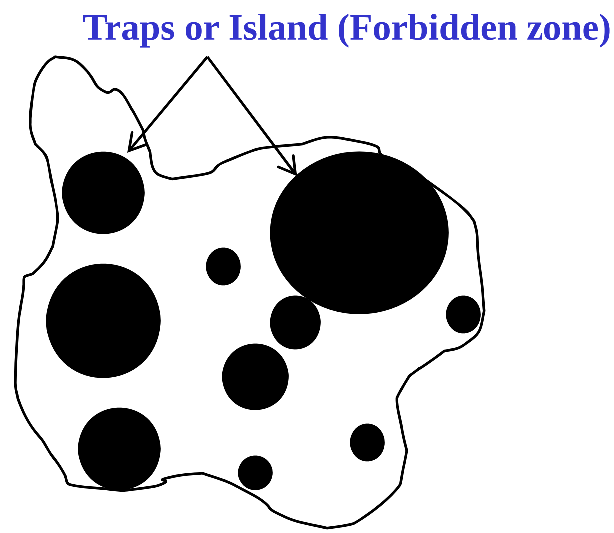
Well if the elementary element (area, volume etc) be not a point quantity?

Well if mass of ball is not a point quantity?

Well if spring is not mass-less?



$$\frac{d}{dt}\phi = \mathrm{GAIN} - \mathrm{LOSS?}$$



$$\frac{d^{\alpha}}{dt^{\alpha}}\{\phi\} = \mathrm{GAIN} - \mathrm{LOSS?}$$

Some are entraps temporarily indicating slow rate of change than *d* / *dt*

The particles cannot have the island paths indicating fast rate of change than *d* / *dt*

Fractional integration-antiderivative

Repeated n-fold integration generalization to arbitrary order

$$I_t^1 f(t) = d_t^{-1} f(t) = \int_0^t f(\tau) d\tau$$

$$I_t^2 f(t) = d_t^{-2} f(t) = \int_0^t \int_0^t f(\tau) d\tau d\tau = \int_0^t (t-\tau) f(\tau) d\tau$$

$$I_t^3 f(t) = d_t^{-3} f(t) = \int_0^t \int_0^t \int_0^t f(\tau) d\tau d\tau d\tau = \frac{1}{2} \int_0^t (t-\tau)^2 f(\tau) d\tau$$

$$I_t^n f(t) = d_t^{-n} f(t) = \underbrace{\int_0^t \int_0^t \ldots\ldots\ldots\ldots\ldots\ldots \int_0^t}_{n} f(\tau) d\tau = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau$$

$$n \in Z^+$$

$$I_t^{\phi} f(t) = d_t^{-\phi} f(t) = \frac{1}{\Gamma(\phi)} \int_0^t (t-\tau)^{\phi-1} f(\tau) d\tau$$

$$\phi \in R^+$$

Fractional derivative

$$\frac{d^n f(x)}{dx^n} = \underbrace{\frac{d}{dx}\frac{d}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\frac{d}{dx}}_{n} f(x)$$

$$\frac{d^n}{dx^n}\{x^m\} = m(m-1)(m-2)\ldots\ldots\ldots\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n}{dx^n}\{x^m\} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

$$\frac{d^{0.5}}{dx^{0.5}}\{x\} = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)}x^{1-0.5} = \frac{\sqrt{x}}{\Gamma(1+0.5)} = \frac{\sqrt{x}}{0.5\Gamma(0.5)} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$\frac{d^{0.5}}{dx^{0.5}}\{C\} \neq 0$$

For positive index the process is differentiation

For negative index the process is integration

Fractional derivative of a constant is not zero

Perin's Brownian Motion & Fick's Diffusion

In discrete time steps of span Δt the walker (particle) is assumed to jump to one nearest neighbour. The one dimensional lattice be taken, then pdf (concentration/ number of particles/flux) at position j at the time $t+\Delta t$ in dependence on population of position of two adjacent sites $j \pm 1$ at time t , is:

$$N_j(t+\Delta t) = \frac{1}{2} N_{j-1}(t) + \frac{1}{2} N_{j+1}(t) \qquad \text{.............(1)}$$

½ is for isotropic case.

In continuum limit with $\Delta t \to 0$ and $\Delta x \to 0$, we have Taylor's expansion:

$$N_j(t+\Delta t) = N_j(t) + \Delta t \frac{\partial N_j}{\partial t} + R_1\left(\left[\Delta t\right]^2\right) \qquad \text{........................(2)}$$

$$N_{j\pm 1}(t) = N_j(x,t) \pm \Delta x \frac{\partial N}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 N}{\partial x^2} + R_2\left(\left[\Delta x\right]^3\right) \qquad \text{........................(3)}$$

Using (2) and (3) and putting in (1) and recognizing that $N(x,t) = N_j(t)$, neglecting $R_1 ; R_2$ we get:

$$\frac{\partial N(x,t)}{\partial t} = \mathbb{D} \frac{\partial^2 N(x,t)}{\partial x^2}$$

$$\mathbb{D} = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t}$$

is Fick's law

A normal diffusion equation & its fractional calculus version

Start with continuity equation $\frac{d}{dt}M(r,t) = -j(r,t)$, where $M(r,t) = \int_0^r dr P(r,t)$ and $j(r,t)$ is the total probability current at *r* from origin. The above equation must be supplemented by constitutive equation relating the current *j* (*r*, *t*) to the probability density function p.d.f *P* (*r*, *t*) i.e.

$$j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r}$$

From these we get diffusion equation in differential form from the above Fick's law.

$$\frac{\partial}{\partial t}P(r,t) = \mathbb{D}_0 \frac{\partial^2}{\partial r^2}P(r,t)$$

The plume is Gaussian and MSD is linear with time, for delta function at origin

$$P(r,t) = \frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}$$

Using Laplace of Gaussian, we get

$$P(r,s) = \mathcal{L}\{P(r,t)\} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}\right\} = \left(\frac{1}{\sqrt{\mathbb{D}_0 s}}\right) e^{\left(-r\sqrt{\frac{s}{\mathbb{D}_0}}\right)}$$

Taking first derivative of above we get

$$\frac{d}{dr}P(r,s) = \frac{-1}{\sqrt{\mathbb{D}_0 s}}\sqrt{\frac{s}{\mathbb{D}_0}} e^{-r\sqrt{s/\mathbb{D}_0}} = -\frac{1}{\mathbb{D}_0} e^{-r\sqrt{s/\mathbb{D}_0}}$$

and then Laplace transforming the constitutive equation with slight manipulation, we get

$$j(r,s) = \sqrt{s\mathbb{D}_0}\,P(r,s)$$

Use $\sqrt{s} \leftrightarrow \mathcal{L}\left\{d^{1/2} / dt^{1/2}\right\}$ and we get time domain expression of current as

$$j(r,t) = \sqrt{\mathbb{D}_0}\,\frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}}$$

Together with constitutive equation, we get fractional calculus version of standard diffusion equation

$$\frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0}\,\frac{\partial P(r,t)}{\partial r}$$

This is standard Brownian process BM with linear MSD

A Fractional Brownian Motion Process (FBM)-a passing remark

Fractional Brownian Motion (FBM) is simplest mathematical model extension of Gaussian stochastic process (random walk) whose variance (MSD) does not scale linearly with time its p.d.f. is:

FBM is natural generalization of BM, here with a stretched exponential

$$P_{FBM}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t^{2/d_w}}} e^{\left(-\frac{x^2}{4\mathbb{D}_0 t^{2/d_w}}\right)}$$

$$\left\langle x^2(t) \right\rangle \equiv 2\mathbb{D}_0 t^{2/d_w} \qquad 1 \le d_w < \infty$$

d_w Anomalous diffusion exponent

Brownian Case is with anomalous diffusion exponent as 2 is following:

$$P_{BM}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t}} e^{\left(-\frac{x^2}{4\mathbb{D}_0 t}\right)} \qquad \left\langle x^2(t) \right\rangle \equiv 2\mathbb{D}_0 t \qquad d_w = 2$$

$$\frac{\partial^{1/2} P_{BM}(x,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0}\,\frac{\partial P_{BM}(x,t)}{\partial |x|}$$

For non-Brownian case:

$$\frac{\partial^{1/d_w} P_{FBM}(x,t)}{\partial t^{1/d_w}} = -A\,\frac{\partial P_{FBM}(x,t)}{\partial |x|}$$

Transport phenomena in complex systems such as random fractal structures exhibit anomalous features which are qualitatively different from the standard regular systems. In the case of fractals such anomalies are due to constraint on the transport process on all lengths scales. These constraints may be seen as temporal correlations existing on time scales. (MSD)

$$X^2 \equiv \left\langle x^2(t) \right\rangle \cong t^{2/d_w}$$

d_w Anomalous diffusion exponent

$$P(x,t) \cong t^{-d_f/d_w} \exp\left[-C\left(x/X\right)^u\right] \qquad u = d_w/(d_w - 1)$$

d_f Fractal dimension.

A Fractional Brownian Motion Process (FBM), represented with memory

A Fractional Brownian Motion Process (FBM), represented with memory, described as integral transform of Brownian Motion (BM). The convolution with memory kernel that is $K(t)$; first proposed by Mandelbrot.

$$x_{\mathrm{FBM}}(t) = \int_{-\infty}^{t} K_M(t-\tau)\, dx_{\mathrm{BM}}(\tau)$$

Where $x_{\mathrm{FBM}}(t)$ and $x_{\mathrm{BM}}(t)$ are the position of the particle undergoing the FBM and BM process respectively.

$$K_M(t-\tau) = \begin{cases} (t-\tau)^{(1/d_w)-(1/2)} - (-\tau)^{(1/d_w)-(1/2)} & ;\tau < 0 \\ (t-\tau)^{(1/d_w)-(1/2)} & ;0 < \tau < t \end{cases}$$

The kernel resembles the singular memory kernel associated with fractional integral (derivatives).

$$I_t^{\phi} f(t) = d_t^{-\phi} f(t) = \frac{1}{\Gamma(\phi)} \int_0^t (t-\tau)^{\phi-1} f(\tau)\, d\tau$$

With anomalous exponent equal to 2 we get a no-memory case with $K_M(t-\tau) = 1$ in this case

$$x_{\mathrm{FBM}}(t) = x_{\mathrm{BM}}(t)$$

A walker undergoing FBM remembers his past, while a walker undergoing BM does not remember its past. Well a walker can remember its past and have preferences in same direction giving persistence walk, or a walker remembering its past can change its direction giving anti-persistence walks, are the cases of anomalous transport. This gives concept of sub or super diffusion, in FBM context.

Note, the variable $x_{\mathrm{FBM}}(t)$ may not be physical distance. It could be

1. Computer net work delay.
2. Could be price of stock market.
3. Could be random returns of insurance system.
4. Could be infected population by swine flu

In general could be variable for physical systems represented by random stochastic process.

Fractional Reactor Kinetic Equation:

$$\frac{1}{v_c}\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) = \mathbb{D}\nabla\phi(x,t) + \left(\gamma\Sigma_f - \Sigma_a\right)\phi(x,t) + \lambda C(x,t)$$

$$\frac{\partial^\alpha}{\partial t^\alpha}C(x,t) = \beta\gamma\Sigma_f\phi(x,t) - \lambda C(x,t) \qquad 0 < \alpha < 1/2$$

$$\phi(x,0) = \phi_0(x) = 1.0$$

$v_c = 220{,}000\text{cm/s}$ $\quad B = \beta\gamma\Sigma_f = 000735\text{cm}^{-1}$ $\quad \mathbb{D} = 0.356\text{cm}^2\text{s}^{-\alpha}$ $\quad \lambda = 0.08\text{s}^{-1}$ $\quad \Sigma = \gamma\Sigma_f - \Sigma_a = 0.005\text{cm}^{-1}$

Time (s)	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.00010	1.04535×10^8	6.21599×10^6	3.58370×10^5	2.00988×10^4	1.13743×10^3
0.00039	1.59622×10^8	1.42279×10^7	1.22266×10^6	1.01416×10^5	8.23638×10^3
0.00068	1.89937×10^8	1.99782×10^7	2.02159×10^6	1.97079×10^5	1.86944×10^4
0.00097	2.12264×10^8	2.48265×10^7	2.78898×10^6	3.01545×10^5	3.16326×10^4
0.00126	2.3039×10^8	2.91405×10^7	2.78898×10^6	3.01545×10^5	3.16326×10^4
0.00155	2.45852×10^8	3.3087×10^7	4.26719×10^6	5.29060×10^5	6.34629×10^4
0.00184	2.59449×10^8	3.67595×10^7	4.98668×10^6	6.50071×10^5	8.19238×10^4
0.00213	2.71652×10^8	4.02168×10^7	5.69632×10^6	7.75079×10^5	1.01890×10^5
0.00242	2.8277×10^8	4.34989×10^7	6.39764×10^6	9.03656×10^5	1.23257×10^5
0.00271	2.93014×10^8	4.66344×10^7	7.09176×10^6	1.03547×10^6	1.45939×10^5
0.00300	3.02536×10^8	4.96447×10^7	7.77958×10^6	1.17024×10^6	1.69866×10^5

The neutron flux growths for the lower values of $\alpha < 0.3$ has saturation tendency with time.
From observations , the inference is drawn that neutron flux multiplication can be obtained at fractional orders values !! $\alpha > 0.3$

This implies that there is possibility of nuclear reactor achieving the desired neutron multiplication factor or criticality at fractional values indicating concept of '**fractional criticality**'. $k_\infty^{(\alpha)}$

Fractional Kinetic Equation-a passing mention:

The fractional time derivative appears- made us to think of Fractional Multiplication *k*

The wait time statistics assumes very-very long wait times, instead nil wait times of walker

Assuming very-very long jump lengths, instead one lattice length jump of walker, will give Fractional Derivative in Space-should make us to think of Fractional geometrical buckling.

The normal assumption is definite average wait times and averaged jump lengths gives Fickian case, with Brownian Motion where as these above assumptions with diverging averages and variances give Fractional Diffusion Kinetic Equation-a case of Fractional Brownian Motion.

Spatial fractional derivative lead to Fractional Divergence principle and flux shape with transcendental cosine function, whereas fractional time derivative give a lingering tailed power law manifestation with higher transcendental Millag-Leffler type (non-exponential) temporal growth!!

Phase Table for the Fractional Diffusion Equation

$$\frac{\partial}{\partial t}\phi(x,t) = {}_0D_t^{1-\alpha}(\mathbb{D}_{\alpha,\mu})\frac{\partial^\mu}{\partial x^\mu}\phi(x,t)$$

We are used to $\alpha = 1, \mu = 2$ The fractional order comes as observation of asymptotic behavior in space time relaxation.

Temporal Fractional Order α	Spatial Fractional Order μ	Type of Walk	Average Waiting Time T	Jump-Length Variance σ^2	Nature of Diffusion
$0 < \alpha < 1$	$0 < \mu < 2$	Long-Jump	∞	∞	Non-Markovian
$\alpha \geq 1$	$0 < \mu < 2$	Long-Jump	$< \infty$	∞	Markovian
$0 < \alpha < 1$	$\mu \geq 2$	Sub-diffusion	∞	$< \infty$	Non-Markovian
$\alpha \geq 1$	$\mu \geq 2$	Brownian	$< \infty$	$< \infty$	Markovian

Clearly we can have argument that massive particle as neutron cannot jump infinitely far. For such massive particles finite velocity of propagation exists making instantaneous long jumps impossible. But in reality we have nuclear reactors where the dimensions are large especially for high power reactors. The neutrons do have ‘coupling’ between spatially distributed ‘point’ reactors. These power reactors are having dimensions much larger than the average diffusion lengths of neutron-and are called coupled core reactors. The spatial heterogeneity in small scales do manifest as fractal dimensions in space which makes the anomalous transport of neutrons contrary to belief that it can only reside and ‘walk’ as local Brownian motion. Long-jumps can therefore take place for these ‘massive’ neutrons-in a ‘fractal’ heterogeneous spatial backdrop, along with long wait times, in heterogeneous lattice. The ‘Fractal background’ helps to have walk-through, making long jumps possible!

Disorder Relaxation in Condense Matter & Fractional Calculus:

Fractional Calculus methods have been invoked (recently) to model relaxation processes in complex systems. This has lead to interesting discussions into nature of transport coefficients appropriately/equations to describe these complex materials. This observation is leading to thought to have 'Universality' of "Disordered Material Relaxation" !!

Relaxation integral I_t^{ϕ} where ϕ characterizes 'degree of intermittency' in the relaxation process-and exact solutions of these integrals (may) describe relaxation in condense matter 'intermittency in relaxation'

Extension in this above model of relaxation , to include β 'dynamic heterogeneity ' arising out from particle clustering (too may be included) which is ubiquitous in condense matter. With $I_t^{\phi,\beta}$ intermittency plus heterogeneity.

$I_t^{\phi} f(t)$ Is Fractional Integration of arbitrary order.

$D_t^{\phi} f(t)$ Is Fractional Differentiation of arbitrary order

Relaxation in complex process some comments:

. Sufficiently high micro structural disorder can lead statistically to macroscopic behavior well approximated by Fractional Calculus.

. Damping (relaxation) behavior of materials if modeled by Linear Differential Equations (LDE); with constant coefficient cannot include ‘long-memory, that fractional order derivatives require.

. Rubber molecules (presumably) cannot remember past here (perhaps) LDE with constant coefficient can be involved. Such systems have ‘exponential-decay’-system without memory. For large times the value goes to zero-(quickly).

. Many materials with ‘complex’ microscopic dissipative mechanisms may macroscopically show Fractional Order Differential Equation behavior. Damping (relaxation) models may involve relatively fewer fitted parameters compared to integer order complex models.

. Fractional Order behavior may be an artifact of many complex internal dissipative mechanisms-each of them with out memory.

Why this name:

Tried to explain mathematics by physical principles
Here I have not tried any theorem statements or lemma proofs, and tried to give simple physical geometrical engineering treatment, remaining out of abstraction, and the explanations to the theories are result of 'thought experiments' which I carried out for long
thus

Mathematico-Physics is thy name!!

However, I am not professional mathematician or physicist, thus I have tried to visualize and physically realize things to see and happen, to explain the abstractions in the theory.

Well simplicity & stupidity are thy name for innovation

*Let us begin by **K**eeping **I**t **S**imple and **S**tupid (**KISS**)*