CIRCUIT ANALYSIS WITH FRACTIONAL CAPACITORS

(Part - 2)

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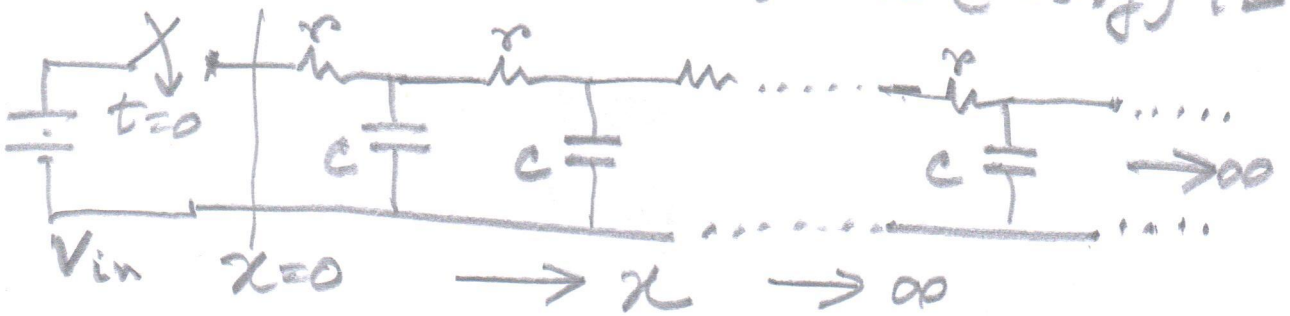
Here we discuss energy transfer to & from fractional capacitor.

First we take an infinite (lossy) TL



r is Ω/cm, c : F/cm are distributed parameters

The voltage diffusion eq^n, (Fick's law) is

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{rc}\frac{\partial^2 v(x,t)}{\partial x^2}$$

$v(x,t)$ is voltage of TL at x and t.

Boundary Conditions $i(0,t) = -\frac{1}{r}\frac{\partial v(0,t)}{\partial x}$

$v(\infty,t) = 0$

The impedance looking from $x=0$ i.e the terminating impedence of T.L is

$$\frac{V(0,s)}{I(0,s)} = \sqrt{\frac{r}{cs}} = Z(0,s), \ \Omega$$

Compare with fractional capacitor impedance $Z(s) = \frac{1}{s^n C_n}$, the TL $Z(0,s) = \sqrt{\frac{r}{c\,s}}$, $\therefore$

$n = 1/2$, $C_n = \sqrt{\frac{c}{r}}$ The TL is half order capacitor (lossy capacitor)

Take the diffusion equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{1}{rc}\frac{\partial^2}{\partial x^2} v(x,t)$$

Do Laplace $t \to s$, to get

$$s\,V(x,s) - V_{init}(x,0) = \frac{1}{rc}\frac{d^2 V(x,s)}{dx^2}$$

The differential equation is

$$\frac{d^2 V(x,s)}{dx^2} - rcs\,V(x,s) = rc\,V_{init}(x,0)$$

The homogeneous solution is via roots of

$$m^2 - rcs = 0 \quad \text{give } m = \pm\sqrt{rcs}$$

The physical solution is by choice of $m = -\sqrt{rcs}$

$$V(x,s) = A e^{-\sqrt{rcs}\,x}$$

at $x=0$, $V(0,s)$ is terminal voltage, so we write

$$V(x,s) = V(0,s)\, e^{-\sqrt{rcs}\,x}$$

Let the TL be excited by step voltage at $x=0$, thus $V(0,s) = \frac{V_{in}}{s}$, therefore the voltage profile accross the TL is

$$V(x,s) = \frac{V_{in}}{s} e^{-(\sqrt{rcs})x}$$

By Laplace table, we write inverse Laplace of above to get time, space expression as

$$v(x,t) = V_{in}\,\mathrm{erfc}\left[\frac{\sqrt{rc}\,x}{2\sqrt{t}}\right]$$

In Laplace domain we have voltage expression as

$$V(x,s) = \frac{V_{in}}{s} e^{-(\sqrt{rcs})x}$$

The impedance is $Z(0,s) = \sqrt{\frac{r}{cs}}$, thus the current is:

$$I(x,s) = \frac{V(x,s)}{Z(0,s)} = \frac{\frac{V_{in}}{s} e^{-(\sqrt{rcs})x}}{\sqrt{\frac{r}{cs}}}$$

$$= \frac{V_{in}}{s}\sqrt{\frac{cs}{r}}\, e^{-(\sqrt{rcs})x}$$

$$= V_{in}\sqrt{\frac{c}{r}} \times \left(\sqrt{s}\, e^{-(\sqrt{rcs})x}\right)$$

Again taking help of Laplace Table we get

$$i(x,t) = \left(V_{in}\sqrt{\frac{c}{r}}\right)\left(\frac{1}{\sqrt{\pi t}} e^{-\frac{rcx^2}{4t}}\right)$$

The power into the line is

$$P_{in}(t) = i(0,t)\, v(0,t)$$

$$= \left[\left(V_{in}\sqrt{\frac{c}{r}}\right)\left(\frac{1}{\sqrt{\pi t}}\right)\right]\left[V_{in}\right]$$

$$= V_{in}^2\sqrt{\frac{c}{\pi r t}}$$

The energy into the line from time 0 to t is $\int_0^t P_{in}(t)\,dt$

$$\mathcal{E}_{in}(t) = \int_0^t V_{in}^2 \sqrt{\frac{c}{\pi r \tau}}\, d\tau = 2V_{in}^2 \sqrt{\frac{ct}{\pi r}}$$

The general expression for stored energy density (per unit length) along the line at any time is

$$d\mathcal{E}_{st}(t) = \frac{1}{2} c(x)\, v^2(x,t)$$

Therefore stored energy (for time 0 to t) at any time in the infinite line ($dx \equiv 0$ to ∞) is integration

$$\mathcal{E}_{st}(t) = \int_0^\infty \frac{1}{2} c(x)\, v^2(x,t)\, dx$$

$$= \frac{c}{2} \int_0^\infty \left[V_{in}\, \mathrm{erfc}\left(\frac{x}{2}\sqrt{\frac{rc}{t}} \right) \right]^2 dx$$

$$= V_{in}^2 \sqrt{\frac{ct}{r}} \left(\frac{2-\sqrt{2}}{\sqrt{\pi}} \right)$$

(erfc integration from tables of integral transf).

Therefore at any time (0 to t), the ratio of the stored energy to input energy is $\eta_{i\to s}$ is,

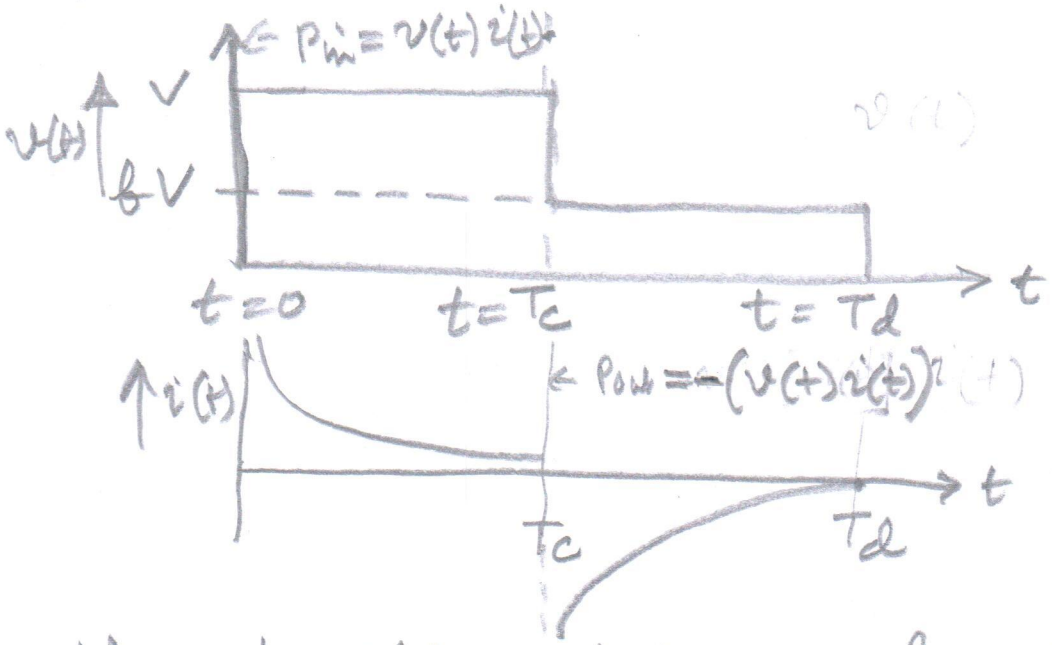
$$\eta_{i\to s} = \frac{\mathcal{E}_{st}(t)}{\mathcal{E}_{in}(t)} = \frac{V_{in}^2 \sqrt{\frac{ct}{r}} \left[\frac{2-\sqrt{2}}{\sqrt{\pi}} \right]}{2V_{in}^2 \sqrt{\frac{ct}{\pi r}}} = \left(1 - \frac{1}{\sqrt{2}}\right)$$

Now if we take out this stored energy from TL the losses will be same & $\eta_{s\to i} = \eta_{i\to s}$; & thus

$$\eta_{i\to o} = \eta_{i\to s} \times \eta_{s\to i} = \left(1-\frac{1}{\sqrt{2}}\right)^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2$$

Now let us pump energy and withdraw the energy to and from fractional capacitor $Z(s) = \frac{1}{s^n C_n}$, and see that $\eta_i \to 0$ for $n = 1/2$, as derived for TL

The scheme is constant voltage excitation (V) for time 0 to T_c, call it charging phase, and thereafter put the fractional capacitor to reduced voltage βV (where $0 \le \beta < 1$), call it discharge phase. The probable voltage & current profiles are



We make $v(t) = 0$ at $t = T_d$, when $i(t) = 0$

For 0 to T_c power is into $Z(s)$ is $v(t)\, i(t)$, for T_c to T_d power out of $Z(s)$ is $v(t)[-i(t)]$

The input voltage for 0 to $t > T_d$ is composed of several step voltages $u(t)$, & is

$$v(t) = V u(t) - V u(t - T_c) + \beta V u(t - T_c) - \beta V u(t - T_d)$$

We take Laplace of input voltage excitation & write as

$$V(s) = \frac{V}{s} - \frac{V}{s}e^{-sT_c} + \frac{bV}{s}e^{-sT_c} - \frac{bV}{s}e^{-sT_d}$$

$$= \frac{V}{s} - \frac{V}{s}(1-b)\,e^{-sT_c} - \frac{bV}{s}e^{-sT_d}$$

using $u(t) \longleftrightarrow \frac{1}{s}$, $u(t-T) \longleftrightarrow \frac{1}{s}e^{-sT}$

The current $I(s) = \frac{V(s)}{Z(s)} = \frac{V(s)}{1/s^n C_n} = s^n C_n V(s)$

Putting $V(s)$ from above we get

$$I(s) = \frac{C_n V}{s^{1-n}} - \frac{C_n V(1-b)}{s^{1-n}}e^{-sT_c} - \frac{C_n V b}{s^{1-n}}e^{-sT_d}$$

Using $\frac{1}{s^{1-n}} \longleftrightarrow \frac{t^{-n}}{\Gamma(1-n)}$, we write inverse Laplace

$$i(t) = \frac{VC_n t^{-n}}{\Gamma(1-n)}u(t) - \frac{VC_n(1-b)(t-T_c)^{-n}}{\Gamma(1-n)}u(t-T_c)$$

$$- \frac{bVC_n(t-T_d)^{-n}}{\Gamma(1-n)}u(t-T_d)$$

For charging period 0 to T_c time to $V = V_{in}$

$$P_{in}(t) = v(t)\,i(t)\,u(t)$$

$$= \frac{V_{in}^2 C_n t^{-n}}{\Gamma(1-n)}u(t)$$

Total energy transferred from 0 to T_c is

$$E_{in}(T_c) = \int_0^{T_c} \frac{V_{in}^2 C_n t^{-n}}{\Gamma(1-n)}dt = \frac{V_{in}^2 C_n t^{1-n}}{(1-n)\Gamma(1-n)}\Bigg|_0^{T_c} = \frac{V_{in}^2 C_n T_c^{1-n}}{\Gamma(2-n)}$$

use $x\Gamma(x) = \Gamma(x+1)$

For the discharging period the applied voltage is reduced to $f V_{in}$ without reversing the polarity. Here current gets reversed. For time T_c to T_d, the $P_{out}(t)$ is

$$P_{out}(t) = -\left(v(t)\left[i(t)\right]\right) u(t-T_c)$$

$$= \left[(f V_{in})\left\{-\left(\frac{V_{in} C_n t^{-n}}{\Gamma(1-n)} - \frac{V_{in} C_n (1-f)(t-T_c)^{-n}}{\Gamma(1-n)}\right)\right]u(t-T_c)$$

$$= -\frac{f V_{in}^2 C_n t^{-n}}{\Gamma(1-n)} u(t-T_c) + \frac{f V_{in}^2 C_n (1-f)(t-T_c)^{-n}}{\Gamma(1-n)} u(t-T_c)$$

The choice of $t = T_d$ is such that, at $t = T_d$ $P_{out} = 0$ current $i(T_d) = 0$. So when power/current goes to zero, we get

$$0 = -\frac{f V_{in}^2 C_n T_d^{-n}}{\Gamma(1-n)} + \frac{f V_{in}^2 C_n (1-f)(T_d - T_c)^{-n}}{\Gamma(1-n)}$$

giv $\quad T_d^{-n} = (1-f)(T_d - T_c)^{-n}$

$$T_d = \frac{T_c}{\left[1-(1-f)^{1/n}\right]}$$

From above $P_{out}(t)$, we have total energy out transfered out of $Z(s) = \frac{1}{s^n C_n}$ is

$$\mathcal{E}_{out}(T_d) = \int_{T_c}^{T_d} P_{out}(t)\, dt$$

$$\mathcal{E}_{out}(T_d) = \int_{T_c}^{T_d} \left[-\frac{b V_{in}^2 C_n t^{-n}}{\Gamma(1-n)} + \frac{b V_{in}^2 (1-b) C_n (t-T_c)^{-n}}{\Gamma(1-n)} \right] dt$$

$$= \left. \frac{-b V_{in}^2 C_n t^{1-n}}{(1-n)\Gamma(1-n)} \right|_{T_c}^{T_d} + \left. \frac{b V_{in}^2 C_n (1-b)(t-T_c)^{1-n}}{(1-n)\Gamma(1-n)} \right|_{T_c}^{T_d}$$

$$= \frac{-b V_{in}^2 C_n (T_d^{1-n} - T_c^{1-n})}{(1-n)\Gamma(1-n)} + \frac{b V_{in}^2 C_n (1-b)(T_d - T_c)^{1-n}}{(1-n)\Gamma(1-n)}$$

$$= \frac{-b V_{in}^2 C_n}{\Gamma(2-n)} \left[T_d^{1-n} - T_c^{1-n} - (1-b)(T_d - T_c)^{1-n} \right]$$

Put $T_d = (T_c) / \left[1 - (1-b)^{1/n} \right]$, to get

$$\mathcal{E}_{out}(T_d) = -\frac{b V_{in}^2 C_n}{\Gamma(2-n)} \left[\frac{T_c^{1-n}}{\left[1-(1-b)^{1/n}\right]^{1-n}} - T_c^{1-n} - (1-b)\left(\frac{T_c}{\left[1-(1-b)^{1/n}\right]} - T_c \right)^{1-n} \right]$$

$$= \frac{-b V_{in}^2 C_n}{\Gamma(2-n)} \left(\left[\frac{T_c}{1-(1-b)^{1/n}} \right]^{1-n} - T_c^{1-n} - (1-b) \left[\frac{T_c (1-b)^{1/n}}{1-(1-b)^{1/n}} \right]^{1-n} \right)$$

The efficiency of the energy transfer is

$$\eta_{i \to o} = \frac{\mathcal{E}_{out}(T_d)}{\mathcal{E}_{in}(T_c)}$$

Therefore the input to output efficiency $\eta_{i\to o}$ is

$$\eta_{i\to o} = \frac{-\dfrac{b V_{in}^2 C_n}{\Gamma(2-n)}\left[\left(\dfrac{T_c}{1-(1-b)^{1/n}}\right)^{1-n} - T_c^{1-n} - (1-b)\left(\dfrac{(1-b)^{1/n}T_c}{1-(1-b)^{1/n}}\right)^{1-n}\right]}{\dfrac{V_{in}^2 C_n T_c^{1-n}}{\Gamma(2-n)}}$$

$$= (-b)\left[\frac{1-\left[1-(1-b)^{1/n}\right]^{1-n} - (1-b)(1-b)^{(1-n)/n}}{\left[1-(1-b)^{1/n}\right]^{1-n}}\right]$$

$$= (-b)\left[\frac{1-\left[1-(1-b)^{1/n}\right]^{1-n} - (1-b)^{1/n}}{\left[1-(1-b)^{1/n}\right]^{1-n}}\right]$$

$$= (-b)\left[\frac{\left[1-(1-b)^{1/n}\right] - \left[1-(1-b)^{1/n}\right]^{1-n}}{\left[1-(1-b)^{1/n}\right]^{1-n}}\right]$$

$$= (-b)\left[\frac{\left[1-(1-b)^{1/n}\right]\left[1-\left[1-(1-b)^{1/n}\right]^{-n}\right]}{\left[1-(1-b)^{1/n}\right]^{-n}\times\left[1-(1-b)^{1/n}\right]}\right]$$

$$= (-b)\left[\frac{1-\left[1-(1-b)^{1/n}\right]^{-n}}{\left[1-(1-b)^{1/n}\right]^{-n}}\right]$$

The efficiency of energy transfer $\eta_{i\to o}$ depends on the b $(0 \le b < 1)$; i.e the fraction of the reduced voltage applied during Pout phase from T_c to T_d time !! Here what is $\eta_{(i\to o)\,max}$ ie max efficiency at what "b"?

Calculating η_{max}!

$$\eta_{i\to o} = (-f)\left[\frac{1-[1-(1-f)^{1/n}]^{-n}}{[1-(1-f)^{1/n}]^{-n}}\right]$$

$$=(-f)\left[(1-(1-f)^{1/n})^{n}-1\right]$$

Put $1-f = x^n$, so $f=(1-x^n)$ call $\eta_{i\to o}=y$

so we have function

$$y=\left[-(1-x^n)\left[(1-x)^n-1\right]\right]$$

$$\frac{dy}{dx}=0$$

$$\frac{d}{dx}(1-x^n)\left[(1-x)^n-1\right]=0$$

$$(1-x^n)\left[-n(1-x)^{n-1}\right]+\left[(1-x)^n-1\right](-nx^{n-1})=0$$

$$-n(1-x)^{n-1}+nx^n(1-x)^{n-1}-nx^{n-1}(1-x)^n+nx^{n-1}=0$$

$$n(1-x)^n\left[-\frac{1}{(1-x)}+\frac{x^n}{(1-x)}-x^{n-1}\right]+nx^{n-1}=0$$

$$n(1-x)^n\left[\frac{x^n-x^{n-1}(1-x)-1}{(1-x)}\right]+nx^{n-1}=0$$

$$n(1-x)^{n-1}\left[x^n-x^{n-1}+x^n-1\right]+nx^{n-1}=0$$

$$(2x^n-x^{n-1}-1)=-\left(\frac{x}{1-x}\right)^{n-1}$$

The LHS & RHS are equal for $x=\frac{1}{2}$

so $x^n=\frac{1}{2^n}$ or $f=1-x^n=1-\frac{1}{2^n}$, is the value of fraction "f" at which η is maximum!

Putting $f = 1 - \frac{1}{2^n}$ in $T_d = \frac{T_c}{1-(1-f)^{1/n}}$, we get

$$T_d = \frac{T_c}{1-(1-f)^{1/n}} = \frac{T_c}{1-\left(1-1+\frac{1}{2^n}\right)^{1/n}}$$

$$= \frac{T_c}{1-\left(\frac{1}{2^n}\right)^{1/n}} = 2T_c$$

Therefore for maximum efficiency at $f = 1-\frac{1}{2^n}$ the $t = T_d$ should be $T_d = 2T_c$. The discharge time should be equal to charge period

Put $f = 1-\frac{1}{2^n}$, in the expression of $\eta_{i\to 0}$

$$\eta_{max} = (-f)\left[\left(1-(1-f)^{1/n}\right)^n - 1\right]$$

$$= \left(\frac{1}{2^n}-1\right)\left(\left[1-\left(1-1+\frac{1}{2^n}\right)^{1/n}\right]^n - 1\right)$$

$$= \left(\frac{1}{2^n}-1\right)\left(\left(1-\frac{1}{2}\right)^n - 1\right) = \left(\frac{1}{2^n}-1\right)\left(\frac{1}{2^n}-1\right)$$

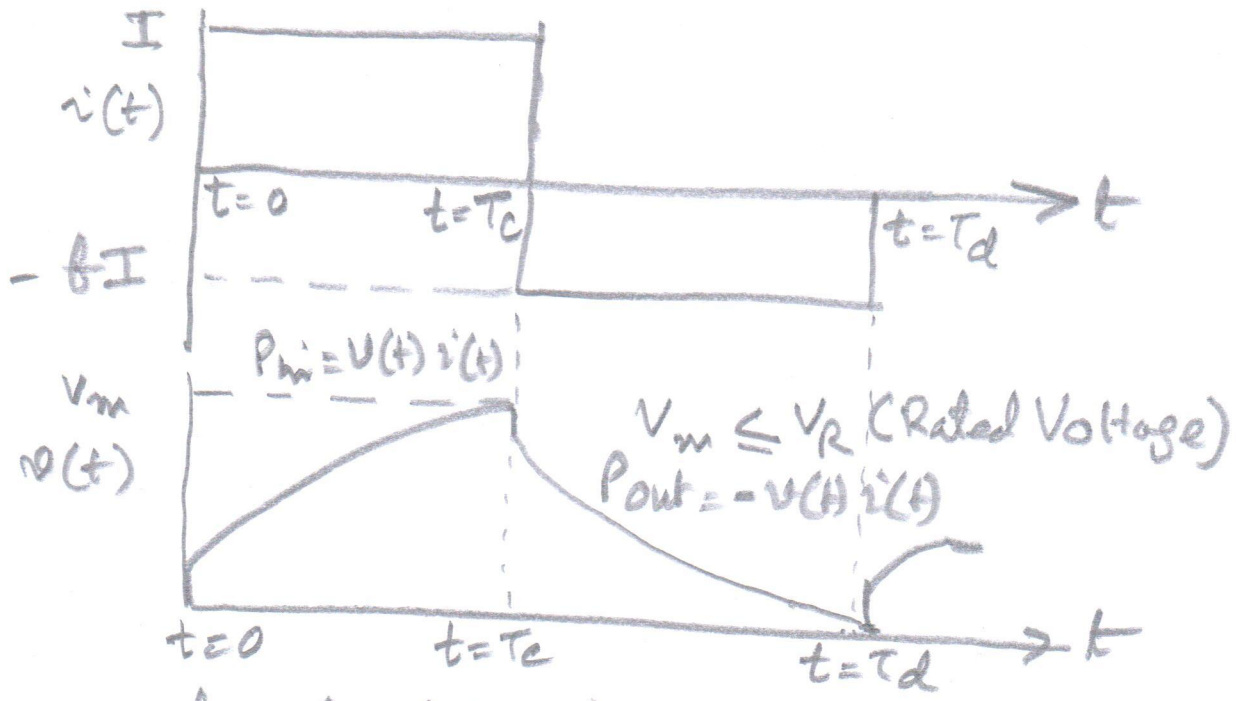
$$= \left(\frac{1-2^n}{2^n}\right)^2$$

For $n = 1/2$, we get

$$\eta_{max} = \left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)^2,$$

what we got for TL case was $\eta_{i\to 0} = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)^2$ which is also $n = 1/2$ an Half order system!

Like we excited charged by step voltage, now we charge the $Z(s) = \frac{1}{s^n C_n}$ by a step current I. This is constant current charging case. While discharge we used a reduced fraction of voltage (but of same polarity), here we would discharge by constant current by reversed current fI, where f is fraction $0 < f \leq 1$. With this constant current excitation & de-excitation we expect following voltage profile across $Z(s) = \frac{1}{s^n C_n}$



The equation for $i(t)$ in terms of step function $u(t)$, $u(t-T_c)$ and $u(t-T_d)$ is

$$i(t) = I\,u(t) - I\,u(t-T_c) - fI\,u(t-T_c) + fI\,u(t-T_d)$$

Taking Laplace of above & rearranging

$$I(s) = \frac{I}{s} - \frac{I}{s}(1+f)e^{-sT_c} + f\frac{I}{s}e^{-sT_d}$$

$$V(s) = Z(s)I(s) = \frac{1}{s^n C_n} I(s)$$

$$= \frac{I}{C_n s^{n+1}} - \frac{(b+1)I}{C_n s^{n+1}} e^{-sT_c} + \frac{bI}{C_n s^{n+1}} e^{-sT_d}$$

Take Laplace inverse

$$v(t) = \frac{I t^n}{C_n \Gamma(n+1)} u(t) - \frac{(b+1)I(t-T_c)^n}{C_n \Gamma(n+1)} u(t-T_c) + \frac{b I (t-T_d)^n}{C_n \Gamma(n+1)}$$

$$P_{in}(t) = v(t)\, i(t)\, u(t) \text{ for } 0 < t < T_c$$

$$= \frac{I^2 t^n}{C_n \Gamma(n+1)} u(t)$$

Total Energy transfered in to the $Z(s) = \frac{1}{s^n C_n}$ upto $t = T_c$

$$\mathcal{E}_{in}(T_c) = \int_0^{T_c} \frac{I^2 t^n}{C_n \Gamma(n+1)} dt = \frac{I^2 T_c^{n+1}}{C_n \Gamma(n+2)}$$

Energy transfered out in discharge period T_c to T_d is integration of P_{out} & P_{out} at any time t is:

$$P_{out}(t) = -v(t)\, i(t)\, u(t-T_c)$$

Note in this time T_c to T_d $i(t) = -bI$, ∴

$$P_{out} = -\left[\frac{I t^n}{C_n \Gamma(n+1)} - \frac{(1+b)I(t-T_c)^n}{C_n \Gamma(n+1)}\right]\left[-bI\right] u(t-T_c)$$

$$= \frac{b I^2 t^n}{C_n \Gamma(n+1)} u(t-T_c) - \frac{b I^2 (1+b)(t-T_c)^n}{C_n \Gamma(n+1)} u(t-T_c)$$

At $t = T_d$ i.e. we want $P_{out}(T_d) = 0$ or $v(T_d) = 0$ then

$$P_{out}(T_d) = 0 = \frac{b I^2 T_d^n}{C_n \Gamma(n+1)} - \frac{b I^2 (1+b)(T_d - T_c)^n}{C_n \Gamma(n+1)}$$

gives

$$T_d^n = (1+b)(T_d - T_c)^n$$

or

$$T_d = \left[\frac{(1+b)^{1/n}}{(1+b)^{1/n} - 1}\right] T_c$$

The total energy transferred out of $Z(s) = \frac{1}{s^n C_n}$ i.e. fractional capacitor is

$$\mathcal{E}_{out}(T_d) = \int_{T_c}^{T_d} \left[\frac{b I^2 t^n}{C_n \Gamma(n+1)} - \frac{b I^2 (1+b)(t - T_c)^n}{C_n \Gamma(n+1)}\right] dt$$

$$= \frac{b I^2}{C_n \Gamma(n+2)} \left[T_d^{n+1} - T_c^{n+1} - (1+b)(T_d - T_c)^{n+1}\right]$$

Put $T_d = \frac{(1+b)^{1/n} T_c}{(1+b)^{1/n} - 1}$ in above & we get

Energy recovered as

$$\mathcal{E}_{out}(T_d) = \frac{b I^2}{C_n \Gamma(n+2)} \left(\left[\frac{(1+b)^{1/n} T_c}{(1+b)^{1/n} - 1}\right]^{n+1} - T_c^{n+1} - (1+b)\left[\frac{(1+b)^{1/n} T_c}{(1+b)^{1/n} - 1} - T_c\right]^{n+1} \right)$$

The input to output energy transfer efficiency

$$\eta_{i\to o} = \frac{\mathcal{E}_{out}(T_d)}{\mathcal{E}_{in}(T_c)}$$

$$= \frac{\dfrac{bI^2}{C_n\Gamma(n+2)}\left(\left[\dfrac{(1+b)^{1/n}T_c}{(1+b)^{1/n}-1}\right]^{n+1} - T_c^{n+1} - (1+b)\left[\dfrac{T_c}{(1+b)^{1/n}-1}\right]^{n+1}\right)}{\dfrac{I^2 T_c^{n+1}}{C_n\Gamma(n+2)}}$$

$$= (b)\left(\frac{[(1+b)^{1/n}]^{n+1} - [(1+b)^{1/n}-1]^{n+1} - (1+b)}{[(1+b)^{1/n}-1]^{n+1}}\right)$$

$\eta_{i\to o}$ depends on "b" the fraction of discharge current

we get optimal efficiency by putting $\frac{d\eta_{i\to o}}{df} = 0$

at $b = 2^n - 1$ and $(\eta_{i-o})_{max} = (2^n-1)^2$

Putting $b = 2^n - 1$ in $T_d = \dfrac{(1+b)^{1/n}T_c}{(1+b)^{1/n}-1}$,

we get $T_d = 2T_c$

For $n = 1/2$ $\eta_{max} = (\sqrt{2}-1)^2 = 17.1\%$

We optimize now

$$y = x\left[\frac{[(1+x)^{1/n}]^{n+1} - [(1+x)^{1/n}-1]^{n+1} - (1+x)}{[(1+x)^{1/n}-1]^{n+1}}\right]$$

Maximization of

$$y = x\left[\frac{\left[(1+x)^{1/n}\right]^{n+1} - \left[(1+x)^{1/n}-1\right]^{n+1} - (1+x)}{\left[(1+x)^{1/n}-1\right]^{n+1}}\right]$$

Put $1+x = z^n$, $x = z^n - 1$, so we get y as

$$y = (z^n-1)\left[\frac{z^{n+1} - (z-1)^{n+1} - z^n}{(z-1)^{n+1}}\right]$$

$$= \frac{(z^n-1)z^{n+1} - (z^n-1)(z-1)^{n+1} - z^n(z^n-1)}{(z-1)^{n+1}}$$

$$= \frac{z^{2n+1} - z^{n+1} - (z^n-1)(z-1)^{n+1} - z^{2n} + z^n}{(z-1)^{n+1}}$$

$$= \frac{z^{2n}(z-1) - z^n(z-1) - (z^n-1)(z-1)^{n+1}}{(z-1)^{n+1}}$$

$$= \frac{z^{2n} - z^n}{(z-1)^n} - z^n + 1$$

$$\frac{dy}{dz} = \frac{1}{(z-1)^n}\left[\frac{d}{dz}(z^{2n} - z^n)\right] + (z^{2n} - z^n)\left[\frac{d}{dz}(z-1)^{-n}\right]$$

$$= \frac{1}{(z-1)^n}\left[2nz^{2n-1} - nz^{n-1}\right] - (z^{2n} - z^n)\, n(z-1)^{-(n+1)}$$

$$= \frac{2nz^{2n-1} - nz^{n-1}}{(z-1)^n} - \frac{n}{(z-1)^{n+1}}\left[z^{2n} - z^n\right] - nz^{n-1}$$

$$\frac{dy}{dz} = \frac{n}{(z-1)^n}\left[2z^{2n-1} - z^{n-1} - \frac{z^{2n} - z^n}{(z-1)}\right] - nz^{n-1}$$

$$= \frac{n}{(z-1)^n}\left[\frac{2z^{2n-1}(z-1) - z^{n-1}(z-1) - z^{2n} + z^n}{(z-1)}\right] - nz^{n-1}$$

$$= \frac{n}{(z-1)^{n+1}}\left[z^{2n} - 2z^{2n-1} + z^{n-1}\right] - nz^{n-1}$$

For $\frac{dy}{dz} = 0$

$$z^{2n} - 2z^{2n-1} + z^{n-1} = (z-1)^{n+1} z^{n-1}$$

$$\frac{z^{2n} - 2z^{2n-1} + z^{n-1}}{z^{n-1}} = (z-1)^{n+1}$$

$$z^{n+1} - 2z^n + 1 = (z-1)^{n+1}$$

$$z^n(z-2) = (z-1)^{n+1} - 1$$

LHS & RHS are equal and zero for $z = 2$

$\therefore$ $z^n = 1 + x$ has max value at

$z = 2$ or $x = 2^n - 1$

For 2^- the $\frac{dy}{dz} > 0$ and for 2^+ $\frac{dy}{dz} < 0$ so the

y is max at $z = 2$ $(x = 2^n - 1)$

The value of y_{max} is

$$y_{max} = (2^n - 1)\left[\frac{2^{n+1} - (2-1)^{n+1} - 2^n}{(2-1)^{n+1}}\right]$$

$$= (2^n - 1)\left[2^{n+1} - 2^n - 1\right]$$

$$= (2^n - 1)\left[2^n(2-1) - 1\right] = (2^n - 1)(2^n - 1)$$

$$= (2^n - 1)^2$$

Got by putting $z = 2$ in $y = \dfrac{(z^n - 1)\left[z^{n+1} - (z-1)^{n+1} - z^n\right]}{(z-1)^{n+1}}$

By above derivation we see

$$\eta = \frac{\mathcal{E}_{out}}{\mathcal{E}_{in}} = f\left[\frac{\left[(1+f)^{1/n}\right]^{n+1} - \left[(1+f)^{1/n} - 1\right]^{n+1} - (1+f)}{\left[(1+f)^{1/n} - 1\right]^{n+1}}\right]$$

attains maximum at $f = 2^n - 1$, gives optimal switching time for pure loss element

$$T_d = \frac{(1+f)^{1/n}\, T_c}{(1+f)^{1/n} - 1} = 2T_c$$

& maximum efficiency is

$$\eta_{max} = \left[2^n - 1\right]^2$$

We have analysed constant current charge discharge case.

We take re trnmissim line and excite now by a step input constant current $I_{in}\, u(t)$ at $x=0$. We have seen earlier that, in Laplace domain the solution of diffusion equation of TL is

$$V(x,s) = e^{-(\sqrt{rcs})x}\, V(0,s),$$ here we put

$$Z(0,s) = \frac{V(0,s)}{I(0,s)} = \sqrt{\frac{r}{cs}}$$ and write

$$V(x,s) = e^{-(\sqrt{rcs})x}\sqrt{\frac{r}{cs}}\; I(0,s)$$

with $I(0,s) = \frac{I_{in}}{s}$, we get

$$V(x,s) = \sqrt{\frac{r}{c}}\,\frac{e^{-(\sqrt{rcs})x}}{\sqrt{s}}\,\frac{I_{in}}{s},$$ Apply inv. Laplace

$$v(x,t) = I_{in}\sqrt{\frac{r}{c}}\left[2\sqrt{\frac{t}{\pi}}\,e^{-\frac{rcx^2}{4t}} - (\sqrt{rc})x\,\mathrm{erfc}\left(\frac{\sqrt{rc}\,x}{2\sqrt{t}}\right)\right]$$

Power into TL at $x=0$ $P_{in}(t) = v(0,t)\,i(0,t)$

$$P_{in}(t)\Big|_{x=0} = 2I_{in}^2\sqrt{\frac{rt}{c\pi}}$$

The energy Xfered at $x=0$

$$\mathcal{E}_{in}(t) = \int_0^t 2I_{in}^2\sqrt{\frac{r\tau}{c\pi}}\,d\tau = \frac{4}{3}I_{in}^2\sqrt{\frac{rt^3}{c\pi}}$$

The general expression for stored energy at any time

$$\mathcal{E}_{st}(t) = \frac{1}{2}\int_0^\infty c(x)\,v^2(x,t)\,dx = \frac{c}{2}\int_0^\infty v^2(x,t)\,dt$$

$$= \frac{I_{in}^2 r}{2}\left[\int_0^\infty\left[2\sqrt{\frac{t}{\pi}}\,e^{-\frac{rcx^2}{4t}} - (\sqrt{rc})x\,\mathrm{erfc}\left(\frac{\sqrt{rc}\,x}{2\sqrt{2}}\right)\right]^2 dx\right.$$

Numerical integration to be employed & then $\eta_{i\to s} = \frac{\mathcal{E}_{st}(t)}{\mathcal{E}_{in}(t)}$ to be evaluated & then $\eta_{i\to o} = (\eta_{i\to s})^2$ to be seen $\sim(\sqrt{2}-1)^2$