Why Fractional calculus approached solutions for Diffusion Problems ?

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Brief Overview of the Subject Thematic

The transport equation assumed as Fractional Differential equation

$$\frac{\partial^{\alpha}(\theta C)}{\partial t^{\alpha}} = \nabla \cdot \theta D \cdot \nabla C - \nabla \cdot \theta \cdot vC + SS,\ 0 < \alpha \leq 1,$$

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \qquad \mathrm{D} = \begin{bmatrix} \mathrm{D}_{xx} & \mathrm{D}_{xy} & \mathrm{D}_{xz} \\ \mathrm{D}_{yx} & \mathrm{D}_{yz} & \mathrm{D}_{yz} \\ \mathrm{D}_{zx} & \mathrm{D}_{zy} & \mathrm{D}_{zz} \end{bmatrix}$$

To be solved in 2D is the objective

The 'local theory' Fickian -Advection Diffusion Equation ADE

$$\frac{\partial}{\partial t} C = \nabla \bullet (\mathbb{D}\, \nabla C - v\, C) = \nabla \bullet \mathbf{J}$$

C : **is solute concentration** v : **local velocity tensor** $\mathbb{D}$: **local dispersion tensor**

To change to 'non-local theory' Fickian Advection Diffusion by fractional calculus

Applicable physical laws in this particular study

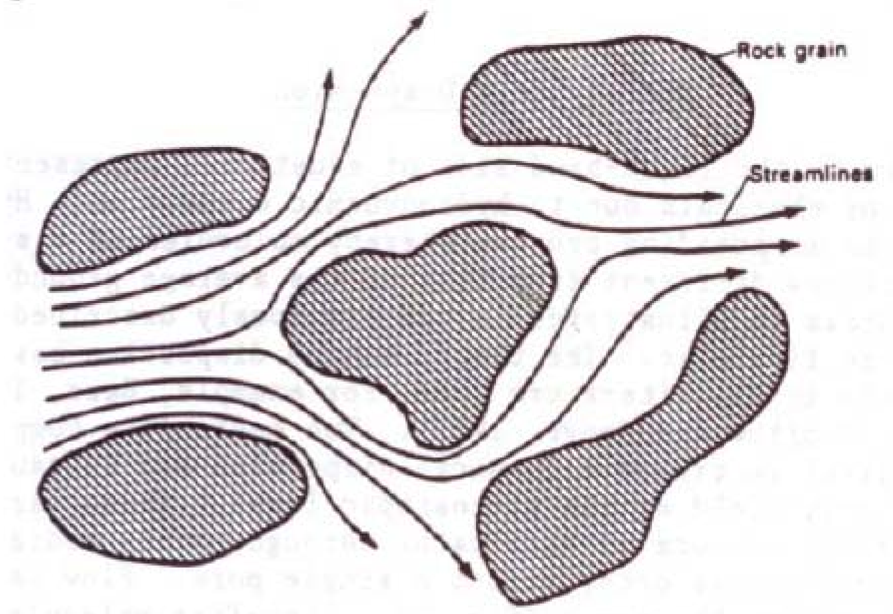
In this study the Darcy mechanism is not considered, so the pressure head's gradient is not considered while study of contamination spreading

Only restricting to Advective Diffusion Equation ADE

So it is anomalous diffusion studies ADE modified to fractional ADE with non-local properties

Transport in Disordered Media

True that 'spatial heterogeneity', roughness gives fractional order time derivative in the transport phenomena.



$$\frac{\partial}{\partial t}\langle c(x,t)\rangle - \mathbb{D}_0 \frac{\partial^2}{\partial x^2}\langle c(x,t)\rangle$$
$$= \frac{32\pi a^3 l \mathbb{D}_0^{1/2}}{L^4}\frac{\partial^{1/2}}{\partial t^{1/2}}\langle c(x,t)\rangle - \frac{160\pi l^3 a^3}{\mathbb{D}_0^{1/2} L^6}\frac{\partial^{3/2}}{\partial t^{3/2}}\langle c(x,t)\rangle + \frac{32\pi a^3 l^5}{\mathbb{D}_0^{3/2} L^8}\frac{\partial^{5/2}}{\partial t^{5/2}}\langle c(x,t)\rangle$$

advection not included here

But Why ?

If disorder is taken as a field with spatial gradient and we evolve basic average transport mechanism via 'perturbative technique' where the 'disorder free Green's function' propagates through several realizations of spatial disorder via fixed point equation resulting in fractional derivative in time!

Also if the spatial heterogeneity, is embedded as stochastic process, and the averaging procedures do give fractional derivative in time

This is mathematically speaking…..

"Evolution of Temporal Fractional Derivative due to Spatial Stochastic Disorder in Transport Phenomena", ***International Journal of Mathematics and Computation (IJMC), Vol. 17, Issue-4, pp. 1-20***

"Formation of Fractional Derivative in Time due to Propagation of Free Greens Function in Spatial Stochastic Disorder Field for Transport Phenomena", ***International Journal of Mathematics and Computation Vol. 17, Issue-4, pp 68-92***

The fractional time derivative -in view of Random Walk

$$\frac{\partial}{\partial t}c(x,t)=\frac{\partial}{\partial x}\left[A(x)c(x,t)\right]+\frac{\partial^2}{\partial x^2}\left[B(x)c(x,t)\right]$$

$$\frac{\partial c(x,t)}{\partial t}=-v\frac{\partial c(x,t)}{\partial x}+\mathbb{D}\frac{\partial^2 c(x,t)}{\partial x^2}$$

ADE Fickian Transport

$$\frac{\partial^\alpha}{\partial t^\alpha}c(x,t)=\frac{\partial}{\partial x}\left[A(x)c(x,t)\right]+\frac{\partial^2}{\partial x^2}\left[B(x)c(x,t)\right];\qquad 0<\alpha\le 1$$

$$\frac{\partial^\alpha c(x,t)}{\partial t^\alpha}=-v\frac{\partial c(x,t)}{\partial x}+\mathbb{D}\frac{\partial^2 c(x,t)}{\partial x^2}$$

The particles undergoing random walk thus evolving as diffusion equation, having <u>‘no-average wait time’</u> between collisions show the fractional time derivative.

The process is <u>non-Markovian</u>…….having temporal memory!!

Functional Fractional Calculus 2nd edition; Springer Verlag

Can the spatial derivative be fractionized?

$$\frac{\partial}{\partial t}c(x,t)=\frac{\partial}{\partial x}\left[A(x)c(x,t)\right]+\frac{\partial^{\beta}}{\partial x^{\beta}}\left[B(x)c(x,t)\right] \qquad \beta<2$$

-in view of Random Walk the diffusing species having infinite moment of ‘jump-lengths’; deviation from standard Brownian Motion (BM); will manifest as fractional spatial derivative operator.

The transport with ‘spatial memory’!!

Solutes that move through aquifers do not generally follow a Fickian second order governing equation because of large deviation from stochastic Brownian motion.

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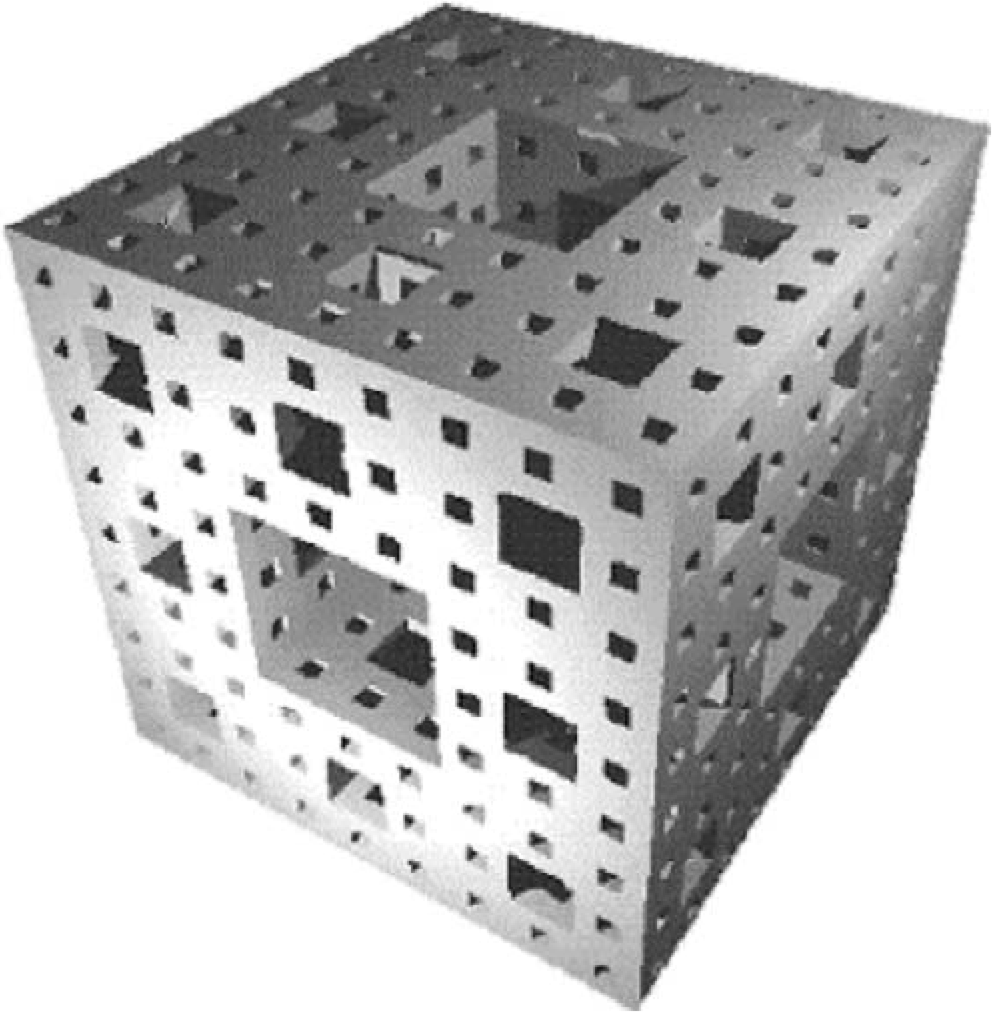
The porous 3D media



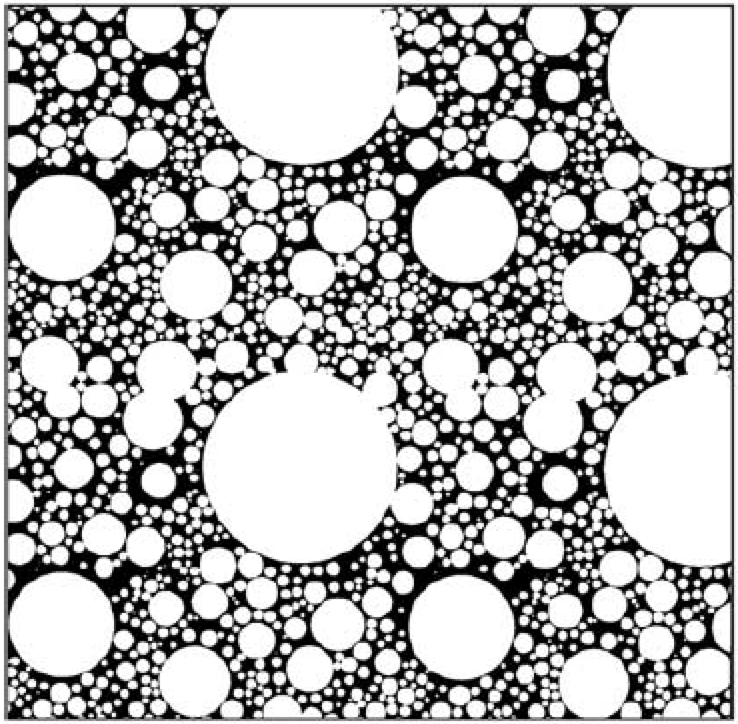
The support media is non Euclidian

$$d_B = 2 + \beta \neq 3$$

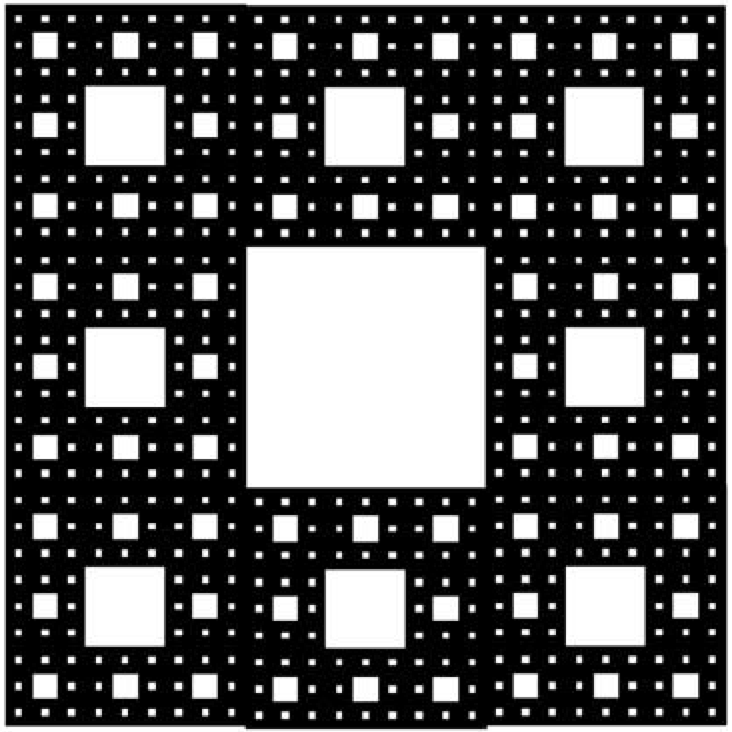
$$0 < \beta < 1$$

The Mengers sponge the archetype of fractal porous media with fractal dimension 2.727

The 2D and 1D support of porous media



Stochastic (Fuller mix)



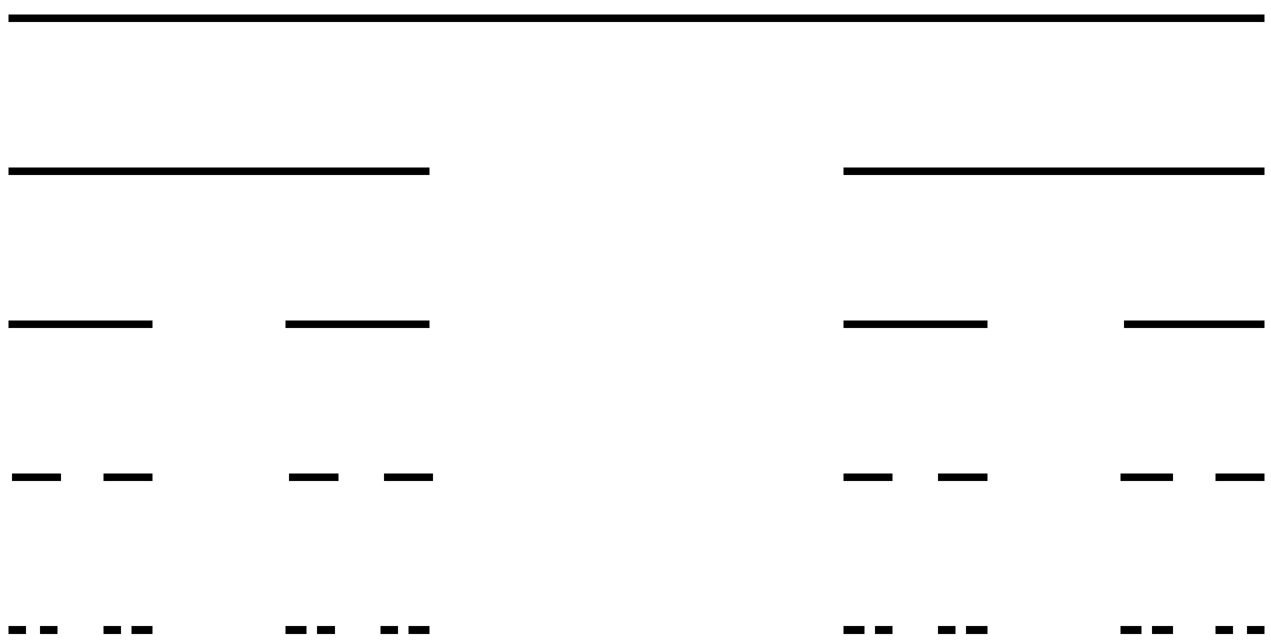
Deterministic (Sierpinski carpet)

$$d_B = 1 + \beta \neq 2$$

$$d_B = \beta \neq 1$$

The Cantor set

The transport happening on fractal a subset of real line!!



Actually the 'support' of space and time are non continuous; and with no cut off the set is virtually empty, having rare points where transport takes and then stops and so on………..requires a different calculus "calculus on subset of real line" ; a very new modern approach……requires fractal functions continuous but no where differentiable.

We continue with first non local method with

Classical Fractional Calculus

then we revert back to method requiring

Calculus on subset of real line

Using non-local

Fractional Calculus

The basic definition of divergence need for fractional divergence

Is it correct in reality where the world is heterogeneous?

$$\mathrm{div}\,\mathbf{J} \triangleq \lim_{V\to 0} \frac{1}{V} \oint_S \mathbf{J} \bullet \mathbf{n}\, \mathrm{d}S \equiv \nabla \bullet \mathbf{J}$$

Physically it implies that divergence is defined as ratio of 'total flux' through a closed surface to the volume enclosed by the surface: "when volume shrinks to ZERO".

This is true only if the flux is indeed a "point" vector quantity relative to the scale of observation; then the limit exists and operator reduces to familiar dot product. (heat flow in homogeneous material). If the support is homogeneous (say with out pores) we have no velocity fluctuations-the transport of flux is thus only due to 'concentration gradient'.

Since velocity itself is variable function of space, as the volume shrinks to zero, the velocity fluctuations and thus the dispersive flux disappears. Therefore the true divergence of the macroscopic solute flux cannot contain a macroscopic dispersive term (due to wide variability in velocity)

$$\mathrm{div}^{\beta}\,\mathbf{J} \triangleq \lim_{V\to REV} \frac{1}{V} \oint_S \mathbf{J} \bullet \mathbf{n}\, \mathrm{d}S \equiv \nabla^{\beta} \bullet \mathbf{J}$$

Functional Fractional Calculus 2nd edition Springer Verlag

Fractional divergence for neutron flux profile in nuclear reactor", *Int. J. Nuclear Energy Science & Technology Vol.3 No.2 pp139-159*

Fractional Divergence need for spatial fractional derivative

$$\mathrm{div}^{\beta}\mathbf{J} \triangleq \lim_{V \to REV} \frac{1}{V} \oint_{S} \mathbf{J} \bullet \mathbf{n}\, \mathrm{d}S \equiv \nabla^{\beta} \bullet \mathbf{J}$$

Making flux not a point quantity as the limit goes to REV instead of ZERO (Representative Elementary Volume)

The divergence is now a non-local operator in space

Requires a spatial memory

This is another reason of taking spatial fractional derivative, apart from a case of diffusing particles having ‘unrestricted jump length distributions’-Levy flight statistics

Conservation of probability to FPE

At any particular time the sum of the probabilities at all locations must be equal to unity So if the probability changes in one location from one moment to next, the probability in the vicinity must also change to conserve the probability.

This is basis of Fokker-Plank Equation (FPE) describing the change of probability of a random function in space-time so naturally it describes solute transport

Derivation of FPE starts with a simple mathematical statement of how a random measure changes state from one moment to next after some event has occurred. In this case we are interested in the probability that a particle has moved from location x_0 to x_2 in time t_0 to t_2. That is $p(x_2 - x_0; t_2 - t_0)$. Particle must have moved through intermediate point x_1, thus summing over all such intermediate x_1, we have.

$$p(x_2 - x_0; t_2 - t_0) = \int p(x_2 - x_1; t_2 - t_1)\, p(x_1 - x_0; t_2 - t_0)\, \mathrm{d}x_1$$

$$p(x - x_0; t + \Delta t) = \int p(x - \zeta; \Delta t)\, p(\zeta - x_0; t - t_0)\, \mathrm{d}\zeta$$

$$\frac{\partial p(x,t)}{\partial t} \equiv \frac{\partial p(x - x_0; t)}{\partial t} = \lim_{\Delta t \to 0} \frac{\left[p(x - x_0; t + \Delta t) - p(x - x_0; t)\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int p(x - \zeta; \Delta t)\, p(\zeta, t)\, d\zeta - p(x,t)\right)$$

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The moments from FPE

The instantaneous transition pdf has following limit, meaning that as the transit time tends to zero the probability that particle does not move goes to unity.

$$\lim_{\Delta t \to 0} p(x_2 - \zeta; \Delta t) = \delta(x - \zeta)$$

$$\lim_{\Delta t \to 0} \Im\left[p(x - \zeta; \Delta t)\right] = \lim_{\Delta t \to 0} \hat{p}(k; \Delta t) = 1$$

The above density has all positive moments zero.
When $\Delta t \neq 0$ **the pdf has higher order moments. The first moment is defined as the expected value of particles new position minus initial position**

$$A(\Delta t) = \int (x - \zeta)\, p(x - \zeta; \Delta t)\,\mathrm{d}x$$

The second moment of many power law pdf (also Levy pdf) is diverging (∞ **); thus we choose a coefficient** $B(\Delta t)$ **that is a measure of spread of pdf 'similar' to second moment of a Gaussian pdf. A very general pdf with finite or infinite variance has thus Fourier Transform as:**

$$\hat{p}(k; \Delta t) = 1 - A(\Delta t)(\mathrm{i}k) + \frac{1}{2} B(\Delta t)(1 + \gamma)(\mathrm{i}k)^{\beta} + \frac{1}{2} B(\Delta t)(1 - \gamma)(-\mathrm{i}k)^{\beta} + \mathcal{O}(\Delta t)$$

forward vs backward weight is γ $\quad -1 \le \gamma \le 1$

scaling exponent in 1-D space is β $\quad 1 < \beta \le 2$

Recovering Fickian ADE

$$\hat{p}(k;\Delta t) = 1 - A(\Delta t)(ik) + \frac{1}{2}B(\Delta t)(1+\gamma)(ik)^{\beta} + \frac{1}{2}B(\Delta t)(1-\gamma)(-ik)^{\beta} + \mathcal{O}(\Delta t)$$

forward vs backward weight is γ $\quad -1 \le \gamma \le 1$

scaling exponent in 1-D space is β $\quad 1 < \beta \le 2$

Finite variance density requires $\beta = 2$

For $\beta = 2$; and $\gamma = 0$

$$\hat{p}(k;\Delta t) = 1 - A(\Delta t)(ik) + B(\Delta t)(ik)^2 + \mathcal{O}(\Delta t)$$

Where $2B(\Delta t)$ **is equal to the second moment of the particle excursion. This special case instantaneous transition density has inverse transform**

$$p(x-\zeta;\Delta t) = \delta(x-\zeta) - A(\Delta t)\delta'(x-\zeta) + B(\Delta t)\delta''(x-\zeta) + \mathcal{O}(\Delta t)$$

substituting above in $\frac{\partial p(x,t)}{\partial t} = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int p(x-\zeta;\Delta t)\,p(\zeta,t)d\zeta - p(x,t)\right)$ **we get**

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int p(x-\zeta;\Delta t)\,p(\zeta,t)\mathrm{d}\zeta - p(x,t)\right)\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\left[\int \delta(x-\zeta)\,p(\zeta,t)\mathrm{d}\zeta - p(x,t)\right] - \int A(\Delta t)\delta'(x-\zeta)\,p(x,t)\mathrm{d}\zeta + \int B(\Delta t)\delta''(x-\zeta)\,p(x,t)\mathrm{d}\zeta\right)\\
&- \lim_{\Delta t\to 0}\frac{A(\Delta t)}{\Delta t}\frac{\partial}{\partial x}p(x,t) + \lim_{\Delta t\to 0}\frac{B(\Delta t)}{\Delta t}\frac{\partial^2}{\partial x^2}p(x,t) = -v\frac{\partial}{\partial x}p(x,t) + \mathbb{D}\frac{\partial^2}{\partial x^2}p(x,t)
\end{aligned}$$

This is BM and its FPE with assumption that for small Δt**, *A* and *B* are linear with transit time** Δt

Recovering non-Fickian ADE

$$\hat{p}(k;\Delta t) = 1 - A(\Delta t)(\mathrm{i}k) + \frac{1}{2}B(\Delta t)(1+\gamma)(\mathrm{i}k)^{\beta} + \frac{1}{2}B(\Delta t)(1-\gamma)(-\mathrm{i}k)^{\beta} + \mathcal{O}(\Delta t)$$

forward vs backward weight is γ $\quad -1 \le \gamma \le 1$

scaling exponent in 1-D space is β $\quad 1 < \beta \le 2$

$$p(x-\zeta;\Delta t) = \delta(x-\zeta) - A(\Delta t)\delta'(x-\zeta) + \frac{1}{2}(1+\gamma)B(\Delta t)D_{+}^{\beta}\delta(x-\zeta) + \frac{1}{2}(1-\gamma)B(\Delta t)D_{-}^{\beta}\delta(x-\zeta) + \mathcal{O}(\Delta t)$$

Use the property of delta function i.e. $\int_b^d \delta(x-c)f(x)\mathrm{d}x = f(c)$ **to get fractional derivatives as**

$$D_{+}^{\beta}\delta(x-\zeta) = \frac{1}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{-\infty}^{x}\delta(\xi-\zeta)(x-\zeta)^{n-\beta-1}\mathrm{d}\xi = \frac{1}{\Gamma(-\beta)}\begin{cases} 0 & x<\zeta \\ (x-\zeta)^{-\beta-1} & x \ge \zeta \end{cases}$$

$$D_{-}^{\beta}\delta(x-\zeta) = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{\infty}\delta(\xi-\zeta)(\xi-x)^{n-\beta-1}\mathrm{d}\xi = \frac{1}{\Gamma(-\beta)}\begin{cases} (\zeta-x)^{-\beta-1} & x \le \zeta \\ 0 & x>\zeta \end{cases}$$

Showing power function density is $\sim\left(|x-\zeta|\right)^{-1-\beta}$

For $\beta < 2$ **i.e. for power law distributions we get by Fourier inverting the above FT equation following similar procedure as for recovering Fickian ADE**

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}vp + \left(\frac{1}{2}+\frac{\gamma}{2}\right)\frac{\partial^{\beta}}{\partial x^{\beta}}\mathbb{D}p + \left(\frac{1}{2}-\frac{\gamma}{2}\right)\frac{\partial^{\beta}}{\partial(-x)^{\beta}}\mathbb{D}p$$

For large number of independent solute particles the $p(x,t)$ **is replaced by** $c(x,t)$ **i.e. concentration**

The fractional ADE

$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x} v c(x,t) + \left(\frac{1}{2} + \frac{\gamma}{2}\right) \frac{\partial^{\beta}}{\partial x^{\beta}} \mathbb{D} c(x,t) + \left(\frac{1}{2} - \frac{\gamma}{2}\right) \frac{\partial^{\beta}}{\partial (-x)^{\beta}} \mathbb{D} c(x,t)$$

For symmetric transitions; $\gamma = 0$ **and defining symmetric operator as**

$$2 \nabla^{\beta} = D_{+}^{\beta} + D_{-}^{\beta}$$

We get the description "fractional ADE" with understanding that only the dispersion term is described by fractional derivatives

$$\frac{\partial}{\partial t} c = -v . \nabla c + \mathbb{D} \nabla^{\alpha} c$$

Fokker Plank Kolmogorov Equation

Basis from random walker theory and with MARKOVIAN model we get, the evolution of probability density as:

$$\frac{\partial}{\partial t} p(x,t) = \frac{\partial}{\partial x}\left[A(x)\, p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[B(x)\, p(x,t)\right]$$

$A(x)$ **is the drift (Advection)** $B(x)$ **is the diffusion function**

For a case of no advection, for force/potential free case and with constant *B*(*x*)=B we have for initial delta function $p(x,0) = \delta(x)$

A Gaussian evolution

$$p(x,t) = \frac{1}{\sqrt{4\pi \mathrm{B}\, t}} e^{-\left(\frac{x^2}{4\mathrm{B}\, t}\right)}$$

A Fractional Fokker Plank Kolmogorov Equation (without Advection)

$$\frac{\partial^\alpha p(x,t)}{\partial t^\alpha} = \frac{\partial^\beta}{\partial x^\beta}\left[\mathrm{B}\, p(x,t)\right]$$

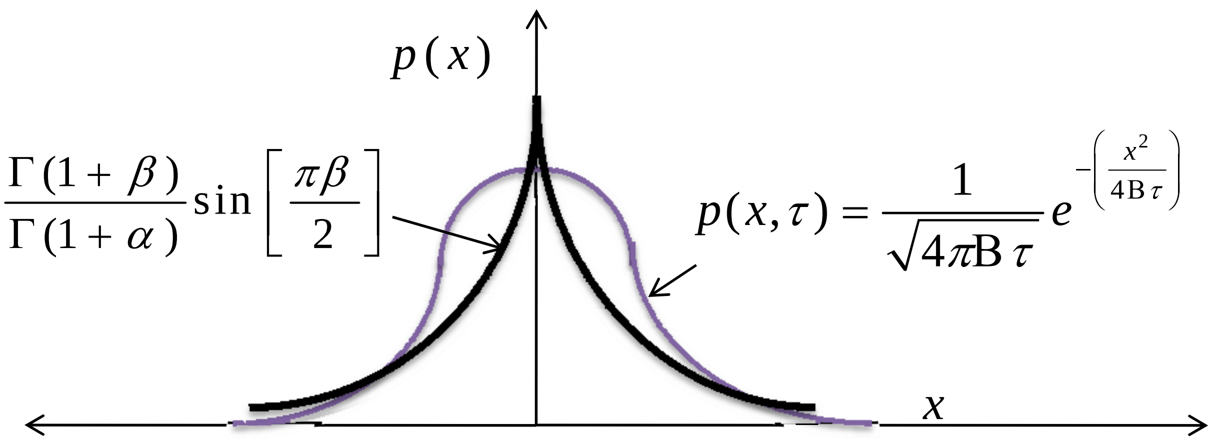
Has Levy stable asymptotic power-law evolution

$$p(x,t) \approx \frac{1}{\pi} \frac{t^{-\alpha}}{x^{\beta+1}} \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)} \sin\left[\frac{\pi\beta}{2}\right]$$

with $\mathrm{B} = 1$

Fokker Plank Kolmogorov Equation & Fractional FPKE

$$p(x,\tau) \approx \frac{1}{\pi}\frac{\tau^{-\alpha}}{x^{\beta+1}}\frac{\Gamma(1+\beta)}{\Gamma(1+\alpha)}\sin\left[\frac{\pi\beta}{2}\right]$$



$$p(x,\tau) = \frac{1}{\sqrt{4\pi B\tau}} e^{-\left(\frac{x^2}{4B\tau}\right)}$$

The Dirac's delta at time zero plumes out at any time τ and evolves. In case of integer order FPKE the shape is dome shaped, where as for fractional order FPKE the cusp shape evolves. Well at very initial stage say at time $\tau = 0.001s$ in integer order cases there exists a large quantity at fairly large distance from origin-stating that the phase velocity requirement is very very large-infinite.

This is due to ideal Fickian case, was modified in 1948 by Catteneo, by having relaxation time constant in the Fick's second law, and thus obtaining telegrapher's equation giving finite phase velocity.

Functional Fractional Calculus 2nd edition; Springer Verlag

Phase Table for fractional ADE

$$\frac{\partial}{\partial t} c(x,t) = {}_0D_t^{1-\lambda}(\mathbb{D}_{\lambda,\beta}) \frac{\partial^{\beta}}{\partial x^{\beta}} c(x,t)$$

We are used to $\lambda = 1, \beta = 2$ **The fractional order comes as observation of asymptotic behavior in space time relaxation.**

Temporal Fractional Order λ	Spatial Fractional Order β	Type of Walk	Average Waiting Time T	Jump-Length Variance σ^2	Nature of Diffusion
$0 < \lambda < 1$	$0 < \beta < 2$	Long-Jump	∞	∞	Non-Markovian
$\lambda \geq 1$	$0 < \beta < 2$	Long-Jump	$< \infty$	∞	Markovian
$0 < \lambda < 1$	$\beta \geq 2$	Sub-diffusion	∞	$< \infty$	Non-Markovian
$\lambda \geq 1$	$\beta \geq 2$	Brownian	$< \infty$	$< \infty$	Markovian

Functional Fractional Calculus 2nd edition; Springer Verlag

Using local

Fractional Calculus

Modern treatment by LFD and calculus on subset of Real line to get FPE

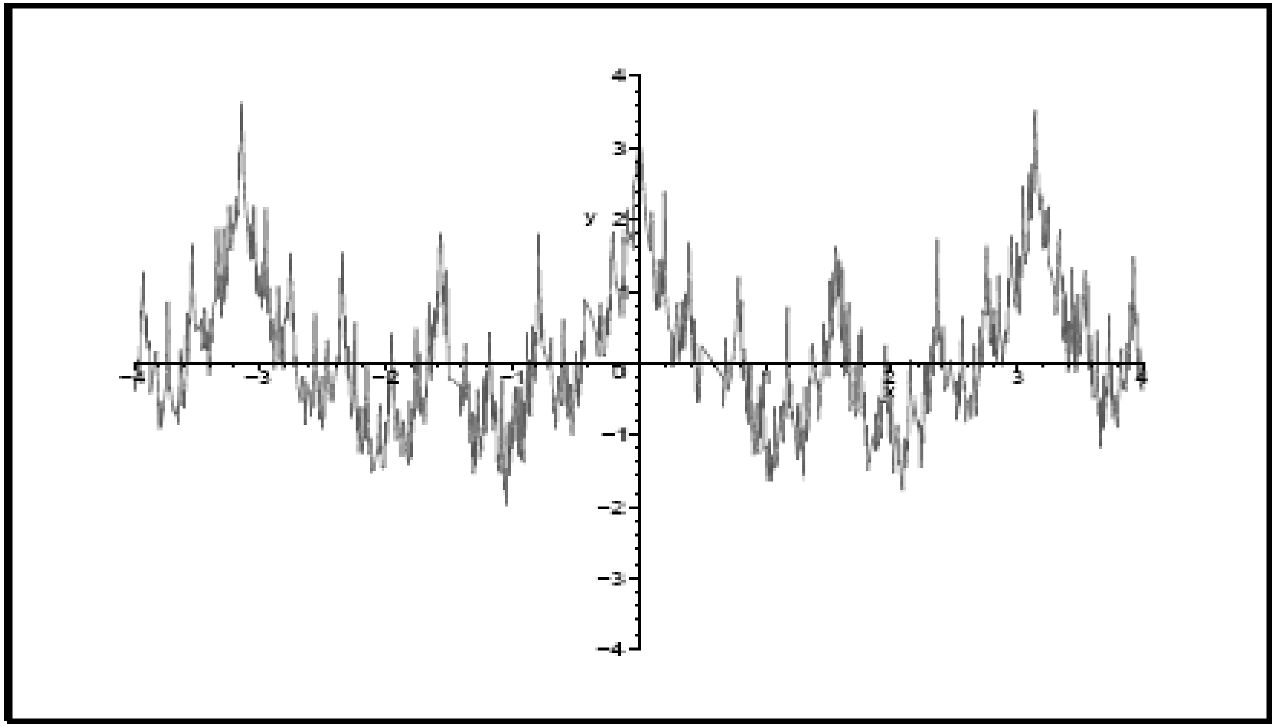
K. M. Kolwankar and A. D. Gangal, “ Fractional differentiability of nowhere differentiable functions”, Chaos 6, (4), (1996) 505-513.
K. M. Kolwankar and A. D. Gangal, “Holder exponents of irregular signals and local fractional derivatives”, Pramana J Phys. 48 (1997) 49-68.
K. M. Kolwankar and A. D. Gangal, “ Local fractional calculus: a calculus for fractals space-time”, In: Proceedings of Fractals: Theory and Applications in Engineering, Springer, pp 171-178, 1999
K. M. Kolwankar and A. D. Gangal, “ Local Fractional Fokker-Planck Equation”, Phys. Rev. Lett, 80 (1998), 214-217
A. Parvate and A. D. Gangal, “ Fractal differential equations and fractal-time dynamical systems”, Pramana J Phys. 64 (2005) 389-409.
A. Parvate and A. D. Gangal, “Calculus on Fractal subset of real line-I: Formulations, Fractals 17 (1) (2009) 53-81.
X. J. Yang and Z.X. Kang, C. H. Liu, “Local fractional Fourier’s transform based on local fractional calculus”, In: Proc. of the 2010 Int. Conf. on Electrical and Controls Engineering (ICECE2010) pp. 1242-1245, 2010
“Functional Fractional Calculus”, 2nd edition Springer-Verlag, Germany 2011.

Local fractional derivatives LFD

Suitable to deal with phenomena taking place in fractal space & time

Suitable in study of local scaling behavior

Suitable to study continuous but no where differentiable processes



In earlier non-local treatment we assumed differentiability of $p\ (x;\ t)$; but if that is not the case then we need to have this method by LFD

LFD defined

$$\mathbf{D}^{q} f(x_0) = \lim_{x\to x_0} \frac{\mathrm{d}^{q}\left[f(x) - f(x_0)\right]}{\left[\mathrm{d}(x-x_0)\right]^{q}} \qquad 0 < q \le 1$$

where $\frac{\mathrm{d}^q f(x)}{\left[\mathrm{d}(x-x_0)\right]^q} = \frac{1}{\Gamma(1-q)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x_0}^{x}\frac{f(y)}{(x-y)^q}\mathrm{d}y$ **is standard Riemann-Liouvelli (RL) derivative**

RL is non-local is made to local operator via the limit; well if limit exits then it is LFD at a point

Fractional Taylor expansion as the coefficient of the power with fractional exponent

$$f(x) = f(x_0) + \frac{\mathbf{D}^{q}_{\pm} f(x_0)}{\Gamma(q+1)}(\pm[x-x_0])^q + \sum_{n=1}^{N}\frac{f^{(n)}}{n!}(x-x_0)^n + \mathcal{R}_q(x_0;[x-x_0])$$

For $0<\alpha<1$ $\quad f(x+h) \approx f(x) + \frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)} \qquad f^{(\alpha)}(x) \triangleq \Gamma(1+\alpha)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h^\alpha}$

For non differentiable f **fractional derivative could give non-linear power law. The** f **is non differentiable as** $\mathrm{d}f \approx \mathrm{d}x^\alpha$ **, so classical derivative is** $\mathrm{d}f/\mathrm{d}x = \infty$ **diverging**

Define $f^{(-\alpha)} \triangleq g(x)$ **then** $\quad g^{(\alpha)}(x) = \Gamma(1+\alpha)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h^\alpha} = f(x)$

Local fractional integration is thus $g(x+h) - g(x) = \int_{x}^{x+h} f(x')\mathrm{d}^\alpha x' = \lim_{h\to 0}\frac{f(x)}{\Gamma(1+\alpha)}h^\alpha$

Local FDE & calculus on subset of Real line

$$\mathbf{D}_x^q f(x) = g(x)$$

$\mathbf{D}_x^q f(x) = \mathrm{C}$ **Does not have finite solution when** $0 < q < 1$

Solution can exist when $g(x)$ **has fractal support**

For instant when $g(x) = \chi_C(x)$ **, membership function of Cantor set {*C*}**

$$g(x) = \chi_C(x) \triangleq \begin{cases} g(x) = 1; & x \in \{C\} \\ g(x) = 0 & \text{otherwise} \end{cases} = F_C^i = \begin{cases} 1 \quad [x_{i+1} - x_i] & \text{contain a point on } \{C\} \\ 0 & \text{otherwise} \end{cases}$$

Solution with initial condition $f(0) = 0$ **if** $q = \alpha = \dim_H \{C\}$

Explicitly we have integration on subset of real line (fractal) as generalization of Riemann integration procedure; call it F^α**-integration**

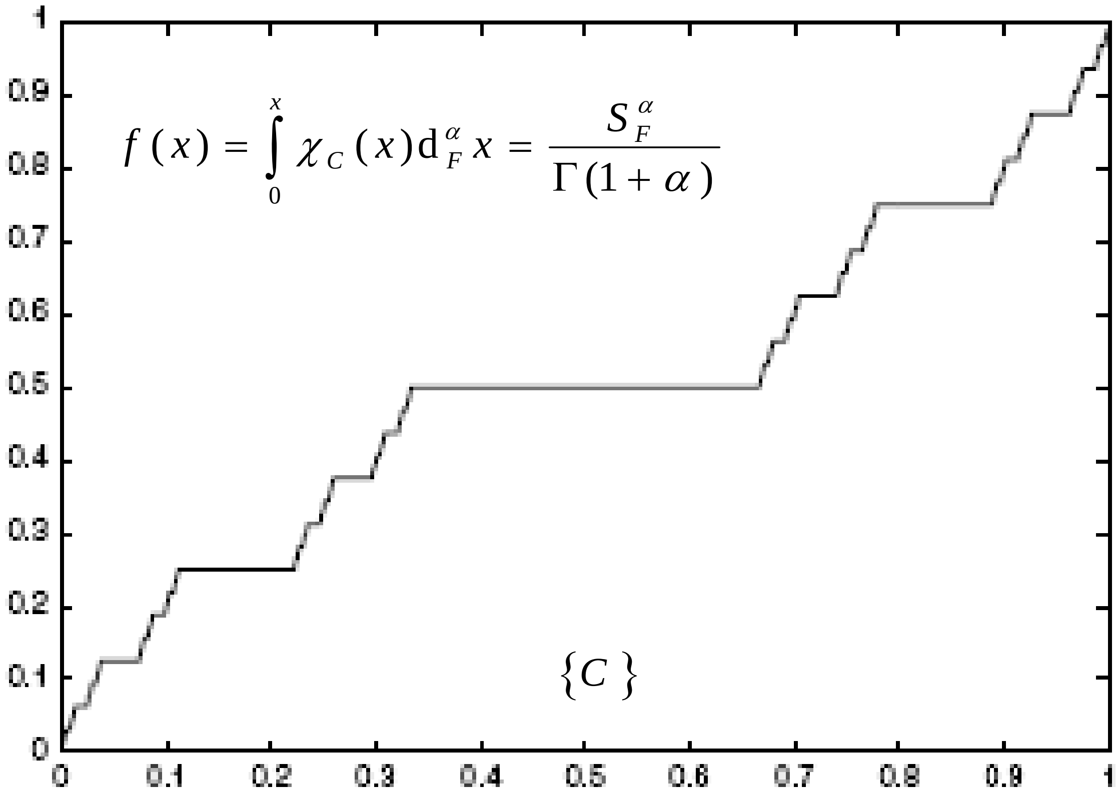
$$f(x) = \int g(x)[\mathrm{d}_F^\alpha x] = \lim_{N \to \infty} \sum_{i=0}^{N-1} F_C^i \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha + 1)} = \frac{S_F^\alpha(x)}{\Gamma(\alpha + 1)} \qquad x \in F = \{C\} \subset \mathbb{R}$$

$S_F^\alpha(x)$ **Devil's stair case function**

Devil's stair case

Devil's stair case is F^{α}-integration on the characteristic function of the Cantor set

$\Gamma(\alpha+1)[S_F^{\alpha}(x)]$



$$f(x) = \int_0^x \chi_C(x)\mathrm{d}_F^{\alpha} x = \frac{S_F^{\alpha}}{\Gamma(1+\alpha)}$$

$\{x\} \in F \subset \mathbb{R}$

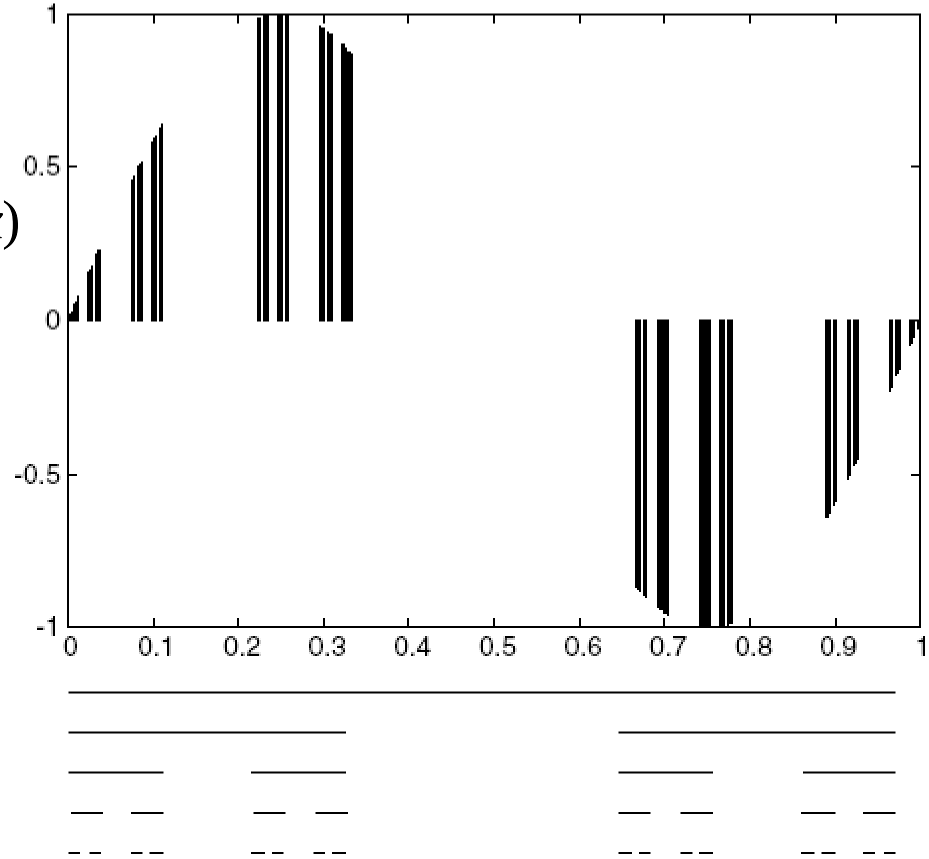
$F = C$

$\dim_B F = \log 2 / \log 3 = 0.68$

$g(x) = \chi_F(x) = 1_C(x)$

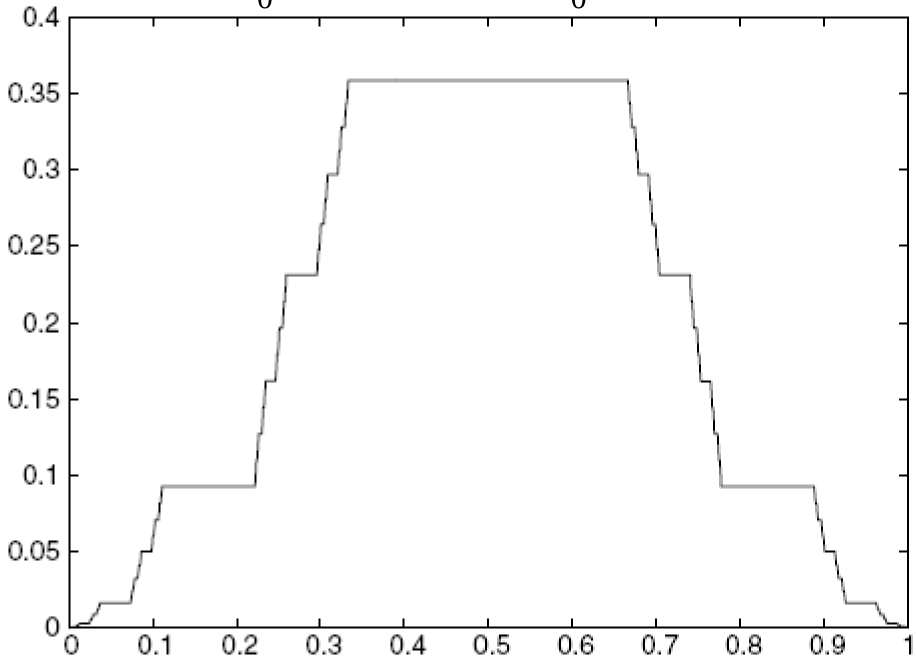
Sine function on Cantor set a subset of real line

$$f(x) = \sin(2\pi x)\chi_C(x)$$



F^{α}-integration of sine function on Cantor set

$$\int_0^x f(x')\mathrm{d}_F^{\alpha}x = \int_0^x \sin(2\pi x)\chi_C(x)\mathrm{d}_F^{\alpha}(x)$$



Start with conservation of probability "Chapman-Kolmogorov"

$$p(x,t+\tau) = \int p(x,t+\tau;x',t)\,p(x';t)\,dx'$$

Take $\Delta = x - x'$ **, and expand via fractional Taylor series**

$$p(x,t+\tau) = p(x,t) + \sum_{n=1}^{N} \frac{1}{n!}\left(\frac{\partial}{\partial(-x)}\right)^{n} \int \mathrm{d}x'\Delta^{n} p(x-\Delta,t+\tau;x,t)\,p(x,t)$$

$$+ \frac{1}{\Gamma(\beta+1)} \mathbf{D}_{x-}^{\beta}\left[\int_{x}^{\infty} \mathrm{d}y\,(y-x)^{\beta}\, p(y,t+\tau;x,t)\,p(x,t)\right]$$

$$+ \frac{1}{\Gamma(\beta+1)} \mathbf{D}_{x+}^{\beta}\left[\int_{-\infty}^{x} \mathrm{d}y\,(x-y)^{\beta}\, p(y,t+\tau;x,t)\,p(x,t)\right] + \mathcal{R}_{\beta}$$

For $0 < \alpha \leq 1$ **the fractional Taylor series in time is**

$$p(x,t+\tau) - p(x,t) = \frac{\tau^{\alpha}\,\mathbf{D}_{t}^{\alpha}\, p(x,t)}{\Gamma(\alpha+1)} + \mathcal{R}_{\alpha}$$

Continue to simplify with transitional moments

$$\frac{\tau^{\alpha}\mathbf{D}_t^{\alpha}\, p(x,t)}{\Gamma(\alpha+1)} = \sum_{n=1}^{N}\left(\frac{\partial}{\partial(-x)}\right)^n \left[\frac{M_n(x,t,\tau)}{n!}\, p(x,t)\right] + \mathbf{D}_{x-}^{\beta}\left[\frac{M_{\beta}^{+}(x,t,\tau)}{\Gamma(\beta+1)}\, p(x,t)\right] + \mathbf{D}_{x+}^{\beta}\left[\frac{M_{\beta}^{-}(x,t,\tau)}{\Gamma(\beta+1)}\, p(x,t)\right]$$

The transitional moments are

$$M_a^{+}(x,t,\tau) = \int_x^{\infty} \mathrm{d}y\,(y-x)^a\, p(y,t+\tau;x,t) \qquad a>0$$

$$M_a^{-}(x,t,\tau) = \int_{-\infty}^{x} \mathrm{d}y\,(x-y)^a\, p(y,t+\tau;x,t) \qquad a>0$$

$$M_a = M_a^{+}(x,t,\tau) + M_a^{-}(x,t,\tau)$$

Simplify further to get LFDE

$$\frac{\tau^{\alpha}\mathbf{D}_t^{\alpha} p(x,t)}{\Gamma(\alpha+1)} = \sum_{n=1}^{N}\left(\frac{\partial}{\partial(-x)}\right)^n \left[\frac{M_n(x,t,\tau)}{n!} p(x,t)\right] + \mathbf{D}_{x-}^{\beta}\left[\frac{M_{\beta}^{+}(x,t,\tau)}{\Gamma(\beta+1)} p(x,t)\right] + \mathbf{D}_{x+}^{\beta}\left[\frac{M_{\beta}^{-}(x,t,\tau)}{\Gamma(\beta+1)} p(x,t)\right]$$

Define $A_{a\mp}^{\beta}(x,t) = \lim_{\tau\to 0}\frac{M_{\beta}^{\pm}(x,t,\tau)\Gamma(\alpha+1)}{\tau^{\alpha}\Gamma(\beta+1)}$ & $A_a^{\beta}(x,t) = A_{a+}^{\beta}(x,t) + A_{a-}^{\beta}(x,t)$

Define linear operator as $\mathbf{L}_{\mathbf{O}}(x,t) \equiv \sum_{n=1}^{N}\left(\frac{\partial}{\partial(-x)}\right)^n A_a^n(x,t) + D_{x-}^{\beta}A_{a-}^{\beta}(x,t) + D_{x+}^{\beta}A_{a+}^{\beta}(x,t)$

We have LFDE $\mathbf{D}_t^{\alpha} p(x,t) = \mathbf{L}_{\mathbf{O}}(x,t)\, p(x,t)$

For $0<\beta<1$

$$\mathbf{L}_{\mathbf{O}} \equiv \mathbf{D}_{x-}^{\beta}A_{a-}^{\beta}(x,t) + \mathbf{D}_{x+}^{\beta}A_{a+}^{\beta}(x,t)$$

For $1<\beta<2$

$$\mathbf{L}_{\mathbf{O}} \equiv -\frac{\partial}{\partial x}A_a^1(x,t) + \mathbf{D}_{x-}^{\beta}A_{a-}^{\beta}(x,t) + \mathbf{D}_{x+}^{\beta}A_{a+}^{\beta}(x,t)$$

This operator can be identified as generalization of the FPE's operator with $\alpha=1; \beta=2$

For $\alpha=1$ **solution is**

$$D_t^1 p(x,t) = \mathbf{L}_{\mathbf{O}}\, p(x,t)$$

$$p(x,t) = e^{t[\mathbf{L}_{\mathbf{o}}(x)]}\, p(x,0)$$

$$p(x,t) = e^{\int_0^t \mathbf{L}_{\mathbf{o}}(x',t')\,dt'}\, p(x,0)$$

F^{α} -Diffusion equation

$$\mathbf{D}_{F,t}^{\alpha} c(x,t) = \frac{\chi_F}{2} \frac{\partial^2}{\partial x^2} c(x,t) \qquad c(x,0) = \delta(x)$$

$$c(x,t) = \frac{1}{\sqrt{2\pi S_F^{\alpha}(t)}} \exp\left(-\frac{x^2}{2S_F^{\alpha}(t)}\right)$$

This can be recognized as sub-diffisive solution, since the stair case function is known to be bounded by kt^{α} ; in simple cases including Cantor set. These equation of LFDE on fractal sets are example of fractal-evolution processes.

Standard integer order diffusion case

$$\mathbf{D}_t^1 c(x,t) = \frac{\partial}{\partial t} c(x,t) = \mathbb{D} \frac{\partial^2}{\partial x^2} c(x,t) \qquad c(x,0) = \delta(x)$$

$$c(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D} t}} \exp\left(-\frac{x^2}{4\mathbb{D}t}\right)$$

BM

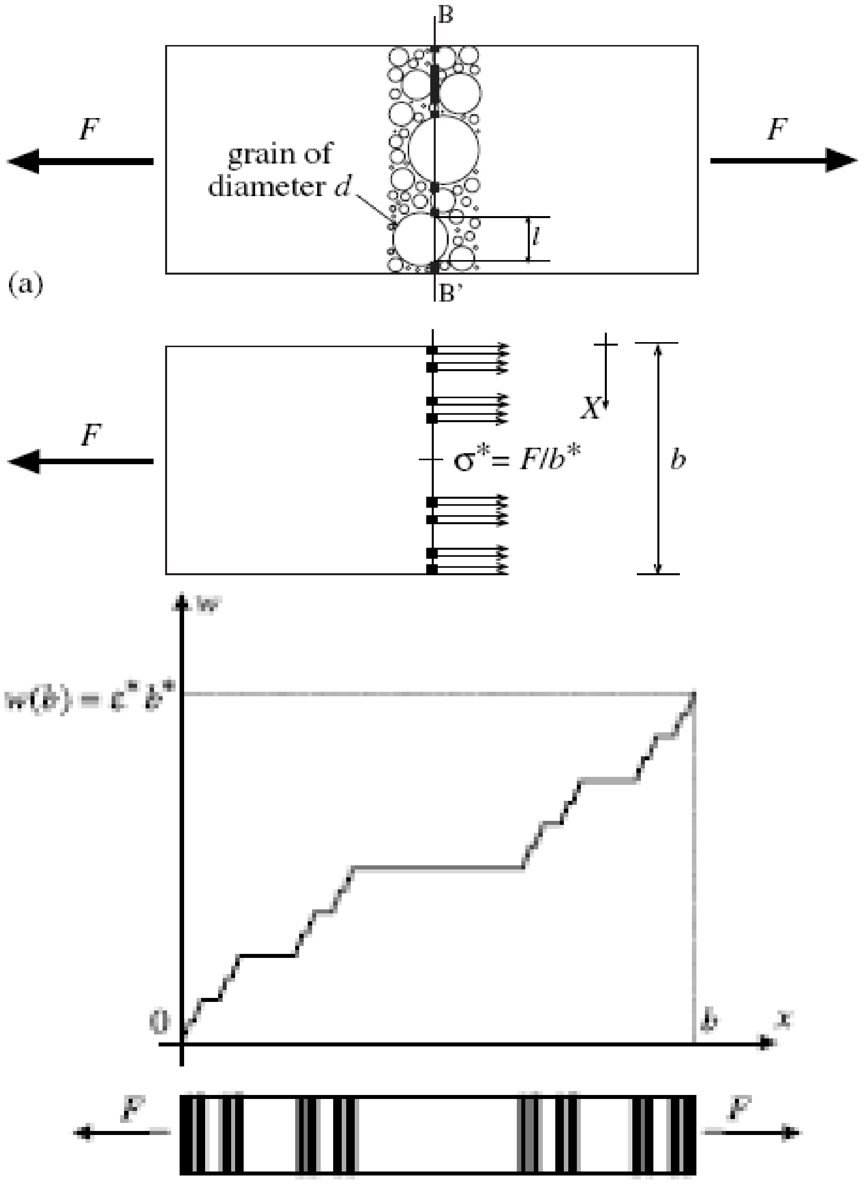
$$\frac{\partial}{\partial t} c_{FBM}(x,t) = \alpha t^{\alpha-1} \mathbb{D} \frac{\partial^2}{\partial x^2} c_{FBM}(x,t) \qquad c(x,0) = \delta(x)$$

One case of anomalous diffusion

$$c_{FBM}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D} t^{\alpha}}} \exp\left(-\frac{x^2}{4\mathbb{D}t^{\alpha}}\right)$$

FBM

LFDE in fractal cracks



Mathematics goes far beyond our physical understanding

$_aD_t^{\alpha}$ $\alpha \in \mathbb{N}$ $\alpha \in \mathbb{Q}$ $\alpha \in \mathbb{R}$

$_xD_{\infty}^{\alpha}$ **non - causal differintegration**

$D_t^{\alpha + i\beta}$ **complex order calculus**

$D_t^{\int_{-\infty}^{\infty} k(q)\mathrm{d}q}$ **continuous order calculus**

$\mathbf{D}_t^{\alpha}$ **local fractional calculus**

........................**and many more have each meaning to the physical world only if we keep understanding the deep philosophy of nature & try to speak natural language**

At the end we still have to revisit 'd / dt' again and again