State Trajectory Control and Control Energy for Fractional Order Multivariate Dynamic System

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SUMMARY

The construction and then analysis of Gramian Matrix for study of controllability for state trajectory is as rigorous for fractional order multivariate system, as for the integer order counter parts. The dynamic system of fractional order can be represented too, as multivariate state space representation; and it becomes easy to construct matrix representation, provided the fractional derivative are 'sequential fractional derivatives' having commensurate order. Here the concept of Gramian for, dynamic state space multivariate fractional order system is discussed, vis-à-vis integer order systems. Interestingly the basic state transition matrix (the Green's function) for fractional order systems with Caputo's formulation has two types and both these types participate in the solution to give state trajectory, is elucidated in this note. These state transition matrices for fractional order systems are from higher transcendental functions in matrix form, called alpha-exponential functions which are shown to be eigenvectors for Caputo and Riemann-Liouvelli derivative based fractional order homogeneous system. These alpha-exponential functions are one-parameter Mittag-Leffler, and Robotnov-Hartley functions. With these state transition matrices, control Gramian, state trajectories, input control vector and control energy are derived, for fractional order system. The differences and modifications for fractional order system vis-à-vis integer order counterparts are elaborated and reasoned out. Several examples are given to compute state transition matrices, Gramian, state trajectory and the control effort.

Keywords: Riemann-Liouvelli fractional derivative, Caputo derivative, sequential fractional derivatives, controllability, Gramian, minimality, state space equation, alpha-exponential function, generalized Wronskian, Control energy

Contents

1. INTRODUCTION

The integer order multivariate dynamic system is described as coupled non-homogeneous, Langevin equations [24]: call them $\dot{x}(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$, A, B are constant Matrices for Linear Time Invariant (LTI) case, $x(t)$ are states of the system, $u(t)$ is the control input that drives the system from initial state $x(t_0)$ to $x(T)$ in time T, a finite positive real number. Notice that in this conventional system, the control input acts instantaneously, to make desired state change, $x(T)$ (or desired rate of state change, $\dot{x}(T)$). If the system is represented as fractional order dynamics $(D^{\alpha}x)(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$, with $\alpha \in [0,1)$ as commensurate fractional order; then in order to have desired state as final one, $x(T)$ or desired rate of change, $\dot{x}(T)$ as final one, the control gets modified as $(D^{-\alpha}u)(t)$ or $(D^{1-\alpha}u)(t)$. This interpretation comes as by inverting the fractional dynamic state equation, to get $x(t)$ or $\dot{x}(t)$. Grossly we call the new control to be proportional to $\sim \int_0^t (t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau$; is (like) fractional integral term (except Gamma function) of order α. Though the fractional dynamics are the system's description, yet while measurement we like to set state (say position, $x(t)$) or integer order rate of state $\dot{x}(t)$ (say velocity). Therefore, fractional dynamics makes the control action as convolute of the instantaneous control signal $u(t)$, wrapping all states of control signal from initial condition till present time via fractional integral (or fractional derivative) of $u(t)$, that is a memorized action [8], [27]. In the integer order system the control energy is defined as [24] $J(t_0, t) = \int_{t_0}^t |u(t)|^2 \,\mathrm{d}t$; similarly our control energy expenditure for fractional order system should be proportional to $\sim \int_0^t \left|(t-\tau)^{\alpha-1}u(\tau)\right|^2 \mathrm{d}\tau$, via this gross idea. This new discussion makes us believe that performance indices to be of fractional integral or fractional derivatives of their integer order counterparts of multivariate dynamic systems. Are we just sake of doing mathematics, making the integer order state equations a fractional order one; the answer to that is in no. In reality the dynamics of system are of fractional order-and integer order is an approximation, the spatial heterogeneity the distributive parameters, which are reality, makes the fractional dynamics a reality [8], [25], [26] and [27]. The minimality, optimality of the fractional order systems are described in [1], [2], [3], [4], [5], [6]. The multivariate description of fractional order control system is depicted in [7], [8], [12], [14], [15], [22], [23], and [27]. The controllability/observability issues of infinite dimensional fractional system are described in [7], for systems with Riemann-Liouvelli derivatives. For positive continuous time linear systems the 'reachability' property is discussed in [14]; these issues become interesting as when we take Caputo derivatives as formulation of the fractional state space equations. Interesting, while the formulation of state space is via Caputo's fractional derivative; the solution of the fractional state space system requires two types of Green's function; we call them state transition matrix. One Green's function (symbol as $\tilde{\Phi}_\alpha$)or state transition matrix corresponds to solution to homogeneous system with Caputo derivative, and other Green's function (solution to homogeneous system with Riemann-Liouvelli derivative symbol as Φ_α). The later one required to be convoluted with forcing function vector comprises of control input vector, for the particular solution of the system of equations, in spite the dynamic system comprises of Caputo derivative. Here we construct Gramian of control vis-à-vis integer order multivariate dynamic system via the obtained Greens functions required for particular solution. Also in order to neutralize the singularity at the terminal point of control for the new control Gramian, we modify by some factor [17]. For fractional order multivariate system we obtain expression for control energy expenditure. We give elaborate examples to elucidate the calculations for state transition matrix, the control Gramian, the state trajectory, the control energy for integer order systems as well as fractional order systems. However, the basis of matrix representation of fractional order dynamic system is when the derivatives are of sequential fractional derivatives; we spend section on these sequential fractional differential equation (SFDE) to start with. For engineering applications these SFDE are of extreme importance, as they give naturally the matrix representation as for multivariate state space. Gramian matrix of functions is $Q(t_0,t)\triangleq\int_{t_0}^{t}\mathrm{F}(t)\mathrm{F}^*(t)\mathrm{d}t$; with F^* denoting transpose of F matrix. The matrix $\mathrm{F}(t)$ is $m\times n$, with functions $f_i(t)$, $i=1,2,3,...m$ as functions, [24]. The function f_i is function of $i-$th row;

and if $f_1, f_2, ... f_m$ are linearly independent functions in the interval $[t_0, t]$, and if this $m \times n$ is constant matrix, then $Q(t_0, t)$ is non-singular, is criteria of controllability [24]. The construction and analysis of the Gramian for control is of utmost importance in multivariate systems analysis; which is as rigorous for integer as well as fractional order state space systems.

2. SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATION & MATRIX REPRESENTATION IN MULTIVARIATE STATE SPACE FORM

The sequential fractional derivative operator ${}_a\mathscr{D}_x^{k\alpha}$ is proposed by Miller-Ross [21]; and a sequential fractional differential equation (SFDE) of order $n\alpha; \quad n \in \mathbb{N}$, is

$$b_0(x)y(x) + b_1(x)\left[{}_a\mathscr{D}_x^{\alpha} y(x)\right] + b_2(x)\left[{}_a\mathscr{D}_x^{2\alpha} y(x)\right] + ... + b_n(x)\left[{}_a\mathscr{D}_x^{n\alpha} y(x)\right] = f(x)$$

Where ${}_a\mathscr{D}_x^{k\alpha}$ is the fractional sequential derivative of 'commensurate order' α.

${}_a\mathscr{D}_x^{\alpha} y(x) = {}_a^*D_x^{\alpha} y(x)$ where ${}_a^*D_x^{\alpha}$ is Riemann-Liouvelli (RL) or Caputo's fractional derivative operator, and ${}_a\mathscr{D}_x^{k\alpha} y(x) = {}_a\mathscr{D}_x^{\alpha}\, {}_a\mathscr{D}_x^{(k-1)\alpha} y(x); \quad k = 2, 3, ...$. Thus for $k = 2; \quad 0 < \alpha < 1/2$, we have

$${}_a\mathscr{D}_x^{2\alpha} y(x) = {}_aD_x^{2\alpha} y(x) - {}_aI_x^{1-\alpha} y(a) \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}$$

Comes from the fact that (generally) for $\alpha > 0, \beta > 0$; $D^{\alpha} D^{\beta} f(x) \neq D^{\alpha+\beta} f(x)$, [18], [21], [27]. A sequential fractional differential equation is 'naturally' be casted in multivariate state space matrix form. Let us take a SFDE as $\mathscr{D}^{n\alpha} y(x) + \sum_{k=0}^{(n-1)} a_k(x) \mathscr{D}^{k\alpha} y(x) = f(x); \quad n \in \mathbb{N}$. It is easy to see that the SFDE reduces to ${}^*D^{\alpha} \mathrm{Y}(x) = \mathrm{A}(x)\mathrm{Y}(x) + \mathrm{B}(x)$, with change of variables as $y_1(x) = y(x); \quad {}^*D^{\alpha} y_j(x) = y_{j+1}(x); \quad j = 1, 2, ..., (n-1)$, and with

$$\mathrm{A}(x) = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ -a_0 & -a_1 & ... & ... & -a_{n-1} \end{pmatrix}, \quad \mathrm{B}(x) = \begin{pmatrix} 0 \\ 0 \\ ... \\ ... \\ f(x) \end{pmatrix}; \quad \mathrm{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ ... \\ ... \\ y_n(x) \end{pmatrix}$$

As an example take $y''(t)+3\left({}_0D_t^{3/2}y(t)\right)+y(t)=f(t)$, and set $y''(t)={}_0\mathscr{D}_t^{4\frac{1}{2}}y(t)$, to get SFDE as ${}_0\mathscr{D}_t^{4\alpha}y(t)+3{}_0\mathscr{D}_t^{3\alpha}y(t)+y(t)=f(t);\quad \alpha=1/2$, this reduces to

$$
{}_0D_t^{\alpha}\mathrm{Y}(t)=\mathrm{AY}(x)+\mathrm{B}(t)\,\text{with }\mathrm{A}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&0&0&-3\end{pmatrix};\quad \mathrm{B}(t)=\begin{pmatrix}0\\0\\0\\f(t)\end{pmatrix};\quad \mathrm{Y}(t)=\begin{pmatrix}y_1(t)\\y_2(t)\\y_3(t)\\y_4(t)\end{pmatrix}
$$

We will giving solutions to this sequential fractional differential equations in later section.

3. FUNDAMENTALS OF CONTROLLABILITY OF MULTIVARIATE INTEGER ORDER SYSTEM

Let us take general differential equation $\dot{x}(t)+a(t)x(t)=bu(t)$, with initial condition as $x(t_0)=x_0$, and constant b. To solve this system we multiply both the sides by $e^{\int a(t)\mathrm{d}t}$ to get in the compact form the differential equation as $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t}x(t)\right]=e^{\int a(t)\mathrm{d}t}bu(t)$. This we integrate from t_0 to t and re-write as $\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[e^{\int a(\tau)\mathrm{d}\tau}x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}e^{\int a(\tau)\mathrm{d}\tau}bu(\tau)\mathrm{d}\tau$; using $\Phi(t)=e^{\int a(t)\mathrm{d}t}$ and then expanding this formulation, we get the following expressions

$$
\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\Phi(\tau)x(\tau)\right]\mathrm{d}\tau=\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau
$$

$$
\Phi(t)x(t)-\Phi(t_0)x(t_0)=\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau
$$

$$
x(t)=[\Phi(t)]^{-1}\Phi(t_0)x_0+[\Phi(t)]^{-1}\int_{t_0}^{t}\Phi(\tau)bu(\tau)\mathrm{d}\tau
$$

For a constant $a(t)=a$, we have $\Phi(t)=e^{at}$ and $\Phi(t_0)=e^{at_0}$, and thus we have

$$
x(t)=e^{-at}e^{at_0}x_0+e^{-at}\int_{t_0}^{t}e^{a\tau}bu(\tau)\mathrm{d}\tau=e^{a(t_0-t)}x_0+\int_{t_0}^{t}e^{a(\tau-t)}bu(\tau)\mathrm{d}\tau
$$

$$
=\Phi(t_0-t)x_0+\int_{t_0}^{t}\Phi(\tau-t)bu(\tau)\mathrm{d}\tau=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)bu(\tau)\mathrm{d}\tau
$$

If $a(t)$ is matrix $\mathrm{A}(t)\in\mathbb{R}^{n\times n}$, from [24] there are several ways to represent $e^{\int \mathrm{A}(t)\mathrm{d}t}$; which will be elucidated in this section with examples. Where we define state transition matrix;

$\Phi(t_0 - t) = e^{\int_t^{t_0} a(t)\mathrm{d}t} = \Phi(t, t_0)$, is also a Green's function of the homogeneous part of the system of differential equation. One way is Matrix exponential as for constant A , as $e^{\int \mathrm{A}(t)dt} = e^{\mathrm{A}t} \cong \mathrm{I} + \mathrm{A}t + \frac{\mathrm{A}^2 t^2}{2} + \frac{\mathrm{A}^3 t^3}{3!} + \ldots$ .The other way is, via inverse Laplace that is $e^{\mathrm{A}t} = \mathcal{L}^{-1}\{(s\mathrm{I} - \mathrm{A})^{-1}\}$. We use this discussion of solving this type of Langevin non-homogeneous equation, to obtain the solutions to the multivariate dynamic system represented as Ω, below.

a. The solution of multivariate dynamic system with state transition matrix in integer order system

For a linear dynamic multivariate system Ω defined by state equations (1)

$$\Omega: \qquad \dot{x}(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t) \qquad y(t) = \mathrm{C}(t)x(t) + \mathrm{D}(t)u(t) \tag{1}$$

State vectors is $x(t) \in \mathbb{R}^{n\times 1}$ is state vector, control input vector $u(t) \in \mathbb{R}^{p\times 1}$, output vector $y(t) \in \mathbb{R}^{q\times 1}$; the real matrices $\mathrm{A} \in \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{n\times p}$, $\mathrm{C} \in \mathbb{R}^{q\times n}$, $\mathrm{D} \in \mathbb{R}^{q\times p}$, whose entries are continuous function of time t over $(-\infty, \infty)$. The solution of (1) is (2); via above example is

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau = \Phi(t, t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0, \tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau \right] \tag{2}$$

Note that in this case $a(t) = -\mathrm{A}(t)$, as compared with example. Where $\Phi(t_0, t) = e^{\int_{t_0}^{t} \mathrm{A}(\tau)\mathrm{d}\tau}$ is state the transition matrix (Green's function) of homogeneous system $\dot{x}(t) = \mathrm{A}(t)x(t)$; x_0 is initial state at initial time $t = t_0$; and $t \geq t_0$ and for linear time invariant (LTI) system, this is $\Phi(t_0, t) = \Phi(t - t_0) = e^{\mathrm{A}(t - t_0)} = G(t - t_0)$. The (2) comprises of homogeneous solution and particular solution. The homogeneous solution is given by $G(t - t_0) = \Phi(t - t_0) = e^{\mathrm{A}(t - t_0)}$; and particular solution is convolution ($\otimes$) of Green's function $G(t - \tau)$ and forcing function vector $\mathrm{B}u(t)$, that is

$$x_p(t) = (G \otimes \mathrm{B}u)(t) = \int_{t_0}^{t} G(t - \tau)\mathrm{B}u(\tau)\mathrm{d}\tau = \int_{t_0}^{t} e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau$$

The output is therefore;

$$y(t) = \mathrm{C}(t)\Phi(t, t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0, \tau)\mathrm{B}(\tau)u(\tau)\mathrm{d}\tau \right] + \mathrm{D}(t)u(t) \tag{3}$$

For linear time invariant (LTI) system the matrices $\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}$ are constant matrices, with $t_0 = 0$, the state transition matrix is $e^{\mathrm{A}t}$ then (2) and (3) are

$$
\begin{aligned}
x(t) &= e^{\mathrm{A}t}x_0 + \int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\mathrm{d}\tau = e^{\mathrm{A}t}\left[x_0 + \int_0^t e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau\right] \\
y(t) &= \mathrm{C}e^{\mathrm{A}t}\left[x_0 + \int_0^t e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\mathrm{d}\tau\right] + \mathrm{D}u(t)
\end{aligned}
\tag{4}
$$

b. Gramian of control and control effort minmality in integer order system

Gramian matrix of functions is $Q(t_0,t) \triangleq \int_{t_0}^{t} \mathrm{F}(t)\mathrm{F}^*(t)\mathrm{d}t$; F^* denoting transpose of F matrix. The matrix $\mathrm{F}(t)$ is $m \times n$, with functions $f_i(t)$, $i = 1,2,3,...m$ as functions, [24]. The function f_i is function of $i-$ th row; and if $f_1, f_2,...f_m$ are linearly independent functions in the interval $\left[t_0,t\right]$, and if this $m \times n$ is constant matrix, then $Q(t_0,t)$ is non-singular, is criteria of controllability [24]. We define the term controllability as following:

If it is possible to transfer any initial state $x(t_0)$ in state space of system Ω to any other state $x(t_1)$ by proper choice of $u(t)$ ($t_0 \le t \le t_1$) in finite time $t_1 - t_0$ where $t_1 > t_0$, then system Ω is controllable. In other words if there exists $u(t)$ in the interval $\left[t_0,t_1\right]$ then the system Ω is controllable else uncontrollable. The controllability in involves matrices A and B which couples input control signal $u(t)$ and system states $x(t)$ of Ω.

The system Ω is controllable at time t_0 if and only if, there exist a finite time, $t_1 > t_0$ such that n rows of the $n \times p$ matrix function $\Phi(t_0,t)\mathrm{B}(t)$ are linearly independent in $\left[t_0,t_1\right]$. Then there exists a control $u(t)$, $t \in \left[t_0,t_1\right]$ to drive the system from any arbitrary $x_0 = x(t_0)$ to arbitrary state $x_1 = x(t_1)$. In control theory [24] the controllability Gramian is used to determine whether or not a linear system is controllable. The Gramian matrix of controllability is defined as

$$Q(t_0,t_1) \triangleq \int_{t_0}^{t_1} \Phi(t_0,t)\mathrm{B}(t)\mathrm{B}^*(t)\Phi^*(t_0,t)\mathrm{d}t \tag{5}$$

The Gramian matrix is non-singular, that is Gramian matrix has full rank; for controllability of Ω with A,B pair. For a LTI system, with $t_0 = 0$ the controllability Gramian is

$$Q(t)=\int_0^t e^{-\mathrm{A}\tau}\mathrm{BB}^*e^{-\mathrm{A}^*\tau}\mathrm{d}\tau=\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{BB}^*e^{\mathrm{A}^*(t-\tau)}\mathrm{d}\tau \tag{6}$$

In above A^* denotes the transpose of matrix A; we deviate from standard symbol A^T since T we use for final time subsequently. The claim here is that the input control vector function

$$u(t)=-\mathrm{B}^*(t)\Phi^*(t_0,t)Q^{-1}(t_0,t_1)\left[x_0-\Phi(t_0,t_1)x_1\right] \tag{7}$$

transfers x_0 to x_1 at $t=t_1$. For verification have the following algebraic exercise, using (2), (5) and (7) as following [24]

$$\begin{aligned} x(t_1)&=\Phi(t_1,t_0)\left[x_0-\left(\int_{t_0}^{t_1}\Phi(t_0,t)\mathrm{B}(t)\mathrm{B}^*(t)\Phi^*(t_0,t)\mathrm{d}t\right)\left(Q^{-1}(t_0,t_1)\left[x_0-\Phi(t_0,t_1)x_1\right]\right)\right]\\ &=\Phi(t_1,t_0)\left[x_0-Q(t_0,t_1)Q^{-1}(t_0,t_1)\left(x_0-\Phi(t_0,t_1)x_1\right)\right]\\ &=\Phi(t_1,t_0)\Phi(t_0,t_1)x_1=x_1 \end{aligned} \tag{8}$$

The input control vector in (7) gives minimal energy control as control energy [24] expenditure defined as

$$J_{(t_0-t)}=\int_{t_0}^{t}u^*(t)u(t)\mathrm{d}t \tag{9}$$

The (9) is minimal for (5), meaning for a system Ω let $u(t)$ be any control for $t\in\left[t_0,t_1\right]$ that transfers the state $x(t_0)$ to the state $x(t_1)$ and $\bar{u}(t)$ be the control input vector described as in (7), then

$$\int_{t_0}^{t_1}u^*(t)u(t)\mathrm{d}t\geq\int_{t_0}^{t_1}\bar{u}^*(t)\bar{u}(t)\mathrm{d}t \tag{10}$$

c. Calculations of state transition matrix control Gramian state trajectory for integer order systems

For example as for illustration sake take the following linear time variant (LTV) system, represented as following state space form

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=\begin{pmatrix}1 & e^{-t}\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}+\begin{pmatrix}0\\ 1\end{pmatrix}u(t)$$

$A=\begin{pmatrix}1 & e^{-t}\\ 0 & -1\end{pmatrix}$, $B=\begin{pmatrix}0\\ 1\end{pmatrix}$, $x(0)=x_0=\begin{pmatrix}0\\ 0\end{pmatrix}$, $x(1)=x_1=\begin{pmatrix}1\\ 1\end{pmatrix}$, control action from $t\in[0,1]$.

Calculation of state transition Matrix Φ for this Linear Time Variant (LTV) system is demonstrated as follows from the two sets of coupled differential equations for states are;

$$\frac{\mathrm{d}}{\mathrm{d}t}x_1=x_1+e^{-t}x_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t}x_2=-x_2+u(t)$$

Let us take the second state and solve the 'homogeneous' system that is $\dot{x}_2=-x_2$. To solve this let us take a general differential equation $\dot{x}(t)+a(t)x(t)=y(t)$. Multiplying both sides by $e^{\int a(t)\mathrm{d}t}$; and with manipulation we get the form as $\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\int a(t)\mathrm{d}t}x\right]=e^{\int a(t)\mathrm{d}t}y$. In our case that is $\dot{x}+x=0$ we have $a(t)=1$ and $y=0$, so $e^{\int a(t)\mathrm{d}t}=e^{t}$. We write the above equation, and integrating from t_0 to t for our case for x_2 as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{t}x_2\right]=e^{t}y \qquad \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{\theta}x_2(\theta)\right]\mathrm{d}\theta=\int_{t_0}^{t}e^{\theta}y(\theta)\mathrm{d}\theta \qquad e^{t}x_2(t)-e^{t_0}x_2(t_0)=\int_{t_0}^{t}e^{\theta}y(\theta)\mathrm{d}\theta$$

Giving $x_2(t)=e^{-t}e^{t_0}x_2(t_0)+e^{-t}\int_{t_0}^{t}e^{\theta}y(\theta)\mathrm{d}\theta=e^{-(t-t_0)}x_2(t_0) \qquad y(\theta)=0$

Putting the solution of the second state's homogeneous equation into the first state equation we obtain the following;

$$\dot{x}_1=x_1+e^{-t}e^{-(t-t_0)}x_2(t_0)=x_1+e^{-2t}\left(e^{t_0}x_2(t_0)\right) \qquad \dot{x}_1-x_1=e^{-2t}\left[e^{t_0}x_2(t_0)\right]$$

Comparing to $\dot{x}+a(t)x=y$, we get $a(t)=-1$, $e^{\int a(t)\mathrm{d}t}=e^{-t}$ and $y(t)=e^{-2t}\left(e^{t_0}x_2(t_0)\right)$. Using the procedure as done for state x_2, we write similarly for state x_1, the following expressions;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-t}x_1\right]=e^{-t}y \qquad \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[e^{-\theta}x_1(\theta)\right]\mathrm{d}\theta=\int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta \qquad e^{-t}x_1(t)-e^{-t_0}x_1(t_0)=\int_{t_0}^{t}e^{-\theta}y(\theta)\mathrm{d}\theta$$

$$x_1(t)=e^{t-t_0}x_1(t_0)+e^{t}\int_{t_0}^{t}e^{-\theta}\left(e^{-2\theta}e^{t_0}x_2(t_0)\right)\mathrm{d}\theta=e^{(t-t_0)}x_1(t_0)+\left[\frac{1}{3}e^{(t-2t_0)}-\frac{1}{3}e^{(t_0-2t)}\right]x_2(t_0)$$

In matrix form the solution of homogeneous system as we obtained for the two state variables is expressed as following;

$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=\begin{pmatrix}e^{(t-t_0)} & \frac{1}{3}e^{(t-2t_0)}-\frac{1}{3}e^{(t_0-2t)}\\ 0 & e^{-(t-t_0)}\end{pmatrix}\begin{pmatrix}x_1(t_0)\\ x_2(t_0)\end{pmatrix}=\Phi(t,t_0)\begin{pmatrix}x_{10}\\ x_{20}\end{pmatrix}$$

Thus from the homogeneous system's solution we obtain the state transition matrix as

$$\Phi(t,\tau)=\begin{pmatrix} e^{t-\tau} & \frac{1}{3}\left(e^{t-2\tau}-e^{-2t+\tau}\right) \\ 0 & e^{-t+\tau} \end{pmatrix}; \qquad \Phi(0,\tau)=\begin{pmatrix} e^{-\tau} & \frac{1}{3}\left(e^{-2\tau}-e^{\tau}\right) \\ 0 & e^{\tau} \end{pmatrix}$$

The Gramian matrix is

$$Q(0,1)=\int_0^1 \Phi(0,\tau)\mathrm{B}\mathrm{B}^*\Phi^*(0,\tau)\mathrm{d}\tau$$

$$=\int_0^1 \begin{pmatrix} \frac{1}{9}\left(e^{-4\tau}-2e^{-\tau}+e^{2\tau}\right) & \frac{1}{3}\left(e^{-\tau}-e^{2\tau}\right) \\ \frac{1}{3}\left(e^{-\tau}-e^{2\tau}\right) & e^{2\tau} \end{pmatrix}\mathrm{d}\tau=\begin{pmatrix} 0.2417 & -0.8541 \\ -0.8541 & 3.1945 \end{pmatrix}$$

$\det Q(0,1)=0.04262\neq 0$, thus system of this example is completely controllable.

$$Q^{-1}(0,1)=\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}$$

The minimal energy input control vector is via (7)

$$\bar{u}(t)=-\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ \frac{1}{3}\left(e^{-2t}-e^{t}\right) & e^{t} \end{pmatrix}\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}-\begin{pmatrix} e^{-1} & \frac{1}{3}\left(e^{-2}-e^{1}\right) \\ 0 & e^{1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$$

$$=-\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ \frac{1}{3}\left(e^{-2t}-e^{t}\right) & e^{t} \end{pmatrix}\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}\begin{bmatrix} 0.4931 \\ -2.7183 \end{bmatrix}$$

$$=5.8384e^{-2t}-0.3026e^{t}$$

The state trajectory due to above obtained control input is obtained by (2)

$$\bar{x}(t)=\begin{pmatrix} e^{t} & \frac{1}{3}\left(e^{t}-e^{-2t}\right) \\ 0 & e^{-t} \end{pmatrix}\left[\int_0^t \begin{pmatrix} \frac{1}{3}\left(e^{-2\tau}-e^{\tau}\right) \\ e^{\tau} \end{pmatrix}\left(5.8384e^{-2t}-0.3026e^{t}\right)\mathrm{d}\tau\right]$$

$$=\begin{pmatrix} 0.3856e^{t}-0.1513-1.9966e^{-2t}+1.4596e^{-2t} \\ -0.1513e^{t}+5.9897e^{-t}-5.8384e^{-2t} \end{pmatrix}$$

Above example elucidates in detail the calculation of the state transition matrix for LTV system, and calculation of minimal control vector and controllability Gramian. The system when is Linear Time Invariant with matrix A a constant matrix, then simpler method is invoked via inverse Laplace transform to get to $\Phi(t)$. The equation $\dot{x}=\mathrm{A}x+\mathrm{B}u$ has fundamental solution for homogeneous system $\dot{x}(t)=\mathrm{A}x(t)$ as $\Phi(t)=e^{\mathrm{A}t}=\mathcal{L}^{-1}\left\{(s\mathrm{I}-\mathrm{A})^{-1}\right\}$; where s is complex frequency (Laplace variable), I is identity matrix [24]. Let us elaborate with following example where, LTI system matrix is

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\Phi(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} s+3 & 2 \\ -1 & s \end{pmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s(s+3)+2}\begin{pmatrix} s & -2 \\ 1 & s+3 \end{pmatrix}\right]$$

$$= \mathcal{L}^{-1}\begin{pmatrix} \frac{2}{s+2} - \frac{1}{s+1} & \frac{2}{s+2} - \frac{2}{s+1} \\ -\frac{1}{s+2} + \frac{1}{s+1} & -\frac{1}{s+2} + \frac{2}{s+1} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-2t} - e^{-t} & 2e^{-2t} - 2e^{-t} \\ -e^{-2t} + e^{-t} & -e^{-2t} + 2e^{-t} \end{pmatrix}$$

This method will be used in the following section to get state transition matrix for the fractional order differential equation system as $\Phi_\alpha(t) = \mathcal{L}^{-1}\left\{(s^\alpha I - A)^{-1}\right\}$, $\alpha \in [0,1)$

4. FRACTIONAL CALCULUS PRELIMINARIES AND GREEN'S FUNCTIONS FOR RIEMANN-LIOUVELLI AND CAPUTO DERIVATIVES

In the integer order calculus, the function $e^{\lambda t}$ plays an important role in solution of ordinary differential equations LTI systems; as it satisfies $(\mathrm{d}e^{\lambda t}/\mathrm{d}t) = \lambda e^{\lambda t}$. Similarly the alpha exponential function-1 satisfies; $x(t) = e_\alpha^{\lambda(t-a)}$, satisfies ${}_aD_t^\alpha x(t) = \lambda x(t)$, with RL derivative; and the alpha exponential function-2 $x(t) = \tilde{e}_\alpha^{\lambda(t-a)}$ satisfies ${}_a^CD_t^\alpha x(t) = \lambda x(t)$ with Caputo derivative. This we develop in this section. In the section-3 we used the notation Φ as 'state transition matrix' (associated Green's function) which is $\Phi(t) = e^{At}$ for LTI system of integer order differential equation system. For fractional order system we can (similarly) define state transition matrix as $\Phi_\alpha(t) = e_\alpha^{At}$, and $\tilde{\Phi}_\alpha(t) = \tilde{e}_\alpha^{At}$ where the notation, e_α^{At} is alpha-exponential function-1, and $\tilde{e}_\alpha^{At}$ is alpha-exponential function-2; which are also Green's function and also eigenvectors for RL and Caputo derivatives based homogeneous linear differential equations. Here we make use of the previous section's development, to get solution of the fractional order sequential linear differential equations (SFDE) in terms of these alpha-exponential functions.

a. The alpha-exponential function-1 and 2.

The alpha-exponential functions follow from the higher transcendental functions of basic types as Mittag-Leffler function [8], [11], [16], [18], [19], [21], [27]. The two parameter Mittag-Leffler function is defined by series as following;

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\qquad \alpha>0\qquad \beta>0\qquad z\in\mathbb{C} \tag{11}$$

$\Gamma(.)$ is Euler Gamma [11] function. For matrix $\mathrm{A}\in\mathbb{R}^{n\times n}$ the (11) is extended for matrix case as [8], [11], [27] following;

$$E_{\alpha,\beta}(\mathrm{A}t^{\alpha})=\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{t^{k\alpha}}{\Gamma(k\alpha+\beta)} \tag{12}$$

Put $\beta=\alpha$ in (12) and we define alpha-exponential function-1 as

$$e_{\alpha}^{\mathrm{A}t}=(t^{\alpha-1})E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})=(t^{\alpha-1})\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{t^{k\alpha}}{\Gamma[(k+1)\alpha]}=\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{13}$$

For $\alpha=1$, we have $E_1(\mathrm{A}t)=e_1^{\mathrm{A}t}=e^{\mathrm{A}t}=\Phi(t)$, is state transition matrix of integer order LTI system with initial time $t_0=0$, as described in previous section. Put $\beta=1$ in (12) we have one parameter Mittag-Leffler function as [8], [11], [16], [18], [19], [21], [27], following

$$E_{\alpha,1}(\mathrm{A}t^{\alpha})=E_{\alpha}(\mathrm{A}t^{\alpha})=\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{14}$$

Define second alpha-exponential function-2 as

$$\tilde{e}_{\alpha}^{\mathrm{A}t}=E_{\alpha,1}(\mathrm{A}t^{\alpha})=\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{15}$$

The alpha-exponential function-1; $e_{\alpha}^{\mathrm{A}t}$ (13) is useful for solving sequential fractional differential equations (SFDE) with Riemann-Liouvelli (RL) fractional derivative, while the alpha-exponential-2; $\tilde{e}_{\alpha}^{\mathrm{A}t}$ (15) is useful for solution of SFDE with Caputo derivative [16].

Since $e_{\alpha}^{\mathrm{A}t}=t^{\alpha-1}E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})$ and each Mittag-Leffler function $E_{\alpha,\alpha}(\lambda z^{\alpha})$, $\lambda>0\quad z\in\mathbb{C}$ is an entire function [16] on the complex plane, we can have an uniquely determined function $f(t)=t^{1-\alpha}F(t)$ such that $e_{\alpha}^{\mathrm{A}t}f(t)=E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})F(t)=\mathrm{I}$ for $t\neq 0$ and $\lim_{t\to 0}e_{\alpha}^{\mathrm{A}t}f(t)=\mathrm{I}$.

For $A = 0$; $e_\alpha^{At} = t^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right) = t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + A\frac{t^\alpha}{\Gamma(2\alpha)} + ...\right) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$

Therefore for this particular case, $f(t) = t^{1-\alpha}\Gamma(\alpha)$, so that identity condition gets satisfied.

b. Fractional Derivatives Riemann-Liouvelli and Caputo and their relation

Let us define a convolution kernel as a power law function and its Laplace depicted as

$$K_\alpha(t) \triangleq \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \qquad \alpha > 0 \qquad \mathcal{L}\{K_\alpha(t)\} = s^{-\alpha} \qquad \mathrm{Re}\, s > 0$$

When $\alpha \to 1$, then $K_\alpha \to H(t)$ the Heaviside step function. When $\alpha \to 0$, then $K_\alpha \to \delta(t)$, the Dirac's delta function. The alpha exponential functions are then

$$e_\alpha^{\lambda t} = (t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^\alpha) = \sum_{k=0}^{\infty}\lambda^k K_{(1+k)\alpha} = t_+^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\lambda t_+^\alpha)^k}{\Gamma[(1+k)\alpha]} \qquad \mathcal{L}\left\{e_\alpha^{\lambda t}\right\} = (s^\alpha - \lambda)^{-1}$$

The alpha exponential function-1 is same as Robotnov-Hartley function; $F_\alpha(\lambda, t)$ [8], [27].

$$\tilde{e}_\alpha^{\lambda t} = E_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty}\lambda^k K_{(1+k\alpha)} = \sum_{k=0}^{\infty}\frac{(\lambda t_+^\alpha)^k}{\Gamma(1+k\alpha)} \qquad \mathcal{L}\left\{\tilde{e}_\alpha^{\lambda t}\right\} = s^{\alpha-1}(s^\alpha - \lambda)^{-1}$$

The alpha-exponential function-2 is one parameter Mittag-Leffler function [8], [11], [27]. The symbol $\otimes$ we use for convolution operation, and find interesting convolution link between these two alpha-exponential functions; that is

$$\tilde{e}_\alpha^{\lambda t} = E_\alpha(\lambda t^\alpha) = \{K_{(1-\alpha)}(t)\} \otimes \{(t^{\alpha-1})E_{\alpha,\alpha}(\lambda t^\alpha)\} = \{K_{(1-\alpha)}\}(t) \otimes \{e_\alpha^{\lambda t}\}$$

The two alpha exponential functions are same for $\alpha = 1$, that is exponential function

$$\tilde{e}_1^{\lambda t} = E_1(\lambda t) = \{e_1^{\lambda t}\} \otimes \{K_0(t)\} = e_1^{\lambda t} = e^{\lambda t} ; \qquad K_0(t) = \delta(t)$$

The causal convolution then defines fractional integration, [8], [16], [18], [19] [21] [27] as ${}_{0+}I_t^\alpha f \triangleq K_\alpha \otimes f$ .The kernel is Heaviside step function for $\alpha = 1$ and Dirac delta function for $\alpha = 0$ (in limit). Above is fractional integral of order α of a continuous causal function f . Interesting observation is that if f , is delta function then we have the convolution as $K_\alpha \otimes \delta = K_\alpha$; meaning that ${}_{0+}I_t^\alpha \delta(t) = K_\alpha = t_+^{\alpha-1}/\Gamma(\alpha)$; the fractional integration of Dirac-delta function; [8], [27]. From this definition of fractional integration, and above convolution relation of two alpha exponentials, we see that ${}_{0+}I_t^{(1-\alpha)} e_\alpha^{\lambda t} = \tilde{e}_\alpha^{\lambda t}$.

While we reverse the sign of α then $K_{-\alpha}$ gets defined as causal distribution as ‘convolute inverse’ of $K_{+\alpha}$ in ‘convolution algebra’ $D'_{+}(\mathbb{R})$ with the use of δ-Dirac distribution-which is neutral element of convolution operation; reads as $K_{+\alpha} \otimes K_{-\alpha} = \delta$. The Laplace of $K_{-\alpha}(t)$ is s^{α} for $\operatorname{Re} s > 0$. With this preliminaries and notations we tend to define Fractional Derivative of order α of continuous causal function as $D^{\alpha} f \triangleq K_{-\alpha} \otimes f$. In order to make this above definition tractable from a analytical point of view it proves useful to define ‘smooth’ fractional derivative operator for continuous f with first derivative f', for $0 < \alpha \le 1$, [8], [16], [18], [19] [21] [27] is as following;

$$d^{\alpha} f \triangleq D^{\alpha} f - f(0^{+})K_{1-\alpha} = D^{\alpha} f - f(0^{+})\frac{t^{-\alpha}}{\Gamma(1-\alpha)} = \int_{0}^{t} K_{1-\alpha}(t-\tau) f'(\tau)\mathrm{d}\tau$$

The difference between D^{α} and d^{α} is exactly the same as the one between derivation in sense of distribution (D^{1}) and classical derivation (d^{1}); namely $D^{1} f = d^{1} f + f(0^{+})\delta$ .First we define the fractional integrals for f ; as Riemann-Liouvelli fractional integration as [8], [16], [18], [19] [21] [27] follows;

$$_{t_0+}I_t^{\alpha} = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} f(\tau)(t-\tau)^{\alpha-1}\mathrm{d}\tau \qquad t > t_0 \qquad \alpha > 0 \tag{16}$$

The above is causal integration (left sided integration), and below we write non causal integration (Weyl’s integration-or right sided integration) [8], [16], [18], [19] [21] [27];

$$_{t}I_{t_1-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{t_1} f(\tau)(\tau-t)^{\alpha-1}\mathrm{d}\tau \qquad t < t_1 \qquad \alpha > 0 \tag{17}$$

Identity operation (I) is defined as $I := {}_{t_0+}I_t^{0} = {}_{t_1-}I_t^{0}$ [8], [16], [18], [19] [21] [27].

The left sided (causal) Riemann-Liouvelli (RL) fractional derivative, for fractional order $\alpha \in \mathbb{R}^{+}$, $\alpha > 0$ and natural number, $n \in \mathbb{N}$ and $n = [\alpha] + 1$; where $[\alpha]$ denoting integer part of fractional number α ; is [8], [16], [18], [19] [21] [27] as follows

$$_{t_0+}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t} f(\tau)(t-\tau)^{n-\alpha-1}\mathrm{d}\tau \tag{18}$$

Similarly the right sided (non-causal) RL fractional derivative is [8], [16], [18], [19] [21] [27];

$$_{t}D_{t_1-}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t}^{t_1} f(\tau)(\tau-t)^{n-\alpha-1}\mathrm{d}\tau \tag{19}$$

Let $\alpha \geq 0$, $n = [\alpha] + 1$ if $\alpha \notin \mathbb{N}_0$; and $n = \alpha$ if $\alpha \in \mathbb{N}_0$. $AC[t_0, t_1]$ is the space of functions that are absolutely continuous on $[t_0, t_1]$ and $AC^n[t_0, t_1]$ denote space of functions $f(t)$ that have continuous derivatives up to order $n-1$ on $[t_0, t_1]$ and such that $f^{(n-1)} \in AC[t_0, t_1]$. If $f \in AC^n[t_0, t_1]$; then Caputo derivatives [8], [16], [18], [19] [21] [27] are for $\alpha \notin \mathbb{N}_0$;

$$ {}_{t_0+}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} f^{(n)}(\tau)(t-\tau)^{n-\alpha-1} \mathrm{d}\tau = {}_{t_0+}I_t^{n-\alpha}\left(f^{(n)}(t)\right) \qquad (20) $$

$$ {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{t}^{t_1} f^{(n)}(\tau)(\tau-t)^{n-\alpha-1} \mathrm{d}\tau = (-1)^n \, {}_{t}I_{t_1-}^{n-\alpha}\left(f^{(n)}(t)\right) \qquad (21) $$

The relation between RL and Caputo's derivatives are [8], [16], [18], [19] [21] [27];

$$ {}_{t_0+}^{C}D_t^{\alpha} f(t) = {}_{t_0+}D_t^{\alpha}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_0)(t-t_0)^k}{k!} \right) \qquad (22) $$

$$ {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = {}_{t}D_{t_1-}^{\alpha}\left(f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t_1)(t_1-t)^k}{k!} \right) \qquad (23) $$

For $0 < \alpha < 1, [\alpha] = 0, n = 1$ we get

$$ {}_{t_0+}^{C}D_t^{\alpha} f(t) = {}_{t_0+}D_t^{\alpha}\left(f(t) - f(t_0)\right) \qquad {}_{t}^{C}D_{t_1-}^{\alpha} f(t) = {}_{t}D_{t_1-}^{\alpha}\left(f(t) - f(t_1)\right) \qquad (24) $$

Expressions (22) - (24) in a way too gives Caputo derivative once RL fractional derivatives are defined via (18) (19), for a several manifolds differentiable function. Here we note that the ${}^{C}D^{\alpha}$ the Caputo derivative is same as d^{α} (derivation in classical sense) and RL derivative that is D^{α} same as derivation in distribution sense, as discussed previously.

Let us re-discuss what we stated in the introduction that $u(t)$ appearing in the system $(D^{\alpha}x)(t) = \mathrm{A}t + \mathrm{B}u(t)$ gets modified as $\hat{u}(t) \sim \int_0^t (t-\tau)^{\alpha-1} u(t)\mathrm{d}t$ to make change in state $x(t)$. Let us take a simple case with Caputo derivative, as ${}_{0+}^{C}D_t^{\alpha} x(t) = u(t)$, $x(t) \in \mathbb{R}^{1\times 1}$ $x(0) = a$, $x(T) = b$. In this system the state matrix $\mathrm{A} = 0$. We integrate both sides by α, and write from (16) the trajectory as $x(t) = a + {}_{0}I_t^{\alpha} u(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\mathrm{d}\tau$. Putting $t = T$, we have

$x(T) = b = a + \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} u(\tau)\mathrm{d}\tau$, or $b - a = \frac{1}{\Gamma(\alpha)} \int_0^T (T-\tau)^{\alpha-1} u(\tau)\mathrm{d}\tau$. Comparing this with integer order counterpart $\dot{x}(t) = u(t)$, with $x(0) = a$ and $x(T) = b$, we have then $b-a=\int_0^T u(\tau)\mathrm{d}\tau$. In a way we have a modified control $\hat{u}(t)=\dfrac{\Gamma(\alpha)}{T}(T-t)^{1-\alpha}(b-a)$, for fractional order system, when we compare these two derivations.

c. Fractional Derivatives of alpha exponential functions

The Caputo derivative of $\tilde{e}_\alpha^{\mathrm{A}(t-t_0)}=E_\alpha(\mathrm{A}(t-t_0)^\alpha)$ we find as follows with Euler expression of Caputo derivative (and RL derivative) of the power function denoted as following. (Note that Caputo derivative of constant function is zero-but for RL derivative it is not)

$$ {}_{t_0+}^{\ C}D_t^\alpha(t-t_0)^\beta=\begin{cases}\dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha} & \beta\neq 0\\ 0 & \beta=0\end{cases} \tag{25}$$

We apply the above (25) to the series-expression (15) and get

$$\begin{aligned} {}_{t_0+}^{\ C}D_t^\alpha\left[\tilde{e}_\alpha^{\mathrm{A}(t-t_0)}\right]&={}_{t_0+}^{\ C}D_t^\alpha\{E_\alpha(\mathrm{A}(t-t_0)^\alpha)\}={}_{t_0+}^{\ C}D_t^\alpha\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{\alpha k}}{\Gamma(k\alpha+1)}\\ &=\sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{(k-1)\alpha}}{\Gamma[(k-1)\alpha+1]}=\mathrm{A}E_\alpha(\mathrm{A}(t-t_0)^\alpha)=\mathrm{A}\tilde{e}_\alpha^{\mathrm{A}(t-t_0)}\end{aligned} \tag{26}$$

Therefore we have useful relation, (26) ${}_{t_0+}^{\ C}D_t^\alpha\tilde{e}_\alpha^{\mathrm{A}(t-t_0)}=A\tilde{e}_\alpha^{\mathrm{A}(t-t_0)}$; similar to exponential function in integer order differential equation where we have; $(\mathrm{d}e^{\lambda t}/\mathrm{d}t)=\lambda e^{\lambda t}$. This alpha-exponential $\tilde{e}_\alpha^{\mathrm{A}t}$ function-2 therefore is useful in solving fractional differential equation with Caputo derivative formulation. It follows that $d^\alpha\tilde{e}_\alpha^{\lambda t}=\lambda\tilde{e}_\alpha^{\lambda t}$ or ${}_{0+}^{\ C}D_t^\alpha[\tilde{e}_\alpha^{\lambda t}]=\lambda\tilde{e}_\alpha^{\lambda t}$, that is $f(t)=\tilde{e}_\alpha^{\lambda t}$ a fundamental solution (eigenvector) of the Caputo system ${}_{0+}^{\ C}D_t^\alpha f(t)=\lambda f(t)$, with λ as eigenvalue.

The RL derivative of $e_\alpha^{\mathrm{A}(t-t_0)}$ is evaluated by applying term by term the Euler formula (25), and also using $\lim_{\alpha\to 0}(\Gamma(\alpha))^{-1}=0$, to the series expression (13) and we get the following

$$\begin{aligned} {}_{t_0+}D_t^\alpha e_\alpha^{\mathrm{A}(t-t_0)}&={}_{t_0+}D_t^\alpha\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\\ &=\sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{k\alpha-1}}{\Gamma(k\alpha)}=\mathrm{A}e_\alpha^{\mathrm{A}(t-t_0)}\end{aligned} \tag{27}$$

Using Weyl derivative formulas [8], [16], [18], [19] [21] [27], that is (28)

$$ {}_tD_{T-}^\alpha(T-\tau)^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(T-\tau)^{\beta-\alpha-1}\qquad \lim_{\beta\to\alpha}{}_tD_{T-}^\beta\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)}=\frac{\Gamma(\alpha)(T-\tau)^{\alpha-1-\beta}}{\Gamma(\beta-\alpha)\Gamma(\alpha)}=0 \tag{28}$$

We get useful eigenvector for Riemann-Liouvelli operator as (29)

$$
\begin{aligned}
{}_tD^{\alpha}_{T-}\left[e_{\alpha}^{\mathrm{A}(T-\tau)}\right] &= {}_tD^{\alpha}_{T-}\left(I\,\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)}+\mathrm{A}\,\frac{(T-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\ldots\right)\\
&= \mathrm{A}\,\frac{(T-\tau)^{\alpha-1}}{\Gamma(\alpha)}+\mathrm{A}^2\,\frac{(T-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\ldots\ldots\ldots\ldots.. \qquad (29)\\
&= (T-\tau)^{\alpha-1}\mathrm{A}\sum_{k=0}^{\infty}\mathrm{A}^k\,\frac{(T-\tau)^{k\alpha}}{\Gamma[(k+1)\alpha]} = \mathrm{A}\left[e_{\alpha}^{\mathrm{A}(T-\tau)}\right]
\end{aligned}
$$

Thus we have useful property, ${}_{t_0+}D^{\alpha}_{t}e_{\alpha}^{\mathrm{A}(t-t_0)}=\mathrm{A}e_{\alpha}^{\mathrm{A}(t-t_0)}$, which states that for fractional differential equation involving RL derivative has solution in terms of alpha-exponential function-1 ($e_{\alpha}^{\mathrm{A}t}$).This alpha-exponential $e_{\alpha}^{\mathrm{A}t}$ function-1 therefore is useful in solving fractional differential equation with Riemann-Liouvelli derivative formulation.

It follows that $D^{\alpha}[e_{\alpha}^{\lambda t}]=\lambda[e_{\alpha}^{\lambda t}]$ or ${}_{0+}D^{\alpha}_{t}[e_{\alpha}^{\lambda t}]=\lambda e_{\alpha}^{\lambda t}$, that is $f(t)=e_{\alpha}^{\lambda t}$ a fundamental solution (eigenvector) of the RL system ${}_{0+}D^{\alpha}_{t}\,f(t)=\lambda f(t)+\delta(t)$, with λ as eigenvalue.

Following interesting relation between the two alpha exponential functions is derived (30)

$$
\begin{aligned}
\int_{t_0}^{t}\mathrm{A}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{d}\tau &= \int_{t_0}^{t}\sum_{k=0}^{\infty}\mathrm{A}^{k+1}\,\frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\,\mathrm{d}\tau\\
&= \sum_{k=1}^{\infty}\mathrm{A}^k\,\frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} = E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})-\mathrm{I} = \tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}-\mathrm{I}
\end{aligned}
$$

$$
\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{I}+\int_{t_0}^{t}\mathrm{A}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{d}\tau \qquad \tilde{\Phi}_{\alpha}(t-t_0)=\mathrm{I}+\int_{t_0}^{t}\Phi_{\alpha}(t-\tau)\mathrm{A}\mathrm{d}\tau \qquad (30)
$$

Denoting $\Phi_{\alpha}(t)=e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t)=\tilde{e}_{\alpha}^{\mathrm{A}t}$ as state transition matrices or the Green's functions for homogeneous fractional multivariate dynamics with Riemann-Liouvelli and Caputo derivative formulations respectively, we have useful expression (30).

5. GENERAL SOLUTION TO SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATION WITH ALPHA EXPONENTIAL FUNCTIONS

The sequential fractional order differential equation we defined in earlier section and again we write a homogeneous SFDE as

$$
\left[{}_a\mathscr{D}_t^{\,n\alpha}+\sum_{k=0}^{n-1}a_k(t)\,{}_a\mathscr{D}_t^{\,k\alpha}\right]x(t)=0 \qquad a\in\mathbb{R} \qquad a_k(t)\in C[a,b] \qquad {}_a\mathscr{D}_t^{\,k\alpha}={}_a\mathscr{D}_t^{\,\alpha}\left({}_a\mathscr{D}_t^{\,(k-1)\alpha}\right)
$$

For $k = 2,3,.....n$. For constant coefficients, $a_k(t) = a_k$ we get LTI system, and the above system has indicial polynomial (characteristic polynomial) as $p(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k$. Also we reiterate that $\mathscr{D}_t^{\alpha}$ can be of RL fractional derivative D_t^{α}; or Caputo type fractional derivative ${}^C D_t^{\alpha}$. This is one important fact about SFDE is that the indicial polynomial are integer order polynomial just as we get indicial polynomial for an integer order differential equation (linear, quadratic, cubic etc). Assume that $x_1(t), x_2(t), ...x_n(t)$ are n functions, defined on $[a,b]$ they are called linearly dependent in $[a,b]$ if there exists constants $c_1, c_2, ..., c_n$ that are not zero simultaneously, such that $c_1 x_1(t) + c_2 x(t) + ... + c_n x_n(t) \equiv 0$, for $a \le t \le b$; else $x_1(t), x_2(t), ...x_n(t)$ are called linearly independent in $[a,b]$. To check the linear dependence in fractional calculus context we use generalized Wronskian [21], as

$$W_{\alpha}(t) = W_{\alpha}(x_1, x_2, ...x_n) = \det \begin{pmatrix} x_1(t) & x_2(t) & ... & x_n(t) \\ {}_aD_t^{\alpha} x_1(t) & {}_aD_t^{\alpha} x_2(t) & ... & {}_aD_t^{\alpha} x_n(t) \\ ... & ... & ... & ... \\ {}_aD_t^{(n-1)\alpha} x_1(t) & {}_aD_t^{(n-1)\alpha} x_2(t) & ... & {}_aD_t^{(n-1)\alpha} x_n(t) \end{pmatrix}$$

Let us assume that the fractional derivatives in SFDE are of RL type. If $x_1(t), x_2(t), ...x_n(t)$ ware solution to our SFDE, then we have ${}_aD_t^{\alpha} W_{\alpha}(t) + a_{n-1} W_{\alpha}(t) = 0 \qquad a \le t \le b$. The general solution is thus $W_{\alpha}(t) = c e_{\alpha}^{-a_{n-1}(t-a)} = c(t-a)^{\alpha-1} E_{\alpha,\alpha}(-a_{n-1}(t-a)^n)$, with c as constant. This we have got from previous sections development. Therefore, the solutions $x_1(t), x_2(t), ...x_n(t)$, of SFDE are linearly dependent in $[a,b]$ if and only if there is a $t_0 \in [a,b]$ for which $W_{\alpha}(t_0) = 0$.

If the indicial polynomial $p(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k$, has n different $\lambda_1, \lambda_2, ...\lambda_n$ simple roots, then the corresponding SFDE with RL formulation will have $x_1(t) = e_{\alpha}^{\lambda_1(t-a)}$ $x_2(t) = e_{\alpha}^{\lambda_2(t-a)}$ .... $x_n(t) = e_{\alpha}^{\lambda_n(t-a)}$ corresponding particular solutions, and the general solution to the SFDE is $x(t) = c_1 e_{\alpha}^{\lambda_1(t-a)} + c_2 e_{\alpha}^{\lambda_2(t-a)} + ... + c_n e_{\alpha}^{\lambda_n(t-a)}$. The $c_1, c_2, ...c_n$ are arbitrary constants. If the indicial polynomial $p(\lambda) = \lambda^n + \sum_{k=0}^{n-1} a_k \lambda^k$, has repeated roots with multiplicity $l, (1 < l \le n)$ then [21] $e_{\alpha}^{\lambda(t-a)}, \frac{\partial}{\partial \lambda} e_{\alpha}^{\lambda(t-a)}, \frac{\partial^2}{\partial \lambda^2} e_{\alpha}^{\lambda(t-a)}, ..., \frac{\partial^{l-1}}{\partial \lambda^{l-1}} e_{\alpha}^{\lambda(t-a)}$ are l linearly independent solutions of SFDE.

Let $\lambda_1,\lambda_2,...,\lambda_k$ be different roots with the multiplicities $l_1,l_2,....,l_k$ respectively, with $l_1+l_2+...+l_n=n$, then general solution of the SFDE is the linear combinations of the following fundamental solutions:

$$\begin{array}{lllll} e_\alpha^{\lambda_1(t-a)}, & \frac{\partial}{\partial\lambda_1}e_\alpha^{\lambda_1(t-a)}, & \frac{\partial^2}{\partial\lambda_1{}^2}e_\alpha^{\lambda_1(t-a)}, & \cdots & \frac{\partial^{l_1-1}}{\partial\lambda_1{}^{l_{1-1}}}e_\alpha^{\lambda_1(t-a)} \\ e_\alpha^{\lambda_2(t-a)}, & \frac{\partial}{\partial\lambda_2}e_\alpha^{\lambda_2(t-a)}, & \frac{\partial^2}{\partial\lambda_2{}^2}e_\alpha^{\lambda_2(t-a)}, & \cdots & \frac{\partial^{l_2-1}}{\partial\lambda_2{}^{l_{2-1}}}e_\alpha^{\lambda_2(t-a)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e_\alpha^{\lambda_k(t-a)} & \frac{\partial}{\partial\lambda_k}e_\alpha^{\lambda_k(t-a)} & \frac{\partial^2}{\partial\lambda_k{}^2}e_\alpha^{\lambda_k(t-a)} & \cdots & \frac{\partial^{l_k-1}}{\partial\lambda_k{}^{l_{k-1}}}e_\alpha^{\lambda_k(t-a)} \end{array}$$

If the SFDE is formed with Caputo derivative then we get similar solutions as in above cases with alpha exponential function-2, that is $\tilde{e}_\alpha^{\lambda(t-a)}=E_{\alpha,1}(\lambda(t-a)^\alpha)$

To illustrate let us take example of equation of motion with fractional damping term as

$$\ddot{x}(t)+\mu\,{}_0D_t^\alpha x(t)+x(t)=0 \qquad \mu>0 \qquad \alpha=1/2 \quad \text{or} \quad \alpha=3/2$$

The above homogeneous equation is called Torvik-Bagley equation, [28] is studied extensively in visco-elasticity. The parameter μ is fractional viscous coefficient. Clearly this equation can be casted into SFDE as

$${}_0\mathscr{D}_t^{4\alpha}x(t)+\mu\,{}_0\mathscr{D}_t^{\alpha}x(t)+x(t)=0 \text{ for } \alpha=1/2$$

$${}_0\mathscr{D}_t^{4\beta}x(t)+\mu\,{}_0\mathscr{D}_t^{3\beta}x(t)+x(t)=0 \text{ for } \alpha=3/2 \text{ and } \beta=1/2=\alpha/3$$

The indicial polynomial reads as $p(\lambda)=\lambda^4+\mu\lambda^k+1$ with $k=1$ for $\alpha=1/2$, and $k=3$ for $\alpha=3/2$. We take case for $k=1$; then $p(\lambda)=0=\lambda^4+\mu\lambda+1$, has four roots (real or complex, distinct or repeated) depending on the value of μ.For a specific case of value μ, we have repeated real roots $\lambda_1=\lambda_2$ and a pair of complex call them $\lambda_{3.4}$. For this combination we write general solution as

$$x(t)=c_1x_1(t)+c_2x_2(t)+c_3x_3(t)+\bar{c}_3\bar{x}_3(t)$$

With, $c_1,c_2\in\mathbb{R}$, $c_3\in\mathbb{C}$ as arbitrary constants. For case of SFDE with $\alpha=1/2$ and with Caputo derivative, $\mathscr{D}^\alpha\equiv{}^CD^\alpha;\alpha=1/2$ we take $x_i(t)=\tilde{e}_{1/2}^{\lambda_i t},(i=1,3)$. Then

$$x_2(t)=\frac{\partial}{\partial\lambda_1}\tilde{e}_{1/2}^{\lambda_1 t}=\frac{t^{1/2}}{\Gamma(3/2)}+\frac{2\lambda_1 t}{\Gamma(2)}+\frac{3\lambda_1^2t^2}{\Gamma(5/2)}+\frac{4\lambda_1^3t^2}{\Gamma(3)}+...$$

To find the general solution $x(t)$ for the SFDE of Torvik-Bagley system, it is required to study the asymptotic behavior of $x(t)$ as $t\to 0^+$. For this mechanical systems it holds

$|\dot{x}(0)| < \infty$ and $|\ddot{x}(0)| < \infty$ definitely, or equivalently the terms involving $t^{1/2}$ and involving $t^{3/2}$ vanishes in $x(t)$; namely

$$\begin{cases} c_1\lambda_1 + c_2 + c_3\lambda_3 + \bar{c}_3\bar{\lambda}_3 = 0 \\ c_1\lambda_1^3 + 3c_2\lambda_1^2 + c_3\lambda_3^3 + \bar{c}_3\bar{\lambda}_3^3 = 0 \end{cases} \qquad \mathrm{Re}(c_3) = \frac{\sqrt[4]{3}}{6}c_2 \qquad \mathrm{Im}(c_3) = \frac{-\sqrt{2}}{12}(3c_1 - 4\sqrt[4]{3}c_2)$$

With such complex number c_3 the arbitrary real constants c_1, c_2 in the general solution $x(t)$ of the Torvik-Bagley SFDE equation are determined by initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

For case of SFDE with $\alpha = 1/2$ and with RL derivative, $\mathscr{D}^{\alpha} \equiv D^{\alpha}; \alpha = 1/2$ we take $x_i(t) = e_{1/2}^{\lambda_i t}, (i = 1, 3)$. Then

$$x_2(t) = \frac{\partial}{\partial\lambda_1} e_{1/2}^{\lambda_1 t} = \frac{1}{\Gamma(1)} + \frac{2\lambda_1 t^{1/2}}{\Gamma(3/2)} + \frac{3\lambda_1^2 t}{\Gamma(2)} + \frac{4\lambda_1^3 t^{3/2}}{\Gamma(5/2)} + \dots$$

To find the general solution $x(t)$ for the SFDE of Torvik-Bagley system, it is required to study the asymptotic behavior of $x(t)$ as $t \to 0^+$. In order that $|x(0)| < \infty$ and $|\dot{x}(0)| < \infty$ the two terms appearing in $x(t)$ that are

$(c_1 + c_3 + \bar{c}_3)\frac{t^{-1/2}}{\Gamma(1/2)}$ $\quad (c_1\lambda_1^2 + 2c_2\lambda_1 + c_3\lambda_3^2 + \bar{c}_3\bar{\lambda}_3^2)\frac{t^{1/2}}{\Gamma(3/2)}$ should be zero. Therefore,

$$\begin{cases} c_1 + c_3 + \bar{c}_3 = 0 \\ c_1\lambda_1^2 + 2c_2\lambda + c_3\lambda_3^2 + \bar{c}_3\bar{\lambda}_3^2 = 0 \end{cases} \qquad \mathrm{Re}(c_3) = -\frac{1}{2}c_1 \qquad \mathrm{Im}(c_3) = \frac{1}{2\sqrt{2}}(c_1 - \sqrt[4]{3}c_2)$$

With such complex number c_3 the arbitrary real constants c_1, c_2 in the general solution $x(t)$ of the Torvik-Bagley SFDE equation are determined by initial condition $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$.

6. FRACTIONAL ORDER SYSTEM

Let us take a fractional differential equation in state variable form as the following:

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \\ y(t) = \mathrm{C}x(t) \end{cases} \qquad x(0) = x_0$$

The fractional differential linear systems with constant coefficients (where A, B, C are constant matrices), are 'time invariant' (LTI) systems. Let us put $\alpha = 1/2$ in above state space representation, with initial condition, zero for state x meaning $x_0^* = [0, 0, \dots 0]^*$, and the control $u(t)$, the following equivalence is obtained as

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}x(t)=\mathrm{A}x(t)+\mathrm{B}u(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}x(t)=\mathrm{A}^2x(t)+\left(\mathrm{AB}u(t)+\mathrm{B}\frac{\mathrm{d}^{1/2}}{\mathrm{d}t^{1/2}}u(t)\right)$$

The above equivalence states exhibits a non instantaneous effect of control $u(t)$ on integer order dynamics of state $x(t)$ at time t, through the convolution from initial time $t=0$ to present time t, via semi derivative of $u(t)$. This equivalence is obtained by writing the Laplace for system as $s^{0.5}x(s)=\mathrm{A}x(s)+\mathrm{B}u(s)$ then rearranging the same by multiplying both sides by $s^{0.5}$, and re-substituting the main equation to get $sx(s)=\mathrm{A}[s^{0.5}x(s)]+\mathrm{B}s^{0.5}u(s)$ $sx(s)=\mathrm{A}[\mathrm{A}x(s)+\mathrm{B}u(s)]+\mathrm{B}s^{0.5}u(s)$. The Laplace inversion of this gives the above time domain equivalent integer order dynamics of state. This is what was put briefly in introduction section, that is to take the modified control in case of fractional order multivariate dynamics with $u(t)$ getting modified as $\sim\int_0^t(t-\tau)^{\alpha-1}u(\tau)\mathrm{d}\tau$, and control energy (performance index) $\int_0^t|u(\tau)|^2\mathrm{d}\tau$, to fractional performance index as $\sim\int_0^t\left|(t-\tau)^{\alpha-1}u(\tau)\right|^2\mathrm{d}\tau$.

a. Solution of fractional order system of differential equation

From the above section's derivations and discussions we employ here to get solution of the fractional differential equations with Riemann-Liouvelli and Caputo type derivatives. The eigenvector as we saw in previous section is $e_\alpha^{\mathrm{A}t}$ the alpha-exponential function-1 for derivative with RL type. We write the following:

A non-homogeneous fractional differential equation with RL fractional derivative and with $0<\alpha\le 1$; $(D^\alpha\mathrm{Y})(t)=\mathrm{AY}(t)+\mathrm{B}(t)$ with $\mathrm{Y}_0=\mathrm{Y}(t_0)$, and A a constant matrix, has solution as

$$\mathrm{Y}(t)=e_\alpha^{\mathrm{A}(t-t_0)}\mathrm{Y}_0+\int_{t_0}^{t}e_\alpha^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\mathrm{d}\tau \ .$$

We can call $G_\alpha(t-\tau)=\Phi_\alpha(t-\tau)=e_\alpha^{\mathrm{A}(t-\tau)}$ Green's function associated with the homogeneous part of the system-is also (fractional) state transition matrix. The particular solution is $\mathrm{Y}_p(t)=(G_\alpha\otimes\mathrm{B})(t)=\int_{t_0}^{t}G_\alpha(t-\tau)\mathrm{B}(\tau)\mathrm{d}\tau$

The above argument is similar to that in the section 2 that is expression (2).

A fractional differential equation with Caputo derivative for $0<\alpha\le 1$

$({}_{t_0+}^{C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)$; with A as constant state-matrix, and with initial condition $\mathrm{Y}(t_0)=b$, a constant, the general solution is following

$$\mathrm{Y}(t)=b+\int_{t_0}^{t}\left(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b\right)\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right)$$

We have noted the RL-Caputo relation for $0<\alpha\le 1$ as $({}_{t_0}^{C}D_t^{\alpha}f)(t)={}_{t_0}D_t^{\alpha}[f(t)-f(t_0)]$, and using this we get; $({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)$, so we get ${}_{t_0}D_t^{\alpha}[\mathrm{Y}(t)-b]=\mathrm{AY}(t)$, now put $\mathrm{Y}(t)=\mathrm{Z}(t)+b$, so we have then $\mathrm{Z}(t_0)=0=\mathrm{Z}_0$. Thus the equivalent equation in RL derivative based fractional differential equation is; $({}_{t_0}D_t^{\alpha}\mathrm{Z})(t)=\mathrm{A}[\mathrm{Z}(t)+b]=\mathrm{AZ}(t)+\mathrm{A}b$, whose solution we know from just above derivation, and we thus write the solution as $\mathrm{Z}(t)=e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Z}_0+\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=\mathrm{Y}(t)-b$. We have thus the result now that is; $\mathrm{Y}(t)=b+\int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}b)\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right)$; as solution to this initial value fractional differential equation with Caputo derivative.

Similarly the for $({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y})(t)=\mathrm{AY}(t)+\mathrm{B}(t)$ with $\mathrm{Y}(t_0)=b$ as constant the solution we write as; $\mathrm{Y}(t)=b+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\left[\mathrm{B}(\tau)+\mathrm{A}b\right]\mathrm{d}\tau=b\left(\mathrm{I}+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\mathrm{d}\tau\right)+\int_{t_0}^{t}e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\mathrm{d}\tau$, which is also

$$\mathrm{Y}(t)=b\tilde{\Phi}_{\alpha}(t)+\int_{t_0}^{t}\Phi_{\alpha}(t-\tau)\mathrm{B}(\tau)\mathrm{d}\tau .$$

Where state transition matrices are; $\Phi_{\alpha}(t)=e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t)=\tilde{e}_{\alpha}^{\mathrm{A}t}$ corresponding to the alpha exponential functions 1 and 2.

Let Λ denote a fractional order system evaluated on $\mathbb{R}$

$$\Lambda:\qquad {}_{0+}^{C}D_t^{\alpha}x(t)=u(t)$$

Where $\alpha\in(0,1]$, with $x(0)=a\in\mathbb{R}$ and $x(T)=b\in\mathbb{R}$, where $T>0$

In terms of system matrix equation ${}^{C}D^{\alpha}x(t)=\mathrm{A}x(t)+\mathrm{B}u(t)$, in this case $\mathrm{A}=0$ and $\mathrm{B}=1$

Here $\Phi_{\alpha}(t)=e_{\alpha}^{\mathrm{A}t}=t^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\mathrm{A}^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right)=t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{A}\frac{t^{\alpha}}{\Gamma(2\alpha)}+...\right)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}\quad ;\mathrm{A}=0$

The integral $\tilde{\Phi}_\alpha(t) = \mathrm{I} + \int_0^t e_\alpha^{\mathrm{A}\tau} \mathrm{A} d\tau = \mathrm{I} = 1$ for $\mathrm{A} = 0$, is the case of system Λ. Therefore the state trajectory of system Λ, for a control input $u(t)$ is

$$x(t)\big|_0^T = x(0)\tilde{\Phi}_\alpha(t) + \int_0^T \Phi(t-\tau)u(\tau)\mathrm{d}\tau = a + \frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}u(t)\mathrm{d}t$$

b. The solution of fractional state equations with Caputo derivative

Here we formalize what we did in previous subsection, and use the results, with alpha exponential functions. Consider a linear time invariant control system denoted by Σ of fractional commensurate order α, where $0 < \alpha \leq 1$.

$$\Sigma: \qquad {}_{0+}^{\;\,C}D_t^\alpha x(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \qquad y(t) = \mathrm{C}x(t) \tag{31}$$

By commensurate order [7], [8], [14], [27] means, for each component the same fractional order of α is used. For a function $x:[0.T] \to \mathbb{R}^{n\times 1}$

$${}_{0+}^{\;\,C}D_t^\alpha x(t) = {}_{0+}^{\;\,C}D_t^\alpha \begin{pmatrix} x_1(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix} = \begin{pmatrix} {}_{0+}^{\;\,C}D_t^\alpha x_1(t) \\ \\ \\ {}_{0+}^{\;\,C}D_t^\alpha x_n(t) \end{pmatrix}$$

Where, $x(t) \in \mathbb{R}^{n\times 1}$ $u(t) \in \mathbb{R}^{m\times 1}$ matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$, $\mathrm{B} \in \mathbb{R}^{n\times m}$, $\mathrm{C} \in \mathbb{R}^{p\times n}$. The Caputo fractional derivative ${}^{C}D_t^\alpha$ is used in (31). The control is $u(t) \in \mathbb{R}^{m\times 1}$, the state is $x(t) \in \mathbb{R}^{n\times 1}$ the output (or observation) is $y(t) \in \mathbb{R}^{p\times 1}$. The forward trajectory of the system Σ starting at $t_0 = 0$ and evaluated at $t \geq 0$ is initial value problem of Fractional Differential equation ${}_{0+}^{\;\,C}D_t^\alpha x(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$, given $x(0) = a$, where $a \in \mathbb{R}^{n\times 1}$ [16], is given as, described in section 4.1, is expressed in (32)

$$x(t) = \left(\mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\mathrm{d}\tau\right)a + \int_0^t \Phi_\alpha(t-\tau)\mathrm{B}u(\tau)\mathrm{d}\tau \tag{32}$$

In (32), $\Phi_\alpha(t) = e_\alpha^{\mathrm{A}t}$. Another way to write (32) is (33) as was derived in section 4.1 is;

$$x(t) = \tilde{\Phi}_\alpha(t)a + \int_0^t \Phi_\alpha(t-\tau)\mathrm{B}u(\tau)\mathrm{d}\tau \;\; ; \;\; \tilde{\Phi}_\alpha(t) = E_\alpha(\mathrm{A}t^\alpha) = \mathrm{I} + \int_0^t \Phi_\alpha(\tau)\mathrm{A}\mathrm{d}\tau \tag{33}$$

Similar to expression (3) [24] and taking into account the output expression of Σ as in (31); the forward output trajectory is as following

$$y(t)=\mathrm{C}x(t)=\mathrm{C}\left(\mathrm{I}+\int_0^t \Phi_\alpha(\tau)\mathrm{A}\mathrm{d}\tau\right)a+\mathrm{C}\int_0^t \Phi_\alpha(t-\tau)\mathrm{B}u(\tau)\mathrm{d}\tau \qquad (34)$$

7. CONTROL GRAMIAN OF FRACTIONAL ORDER SYSTEM

Let $T>0$, the system Σ is controllable on $[0,T]$ if for any $a\in\mathbb{R}^{n\times 1}$ and $b\in\mathbb{R}^{n\times 1}$, there is control signal $u(.)$ defined on $[0,T]$ which steers the initial state $x(0)=a$ to final state $x(T)=b$.

Our integer order Gramian of control is $Q(t_0,t_1)\triangleq\int_{t_0}^{t_1}\Phi(t_0,t)\mathrm{B}(t)\mathrm{B}^*(t)\Phi^*(t_0,t)\mathrm{d}t$, obtained from (6), or $Q(t)=\int_0^t e^{-\mathrm{A}\tau}\mathrm{B}\mathrm{B}^*e^{-\mathrm{A}^*\tau}\mathrm{d}\tau=\int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}\mathrm{B}^*e^{\mathrm{A}^*(t-\tau)}\mathrm{d}\tau$. In this replacing $e^{\mathrm{A}t}$ by $e_\alpha^{\mathrm{A}t}$ we get the $\int_0^t e_\alpha{}^{\mathrm{A}(t-\tau)}\mathrm{B}\mathrm{B}^*e_\alpha{}^{\mathrm{A}^*(t-\tau)}\mathrm{d}\tau=\int_0^T(T-\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\mathrm{A}(T-\tau)^\alpha)^k}{\Gamma((1+k)\alpha)}\mathrm{B}\mathrm{B}^*(T-\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(\mathrm{A}^*(T-\tau)^\alpha)^k}{\Gamma((1+k)\alpha)}\mathrm{d}\tau$, this integral as similar control Gramian. Surely this diverges as $\tau\to T$. In order to have converging integral we multiply the above integral by $(T-\tau)^{2(1-\alpha)}$, and define the new control Gramian for Σ system, [17] [25]; as following.

$$\begin{aligned}Q_\alpha(0,T)&=\int_0^T(T-t)^{2(1-\alpha)}e_\alpha^{\mathrm{A}(T-t)}\mathrm{B}\mathrm{B}^*e_\alpha^{\mathrm{A}^*(T-t)}\mathrm{d}t\\&=\int_0^T\left[(T-t)^{2(1-\alpha)}\right]\Phi_\alpha(T-t)\mathrm{B}\mathrm{B}^*\Phi_\alpha^*(T-t)\mathrm{d}t\end{aligned} \qquad (35)$$

The controllability Gramian, of fractional order α , on the time interval, $[0,T]$, corresponding to the system Σ in (31). The term $(T-t)^{2(1-\alpha)}$ in (35) is called a neutralizer of singularity at $t=T$ [17], required for convergence of (35).

8. STATE TRAJECTORY AND CONTROL EFFORT FOR FRACTIONAL ORDER MULTIVARIATE DYNAMIC SYSTEM

a. Proposition for minimal effort control for fractional order system

Let $T > 0$ and the control Gramian (35) for system Σ, given by (31) Q_α is nonsingular. Then

(a) For any states $a,b \in \mathbb{R}^{n\times 1}$, the control law

$$\bar{u}(t) = -(T-t)^{2(1-\alpha)} \mathrm{B}^* \Phi_\alpha^* (T-t) Q_\alpha^{-1} f_T(a,b) \qquad t \in [0,T) \tag{36}$$

Where $f_T(a,b) = \left(\mathrm{I} + \int_0^T \Phi_\alpha(t)\mathrm{A}dt \right) a - b = -b + \tilde{\Phi}_\alpha(T)a$ and $\bar{u}(T) = 0$ drives point a to b.

This (36) is similar to (7) which is $u(t) = -\mathrm{B}^*(t)\Phi^*(t_0,t)Q^{-1}(t_0,t_1)\left[x_0 - \Phi(t_0,t_1)x_1\right]$, with extra factor as $(T-t)^{2(1-\alpha)}$, for neutralizing singularity at T.

(b) Among all possible controls driving a to b in time T the control $\bar{u}(t)$ defined in (36) minimizes the integral

$$J_{m(0-T)} = \int_0^T \left| (T-t)^{\alpha-1} u(t) \right|^2 \mathrm{d}t \tag{37}$$

The (37) is similar to (9)which is $J_{(t_0-t)} = \int_{t_0}^{t} u^*(t)u(t)\mathrm{d}t$, with extra term $(T-t)^{2(1-\alpha)}$; as we discussed at length that from integer order multivariate control the instantaneous control $u(t)$ gets as convoluted action in case of fractional order system $\sim (T-t)^{(1-\alpha)}u(t)$.

Here we define inner product as $\langle f, g \rangle \triangleq \int_{t_0}^{t} f^*(t)g(t)\mathrm{d}t$; f^* denoting transpose of matrix f.

Moreover, $\int_0^T \left| (T-t)^{\alpha-1} u(t) \right|^2 \mathrm{d}t = \left\langle Q_\alpha^{-1} f_T(a,b),\, f_T(a,b) \right\rangle$.

b. Proof of the proposition

(a) From the expression (36) of $\bar{u}$ we have

$$\begin{aligned} x(T) &= f_T(a,b) + b - \left(\int_0^T (T-t)^{2(1-\alpha)} \Phi_\alpha(T-t)\mathrm{B}\mathrm{B}^*\Phi_\alpha^*(T-t)\mathrm{d}t \right) Q_\alpha^{-1} f_T(a,b) \\ &= f_T(a,b) + b - Q_\alpha Q_\alpha^{-1} f_T(a,b) = b \end{aligned} \tag{38}$$

Also we have $\lim_{t\to T_-}\bar{u}(t)=\bar{u}(T)$

(b) Take $\eta(t)=\left|(T-t)\right|^{2(1-\alpha)}$, and we observe following steps;

$$\begin{aligned}\int_0^T\left|(T-t)^{\alpha-1}\bar{u}(t)\right|^2\mathrm{d}t&=\int_0^T\left|(T-t)^{1-\alpha}\mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b)\right|^2\mathrm{d}t\\&=\int_0^T\eta(t)\left\langle \mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b),\mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b)\right\rangle\mathrm{d}t\\&=\left\langle\int_0^T\eta(t)\Phi_\alpha(T-t)\mathrm{B}\mathrm{B}^*\Phi_\alpha^*(T-t)\mathrm{d}t,Q_\alpha^{-1}f_T(a,b)\right\rangle\\&=\left\langle Q_\alpha Q_\alpha^{-1}f_T(a,b),Q_\alpha^{-1}f_T(a,b)\right\rangle=\left\langle f_T(a,b),Q_\alpha^{-1}(a,b)\right\rangle\end{aligned}\tag{39}$$

Take another control signal $u(t)$ for which $(T-t)^{\alpha-1}u(t)$ is square integrable on $\left[0,T\right]$ and $x(T)=b$, then

$$\begin{aligned}&\int_0^T(T-t)^{2(1-\alpha)}\left\langle u(t),\bar{u}(t)\right\rangle\mathrm{d}t\\&=-\int_0^T\eta(t)\left\langle u(t),(T-t)^{2(1-\alpha)}\mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b)\right\rangle\mathrm{d}t\\&=-\int_0^T\left\langle u(t),\mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b)\right\rangle\mathrm{d}t=\left\langle f_T(a,b),Q_\alpha^{-1}f_T(a,b)\right\rangle\end{aligned}\tag{40}$$

Therefore

$$\int_0^T(T-t)^{2(1-\alpha)}\left\langle u(t),\bar{u}(t)\right\rangle\mathrm{d}t=\int_0^T(T-t)^{2(1-\alpha)}\left\langle \bar{u}(t),\bar{u}(t)\right\rangle\mathrm{d}t=\int_0^T(T-t)^{2(1-\alpha)}\left|u(t)\right|^2\mathrm{d}t\tag{41}$$

And from above we get

$$\int_0^T(T-t)^{2(1-\alpha)}\left|u(t)\right|^2\mathrm{d}t=\int_0^T(T-t)^{2(1-\alpha)}\left|\bar{u}(t)\right|^2\mathrm{d}t+\int_0^T(T-t)^{2(1-\alpha)}\left|u(t)-\bar{u}(t)\right|^2\mathrm{d}t\tag{42}$$

Which gives the minimality property for the integral

c. Calculations of state transition matrix, Gramian, state trajectory, control effort of Fractional Order Multivariate System

Let system $\sum 1$ defined as

$$\sum 1:\quad \begin{cases}{}^{C}D_{0+}^{0.5}x_1(t)=x_2(t)\\ {}^{C}D_{0+}^{0.5}x_2(t)=u(t)\end{cases}\qquad \mathrm{A}=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad \mathrm{B}=\begin{pmatrix}0\\1\end{pmatrix}\tag{43}$$

Take initial point $a=(1,0)^*$, final point as origin $b=(0,0)^*$.

$$\Phi_\alpha(t)=\mathcal{L}^{-1}\left\{(s^{0.5}\mathrm{I}-\mathrm{A})^{-1}\right\}=\mathcal{L}^{-1}\left\{\left(\begin{bmatrix}\sqrt{s} & 0\\ 0 & \sqrt{s}\end{bmatrix}-\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right)^{-1}\right\}=\mathcal{L}^{-1}\left\{\begin{pmatrix}\sqrt{s} & -1\\ 0 & \sqrt{s}\end{pmatrix}^{-1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}\begin{pmatrix}\sqrt{s} & 1\\ 0 & \sqrt{s}\end{pmatrix}\right\}=\mathcal{L}^{-1}\begin{pmatrix}\frac{1}{\sqrt{s}} & \frac{1}{s}\\ 0 & \frac{1}{\sqrt{s}}\end{pmatrix}$$

Using $\mathcal{L}^{-1}\{s^{-\alpha}\}=t^{\alpha-1}/\Gamma(\alpha)$; $\mathcal{L}^{-1}\{s^{-0.5}\}=1/\sqrt{\pi t}$, and $\mathcal{L}\left\{\begin{cases}1 & t>0\\ 0 & t<0\end{cases}\right\}=\frac{1}{s}$, we write

$$\Phi_\alpha(t)=\begin{pmatrix}\frac{1}{\sqrt{\pi t}} & 1\\ 0 & \frac{1}{\sqrt{\pi t}}\end{pmatrix}$$

$$\tilde{\Phi}_\alpha(t)=\mathrm{I}+\int_0^t\Phi_\alpha(\tau)\mathrm{A}\,\mathrm{d}\tau=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\int_0^t\begin{pmatrix}\frac{1}{\sqrt{\pi\tau}} & 1\\ 0 & \frac{1}{\sqrt{\pi\tau}}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\mathrm{d}\tau=\begin{pmatrix}1 & 2\sqrt{\frac{t}{\pi}}\\ 0 & 1\end{pmatrix}$$

Using formulas (33) we get state trajectory as

$$x(t)=\begin{pmatrix}1 & \frac{2\sqrt{t}}{\sqrt{\pi}}\\ 0 & 1\end{pmatrix}a+\int_0^t\begin{pmatrix}\frac{1}{\sqrt{\pi(t-\tau)}} & 1\\ 0 & \frac{1}{\sqrt{\pi(t-\tau)}}\end{pmatrix}\mathrm{B}u(\tau)\mathrm{d}\tau$$

Taking $u(t)\equiv 1$, then for a given a, $x(t)=\begin{pmatrix}1+t & 2\sqrt{\frac{t}{\pi}}\end{pmatrix}^*$. Meaning for a constant $u(.)\equiv 1$ for $t>0$ we cannot steer the given initial condition a to the origin.

Let $f_T(a,b)=\tilde{\Phi}_\alpha(T)a-b=a$. The control Gramian form is

$$Q_\alpha=\int_0^T\left[(T-t)^{2(1-\alpha)}\right]\Phi_\alpha(T-t)\mathrm{BB}^*\Phi_\alpha^*(T-t)\mathrm{d}t=\begin{pmatrix}\frac{T^2}{2} & \frac{2T^{3/2}}{3\sqrt{\pi}}\\ \frac{2T^{3/2}}{3\sqrt{\pi}} & \frac{T}{\pi}\end{pmatrix}$$

And the control

$$\bar{u}(t)=-(T-t)^{2(1-\alpha)}\mathrm{B}^*\Phi_\alpha^*(T-t)Q_\alpha^{-1}f_T(a,b)=-\frac{18(T-t)}{T^2}+\frac{12\sqrt{T-t}}{T^{3/2}}$$

Drives the point a to b with modified energy

$$J_{m(0-T)}=\int_0^T\left|(T-t)^{-0.5}\overline{u}(t)\right|^2\mathrm{d}t=\frac{18}{T^2}$$

For $0<\alpha<1$ a system in $\mathbb{R}^3$ that is $\sum 2$ is described as

$$\sum 2:\qquad \begin{cases} {}^{C}D_{0+}^{\alpha}x_1(t)=x_2(t) \\ {}^{C}D_{0+}^{\alpha}x_2(t)=-x_1(t)+u(t)\end{cases}\qquad \mathrm{A}^0=\mathrm{I}\qquad \mathrm{A}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\qquad \mathrm{A}^2=-\mathrm{I}$$

The system matrix is skew symmetric; hence $\mathrm{A}^k=\mathrm{I}$ for $k=0,4,8,..$, $\mathrm{A}^k=\mathrm{A}$ for $k=1,5,9,...$ and $\mathrm{A}^k=-\mathrm{I}$ for $k=3,7,11,...$. Also

$$\Phi_\alpha(t)=t^{\alpha-1}\left(\mathrm{I}\frac{1}{\Gamma(\alpha)}+\mathrm{A}\frac{t^\alpha}{\Gamma(2\alpha)}-\mathrm{I}\frac{t^{2\alpha}}{\Gamma(3\alpha)}-\mathrm{A}\frac{t^{3\alpha}}{\Gamma(4\alpha)}+...\right)$$
$$=\mathrm{I}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}-\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}+..\right)+\mathrm{A}\left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{t^{4\alpha-1}}{\Gamma(4\alpha)}+...\right)$$

Use the following notation, to simplify above expression as following

$$\sin_\alpha t=\left(\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{t^{4\alpha-1}}{\Gamma(4\alpha)}+...\right)\qquad \cos_\alpha t=\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}-\frac{t^{3\alpha-1}}{\Gamma(3\alpha)}+...\right)$$

$$\Phi_\alpha(t)=I\sin_\alpha t+A\cos_\alpha t=\begin{pmatrix}\sin_\alpha t & \cos_\alpha t\\ -\cos_\alpha t & \sin_\alpha t\end{pmatrix}$$

With $B=\begin{pmatrix}0 & 1\end{pmatrix}^*$; we write Gramian of control as

$$Q_\alpha=\int_0^T(T-t)^{2(1-\alpha)}\begin{pmatrix}\sin_\alpha^2(T-t) & \sin_\alpha(T-t)\cos_\alpha(T-t)\\ \sin_\alpha(T-t)\cos_\alpha(T-t) & \cos_\alpha^2(T-t)\end{pmatrix}$$

We note here that to find exact formula of Q_α and Q_α^{-1} is difficult, thus to find the control. We simplify for $\alpha=1/2$, with some approximation can be carried out like following

$$\sin_{1/2}t=\left(\frac{t^0}{\Gamma(1)}-\frac{t^1}{\Gamma(2)}+......\right)=e^{-t}$$

$$\cos_{1/2}t=\left(\frac{t^{-1/2}}{\Gamma(1/2)}-\frac{t^{1/2}}{\Gamma(3/2)}+..\right)=\left(\frac{t^{-1/2}}{\Gamma(1/2)}\right)(1-2t+...)=\left(\frac{1}{\sqrt{\pi t}}\right)(1-2t+..)$$

$$\cong c_N(t)=\frac{1}{\sqrt{\pi t}}\left(1-\sum_{k=1}^{N}\frac{2^k t^{2k-1}}{\prod_{i=1}^{k}(2i-1)}\right)\qquad ,N=1,2,..$$

This was we have approximate and state transition matrix as Gramian as (for $\alpha=1/2$)

$$Q_{1/2}(0,T)=\int_0^T (T-t)\begin{pmatrix} e^{-2(T-t)} & c_N(T-t)e^{-(T-t)} \\ c_N(T-t)e^{-(T-t)} & c_N^2(t) \end{pmatrix} \qquad \Phi_{1/2}(t)=\begin{pmatrix} e^{-t} & c_N(t) \\ -c_N(t) & e^{-t} \end{pmatrix}$$

The above simplified control Gramian and state transition matrix gives approximations for different choices of N. With $N=1$, we get $c_1=(\pi t)^{-1/2}(1-2t)$, and so on. This eases the calculations; and one can get for the system $\sum 2$ the trajectory, control effort (energy) for given start and end point.

9. DISCUSSION WITH COMPARISION OF FRACTIONAL ORDER & INTEGER ORDER SYSTEM

In the classical theory of integer order multivariate state space dynamic systems, it is true that $\operatorname{rank}\mathrm{B}=n$ is sufficient to decide on controllability of system. For fractional order state space we discussed grossly and then via Gramian formulation that one needs to construct special control $u(t)$. In section 3.1 we stated about one such construction for having identity relation for $e_\alpha^{\mathrm{A}t}$. We elucidate this modified control via following system

Let Λ denote a fractional order system evaluated on $\mathbb{R}$

$$\Lambda: \qquad {}_{0+}^{C}D_t^{\alpha}x(t)=u(t)$$

Where $\alpha\in(0,1]$, with $x(0)=a\in\mathbb{R}$ and $x(T)=b\in\mathbb{R}$, where $T>0$

In terms of system matrix equation ${}^{C}D^{\alpha}x(t)=\mathrm{A}x(t)+\mathrm{B}u(t)$, in this case $\mathrm{A}=0$ and $\mathrm{B}=1$

Here $\Phi_\alpha(t)=e_\alpha^{\mathrm{A}t}=t^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\mathrm{A}^k t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right)=t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)}+\mathrm{A}\frac{t^{\alpha}}{\Gamma(2\alpha)}+...\right)=\frac{t^{\alpha-1}}{\Gamma(\alpha)} \qquad ;\mathrm{A}=0$

The integral $\tilde{\Phi}_\alpha(t)=\mathrm{I}+\int_0^t e_\alpha^{\mathrm{A}\tau}\mathrm{A}d\tau=\mathrm{I}=1$ for $\mathrm{A}=0$, is the case of system Λ. Therefore the state trajectory (32) of system Λ, for a control input $u(t)$ is, with $\mathrm{B}=1$ is as follows;

$$x(t)\big|_0^T=x(0)\tilde{\Phi}_\alpha(t)+\int_0^T\Phi(t-\tau)u(\tau)\mathrm{d}\tau=a+\frac{1}{\Gamma(\alpha)}\int_0^T(T-t)^{\alpha-1}u(t)\mathrm{d}t$$

In this system $\mathrm{B}=1$ and $\mathrm{B}^*=1$ with $\operatorname{rank}\mathrm{B}=1$; also $\mathrm{BB}^*=\mathrm{I}$. Let $f(t)=t^{1-\alpha}F(t)$ such that $e_\alpha^{\mathrm{A}t}f(t)=E_{\alpha,\alpha}(\mathrm{A}t^{\alpha})F(t)=\mathrm{I}$ for $t\neq 0$ and $\lim_{t\to 0}e_\alpha^{\mathrm{A}t}=\mathrm{I}$, this is as per section 3.1. Then the modified control

$$\hat{u}(t)=\frac{1}{T}\mathrm{B}^* f(T-t)[b-\tilde{\Phi}_\alpha(T)a],\ t\in[0,T]$$

will transfer state from a to state b in $T>0$. This is verified as from direct calculations of using state trajectory expressions (32) and identity properties of $f(t)$ and B, as described below;

$$x(T)=\tilde{\Phi}_{\alpha}(T)a+\frac{1}{T}\int_{0}^{T}\Phi_{\alpha}(T-t)\mathrm{BB}^{*}f(T-t)[b-\tilde{\Phi}_{\alpha}(T)a]\mathrm{d}t$$

$$=\tilde{\Phi}_{\alpha}(T)a+\frac{1}{T}\int_{0}^{T}e_{\alpha}^{\mathrm{A}(T-t)}(\mathrm{I})f(T-t)[b-\tilde{\Phi}_{\alpha}(T)a]\mathrm{d}t$$

$$=\tilde{\Phi}_{\alpha}(T)a+\frac{1}{T}\int_{0}^{T}(\mathrm{I})(\mathrm{I})[b-\tilde{\Phi}_{\alpha}(T)a]\mathrm{d}t=\frac{1}{T}\int_{0}^{T}[b-\tilde{\Phi}_{\alpha}(T)a]\mathrm{d}t=b$$

Our, $f(t)=t^{1-\alpha}\Gamma(\alpha)$ is described in section 3.1, for a case with $\mathrm{A}=0$; so the modified control is obtained by putting $\mathrm{B}^{*}=1$, $\tilde{\Phi}_{\alpha}(T)=1$, $f(T-t)=(T-t)^{1-\alpha}\Gamma(\alpha)$, the values as $\hat{u}(t)=\frac{1}{T}\mathrm{B}^{*}f(T-t)[b-\tilde{\Phi}_{\alpha}(T)a]=\frac{\Gamma(\alpha)}{T}(T-t)^{1-\alpha}(b-a)$ for system Λ, transferring the state from a to b, with modified energy as per (37) is $J_{m}=\frac{[\Gamma(\alpha)]^{2}(b-a)^{2}}{T}$

This gives discussion as to why we needed modified control, for fractional order system.

10.CONCLUSIONS

We demonstrated the use of sequential fractional differential equation in matrix form, and its solutions, which are useful in engineering aspects of controllability, for multivariate fractional state space dynamic systems. The solutions are as rigorous as their integer order counterparts. We have derived the basic state transition matrix for fractional order systems with Caputo’s formulation and shown that it has two types; both participate in the solution to give state trajectory calculations. Thereby with these state transition matrices, we derived control Gramian, state trajectories, input control vector and control energy for fractional order system. We stressed the need for modified control and thus modified controller effort, performance index for the fractional order system vis-à-vis integer order systems, as we showed inherently the control action in fractional order system is convolute from starting time to present time-inheriting memory.

11.REFERENCES

[1] O. P Agarwal, D. Baleanu, 2007, “A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems”, J. Vib. Control **13** No. 9-10, pp. 1269-1281.

[2] R. Almeida, A. B. Malinowska, D. F. M. Torres, 2010 “ A fractional calculus of variations for multiple integrals with application to vibrating string”, J. Math. Phys. **51**, No. 3, pp 1-12.

[3] R. Almeida, D. F. M. Torres, 2009, “Calculus of variations with fractional derivatives and fractional integrals”, Appl. Math. Lett. **22** , No. 12, pp. 1816-1820.

[4] D. Baleanu, O. Defterli, O. P. Agarwal, 2009 “A central difference numerical scheme for fractional optimal control problem”, J. Vib. Control **15** No. 4, pp. 583-597.

[5] D. Baleanu, 2008, “New applications of fractional variation principles”, Rep. Math. Phys. **61** No. 2, pp. 199-206.

[6] M. Benchora, S. Hamani, S. K. Ntouyas, 2008, “Boundary value problems for differential equations with fractional order”, Surv. Math. Appl. **3** , pp. 1-12.

[7] M. Bettayeb, S. Djennoune. 2008, “New results on controllability and observability of fractional dynamic systems”, J. Vib. Control **14**, No. 9-10, pp. 1531-1541.

[8] Shantanu Das, 2007, “Functional Fractional Calculus for system identification and controls” Springer-Verlag.

[9] R. A. El-Nabulsi, D. F. M. Torres,2007 “ Necessary optimality conditions for fractional action like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β) ”, Math. Methods Appl. Sci. **30** No. 15, pp. 1931-1939.

[10] G. S. F. Frederico, D. F. M. Torres, 2008, “Fractional conservation laws in optimal control theory”, Nonlinear Dynamics **53**, No. 3, pp. 215-222.

[11] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, 1955, “ Higher transcendental functions”, Vol. III, Mc Graw-Hill, New York.

[12] S. Guermah, S. Djennoune, M. Bettayeb, 2008, “Controllability and observability of linear discrete time fractional order systems”, Int. J. Appl. Math. Comput. Sci. **18** No. 2, pp. 213-222.

[13] R. Hotzel, M Fliess, 1998, “On linear systems with fractional derivation; introductory theory and examples”, Math. Comput. Simulation **45**, No. 3-4, pp. 385-395.

[14] T. Kaczorek, 2009, “Reachability of cone fractional continuous-time linear systems”, Int. J. Appl. Math. Comput. Sci. **19**, No.1, pp.89-93.

[15] T. Kaczorek, 2008, “Fractional positive continuous time linear systems”, Int. J. Appl. Math. Comput. Sci. **18**, No.2, pp. 223-228.

[16] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, 2006, “Theory and Applications of Fractional Differential Equations”, Elsevier, Amsterdam.

[17] D. Matgnon, B, d’Andrea-Novel, 1996, “Some results on controllability and observability of finite dimensional fractional differential systems”, IMACS, IEEE-SMC Proceedings’ Conference, Lille, France, 952-956.

[18] K. B. Oldham and J. Spanier, 1974, “Fractional calculus”. Academic Press San Diego.

[19] Igor Podlubny, 1999, “Fractional differential equations”, Academic Press San Diego.

[20] J. Sabatier, O. P. Agarwal, J. A. Tenreiro Machado (Eds.), 2007, “ Advances in Fractional Calculus”, Springer, Dordrecht.

[21] K.S. Miller and B. Ross. 1993, “An Introduction to the Fractional Calculus and Fractional Differential Equations”, Wiley, New York.

[22] D. Sierociuk, A. Dzielinski. 2006, “Fractional Kalman filter algorithm for states, parameters and order of fractional system estimation”, Int. J. Appl. Math. Comput. Sci. **16**, No. 1, pp.129-140.

[23] B. M. Vinagre, C. A. Monje, A. J. Caldero, 2002, “Fractional order systems and fractional order actions, in Tutorial Workshop N0. 2, Fractional Calculus Applications in Automatic Control and Robotics, 41st IEEE CDC, Las Vegas.

[24] M Gopal,1994, “Modern Control System 2nd Edition”, Willey Eastern Ltd.

[25] Shantanu Das, 2012, “Evolution of Temporal Fractional Derivative due to Spatial Stochastic Disorder in Transport Phenomena", International Journal of Mathematics and Computation, Vol. **17**, Issue-4, pp. 1-20.

[26] Shantanu Das, 2012, "Formation of Fractional Derivative in Time due to Propagation of Free Greens Function in Spatial Stochastic Disorder Field for

Transport Phenomena", International Journal of Mathematics and Computation Vol. **17**, Issue-4, pp. 68-92.

[27] Shantanu Das, 2011, "Functional Fractional Calculus 2nd Edition"; Springer Verlag .

[28] R L Bagley, P. J. Torvik, "On appearance of the fractional derivative in behavior of materials", ASME J. Appl. Mech, 1984; 51 (2): 294-8

[29] D F M Torres et al, "Minimum modified energy control for fractional linear control with Caputo derivative", arXiv:1004.3113v1 [math OC] 2010