Essence of FRACTIONAL CALCULUS in applied sciences

Part-I

WORK SHOP ON
FRACTIONAL ORDER SYSTEM
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IEEE KOLKATA CHAPTER
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Salute to Indian Mathematicians of Fractional Calculus

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…………

………………and to all exponents around the globe to have given this wonderful subject to us applied scientists and engineers, a language what nature understands the best, to communicate with nature in better and efficient way.

<u>Essence of fractional calculus</u> is…….

………….in understanding nature better.

………….in making effort to have this subject as Popular Science.

………….in simple teaching and evolving the future methods in mathematics and making working systems

……………in realizing that our physical understanding is limited and mathematical tools go far beyond our understanding

……………in appreciating the wonderful world of mathematics that lays between integer order differentiation and integration.

What is not FRACTIONAL CALCULUS

Fractional Calculus does not mean the calculus of fractions, nor doest it mean a fraction of any calculus, differentiation, integration or calculus of variations.

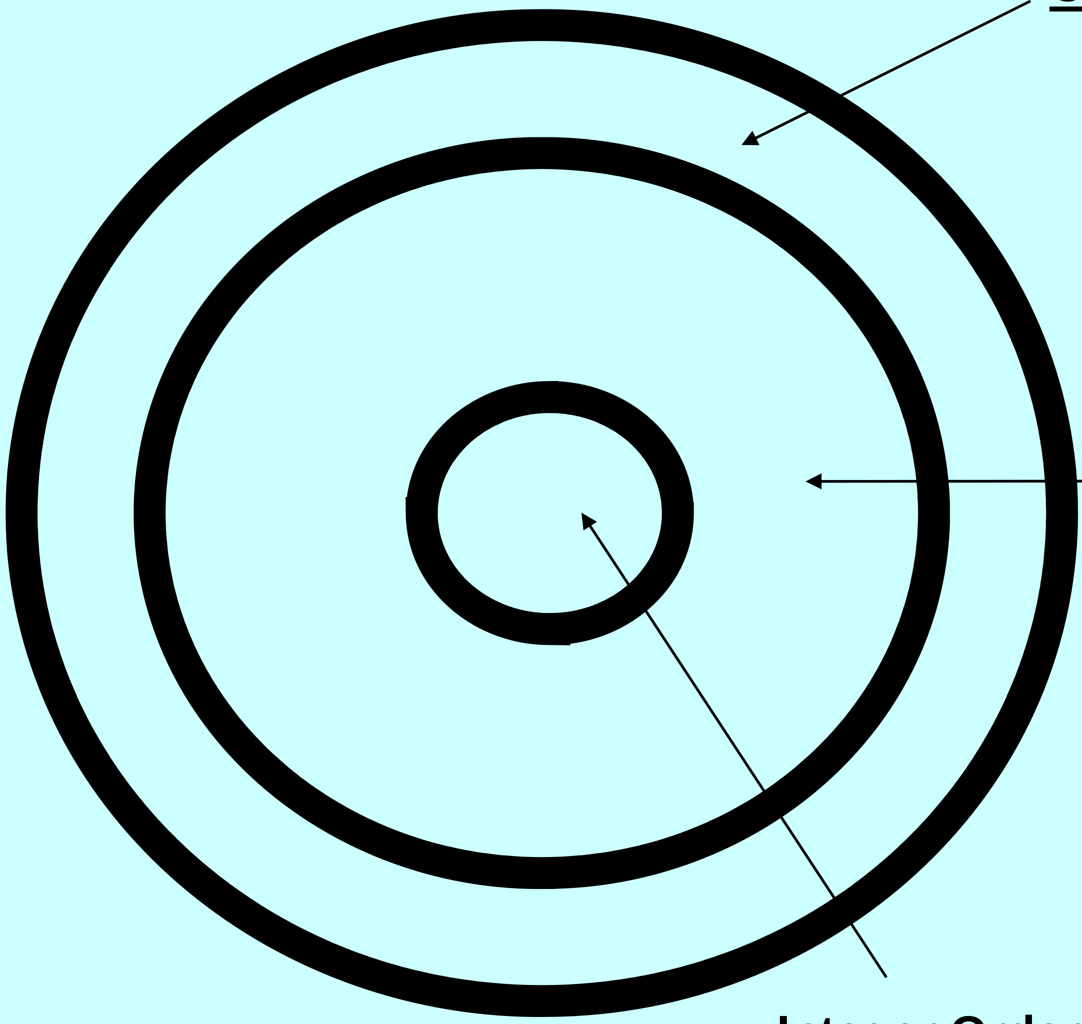
The FRACTIONAL CALCULUS is a name of theory of integration and derivatives of arbitrary order, which unify and generalize the notion of integer order n-fold repeated differentiation and n-fold repeated integration.

FRACTIONAL CALCULUS is

GENERALIZED differentiation and integration.

GENERALIZED DIFFERINTEGRATIONS

THE GENERALIZED CALCULUS



Complex-Order

$$\frac{d^{\alpha + i\beta} f(z)}{dz^{\alpha + i\beta}}$$

$$\alpha, \beta \in \mathbb{R}$$

Fractional order

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}}$$

$$\alpha \in \mathbb{R}$$

Non-local
Distributed
History/heredity
Non-Markovian

Integer Order
Newtonian
Point property

$$\frac{d^{n} f(x)}{dx^{n}}$$

$$n \in \mathbb{Z}$$

Generalization of theory of numbers and calculations

$2^3 = 2 \times 2 \times 2 = 8$ **Can be visualized**

$2^{0.5} = \exp\{(0.5)\ln 2\} = 1.414$ **Number exists but hard to visualize how.**

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ **Is a visualized quantity, but what about** $(5.5)!$

Generalized factorial as GAMMA FUNCTION $(5.5)! = \Gamma(1+5.5) = \Gamma(6.5) = 287.88$

$$x^r = e^{r \ln x}, r \in \mathbb{R}$$

$$x! = \Gamma(x+1) = x\Gamma(x)$$

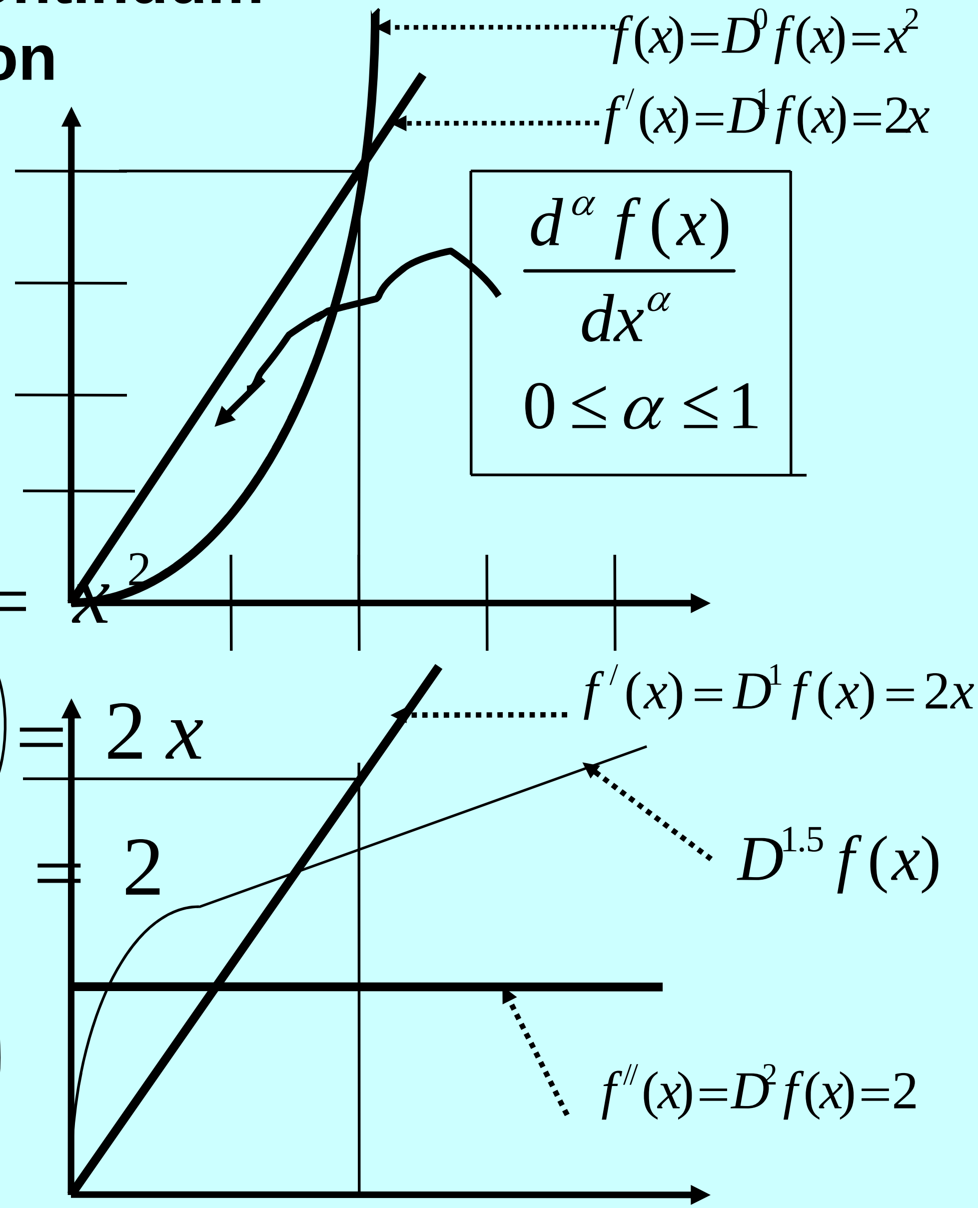
$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

$$\Gamma(x) = \lim_{n \to \infty} \frac{(n!)n^x}{x(x+1)(x+2)\dots(x+n)}$$

Wonderful universe of mathematics lays in between
One full integration and one full differentiation

$$\iint f dx \qquad \int f dx \qquad f \qquad \frac{df}{dx} \qquad \frac{d^2 f}{dx^2} \qquad \frac{d^3 f}{dx^3}$$

Fractional calculus gives continuum between full differ-integration



$$f(x) = x^2$$

$$f^{/}(x) = 2x$$

$$f^{//}(x) = 2$$

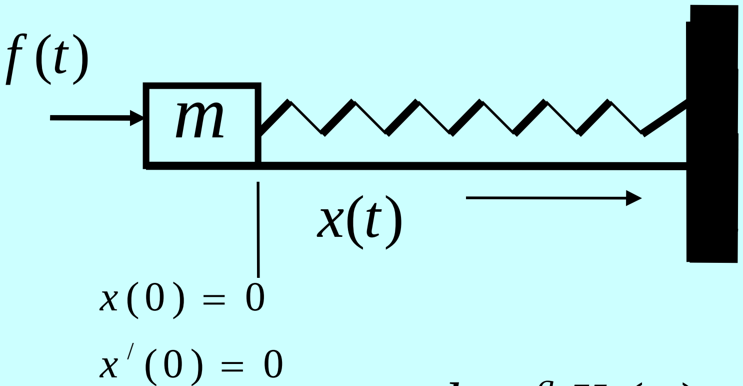
Curve fitting will be effective by use of fractional differential equation, as compared with polynomial regression and integer order differential equation. The reason is extra freedom to closely track the the curvature in continuum.

Could be a magnifier tool to observe the formation of discontinuity.

Application-I

Generalization of Newtonian mechanics and differential equations

$$mx^{//}(t)+b_0x^{/}(t)+kx(t)=f(t)$$



Mass concentrated at point
Mass less spring
Frictionless spring
Infinite wall

$$(ms^2+b_0s+k)X(s)=F(s)$$

Spring with friction →

$$k_q s^q X(s)=F_{sp}(s)$$

$$0\le q\le 1$$

$$(ms^2+b_0s+k_qs^q+k)X(s)=F(s)$$

$$(ms^2+b_0s+k_{q_n}s^{q_n}+k_{q_{n-1}}s^{q_{n-1}}+\ldots+k_{q_1}s^{q_1}+k_{q_0})X(s)=F(s)$$

$$\left(\sum_{n=0}^{N=2}k_ns^{q_n}\right)X(s)=F(s)$$

$$\left(\int_0^2 k(q)s^q\right)X(s)=F(s)$$

Distributed mass
Spring with mass
Spring with friction
Damping with spring action
Non conservation system
Leaky wall/termination

Application-II

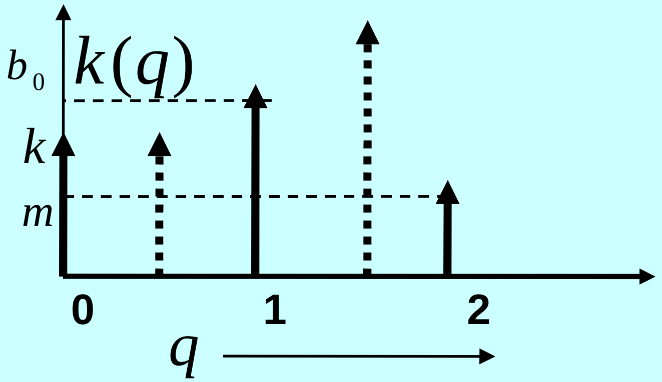
System Identification & order distribution

Integer Order:

$$mx''(t)+b_0x'(t)+kx(t)=f(t)$$

$$(ms^2+b_0s+k)X(s)=F(s)$$

$$\{\int_0^\infty[m\delta(q-2)+b_0\delta(q-1)+k\delta(q)]s^q dq\}X(s)=F(s)$$



Fractional Order

$$(ms^2+b_1s^{3/2}+b_0s+k_1s^{1/2}+k_0)X(s)=F(s)$$

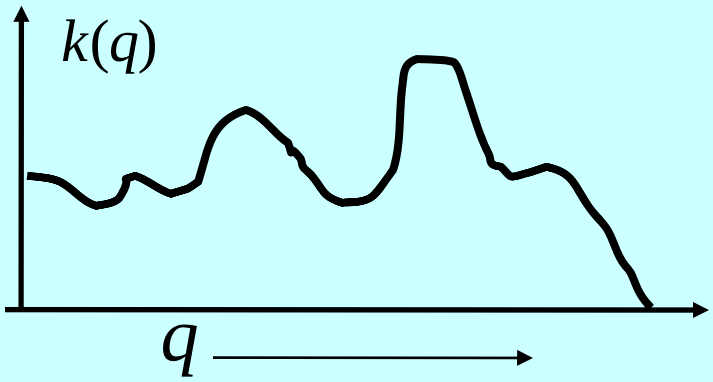
$$\{[m\delta(q-2)+b_1\delta(q-1.5)+b_0\delta(q-1)+k_1\delta(q-0.5)+k_0\delta(q)]s^q dq\}X(s)=F(s)$$

$$m\frac{d^2x(t)}{dt^2}+b_1\frac{d^{3/2}x(t)}{dt^{3/2}}+b_0\frac{dx(t)}{dt}+k_1\frac{d^{1/2}x(t)}{dt^{1/2}}+k_0=f(t)$$

Continuous Order

$$\left(\int_0^\infty k(q)s^q dq\right)X(s)=F(s)$$

$$\{\Im^{-1}\left(\int_0^\infty k(q)s^q dq\right)\}*x(t)=f(t)$$



Application-III

Order distribution based feed back control system

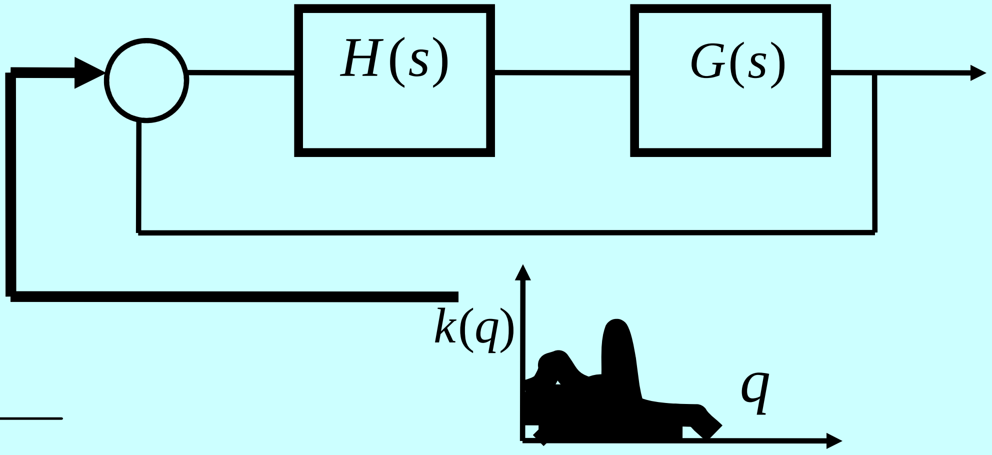
Reaction of a system depends on order value.
Reaction of a system depends on amplitude of order
A first (integer) order system cannot go into oscillations.
Presence of fractional order and its strength can give oscillations.
Why not control system order and its strength?

A futuristic automatic controller

$$G(s)=\frac{1}{s^2+a}, H(s)=\int_0^{\infty=2} k(q)s^q dq$$

$$T(s)=\frac{H(s)G(s)}{1+H(s)G(s)}$$

$$T(s)=\frac{\int_0^2 k(q)s^q dq}{s^2+1+(\int_0^2 k(q)s^q dq)}$$



Demanded order distribution-

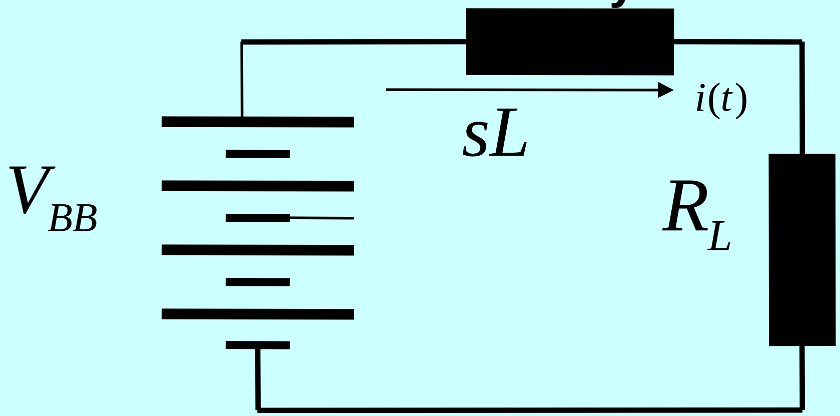
Application-IV

Circuit theory

Fractional order source

Fractional order load

Fractional order connectivity



$$L\frac{di(t)}{dt} + R_L i(t) = V_{BB}$$

Inside battery

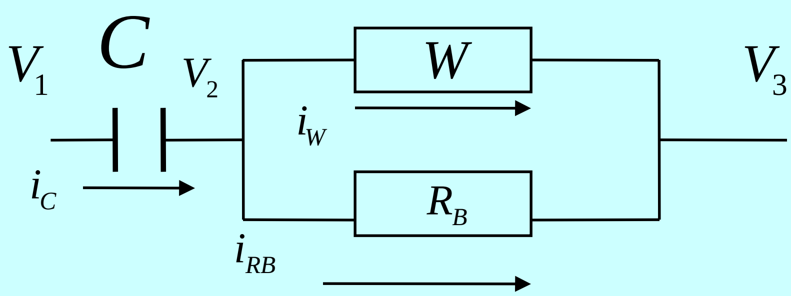
$$i(t) = i_C(t)$$

$$v_1(t) - v_2(t) = \frac{1}{C}\int_0^t i_C(t)dt + v_{1-2}(0)$$

$$i_C(t) = i_W(t) + i_{RB}(t)$$

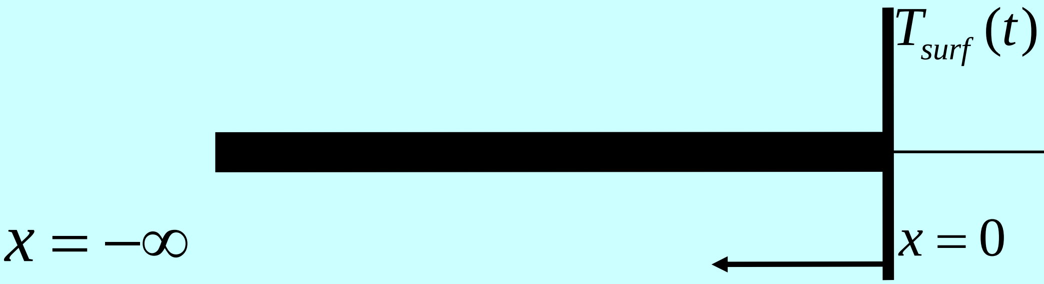
$$i_W(t) = {}_0D_t^{1/2}[v_2(t) - v_3(t)]$$

$$i_{RB}(t) = \frac{v_2(t) - v_3(t)}{R_B}$$



Application-V

Heat flux and temperature for semi infinite heat conductor.



$$c\rho \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$$c\rho \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$u(t,x) = T(t,x) - T_0$$

$$T(0,x) = T_0, u(0,x) = 0$$

$$T(t,0) = T_{surf}(t), u(t,0) = T_{surf}(t) - T_0$$

$$Q(t) = \frac{\partial T(t,0)}{\partial x}$$

$$Q(t) = \frac{k}{\sqrt{\alpha}} D_t^{1/2} [T_{surf}(t) - T_0]$$

$$\alpha = \sqrt{\frac{k}{c\rho}}$$

Application-VI

Impedance RC distributed semi infinite transmission line

$$\frac{\partial v(x,t)}{\partial x} = i(x,t)R$$

$$\frac{\partial i(x,t)}{\partial x} = C\frac{\partial v(x,t)}{\partial t}$$

$$\frac{\partial^2 v}{\partial x^2} = R\frac{\partial i}{\partial x} = RC\frac{\partial v}{\partial t}$$

$$v(0,t) = v_1(t), v(\infty,t) = 0$$

$$i(t) = \frac{1}{R\sqrt{\alpha}} \cdot {}_a D_t^{1/2} v(t)$$

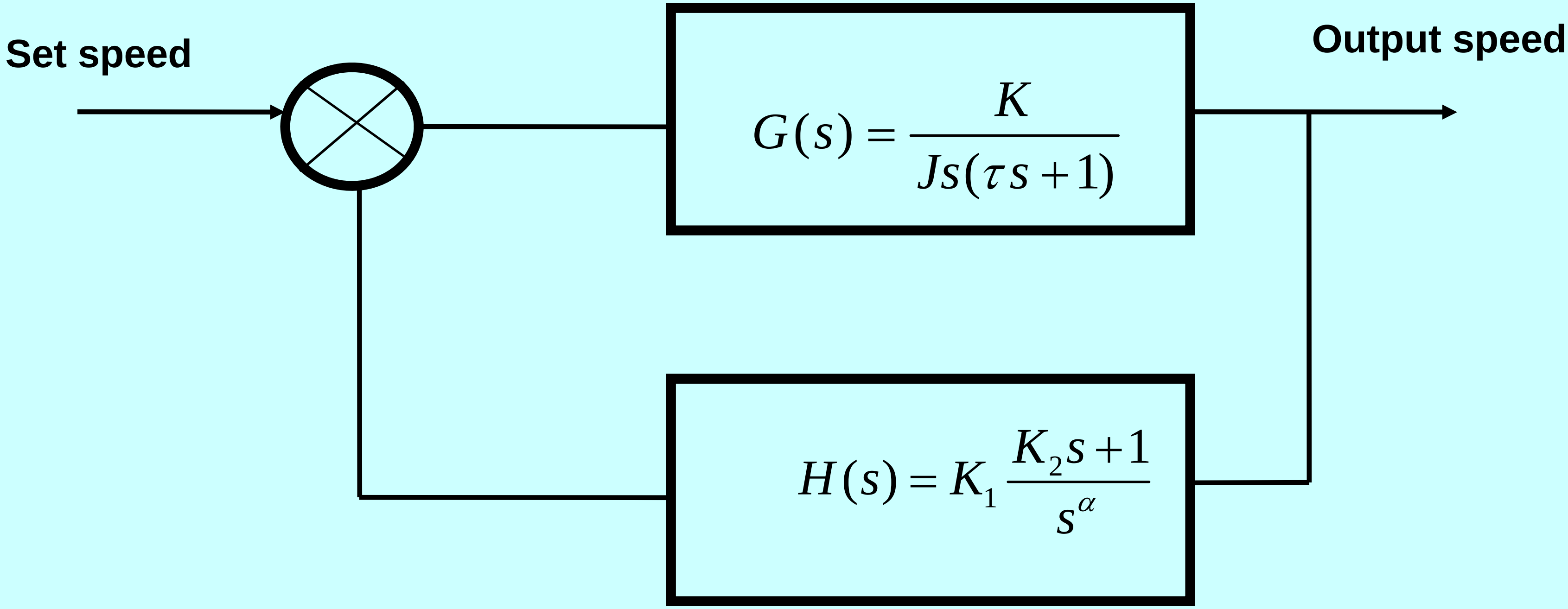
$$\alpha = \frac{1}{RC}$$

$$Z(s) = \sqrt{\frac{R}{C}}\left(\frac{1}{s^{1/2}}\right)$$

Basic building block for fractional order immittance realization of arbitrary order to make fractional order analog function generator and fractional order analog PID controller.

Application-VII

Fuel efficient control system



The constant close loop phase gives a feature of ISO-DAMPING where the peak overshoot is invariant on parametric spreads, giving fuel efficiency, avoidance of plant spurious excursions and trips, enhances safety and increases plant operational longevity.

Application-VIII

Fractional Divergence

To define non-local flux of material flowing through an isotropic media, loss volume and heterogeneous ambient.

Non Fickian diffusion phenomena

Anomalous diffusion

Anomalous random walk with unrestricted jump length per time.

$$\left(div^{\alpha} J\right) = \nabla^{\alpha} J \equiv \lim_{\Delta V \to REV} \frac{1}{V} \oint_{S} J \,.\, \vec{n} dS = \Phi(x)$$

$$\frac{d^{\beta}\Phi(x)}{dx^{\beta}} + B^{2}\Phi(x) = 0$$

$$\beta = 1 + \alpha$$

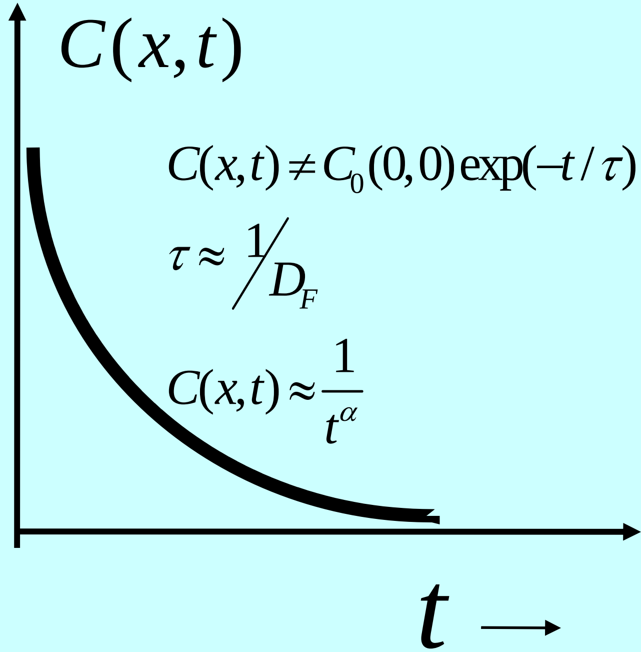
$$1 \leq \beta \leq 2$$

Application-IX

Electrode Electrolyte interface, derivation of Warburg law
Application in Electrochemistry.
Non-Fickian reaction kinetics.
Power law in anomalous diffusion
Time constant aberration
Magnetic flux diffusion studies in geophysics

$$ {}_0D_t^{\alpha} C(x,t) = D_{NF} \frac{\partial^2}{\partial x^2} C(x,t) $$

$$ C(x,t) \approx \frac{1}{t^{\alpha}} $$



Reaction to impulse excitation
Non exponential reaction

Application-X

Fractional Curl
In between dual solution in electrodynamics

$$\rho_e \leftrightarrow \rho_m$$

$$(E, H, D, B, \mu, \varepsilon) \leftrightarrow (H, -E, B, -D, \varepsilon, \mu)$$

$$(E, \eta H) \leftrightarrow (\eta H, -E)$$

$$E_{fd} = \frac{1}{(ik)^{\alpha}} (\nabla \times)^{\alpha} E$$

$$\eta H_{fd} = \frac{1}{(ik)^{\alpha}} (\nabla \times)^{\alpha} H$$

Future R&D in in-between mapping of Right Handed Maxwell systems and Left Handed Maxwell Systems (RHM)-(LHM)

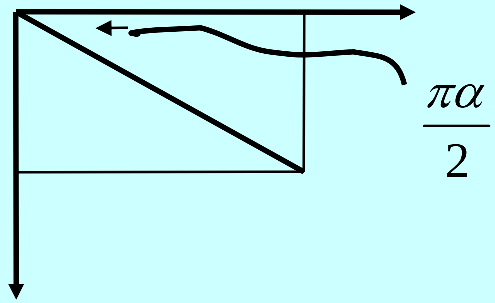
Application-XI

Electrodynamics
Wave propagation in media with losses.

$$\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} + \mu_0 \varepsilon_0 \chi_0 \frac{\partial^\alpha E}{\partial t^\alpha} + \frac{\partial^2 E}{\partial x^2} = 0$$

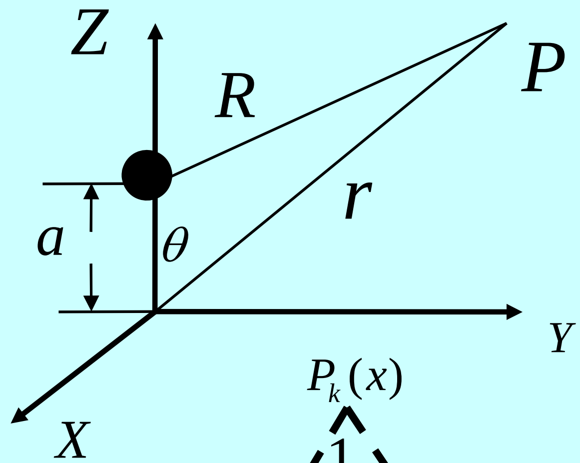
$$1 \le \alpha \le 2$$

$$D^\alpha E \sin \omega t = E \omega^\alpha \sin\left(\omega t + \frac{\pi\alpha}{2}\right)$$



Power factor modeling in AC machines, a new field of R&D.

Application-XII



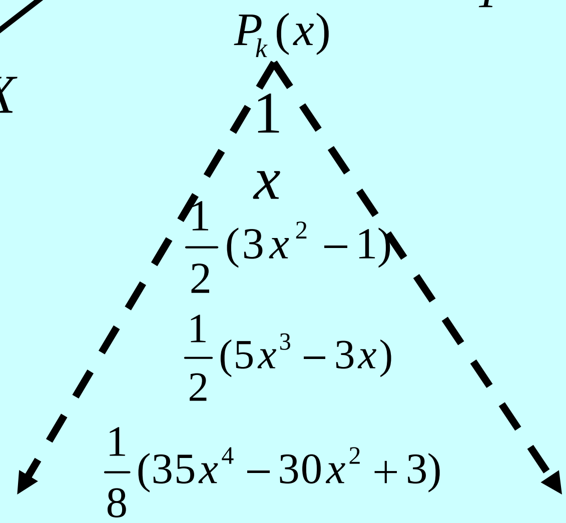
Electrodynamics

Multipole expansion

$$\Phi(r,\theta) \propto \frac{q}{R} = \frac{q}{\sqrt{r^2 + a^2 - 2ar\cos\theta}}$$

$$\rightarrow \frac{q}{r} + \frac{qa}{r^2}(\cos\theta) + \frac{qa^2}{2r^3}(3\cos^2\theta - 1) + \ldots$$

$$= \frac{q}{r}\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^k P_k(\cos\theta)$$



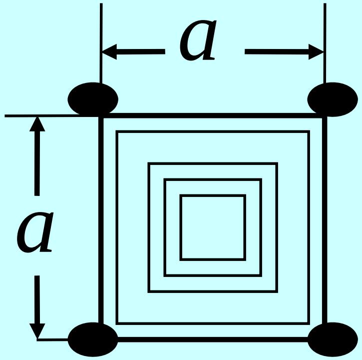
Fractional mutipole

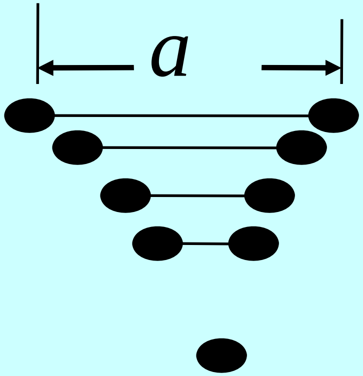
Fractal charge distribution

$\alpha = 0 \rightarrow 2^0$		**Mono**
$\alpha = 1 \rightarrow 2^1$		**Dipole**
$\alpha = 2 \rightarrow 2^2$		**Quadra**

$$\Phi = \left(\frac{qa^\alpha}{4\pi\varepsilon_0}\right)\left({}_{-\infty}D^\alpha\right)\left(\frac{1}{r}\right)$$

$$\Phi = \frac{qa^\alpha\Gamma(1+\alpha)}{4\pi\varepsilon_0(r)^{1+\frac{\alpha}{2}}}P_\alpha(\cos\theta)$$



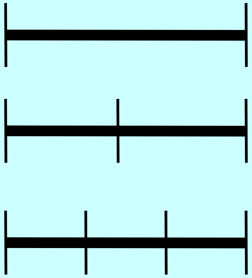


Fractional Legendre polynomial, Fractional Poles, dipole, monopole

Self similarity-fractal distribution

Application-XIII
Fractal Geometry & Fractional Calculus

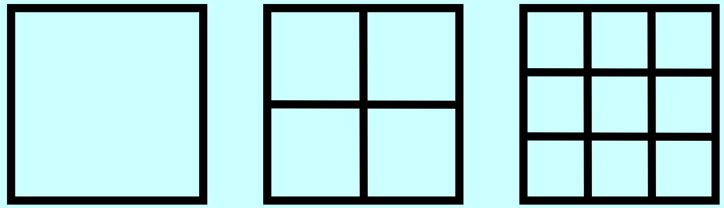
$$D_F = \lim_{\varepsilon \to 0} \frac{\log N}{\log\left(\frac{1}{\varepsilon}\right)}$$



$N = 2$ $\quad r = 2, \varepsilon = 1/2$

$N = 3$ $\quad r = 3, \varepsilon = 1/3$

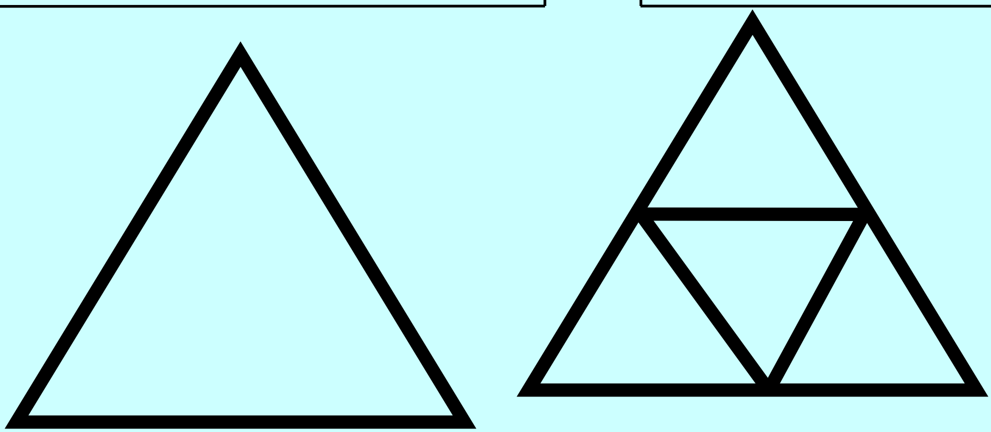
$$D_F = 1$$



$N = 4$ $\quad r = 2, \varepsilon = 1/2$

$N = 9$ $\quad r = 3, \varepsilon = 1/3$

$$D_F = 2$$



$N = 3$

$r = 2, \varepsilon = 1/2$

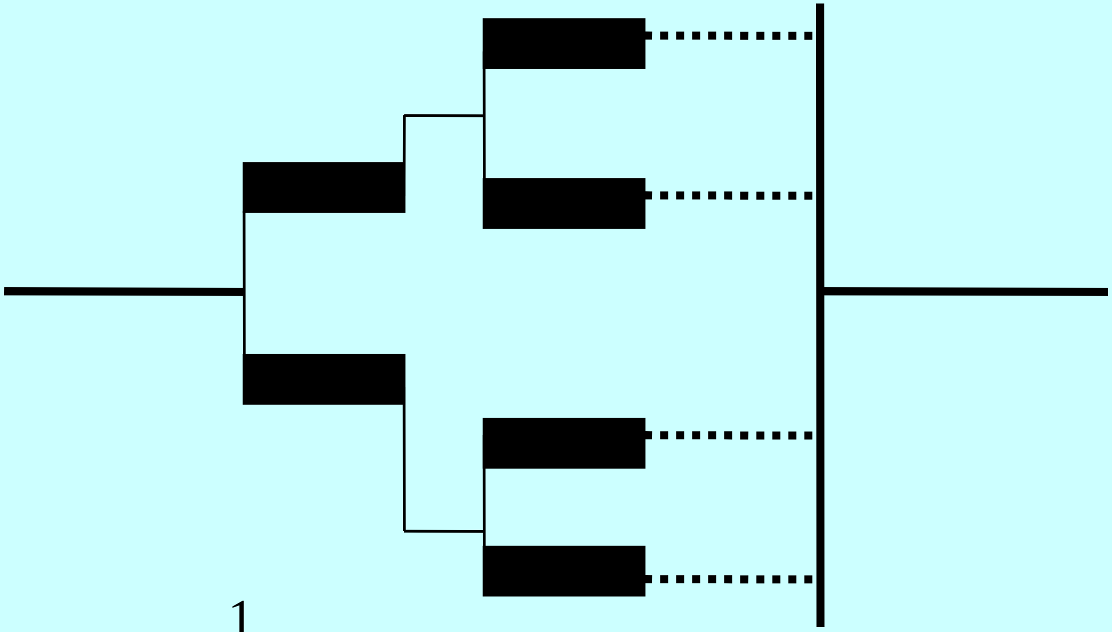
$$D_F = \frac{\log 3}{\log\left(\frac{1}{1/2}\right)} = \log 3 / \log 2 = 1.585$$

Application to graph theory and reliability analysis of software, data structure, cancer cell growth as future R&D topic on use of Local Fractional Calculus.

Application-XIV

Relation of fractal dimensions and order of fractional calculus

Time constant aberration and transfer function of flow through a Fractal structure and relation to its fractal dimension.



$$G_1(s) = \frac{1}{1 + \tau s}$$

$$G_2(s) = \frac{1}{1 + (\tau s)^{\lambda}}$$

$\lambda \leftrightarrow D$ **Relation of order to the fractal dimension**

Application-XV

Fractional calculus and multifractal functions

Fractals and multifractal functions and corresponding curves or surfaces are found in numerous non-linear, non-equilibrium phases like low viscous turbulent fluid motion, self similar and scale independent processes, continuous but nowhere differentiable curves.

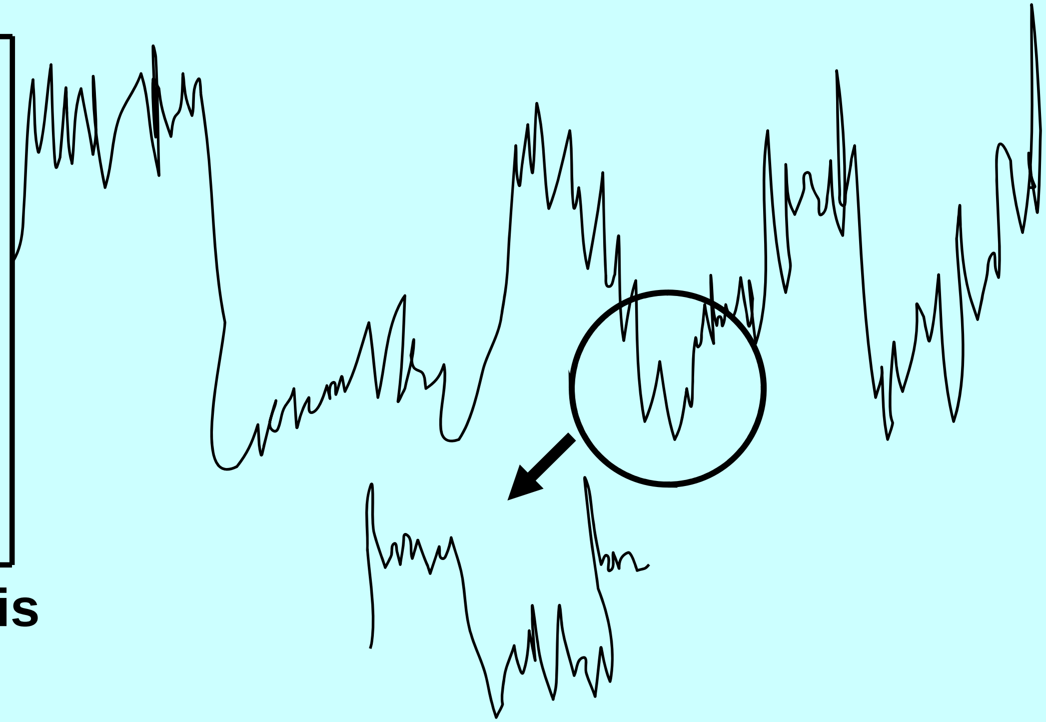
Weistrauss

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi n)$$

$$0 < a < 1, b > 0, ab > 1 + \frac{3}{2}\pi$$

$$D = \frac{\log a}{\log b + 2}$$

Fractality implies D>1 and it is scale independent, has no smaller scale



Application XV
Viscoelasticity

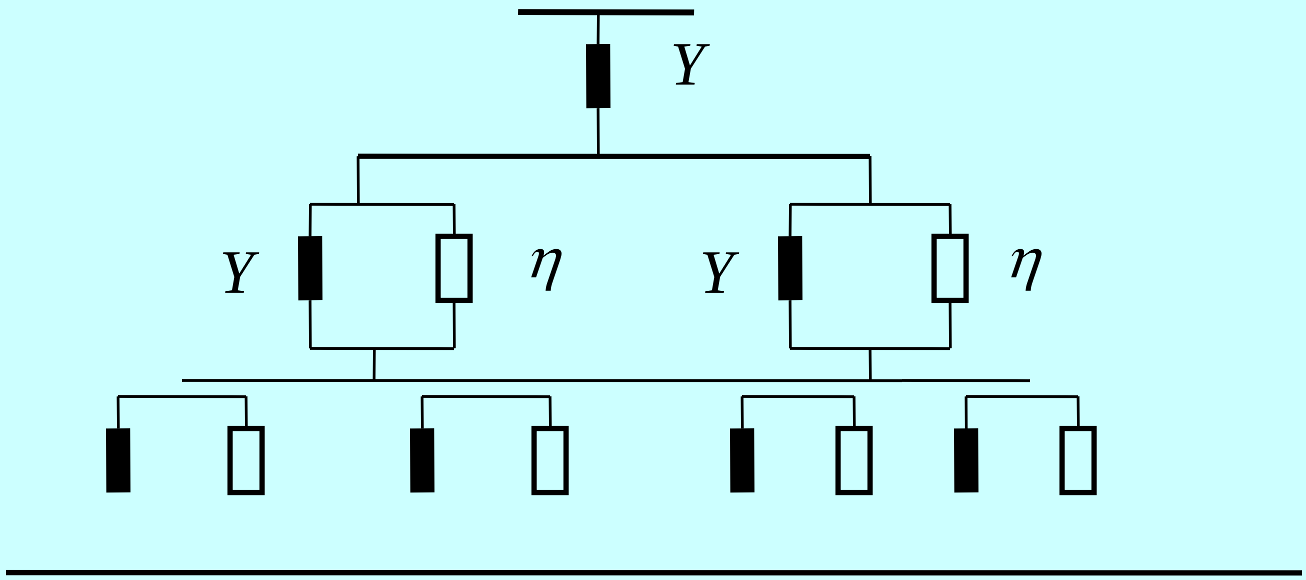
$$\sigma(t) = K_0 D_t^{\alpha} \varepsilon(t)$$

$$0 < \alpha < 1$$

$$\sigma(t) = Y\varepsilon(t) = Y_0 D_t^{0} \varepsilon(t)$$ **Pure solid Hook's law**

$$\sigma(t) = \eta \frac{d}{dt}\varepsilon(t) = \eta_0 D_t^{1} \varepsilon(t)$$ **Newtonian fluid**

Ideally no matter is pure solid nor is pure fluid



Application-XVI
Biology

Muscles and joint tissues in musco-skeletal system seem to behave as visco-elastic material, with fractional integrator, then this could be compensated by fractional order differentiator dynamics of neurons.

Membrane reaction relation as power law to frequency of current

$$X(\omega) = X_0 \omega^{-\alpha}$$

$$G(s) = X_0^{/} s^{-\alpha}$$

Motor discharge rate to rate of change of position

$$\frac{R(s)}{V(s)} = \frac{\tau_1 (s\tau_2 + 1)}{s\tau_1 + 1} s^{\beta - \alpha}$$

And several more……..

Observations

Distributed systems behave as fractional order
Representation of distributed system is better with fractional calculus.
Distribution can be in space or in time.
Almost all semi-infinite system gets representations in half derivative.
Good field of study as to why?
Can ambient changes manifest the order of calculus from say half to other value?
What is the physics behind that change?
This order value changes can be instrumented to study or make the instruments or instrumentation systems for measurement and control.

Generalized repeated differ-integration of monomial

$$f(x) = x^m$$

$$\frac{d}{dx}(x^m) = mx^{m-1}$$

$$\frac{d^2}{dx^2}(x^m) = m(m-1)x^{m-2}$$

$$\int x^m dx = \frac{1}{m+1}x^{m+1}$$

$$\iint x^m dxdx = \frac{1}{(m+1)(m+2)}x^{m+2}$$

Euler formulation (1730)

$$\frac{d^n}{dx^n}(x^m) = m(m-1)(m-2)...(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)....(m-n+1)\Gamma(m-n+1)$$

$$\frac{\Gamma(m+1)}{\Gamma(m-n+1)} = m(m-1)...(m-n+1)$$

For any arbitrary index $m,n \in \mathbb{R}$ **Differ-integration is:**

$$\boxed{\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}}$$

Examples of Euler formula:

$$\frac{d^{0.5}}{dx^{0.5}}(x) = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)}x^{1-0.5} = \frac{\Gamma(2)x^{0.5}}{\Gamma(1+0.5)} = \frac{1}{0.5\Gamma(0.5)}\sqrt{x} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$\frac{d^{-0.5}}{dx^{-0.5}}\sqrt{x} = \frac{\Gamma(0.5+1)x^{0.5-(-0.5)}}{\Gamma(0.5-\{-0.5\}+1)} = \frac{0.5\Gamma(0.5)}{\Gamma(2)}x = \frac{\sqrt{\pi}}{2}x$$

$$\frac{d}{dx}x = \frac{\Gamma(1+1)x^{1-1}}{\Gamma(1-1+1)} = \frac{\Gamma(2)}{\Gamma(1)} = 1, \quad \frac{d^{-1}}{dx^{-1}}x = \frac{\Gamma(1+1)x^{1-(-1)}}{\Gamma(1-\{-1\}+1)} = \frac{x^2}{2}$$

Using monomial integration in solving differential equation

Example classical oscillator

$$x^{//}(t)+x(t)=f(t)$$

$$f(t)=\delta(t)$$

$$x(0)=0, x^{/}(0)=0$$

$$x(t)=\sin t$$

$$x^{//}(t)=f(t)-x(t)$$

$$\iint\limits_{0\to t} x^{//}(t)=\iint\limits_{0\to 0+} f(t)dt-\iint\limits_{0+\to t} x(t)dt$$

$$x(t)-x(0)-tx^{/}(0)=\iint\limits_{0\to 0+} f(t)dt-\iint\limits_{0+\to t} x(t)dt$$

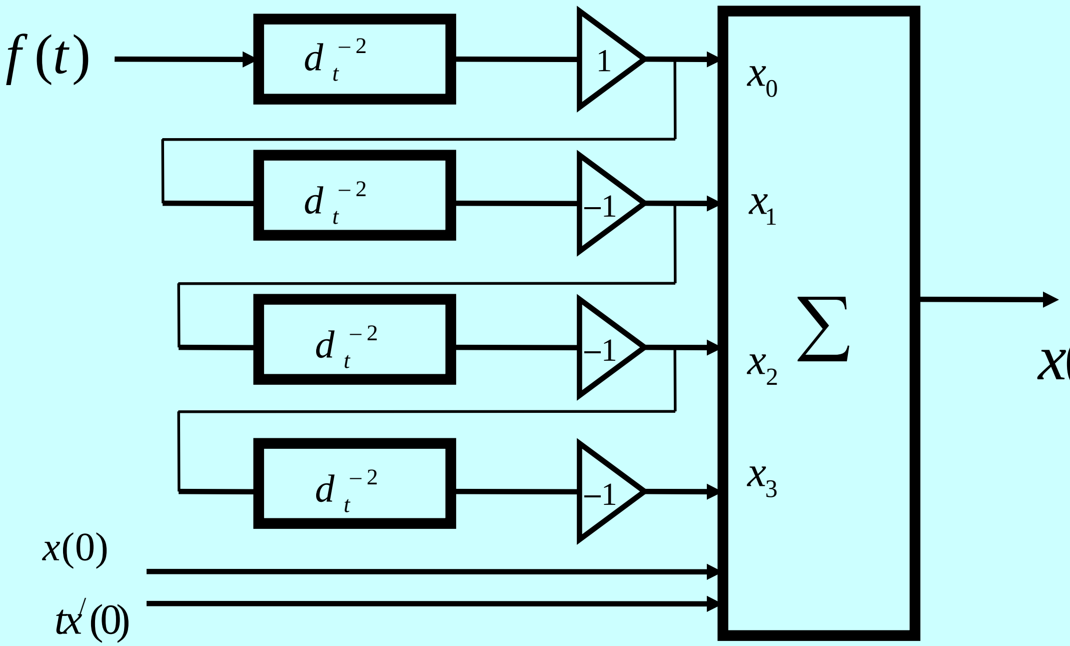
$$x(t)=x(0)+t[d_t^1 x(t)]_{@t=0}+\left(d_t^{-2}f(t)\right)-d_t^{-2}(d_t^{-2}f(t))+d_t^{-2}(d_t^{-2}(d_t^{-2}f(t)))-..$$

$$d^{-1}\delta(t)=1$$

$$d^{-2}\delta(t)=t$$

$$d^{-3}\delta(t)=\frac{t^2}{2}$$

$$d^{-4}\delta(t)=\frac{t^3}{3\times 2}$$



$$x(t)=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+..\approx\sin t$$

$$f(t)=\delta(t), x(0)=0, x^{/}(0)=0$$

Using monomial differ-integration to solve fractional Differential equation:

Example oscillator with fractional loss component

$$x''(t)+d^{1/2}x(t)+x(t)=f(t)$$

$$x(t)=x(0)+tx'(t)+d^{-2}f(t)-d^{-2}x(t)-d^{-3/2}x(t)$$

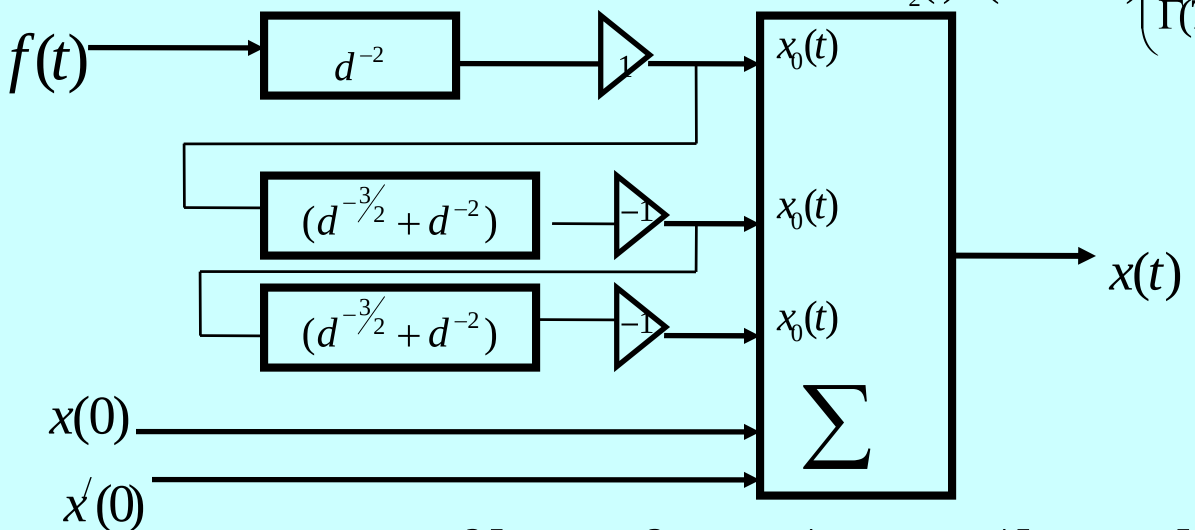
$$x(0)=0, x'(0)=0, f(t)=\delta(t)$$

Euler's generalization $d^n x^m=\dfrac{\Gamma(m+1)x^{m-n}}{\Gamma(m-n+1)}$

$$x_0(t)=d^{-2}\delta(t)=t$$

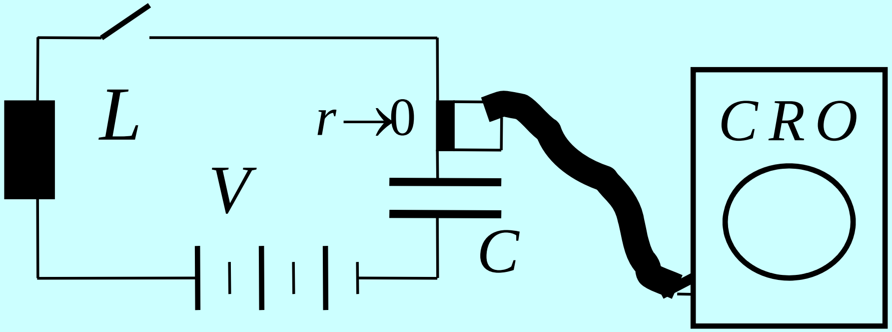
$$x_1(t)=-(d^{-3/2}+d^{-2})(t)=-\left(\frac{t^{5/2}}{\Gamma(7/2)}+\frac{t^3}{\Gamma(4)}\right)$$

$$x_2(t)=(d^{-3/2}+d^{-2})\left(\frac{t^{5/2}}{\Gamma(7/2)}+\frac{t^3}{\Gamma(4)}\right)=\frac{t^4}{\Gamma(5)}+2\frac{t^{9/2}}{\Gamma(11/2)}+\frac{t^5}{\Gamma(6)}$$



$$x(t)=t-\frac{t^{2.5}}{\Gamma(3.5)}-\frac{t^3}{\Gamma(4)}+\frac{t^4}{\Gamma(5)}+2\frac{t^{4.5}}{\Gamma(4.5)}+\frac{t^5}{\Gamma(6)}+..$$

Fractional oscillator an example:

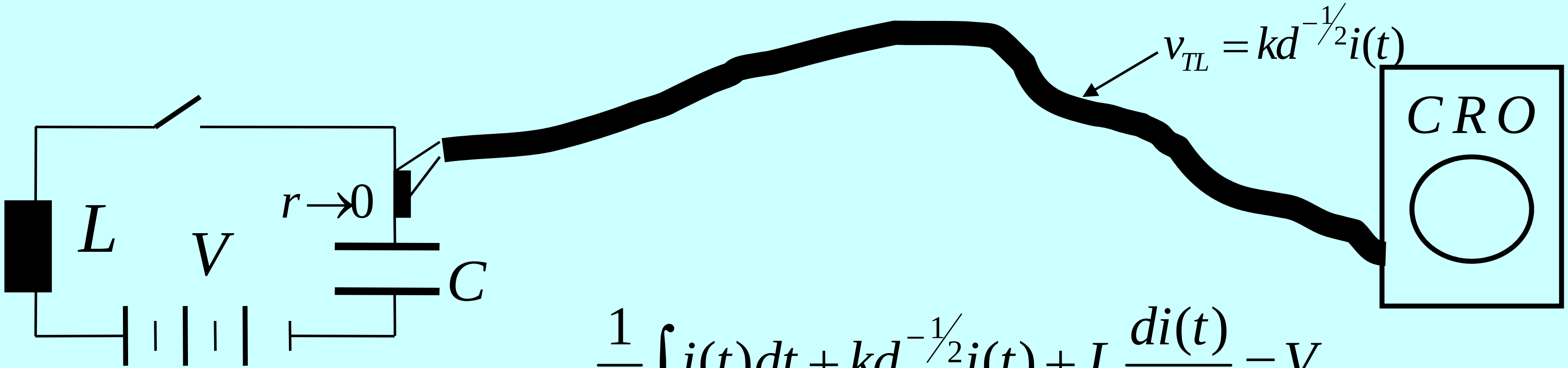


Short CRO cable circuit as oscillator

$$\frac{1}{C}\int i(t)\,dt + \lim_{r\to 0} r\,i(t) + L\frac{di(t)}{dt} = V$$

$$\frac{1}{C}i(t) + L\frac{d^2 i(t)}{dt^2} = \frac{dV}{dt} = V\delta(t)$$

Long CRO cable as Semi infinite TL half derivative



$$\frac{1}{C}\int i(t)\,dt + kd^{-1/2}i(t) + L\frac{di(t)}{dt} = V$$

$$\frac{1}{C}i(t) + k\frac{d^{1/2}i(t)}{dt^{1/2}} + L\frac{d^2 i(t)}{dt^2} = \frac{dV}{dt} = V\delta(t)$$

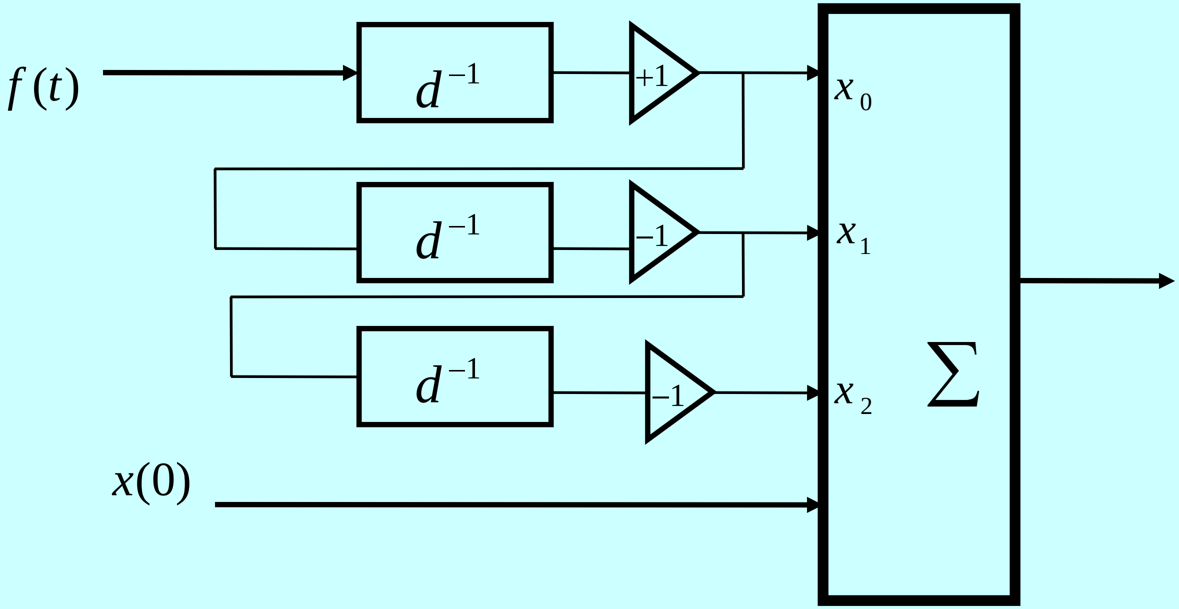
First order system and monomial integration

$$x^{/}(t)+x(t)=f(t)$$

$$d^{-1}d^{1}x(t)+d^{-1}x(t)=d^{-1}f(t)$$

$$x(t)-x(0)=d^{-1}f(t)-d^{-1}d^{-1}f(t)+d^{-1}d^{-1}d^{-1}f(t)-..$$

$$x(0)=0,\ f(t)=\delta(t)$$



$$x(t)=1-t-\frac{t^2}{2!}+\frac{t^3}{3!}-...\approx \exp(-t)$$

First order system with fractional loss term monomial solution

$$x'(t) + d^{1/2}x(t) + x(t) = f(t)$$
$$f(t) = \delta(t),\ x(0) = 0$$

Euler relation $\dfrac{d^n}{dx^n}x^m = \dfrac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$

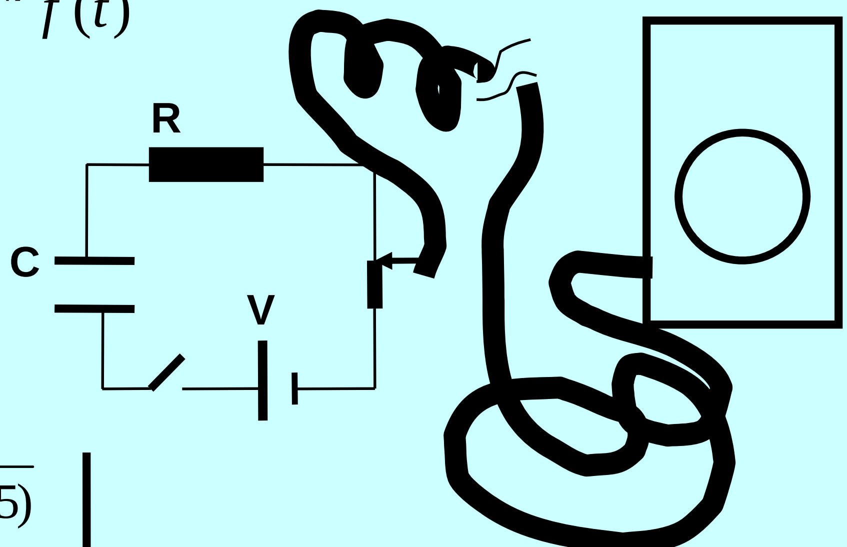
$$x(t) = x(0) + d^{-1}f(t) + \sum_{n=1}^{\infty}(-1)^n\left(d^{-1} + d^{-1/2}\right)^n f(t)$$

$$x_0 = d^{-1}\delta(t) = 1$$

$$x_1 = (d^{-1} + d^{-1/2})x_0 = d^{-1}(1) + d^{-1/2}(1) = t + \frac{t^{1/2}}{\Gamma(1.5)}$$

$$x_2 = (d^{-1} + d^{-1/2})x_1 = d^{-1/2}t + d^{-1/2}\frac{t^{1/2}}{\Gamma(1.5)} + d^{-1}t + d\frac{t^{1/2}}{\Gamma(1.5)}$$
$$= \frac{t^{3/2}}{\Gamma(2.5)} + \frac{t}{\Gamma(2)} + \frac{t^2}{2} + \frac{t^{3/2}}{\Gamma(2.5)}$$

$$x(t) = 1 - t - \frac{t^{1/2}}{\Gamma(1.5)} + \frac{t}{\Gamma(2)} + \frac{2t^{3/2}}{\Gamma(2.5)} + \frac{t^2}{2} - \cdots$$



$$\frac{1}{C}\int i(t)dt + Ri(t) + kd^{-1/2}i(t) = V$$

$$R\frac{di(t)}{dt} + kd^{1/2}i(t) + \frac{1}{C}i(t) = \frac{dV}{dt} = V\delta(t)$$

Distributed effect of long TL comes as fractional derivative/integral term. behaves as half order element , will it give II order response for I order system?

Poles in first order system with fractional loss

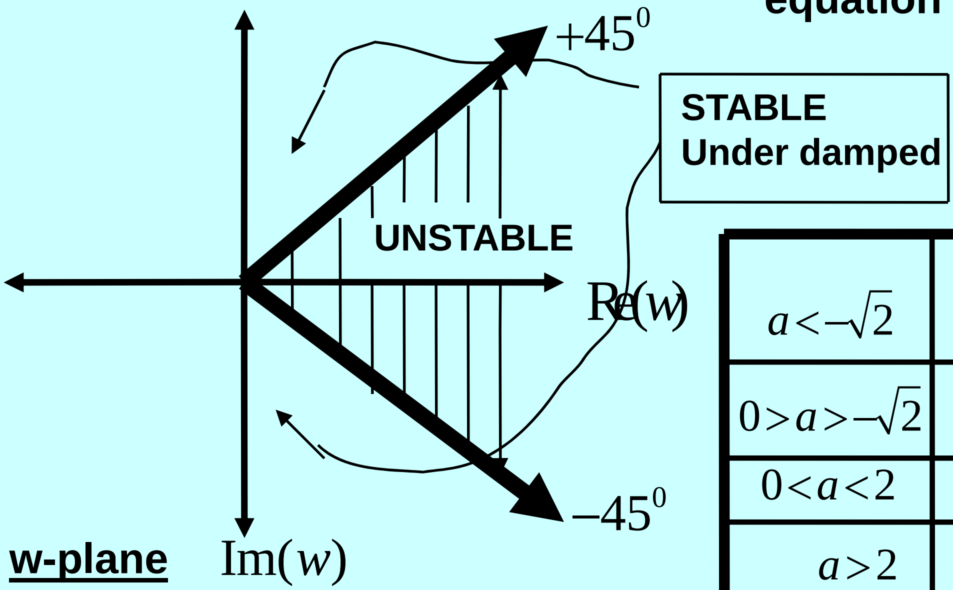
Concept of w-plane conformal mapping

$$dx(t) + ad^{1/2}x(t) + x(t) = f(t)$$

$$sX(s) + as^{1/2}X(s) + X(s) = F(s)$$

Characteristic equation is: $s + a\sqrt{s} + 1$ **in s-plane**

let $s^{1/2} = w$ **then** $w^2 + aw + 1$ **Is characteristic equation in w-plane.** $\arg w = \frac{1}{2}\arg s, \mathrm{mod}(w) = \sqrt{\mathrm{mod}(s)}$



$a < -\sqrt{2}$	$\arg(w) < \pm 45^0$	$\arg(s) < \pm 90^0$	**Unstable**
$0 > a > -\sqrt{2}$	$\pm 45^0 - \pm 90^0$	$\pm 90^0 - \pm 180^0$	**Stable**
$0 < a < 2$	$\pm 90^0 - \pm 180^0$	$\pm 180^0 - \pm 360^0$	**Hyperdamped**
$a > 2$	$\arg(w) > \pm 180^0$	$\arg(s) > \pm 360^0$	**Ultradamped**

A first order system with fractional term may become unstable can have oscillatory behavior and can behave as stable second order stable under damped systems

Classical order definition with number of energy storage element and or number of initial condition can give misleading information about the response In presence of fractional order terms.

Comment regarding system order

On contrary to widely accepted opinion in integer order theory, the first order system cannot go into instability or oscillations, the presence of fractional order elements in the first order system can give a counterintuitive result.

On contrary to widely accepted opinion that chaos cannot occur in continuous-time system of order less than three (in presence of non-linearity as feed back), fractional order system of order less than three can display chaotic behavior, with non linear feed back.

Order definition in classical theory saying the order is number of energy storage elements, or number of initialization constants required or the nature of output of damped nature, is not therefore valid in the presence of fractional order element.

Power series functions used in fractional calculus

Exponential function forms basis in the integer order calculus so is MITTAG LEFFLER function for the fractional calculus

Mittag-Leffler

$$E_q(at^q)=\sum_{n=0}^{\infty}\frac{a^n t^{nq}}{\Gamma(nq+1)}\leftrightarrow\frac{s^q}{s(s^q-a)}=\frac{s^{q-1}}{s^q-a}$$

Agarwal

$$E_{a,b}(t^q)=\sum_{n=0}^{\infty}\frac{t^{\left(n+\frac{b-1}{a}\right)}}{\Gamma(na+b)}\leftrightarrow\frac{s^a}{s^b(s^a-1)}=\frac{s^{a-b}}{s^a-1}$$

Erdelyi

$$E_{a,b}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(an+b)}\leftrightarrow\sum_{n=0}^{\infty}\frac{\Gamma(m+1)}{\Gamma(na+b)s^{m+1}}$$

Robotnov-Hartley

$$F_q(a,t)=\sum_{n=0}^{\infty}\frac{a^n t^{(n+1)q-1}}{\Gamma(\{n+1\}q)}\leftrightarrow\frac{1}{(s^q-a)}$$

Many more like Miller-Ross, Generalized G, Generalized R, Fox function

$$E_{1,1}(t)=\exp(t)$$

$$R_{1,0}(a,0,t)=\exp(at)$$

$$aR_{2,0}(-a^2,0,t)=\sin at$$

$$E_q(-at^q)=R_{a,q-1}(-a,0,t)$$

Solution of fractional differential equation (in ML function)

Fractional differential equation of broacher (tracking filter)

$$\frac{d^{0.25}}{dt^{0.25}}y(t) + y(t) = x(t)$$

$$y(0) = 0$$

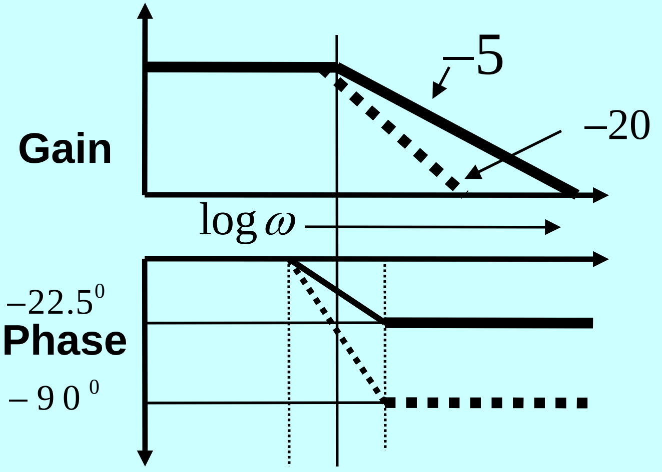
$$s^{0.25}Y(s) + Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^{0.25}+1}X(s)$$ **For step excitation** $X(s) = 1/s$

$$Y(s) = \frac{1}{s}\left(\frac{1}{s^{0.25}+1}\right) = \frac{1}{s}\left(1 - \frac{s^{0.25}}{s^{0.25}+1}\right) = \frac{1}{s} - \frac{s^{0.25}}{s(s^{0.25}+1)} = \frac{1}{s} - \frac{s^{0.25-1}}{s^{0.25}+1}$$

$$y(t) = \Im^{-1}Y(s) = \Im^{-1}\left(\frac{1}{s}\right) - \Im^{-1}\left(\frac{s^{0.25-1}}{s^{0.25}+1}\right)$$

$$\boxed{y(t) = 1 - E_{0.25}(-t^{0.25})}$$



For first order solution is:

$$y(t) = 1 - E_1(-t) = 1 - \exp(-t)$$

Salient points observed in the discussion:

The distributed effect of parameters distributed over large space gives half order of derivative or integration.

Can this be taken as general rule that semi infinite distributed self similar structures behave with half order of calculus?

If the distribution in space gives order of derivative as fractional order suggesting non-local behavior, can we say event distributed in time (historical behavior hereditary character temporal memory behavior be represented with fractional differ-integration of time?

The solution seems to have self similar pattern, time/space power series with fractional power real power.

Reality of systems are naturally not point quantity thus fractional calculus is the language what nature understands the best.

End of part-I

Essence of FRACTIONAL CALCULUS in applied sciences

Part-II

WORK SHOP ON
FRACTIONAL ORDER SYSTEM
28-29 March, 2008

IEEE KOLKATA CHAPTER
DRDL HYDERABAD
BRNS(DAE) MUMBAI

Shantanu Das
Reactor Control Division
BARC

Reimann Liouvelli (RL) fractional integration:

Repeated n-fold integration generalization to arbitrary order

$$d_t^{-1} f(t) = \int_0^t f(\tau) d\tau$$

$$d_t^{-2} f(t) = \int_0^t \int_0^t f(\tau) d\tau d\tau = \int_0^t (t-\tau) f(\tau) d\tau$$

$$d_t^{-3} f(t) = \int_0^t \int_0^t \int_0^t f(\tau) d\tau d\tau d\tau = \frac{1}{2} \int_0^t (t-\tau)^2 f(\tau) d\tau$$

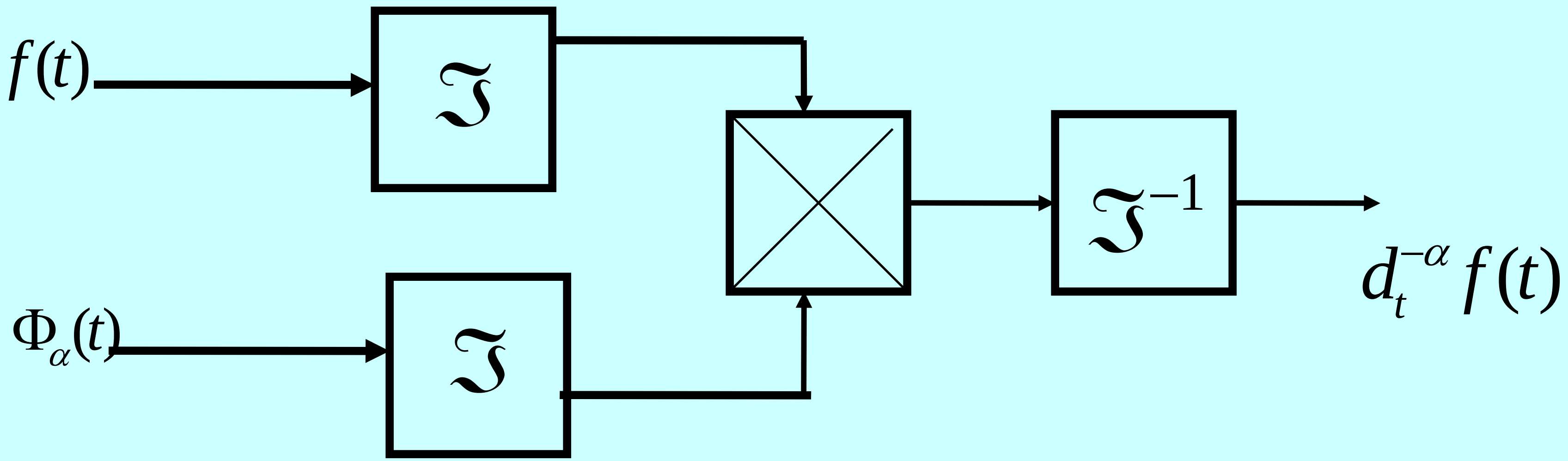
$$d_t^{-n} f(t) = \int_0^t \int_0^t \underbrace{\cdots\cdots\cdots\cdots \int_0^t f(\tau) d\tau}_{n} = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau$$

$$d_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

Convolution with power function RL fractional integration:

$$d_t^{-\alpha} f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau = \left[f(t) \right] * \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) = f(t) * \Phi_\alpha(t)$$

$$\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$



Fractional derivative the Euler (1730) formula for monomial

$$\frac{d^n f(x)}{dx^n} = \underbrace{\frac{d}{dx}\frac{d}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\frac{d}{dx}}_{n} f(x)$$

$$\frac{d^n}{dx^n}\left\{x^m\right\} = m(m-1)(m-2)\ldots\ldots\ldots\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)\Gamma(m-n+1)$$

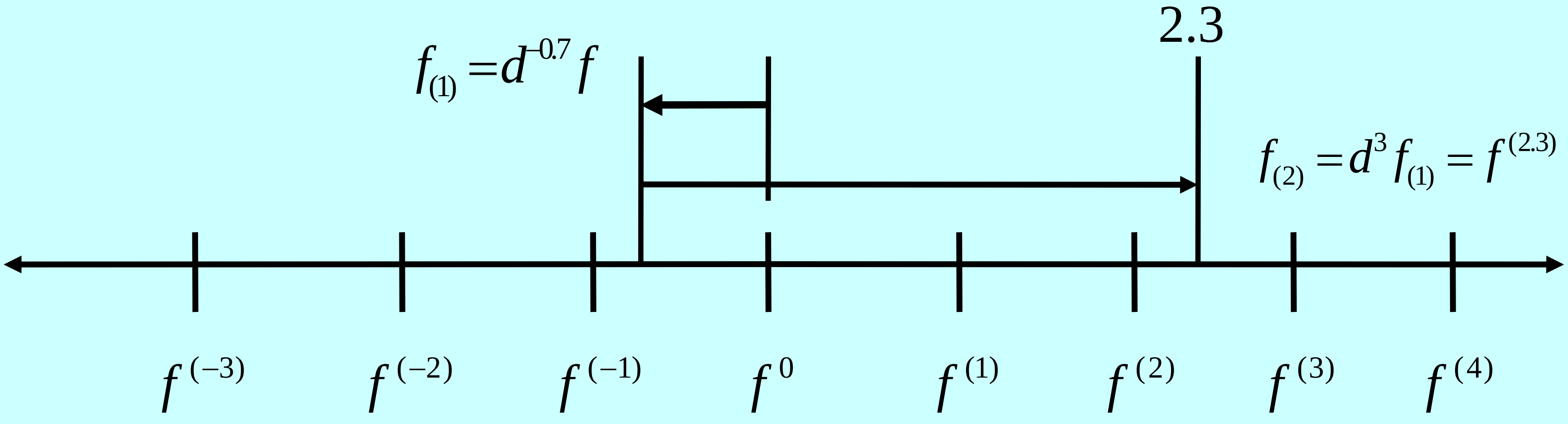
$$\frac{d^n}{dx^n}\left\{x^m\right\} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

$$\frac{d^{0.5}}{dx^{0.5}}\left\{x\right\} = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)}x^{1-0.5} = \frac{\sqrt{x}}{\Gamma(1+0.5)} = \frac{\sqrt{x}}{0.5\Gamma(0.5)} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

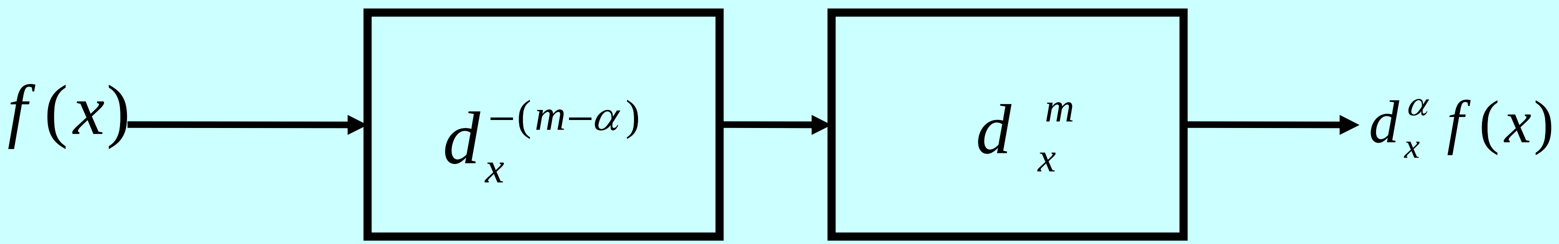
For positive index the process is differentiation
For negative index the process is integration

Reimann Liouvelli (RL) Fractional derivative

Left Hand Definition (LHD)



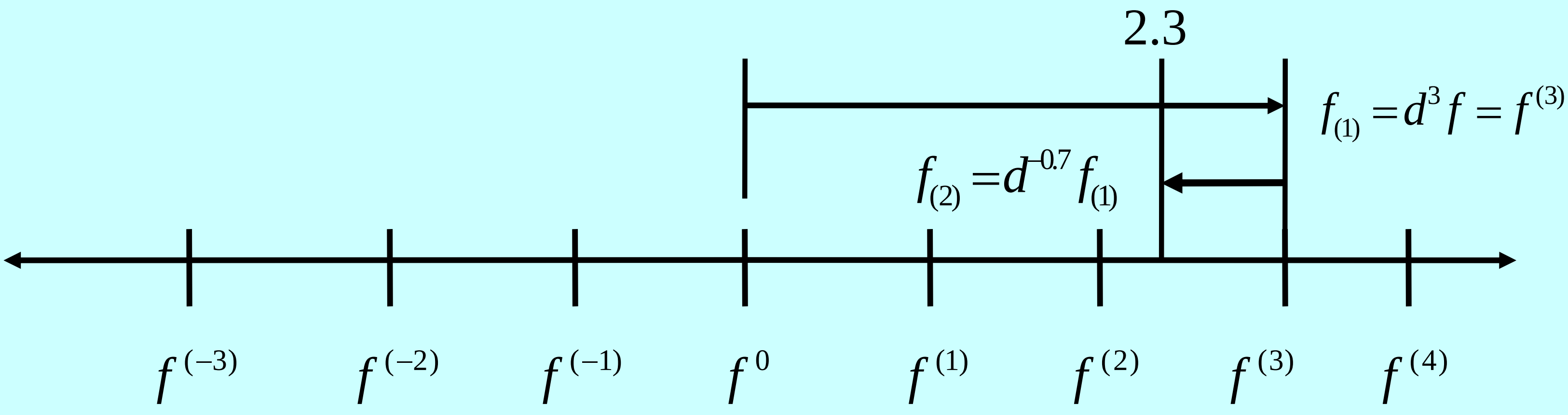
Here 'm' is the integer just greater than fractional order of derivative



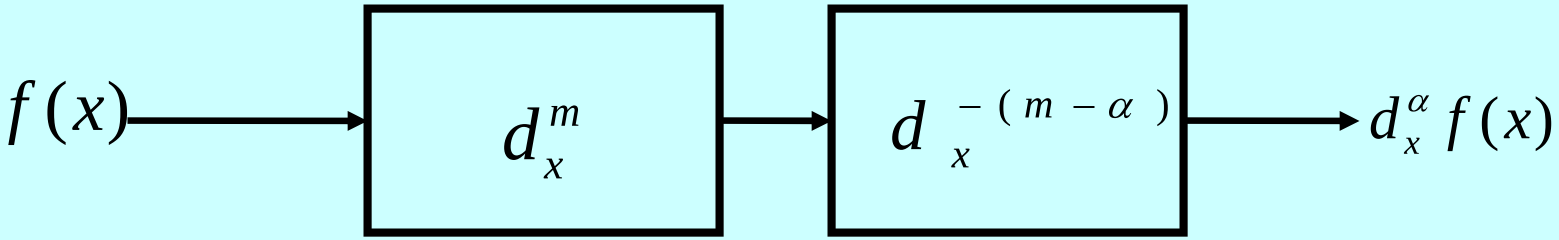
$$d_t^{\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{-\alpha-1+m} f(\tau)d\tau\right]$$

Caputo (1967) Fractional derivative

Right Hand Definition (RHD)



Here 'm' is the integer just greater than the fractional order derivative



$$d_t^\alpha f(t) = \left[\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{-\alpha-1+m} \frac{d^m f(\tau)}{d\tau^m} d\tau \right]$$

Duality

For LHD fractional derivative of constant is not zero $d_x^{\alpha} C \neq 0 = C\left[\Gamma(1-\alpha)\right]^{-1} x^{-\alpha}$
This fact lead to RL or LHD approach to consider "limit of differentiation" (lower terminal) to minus infinity. The physical significance of this minus infinity is starting the physical processes at time immemorial!! However lower limit to minus infinity is necessary abstraction for steady state (sinusoidal) response. For LHD $d_x^{\alpha-1} f(0), d_x^{\alpha-2} f(0)$ **are required. This posses physical interpretability.**

For RHD the fractional derivative of the constant is zero. But this requires also $f(0) = 0,$ **with** $f^{(1)} = f^{(2)} = \ldots f^{(m)} = 0$ **in mathematical world this posses a problem.**

Our mathematical tools go far beyond our physical understanding

Standardization of symbols for fractional differintegrals

Initialized differintegration

$${}_{c}D_{t}^{\pm q}$$

Uninitialized differitegrations

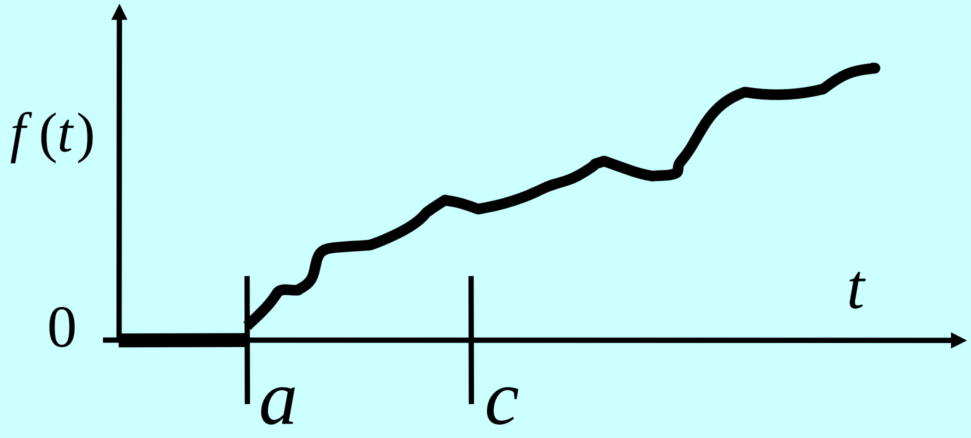
$${}_{c}d_{t}^{\pm q}$$

$${}_{c}D_{t}^{\pm q} = {}_{c}d_{t}^{\pm q} + \psi(f, \pm q, a, c, t)$$

Initialization function

$$\psi(t) = \psi(f, \pm q, a, c, t)$$

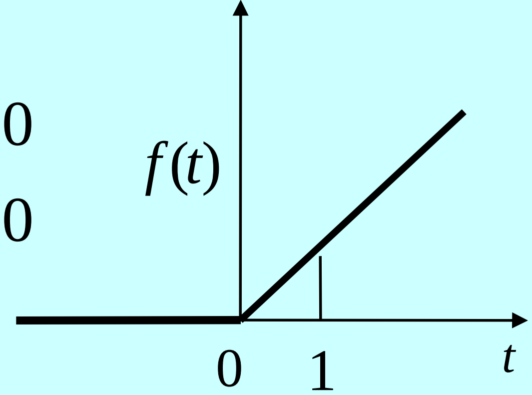
For a function $f(t)$ **born at time (space)** $= a$ **and the differintegration starts at time (space)** $= c$



Initialized fractional integration

$$ {}_{1}D_t^{-1/2}\{t\} = {}_{1}d_t^{-1/2} + \psi(t) $$

$$ f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} $$



$$ \psi(t) = \psi\{ f(t) = t, -1/2, 0, 1, t\} $$

$$ {}_{0}D_t^{-1/2}\{t\} = {}_{0}d_t^{-1/2} + \{\psi(t) = 0\} = \frac{1}{\Gamma(0.5)}\int_0^t (t-\tau)^{0.5-1}\tau\, d\tau = \frac{4t^{3/2}}{3\sqrt{\pi}} $$

$$ {}_{1}d_t^{-1/2}\{t\} = \frac{1}{\Gamma(0.5)}\int_1^t (t-\tau)^{0.5-1}\tau\, d\tau = \frac{2(t-1)^{1/2}(2t+1)}{3\sqrt{\pi}} $$

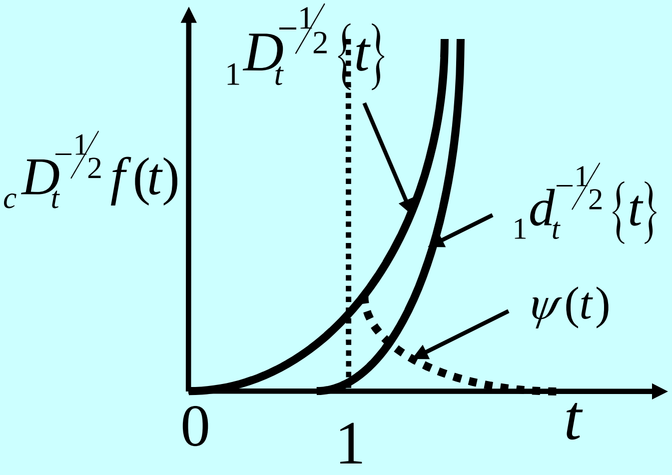
$$ \psi(t) = \frac{1}{\Gamma(0.5)}\int_0^1 (t-\tau)^{0.5-1}\tau\, d\tau = \frac{2}{3\sqrt{\pi}}\left[2t^{3/2} - (t-1)^{1/2}(2t+1)\right] $$

$$ {}_{0}D_t^{-1/2}\{t\} = {}_{1}D_t^{-1/2}\{t\} = {}_{1}d_t^{-1/2}\{t\} + \psi(t) $$

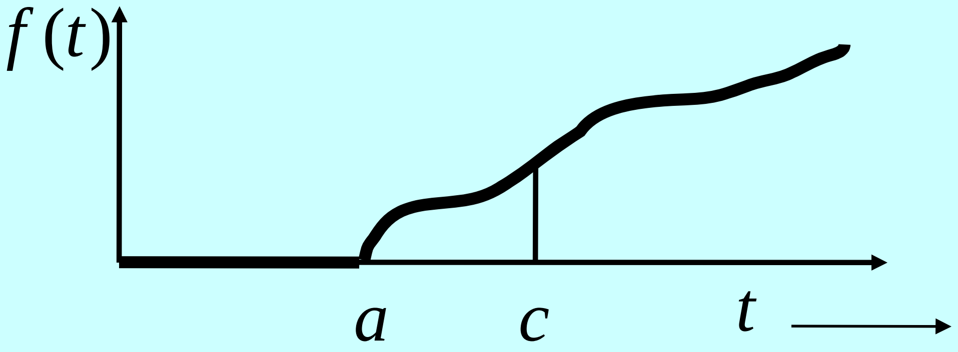
$\psi(t)$ **Is the history of the functional process since birth and the history effect decays with time, memory is lost!!**

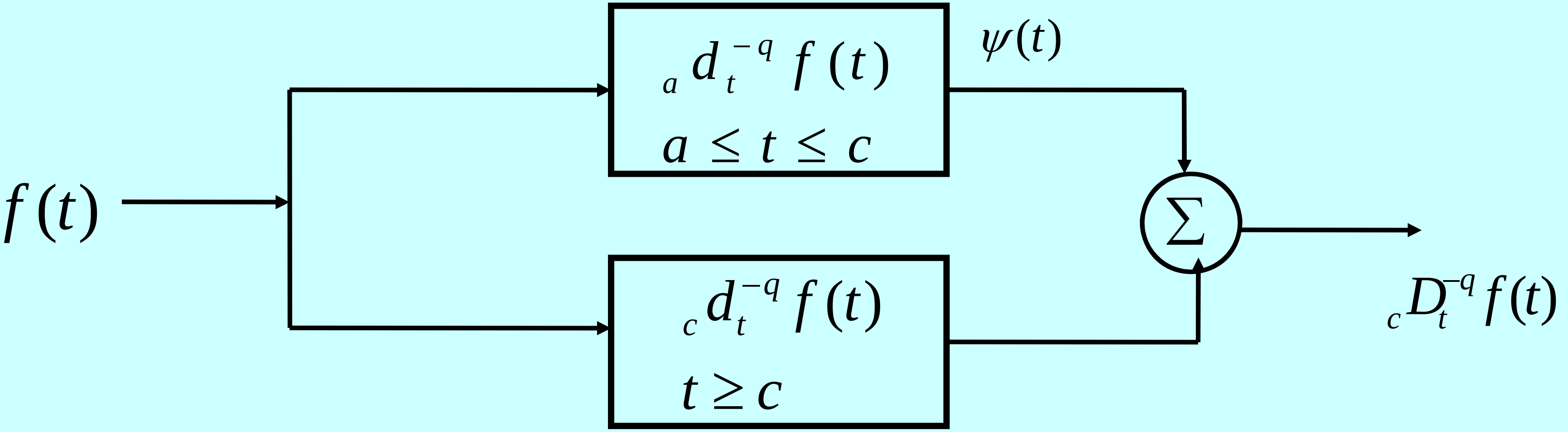
$$ {}_{c}D_t^{-q} f(t) = {}_{a}D_t^{-q} f(t) $$

$$ t \geq c \geq a $$



Initialization function fractional integration





Solution of FDE

$$ {}_0D_t^{1/2} f(t) + bf(t) = 0 $$

$$ t > 0, [{}_0D_t^{-1/2} f(t)]_{@t=0} = C $$

$$ \Im\left[f^{(n)}(t) \right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots $$

$$ \Im\left[f^{\left(1/2\right)}(t) \right] = s^{1/2} F(s) - s^{1/2-1} f(0) = s^{1/2} F(s) - s^{-1/2} f(0) $$

$$ \Im\left[f^{\left(1/2\right)}(t) \right] = s^{1/2} F(s) - [{}_0D_t^{-1/2} f(t)]_{@t=0} $$

$$ F(s) = \frac{C}{s^{1/2} + b} $$

$$ f(t) = Ct^{-1/2} E_{0.5,0.5}(-b\sqrt{t}) $$

$$ f(t) = C\left(\frac{1}{\sqrt{\pi t}} - \exp(t) erfc(\sqrt{t}) \right), b = 1 $$

Solution of FDE with initialization function

$$ {}_0D_t^{1/2} f(t) + bf(t) = 0 $$

$$ {}_0d_t^{1/2} f(t) + \psi(t) + bf(t) = 0 \quad \textbf{for} \quad t>0 $$

$$ s^{1/2}F(s) + \psi(s) + bF(s) = 0 $$

$$ F(s) = -\frac{\psi(s)}{s^{1/2} + b} $$

For $\psi(t) = -C\delta(t)$

$$ F(s) = \frac{C}{s^{1/2} + b} \qquad f(t) = \frac{C}{\sqrt{t}} E_{\frac{1}{2},\frac{1}{2}}(-b\sqrt{t}) $$

$$ \Im^{-1}\left[\frac{1}{s^{1/2} + b}\right] \leftrightarrow R_{1/2,0}(-b,0,t) $$

$$ f(t) = -\int_0^t R_{\frac{1}{2},0}(-b,0,t-\tau)\psi(\tau)\,d\tau $$

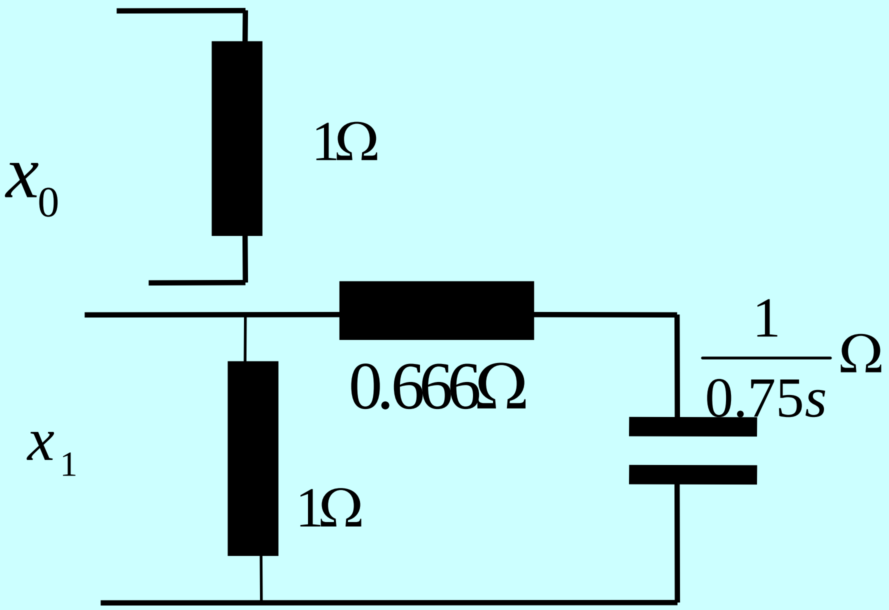
General solution

Formal methods to solve fractional differential equation

1. Laplace Transforms
1. Fractional Greens function.
1. Mellin Transforms
1. Power Series Method.
1. Babenko's Symbolic calculus method.
1. Orthogonal Polynomial decomposition.
1. Adomian Decomposition.
1. Numerical

Synthesis of fractional order immittances

Newton method of root evaluation



$$x = (a)^{1/n}, x_0 = 1a$$

$$x_k = x_{k-1}\frac{(n-1)(x_{k-1})^n + (n+1)a}{(n+1)(x_{k-1})^n + (n-1)a}$$

$$n = 3, a = \frac{1}{s}, x_0 = 1,$$

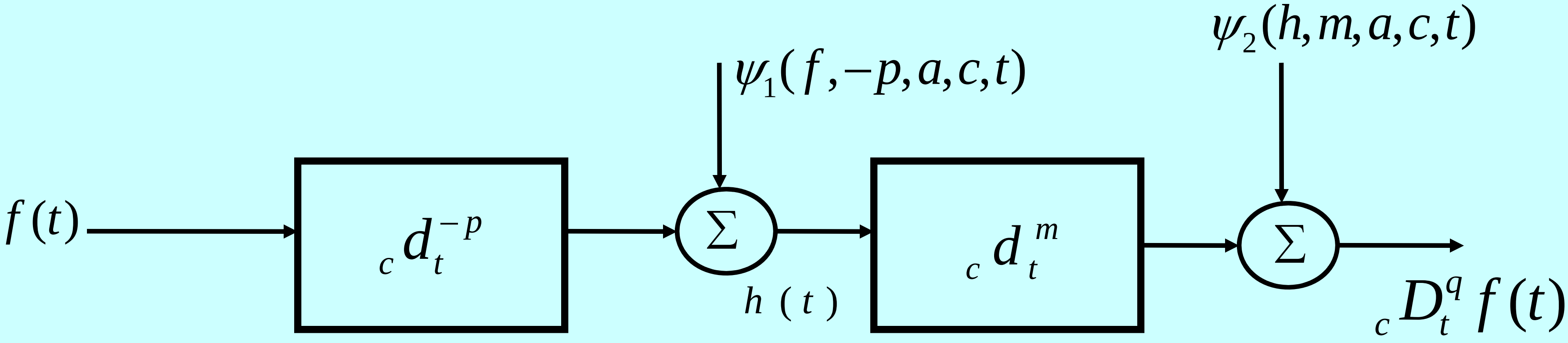
$$x_1 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s+2}{2s+1}$$

$$x_2 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s^5 + 24s^4 + 80s^3 + 92s^2 + 42s + 4}{4s^5 + 42s^4 + 92s^3 + 80s^2 + 24s + 1}$$

$$s^{0.5} = \frac{s+3}{3s+1}, \left(\frac{1}{s^{0.5}}\right) = \frac{3s+1}{s+3}, \frac{1}{s^{0.25}} = \frac{3s+5}{5s+3}, \frac{1}{s^{0.15}} = \frac{1+1.35s}{1.35s+s}$$

Initialization of fractional derivative

Riemann-Liouvelli derivative



$$q = (m - p)$$

$${}_c D_t^q f(t) = {}_a D_t^q f(t)$$

$${}_c D_t^q f(t) = {}_c d_t^m \left\{ h(t) \right\} + \psi_2(h,m,a,c,t)$$

$${}_c D_t^q f(t) = {}_c d_t^m \left\{ {}_c d_t^{-p} f(t) + \psi_1(f,-p,a,c,t) \right\} + \psi_2(h,m,a,c,t)$$

$${}_c D_t^q f(t) = {}_c d_t^q f(t) + \psi_1^{(m)}(t) + \psi_2(t)$$

$${}_c D_t^q f(t) = {}_c d_t^q f(t) + \psi(f,q,a,c,t)$$

For terminal initialization $\psi_2 = 0$

For side initialization ψ_2 **is arbitrary**

Integer order calculus in fractional context

RL derivative

$$ {}_aD_t^1 f(t) = \frac{d}{dt} f(t) = \frac{1}{\Gamma(1)} \frac{d^2}{dt^2} \int_a^t (t-\tau)^{2-1-1} f(\tau) d\tau = \frac{d^2}{dt^2} \int_a^t f(\tau) d\tau $$

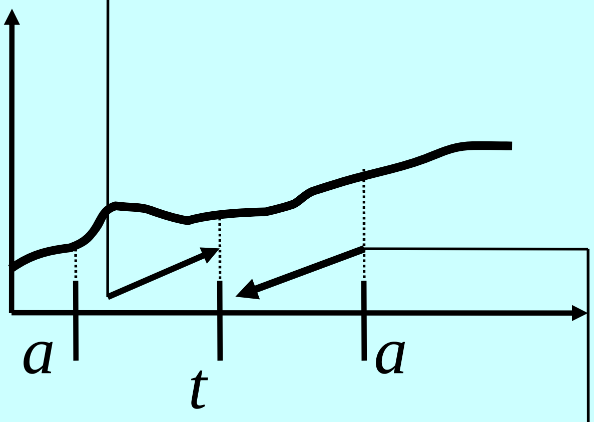
Integrate the function from a to t and then obtain second derivative.

Obtaining the differentiation in fractional context imbibes history (hereditary) of the function from start of the differentiation process.

This also describes the ‘non-local’ behavior in space or time.

Forward and backward differentiation integer order derivative in fractional context

RL derivative



Forward RL

$$ {}_aD_t^q\{f(t)\} = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_a^t (t-\tau)^{m-q-1} f(\tau)d\tau $$

$$ {}_aD_t^1 f(t) = \frac{d}{dt} f(t) = \frac{d^2}{dt^2}\left[\int_a^t f(\tau)d\tau\right] $$

$$ = \frac{d^2}{dt^2}\left[\frac{1}{2}\{f(t)+f(a)\}(t-a)\right] $$

$$ = f'(t) + \frac{1}{2}(t-a)f''(t) $$

$$ \to f'(t) $$

Backward RL

$$ {}_tD_a^q f(t) = \frac{1}{\Gamma(m-q)}(-1)^m\frac{d^m}{dt^m}\int_t^a (\tau-t)^{m-q-1} f(\tau)d\tau $$

$$ {}_tD_a^1 f(t) = \frac{1}{\Gamma(1)}(-1)^2\frac{d^2}{dt^2}\int_t^a (\tau-t)^{2-1-1} f(\tau)d\tau $$

$$ = -{}_aD_t^1 f(t) $$

If forward and backward derivatives are equal (with sign) then fractional derivative at a POINT exist, meaning to get fractional derivative at point entire character of function be known!

Grunwald-Letnikov(GL) fractional differintegration

$$f^{/}(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f^{//}(x)=\lim_{h_1\to 0}\frac{\lim\limits_{h_2\to 0}\frac{f(x+h_1+h_2)-f(x+h_1)}{h_2}-\lim\limits_{h_2\to 0}\frac{f(x+h_1)-f(x)}{h_2}}{h_1}$$

$$f^{//}(x)=\lim_{h\to 0}\frac{f(x+2h)-2f(x+h)-f(x)}{h^2}$$

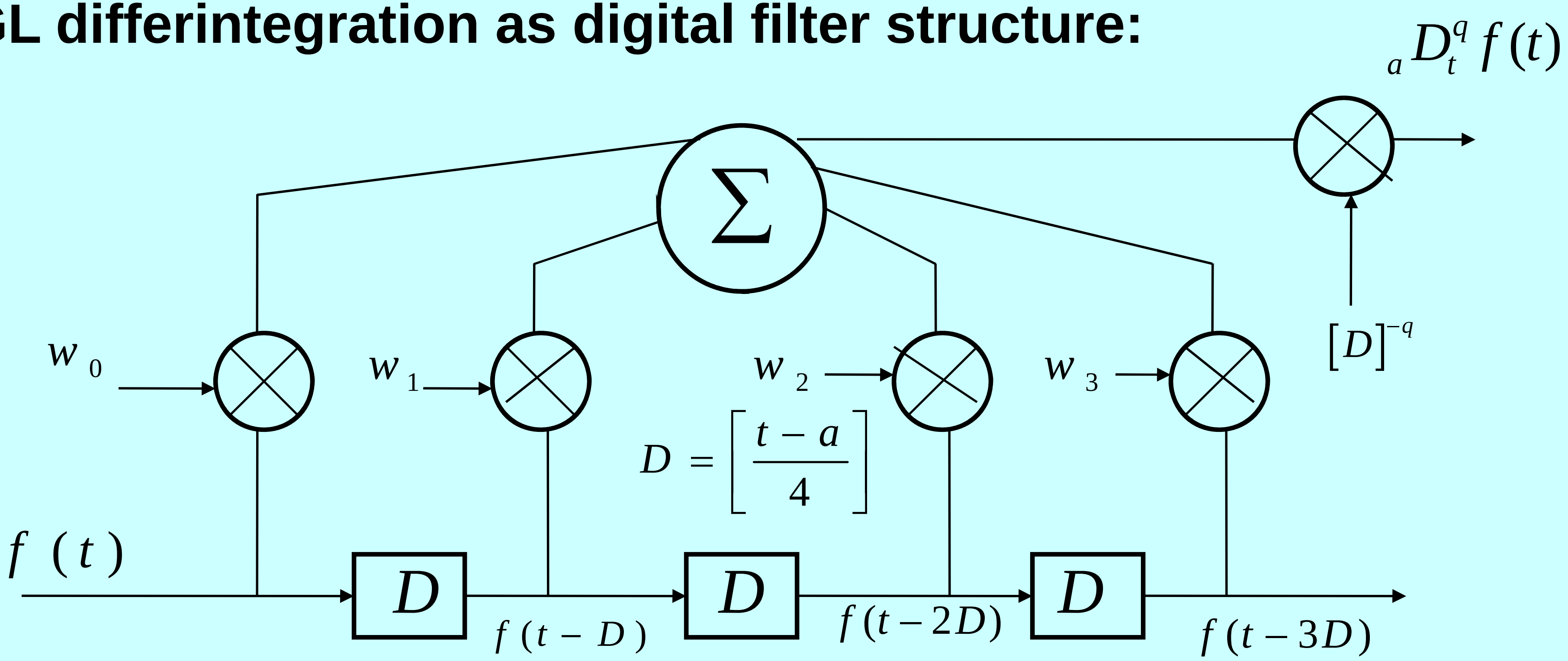
$${}_aD_x^{\alpha} f(x)=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{m=0}^{\left[\frac{x-a}{h}\right]}(-1)^m\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}f(x-mh)$$

$$\binom{\alpha}{m}=\frac{\alpha!}{m!(\alpha-m)!}\leftrightarrow\frac{\Gamma(\alpha+1)}{m!\Gamma(\alpha-m+1)}$$

$${}_aD_x^{-\alpha} f(x)=\lim_{h\to 0}h^{\alpha}\sum_{m=0}^{\left[\frac{x-a}{h}\right]}(-1)^m\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}f(x-mh)$$

$$\begin{bmatrix}-\alpha\\ m\end{bmatrix}=\binom{-\alpha}{m}=\frac{-\alpha(-\alpha-1)..(-\alpha-m+1)}{m!}=(-1)^m\frac{(\alpha+m-1)!}{m!(\alpha-1)!}\leftrightarrow(-1)^m\frac{\Gamma(\alpha+m)}{m!\Gamma(\alpha)}$$

GL differintegration as digital filter structure:



$$ {}_aD_t^q f(t) = \lim_{\Delta T \to 0} \frac{(\Delta T)^{-q}}{\Gamma(-q)} \sum_{k=0}^{N-1} \frac{\Gamma(k-q)}{\Gamma(k+1)} f(t-k\Delta T) = \lim_{D \to 0} [D]^{-q} \sum_{k=0}^{N-1} w_k f(t-kD) $$

Digital filter FIR/IIR
Tustin Discretization with Generating Function
Matrix approach FFT for weights
Short Memory Principle

About weights of GL in fractional differintegration

$$D_{+}^{q} f(x) = \lim_{\Delta x \to 0} \frac{1}{(\Delta x)^{q}} \sum_{k=0}^{\infty} w_k f(x - k\Delta x)$$

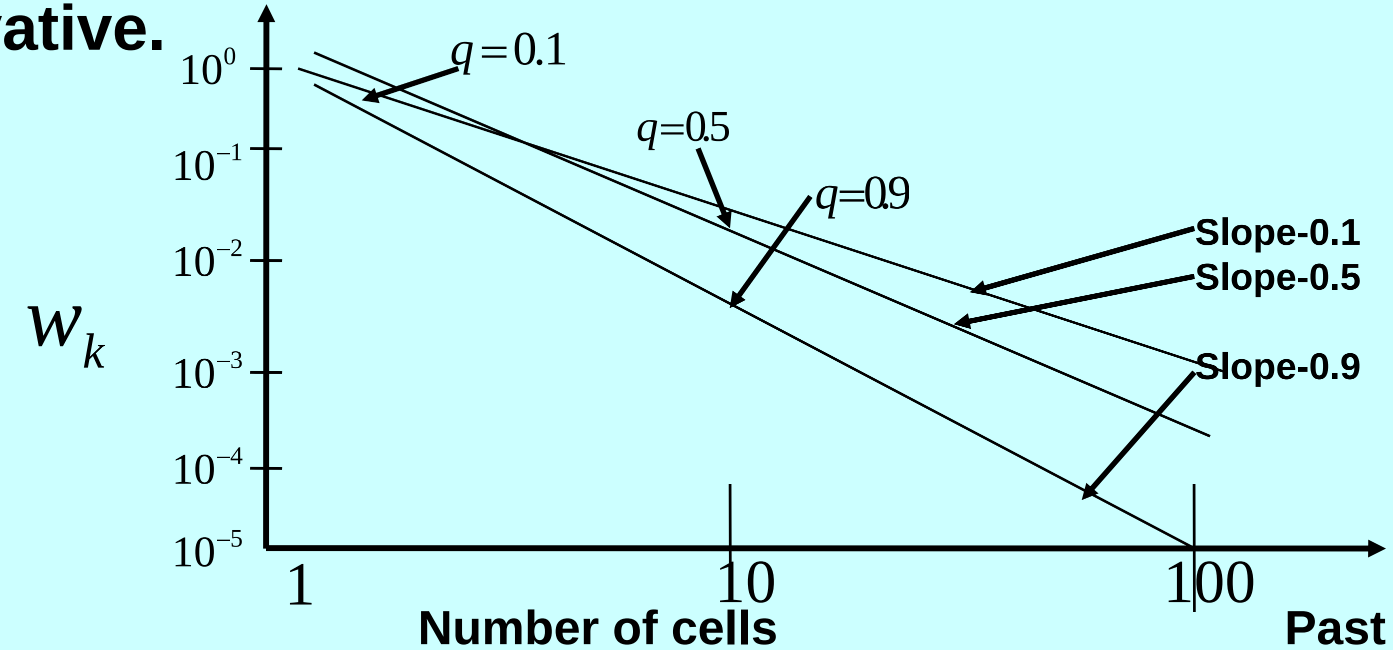
$$w_k = \frac{\Gamma(k-q)}{\Gamma(-q)\Gamma(k+1)}$$

Is apparent that fractional derivative is limit of a weighted average of the values over the function from minus infinity to point of interest (x), these weights corresponds (in limit) to a power function defined by the order of the fractional derivative (q). This averaging is for forward derivative. For backward derivative, this is limit of a average of values over the function from point of interest (x) to plus infinity. Therefore the forward fractional derivative operator has memory of the function from minus infinity to x, and backward derivative has memory of the function from x to plus infinity.

Thus point fractional derivative at a point x has a unique power law 'memory' both forward and backward on function

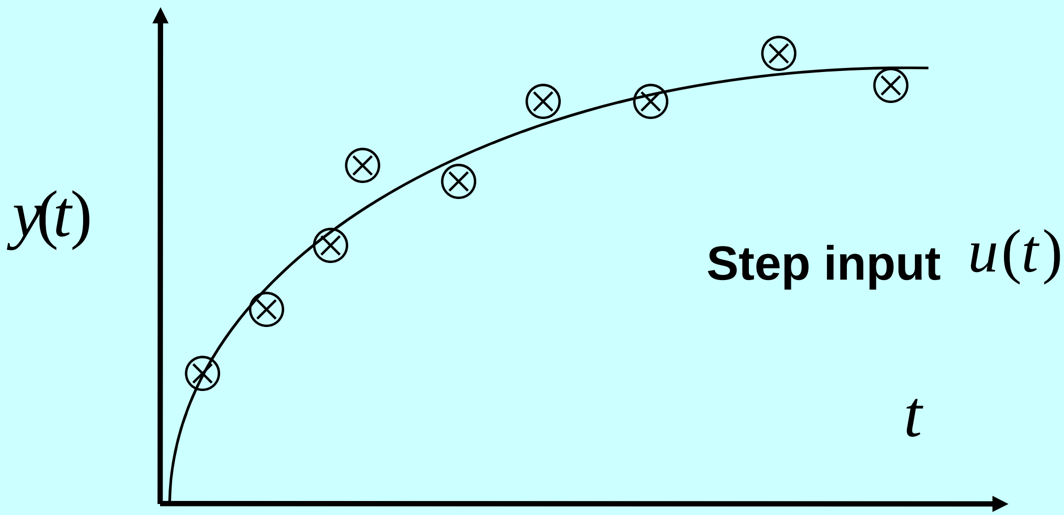
Local fractional Derivative $D^{q} = \frac{1}{2} D_{+}^{q} + \frac{1}{2} D_{-}^{q}$ **at a point depends on the character of entire function. Integer order derivative depends only on local behavior meaning slope of function at point. Fractional derivative is non-local phenomena**

Strength of weights and power law exponents of fractional derivative.



Log-log plot demonstrating power law decay in weights placed on the 100 closest cells in calculating q-th derivative. Weights depending on fractional derivative for 0.1, 0.5, 0.9.
The larger order derivative place more weights on proximal cells and dependence on distal cells decrease very quickly as distance x increases.
The lower order derivatives place relatively less weight on proximal cell and dependence on distal cell decrease very slowly as x increases.

Curve fitting-A
System identification



⊗ **Set of measured values** $y_i^*\,(i = 0, M)$ **,average error margin** $Q = \dfrac{\sum_{i=0}^{M} (y_i^* - y_i)}{M+1}$

$$1.8675 \frac{d^2}{dt^2} y(t) + 5.518 \frac{d}{dt} y(t) + 0.0063 y(t) = u(t)$$

$$Q \sim 3 \times 10^{-3}$$

$$0.7943 \frac{d^{2.571}}{dt^{2.571}} y(t) + 5.2385 \frac{d^{0.83}}{dt^{0.83}} y(t) + 1.5960 y(t) = u(t)$$

$$Q \sim 10^{-4}$$

$$6.288 \frac{d^{1.0315}}{dt^{1.0315}} y(t) + 1.8508 y(t) = u(t)$$

$$Q \sim 4 \times 10^{-4}$$

Curve fitting-B

Life span estimation, Predictive Maintenance, Reliability analysis…

. During a certain period, after installation of a wire on load, an enhancement of its properties is observed. Say yield point.

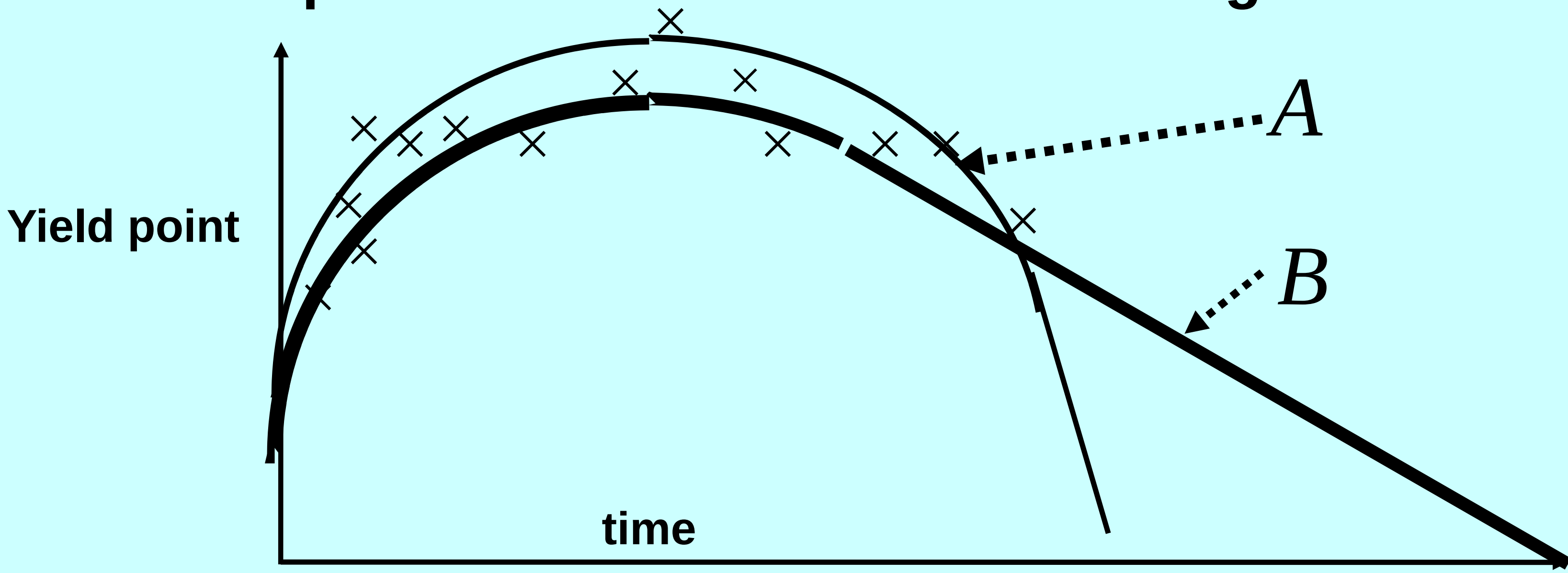
. Then properties of wires become worse and worse until it breaks down.

. The period of enhancement is shorter than the period of decrease of Property and the general shape of the process curve is not symmetric.

Set of experimental measurements $y_1, y_2, \ldots, y_n$ is fitted with fractional differential equation $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots - a_{m0} D_t^{-\alpha} y(t)$ with $(0 < \alpha \leq m)$.

$a_0, a_1, a_2, \ldots a_{m-1}$ initial values of fitted function and (m-1) derivatives. The fractional integration and its fractional order represents the cumulative impact of the previous history loading on the present state of wire. The order of fractional integration is related to shape of memory function of wire material.

Experimental fit quadratic and fractional order regression

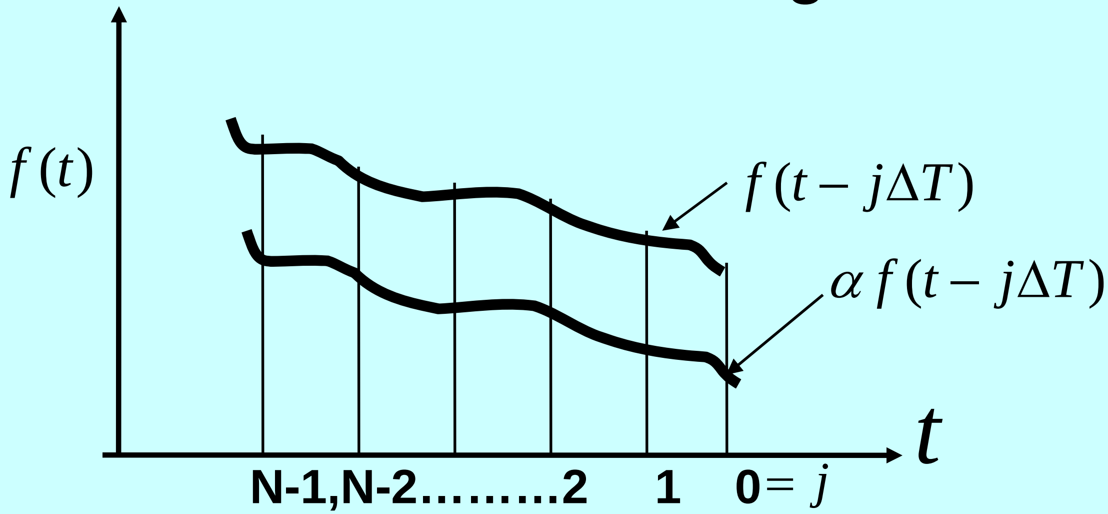


$A \qquad y(t) = 0.033t^2 + 0.562t + 10.723$

$B \qquad y(t) = -0.046\, {}_0D_t^{-1.32} y(t) + 1.2760t + 10.1955$

It is obvious that the order of fractional integration would be different for different wires because they work in different conditions. Thus it is necessary to apply this regression in each case separately. Main problem is that each particular wire changes its property due to certain very peculiar causes(heredity/history). The order 1.32 is for this particular wire of 2.4mm diameter at this loading, a 2.8mm diameter wire will have different order

Infinitesimal element fractional integration



$$ {}_aD_t^{-1} f(t) = \lim_{\Delta T\to 0}\left\{ \ldots\ldots\ldots\ldots + \Delta T\left[f(t-j\Delta T) + f(t-(j+1)\Delta T)\ldots\ldots\ldots\ldots\right]\right\} $$

$$ {}_aD_t^{-q} f(t) = \lim_{\Delta T\to 0}\left\{ \ldots\ldots\ldots\ldots + \Delta T^{q}\left[\alpha f(t-j\Delta T) + \beta f(t-(j+1)\Delta T)\ldots\ldots\ldots\ldots\right]\right\} $$

$$ 0<q<1, \alpha = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}, \beta = \frac{\Gamma(j+1-q)\Gamma(j+1)}{\Gamma(j-q)\Gamma(j+2)} $$

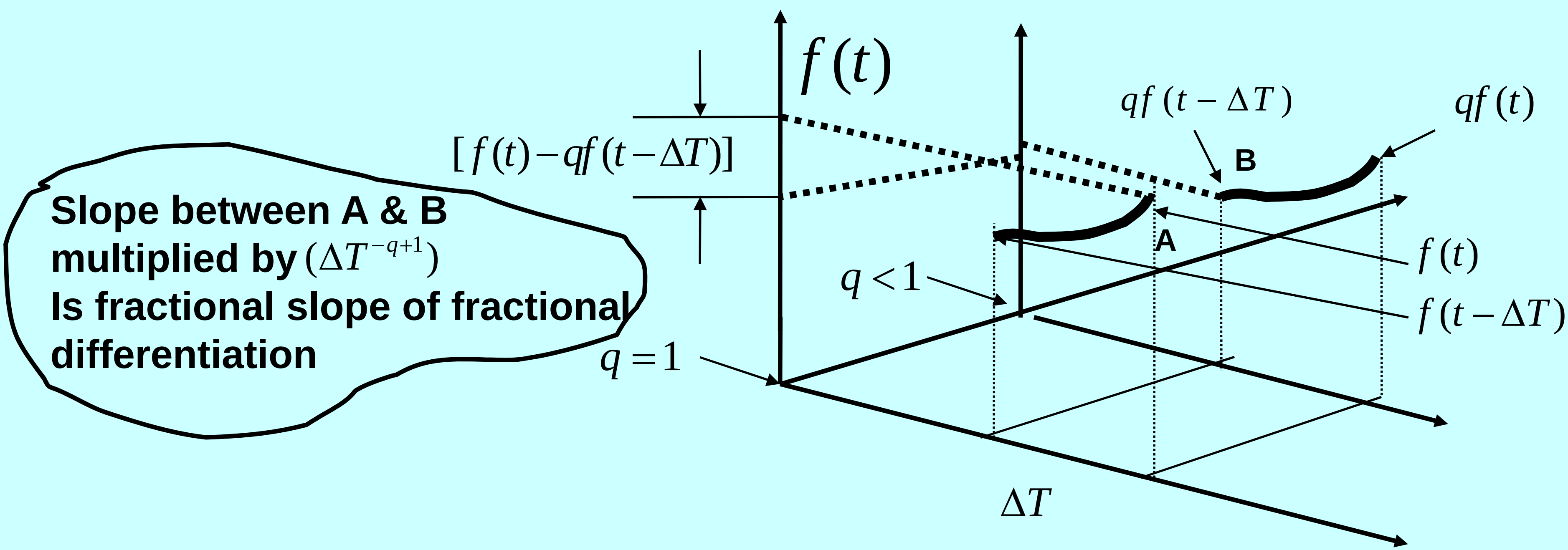
Fractional integration can be viewed a area under the curve $\alpha f(t-j\Delta T)$
Multiplied by ΔT^{q-1}
In between volume $[\alpha f(t-j\Delta T).\Delta T].\Delta T$ **and area** $\left\{\alpha f(t-j\Delta T).\Delta T\right\}$

Infinitesimal element fractional differentiation

$$ {}_aD_t^q f(t) = \lim_{\Delta T \to 0} \frac{f(t) - qf(t-\Delta T)}{\Delta T^q} + \ldots\ldots $$

$$ {}_aD_t^1 f(t) = \lim_{\Delta T \to 0} \frac{f(t) - f(t-\Delta T)}{\Delta T} $$

Fractional derivative can be viewed as fractional slope, fractional rate of change. Fractional derivative is slope between $f(t)$ **and** $qf(t-\Delta T)$ **i.e. equal to** $\frac{f(t)-qf(t-\Delta T)}{\Delta T}$ **multiplied by** $\left(\frac{1}{\Delta T^{q-1}}\right)$



A practical challenging instrumentation problem
(Dr U Paul NPD/BARC)

Total Absorption Gamma Calorimeter International Project

Observation:
Energy resolution of the detector with long pencil (1cmX2cmX20cm) crystal depends on interaction point of incident Gamma photon. Crystal defects inhomogenity (along the length of 1D crystal) is responsible for observed behavior. Scintillating light photons propagates through inhomogeneous medium before being collected by read out device PMT

Requirement
Energy resolution independent of interaction point in crystal
Development of technique and instrument which can compensate the resolution by fractal technique.

New Science application in fractional calculus
Application of flow of matter/energy through fractal defected porous path.

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