Tutorial Notes

L'Hopital's question
What is ½ derivative of $f(x) = x$ i.e. $d^{1/2} f(x) / dx^{1/2}$?

for

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Dedicated
to

L'Hopital

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We have used in derivation of fractional curl concept of fractional derivative, without any rigor of fractional calculus, assuming such fractional derivative operator exists. In this short tutorial we will be introducing the concept of fractional order differentiation and fractional order integration; by trying to answer the question posed by L'Hopital in 1695 that is what is ½ derivative of the function $f(x) = x$ *. If we are climbing a mountain and were asked what is the slope of the point where you are standing? We immediately recollect that if we have a theodolite with us we can measure the angle at that point and then calculate the slope and that is* d/dx *of that point. Now we were asked to find out what is* $d^{1/2}/dx^{1/2}$ *, then we would require all the values of mountain height right from the base of the hill as-this semi derivative is like 'integration' operator requiring the values of the all the point of the function from start point-thus we call it a non-local operator; unlike integer order derivative which is a local operator. Recalling Huygen's principle of electromagnetic wave propagation, where it states that every point of a wave front may be considered as the source of secondary wavelets that spreads out in all directions with speed of propagation of the waves; in a way states about non-locality. The fractional derivative is similar in that way where all the points of the function are needed. In a way this fractional derivative operator thus imbibes history and memory! We shall try and find out what are semi-derivative and semi-integration; and in a way introduce this fractional calculus. This is tip of iceberg.*

Contents

1. Question posed by L'Hopital to Leibniz

In a letter to L'Hopital in 1695 September 30, Leibniz raised the possibility of generalizing the operation of differentiation to non-integer orders, and L'Hopital asked what would be the result of half-differentiating x; that is:

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}(x) = ? \qquad \text{or} \qquad \mathrm{D}_x^{1/2}[x] = ?$$

Leibniz replied "It leads to a paradox, from which one day useful consequences will be drawn".

The paradoxical aspects are due to the fact that there are several different ways of generalizing the differentiation operator to non-integer powers, leading to in equivalent results.

We can say this query dated 30th September, 1695 gave birth to "Fractional Calculus"; therefore this subject of fractional calculus with half derivative and integrals etc, are as old as conventional Newtonian or Leibniz's calculus. However, this subject was dormant till the beginning of the century, and only now have started finding the applications.

2. Let us try to find L'Hopital's query

Let us try and find out if we can differentiate the function $f(x)$ by ½ to get

$$\frac{\mathrm{d}^{1/2} f(x)}{\mathrm{d}x^{1/2}} = ? \qquad \text{for} \qquad f(x) = x$$

In other words we try to find answers to L'Hospital's querry.

Differentiation and integration are usually regarded as discrete operations, in the sense that we differentiate or integrate a function once, twice, or any whole number of times. However, in some circumstances it's useful to evaluate a fractional derivative. Half derivative Quarter derivative semi-integration etc.

3. Generalization of Differentiation and Integration

In some ways the most natural and appealing generalization is based on the exponential function $f(x) = e^{ax}$ whose nth derivative is simply $a^n e^{ax}$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{ax} = a^n e^{ax} \qquad n \in \mathbb{Z}^+$$

This is n-fold repeated differentiation or integer order derivative of $f(x) = e^{ax}$. This immediately suggests defining the derivative of order υ (not necessarily an integer) as

$$\frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon} e^{ax} = a^\upsilon e^{ax} \qquad \upsilon \in \mathbb{R}^+$$

Negative values of υ represent integrations (anti-derivative) and we can even extend this to allow complex values of υ, or even to a continuous distribution of this order in some interval.

$$\frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon} f(x) \qquad \frac{\mathrm{d}^{-\upsilon}}{\mathrm{d}x^{-\upsilon}} f(x) \qquad \frac{\mathrm{d}^{p+iq}}{\mathrm{d}x^{p+iq}} f(x) \qquad \frac{\mathrm{d}^{\int_a^b k(q)\mathrm{d}q}}{\mathrm{d}x^{\int_a^b k(q)\mathrm{d}q}} f(x)$$

Any function expressible as a sum of exponential functions can then be differentiated in the same way. For example, the generalized derivative of the cosine function according to this approach is given by

$$\begin{aligned}\frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon}\cos(x) &= \frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon}\left(\frac{e^{ix}+e^{-ix}}{2}\right) = \frac{(i)^\upsilon e^{ix} + (-i)^\upsilon e^{-ix}}{2} \\ &= \frac{(e^{i\upsilon\pi/2})e^{ix} + (e^{-i\upsilon\pi/2})e^{-ix}}{2} = \frac{e^{i\left(x+\frac{\upsilon\pi}{2}\right)} + e^{-i\left(x+\frac{\upsilon\pi}{2}\right)}}{2} \\ &= \cos\left(x + \frac{\upsilon\pi}{2}\right)\end{aligned}$$

Since $(\pm i)^\upsilon = (e^{\pm i\pi/2})^\upsilon = e^{\pm i\upsilon\pi/2}$, we have the nice result

$$\frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon}\cos(x) = \cos\left(x + \frac{\upsilon\pi}{2}\right)$$

Thus the generalized differential operator simply shifts the phase of the cosine function (and likewise the sine function by $\upsilon \times (90)^0$), that is in proportion to the order of the differentiation.

For differentiation the process advances the phase, needless to say the integration makes the phase lagged. After all for $\upsilon = 1$ and $\upsilon = -1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = \cos\left(x + \frac{\pi}{2}\right) \qquad \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}}\cos(x) = \cos\left(x - \frac{\pi}{2}\right)$$

Needless to say, this approach can be applied to the exponential, i.e. ½ derivative of e^{ax} should be

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} e^{ax} = (a^{1/2}) e^{ax}$$

This approach is correct in some cases as we will see in subsequent section. This approach is also called Liouvelli's approach.

The exponential approach seems to give a very satisfactory way of defining fractional derivatives… but we have yet to answer L'Hopital's question, which was to determine the half-derivative of $f(x) = x$.

4. Paradox stems from unification of differentiation and integration which makes the differentiation a non-local operation

There is no Fourier representation of this open-ended function $f(x) = x$, so it has no well-defined 'spectral decomposition'. Of course, we can find the Fourier representation of x over some finite interval, but what interval should we choose? This ambiguity gives a hint of why Leibniz considered the subject to be paradoxical. Leibniz was well aware that the result of integrating a function is neither unique nor local, because it depends on how the function behaves over the range for which the integration is performed, not just at a single point. But he was used to thinking of differentiation as both unique and local, because whole derivatives $\mathrm{d}/\mathrm{d}x$, $\mathrm{d}^2/\mathrm{d}x^2$, $\mathrm{d}^3/\mathrm{d}x^3 .... \mathrm{d}^n/\mathrm{d}x^n$ happen to possess both of those attributes. These conventional derivatives are local operator gives a slope at a point, i.e. depend on local point, whereas integration $\mathrm{d}^{-1}/\mathrm{d}x^{-1}$, $\mathrm{d}^{-2}/\mathrm{d}x^{-2} .... \mathrm{d}^{-n}/\mathrm{d}x^{-n}$ depends on the entire interval, hence non-local in character; so does ½ , ¼, ….derivatives are. This we shall deal while formulating the fractional differentiation.

The apparent paradoxes of fractional derivatives arise from the fact that, in general, differentiation is non-unique and non-local, just as is integration. This shouldn't be surprising, since the generalization essentially unifies integrals and derivatives into a single operator. If anything, we ought to be surprised at how this operator takes on uniqueness and locality for positive integer arguments.

5. To get formulation for generalized derivative

To get a clearer idea of the ambiguity in the concept of a generalized derivative, it's useful to examine a few other approaches, and compare them with the exponential approach described above. The most fundamental approach may be to begin with the basic definition of the whole derivative of a function $f(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\varepsilon\to 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

Repeating n-times of this operation leads to a binomial series of following type

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - j\varepsilon) \tag{1}$$

for any positive integer n. To illustrate, this formula gives the second derivative ($n = 2$) of the function $f(x) = x^4$ as

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x)^4 &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \sum_{j=0}^{2} (-1)^j (x - j\varepsilon)^4 \\ &= \lim_{\varepsilon\to 0} \frac{1}{h^2}\left[x^4 - 2(x-\varepsilon)^4 + (x-2\varepsilon)^4\right] \\ &= \lim_{\varepsilon\to 0}\left[12x^2 - 24x\varepsilon + 14\varepsilon^2\right] = 12x^2\end{aligned}$$

We can generalize equation (1) for non-integer orders, but to do this we must not only generalize the binomial coefficients, we also need to determine the appropriate generalization of the upper summation limit, which we wrote as n in equation (1).

To clarify the situation, let us go back and derive "from scratch" the operations of differentiation and integration in a unified context. Consider an arbitrary smooth function $f(x)$ as shown in the figure below.

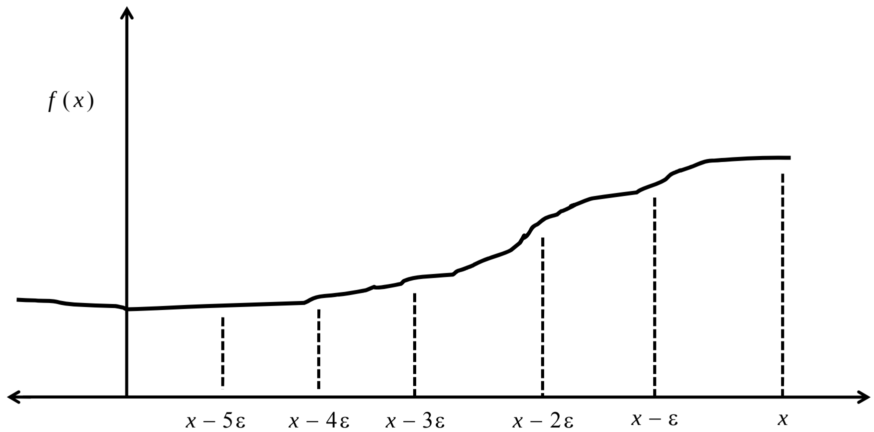

Figure-1: Dividing the function interval into small slices of ε

In addition to the point at x, we've also marked six other equally-spaced values on the interval from 0 to x, each a distance ε from its neighbors. The number k of these points is related to the values of x and ε by $x = \mathrm{k}\varepsilon, \quad \mathrm{k} = 0, 1, 2, 3....$ For convenience, we define a (backward) shift operator E_ε such that $\mathrm{E}_\varepsilon f(x) \triangleq f(x-\varepsilon)$.

With this back ward shift operator we get differentiation as

$$\mathrm{D}\left[f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\varepsilon\to 0} \frac{f(x) - f(x-\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{f(x) - \mathrm{E}_\varepsilon[f(x)]}{\varepsilon}$$
$$= \lim_{\varepsilon\to 0} \left(\frac{1-\mathrm{E}_\varepsilon}{\varepsilon}\right) f(x)$$

With the help of the series expansion as

$$\frac{1}{1-x} \cong 1 + x + x^2 + x^3 + ...$$

and the shift operator as E_ε^N indicating N backward shift till $f(0)$ the start point of the function, that is:

$$\mathrm{E}_\varepsilon^N = f(x - N\varepsilon) \qquad N = 0,1,2,3,....$$

We obtain the following

$$\mathrm{D}^1\left[f(x)\right] = \lim_{\varepsilon\to 0}\left(\frac{1-\mathrm{E}_\varepsilon}{\varepsilon}\right)^1 f(x) = \lim_{\varepsilon\to 0}\frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

$$\mathrm{D}^{-1}\left[f(x)\right] = \lim_{\varepsilon\to 0}\left(\frac{1-\mathrm{E}_\varepsilon}{\varepsilon}\right)^{-1} f(x)$$

$$= \lim_{\varepsilon\to 0}\varepsilon\left[1 + \mathrm{E}_\varepsilon + \mathrm{E}_\varepsilon^2 + ..\right] f(x) = \lim_{\varepsilon\to 0}\varepsilon\left[f(x) + f(x-\varepsilon) + f(x-2\varepsilon) + ... f(0)\right]$$

As in limit ε goes to zero the D^1 is a simply a differentiation operator and D^{-1} is a simple integration operator. We can do n times the above and write the following:

$$D^n\left[f(x)\right] = \lim_{\varepsilon\to 0}\left(\frac{1-E_\varepsilon}{\varepsilon}\right)^n f(x)$$

This reproduces the ordinary whole multiple derivatives. For example, the second derivative of $f(x)$

$$\mathrm{D}^2\left[f(x)\right] = \left(\frac{1-\mathrm{E}_\varepsilon}{\varepsilon}\right)^2 f(x) = \frac{f(x) - 2f(x-\varepsilon) + f(x-2\varepsilon)}{\varepsilon^2}$$

in the limit as ε goes to zero, which illustrated how we recover the binomial equation (1) for any whole number of differentiations. However, strictly speaking, this context makes it clear that we should actually write the second derivative as

$$\mathrm{D}^2\left[f(x)\right] = \left(\frac{1-\mathrm{E}_\varepsilon}{\varepsilon}\right)^2 f(x) = \frac{(1)f(x) - (2)f(x-\varepsilon) + (1)f(x-2\varepsilon) - (0)f(x-3\varepsilon) + ...(0)f(0)}{\varepsilon^2}$$

It just so happens that, if n is a positive integer, all the binomial coefficients after the first $n+1$ are identically zero

$$\binom{n}{j} = {}^{n}C_j = 0 \qquad\qquad j > n$$

so we can truncate the series, but for any negative or fractional positive values of n , the binomial coefficients are non-terminating, so we must include the entire summation over the specified range.

Consequently, the upper summation limit in (1) should actually be $(x - x_0)/\varepsilon$, where x_0 is the lower bound on the range of evaluation. We often choose $x_0 = 0$ by convention, but it is actually arbitrary, and we will see below some circumstances in which the lower bound is not zero.

In any case, we can re-write equation (1) in the more correct form that does not rely on n being a positive integer

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \sum_{j=0}^{\left\lfloor \frac{x - x_0}{\varepsilon} \right\rfloor} (-1)^j \binom{n}{j} f(x - j\varepsilon) \tag{1a}$$

The bracket $\lfloor \ldots \rfloor$ is floor operator makes the value to nearest lower integer.

6. Defining factorial and binomial coefficients for non-integer and getting series formulation for fractional derivative

To define the binomial coefficient for non-integer values of n, recall that for integer arguments these coefficients are defined as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

so we need a way of evaluating the factorial function for non-integer arguments $n \in \mathbb{R}$.

Notice that for any positive integer n we have the definite integral

$$\int_{-1}^{+1} (1 - x^2)^n \,\mathrm{d}x = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

From above we get

$$n! = \sqrt{\frac{(2n+1)!}{2^{2n+1}} \int_{-1}^{+1} \mathrm{d}x (1-x^2)^n}$$

The argument of the factorial on the right side is $2n+1$, so the right hand expression is well-defined for half-integer value of n such that $2n+1$ is non-negative. Hence this is a well-defined expression for the factorial of any such half-integer argument. For example, setting $n = -1/2$ and using $\int_{-1}^{+1} \mathrm{d}x (1-x^2)^{-1/2} = \pi$ in above expression we get

$$\left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

Furthermore, now that the factorial of all (positive) half-integers is defined, the above formula allows us to compute the factorial of any quarter-integer, and then every sixteenth, and so on. Hence, using the binary representation of real numbers, and using the identity $(x+1)! = (x+1)(x!)$, we now have a well-defined factorial function for any real number.

This is traditionally called the gamma function, with the argument offset by 1 relative to the factorial notation, so we have

$$\Gamma(n) = (n-1)!$$

Therefore

$$\sqrt{\pi} = \left(-\frac{1}{2}\right)! = \left(\frac{1}{2} - 1\right)! = \Gamma\left(\frac{1}{2}\right)$$

for any positive integer n.

The gamma function has several formulations and its integral representation is

$$\Gamma(\upsilon) = \int_{0}^{\infty} u^{\upsilon-1} e^{-u} \mathrm{d}u$$

The fundamental recurrence formula for the gamma function is therefore $\Gamma(\upsilon+1) = \upsilon\Gamma(\upsilon)$

Note the reflection relation is

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

We have the following values for positive half-integer arguments

$$\Gamma\left(-\frac{3}{2}\right)=\frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right)=\frac{3}{4}\sqrt{\pi}$$

Now that we have a general way of expressing "factorials" for non-integers, we can re-write equation (1a) in generalized form, replacing each appearance of the integer n with the real number υ. This gives

$$\frac{\mathrm{d}^{\upsilon}}{\mathrm{d}x^{\upsilon}} f(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{\upsilon}}\sum_{j=0}^{\left\lfloor\frac{x-x_0}{\varepsilon}\right\rfloor}(-1)^j\frac{\Gamma(\upsilon+1)}{j!\Gamma(\upsilon+1-j)} f(x-j\varepsilon) \tag{2}$$

If υ is an integer n, the vanishing of the binomial coefficients for all j greater than n implies that we don't really need to carry the summation beyond $j=n$, and in the limit as ε goes to zero the n values of $f(x-j\varepsilon)$ with non-zero coefficients all converge on x so the derivative is local.

7. Non-locality of fractional derivatives and similarity to Huygen's principle

However, in general, the binomial expansion has infinitely many non-zero coefficients, so the result depends on the values of x all the way down to x_0. We typically choose $x_0=0$ so we are effectively evaluating the "derivative" (which is not the same as the "slope") for the interval from 0 to x. Thus, as mentioned previously, the generalized derivative is a non-local operation, just as is integration. The general derivative depends on the value of the function f over the whole range from x_0 to x. This can be seen from the factor

$f(x - j\varepsilon)$ in the summation in equation (2), showing that as j ranges from zero to $(x - x_0)/\varepsilon$ the argument of f ranges from x down to zero (the start point of origination of function).

It just so happens that this non-locality disappears for positive whole derivatives, when n is whole number like 1, 2, 3…. . But for $n = \upsilon \in \mathbb{R}$ the property in non-local, thus requiring all the points of the function to construct the generalized derivative of the function. We may also state that fractional derivative is a non-local quantity, whereas the normal integer order derivative is local or point property. The fractional derivative thus depends on history in the sense if we consider the definition of shift operation as forward difference

This simple mathematical fact has an important consequence in the strong form of Huygen's Principle which accounts for the sharp propagation of light and other wavelike phenomena in three dimensional spaces. The Huygen's Principle states that every point of a wave front may be considered as the source of secondary wavelets that spreads out in all directions with speed of propagation of the waves. What this means is that you have a wave, you can view the "edge" of the wave as actually creating a series of circular waves. These waves combine together in most cases to just continue the propagation, but in some cases there are significant observable effects. The wave fronts can be viewed as the line 'tangent' to all circular waves. These results can be obtained separately from Maxwell's equations, though the Huygen's principle which came first is a useful model and often convenient for calculations of wave phenomena. It is intriguing that Huygen's work preceded Maxwell's by about two centuries, and yet seemed to anticipate it, without the solid theoretical basis that Maxwell provided. Ampere's law and faraday's law predict that every point in an electromagnetic wave acts as a source of the continuing wave, which is perfectly in line with Huygen's analysis. This is a totally non-local phenomenon involving all the points in space!

8. Try to answer L'Hopital's question now!

Choosing $x_0 = 0$ as the low end of our differentiation interval, the formula (2) for the general derivative becomes

$$\frac{d^{\upsilon}}{dx^{\upsilon}} f(x) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\upsilon}} \sum_{j=0}^{\lfloor x/\varepsilon \rfloor} (-1)^j \frac{\Gamma(\upsilon+1)}{j!\Gamma(\upsilon+1-j)} f(x-j\varepsilon) \tag{3}$$

With this, we are finally equipped to attempt to answer L'Hospital's question. Taking the function $f(x)=x$ with $\upsilon = 1/2$, signifying the half-derivative, this formula gives

$$\frac{d^{1/2}}{dx^{1/2}}(x) = \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} \begin{bmatrix} 1(x) - \frac{1}{2}(x-\varepsilon) - \frac{1}{8}(x-2\varepsilon) - \frac{1}{16}(x-3\varepsilon)... \\ -\frac{5}{128}(x-4\varepsilon)... - \begin{pmatrix} 1/2 \\ \lfloor x/\varepsilon \rfloor - 2 \end{pmatrix}(2\varepsilon) - \begin{pmatrix} 1/2 \\ \lfloor x/\varepsilon \rfloor - 1 \end{pmatrix}(\varepsilon) \end{bmatrix}$$

In this equation the binomial coefficients symbol is understood to denote the generalized function, with the factorials expressed in terms of the gamma function. As explained previously, the coefficients in the above expression are just the coefficients in the binomial expansion of $(1-E_{\varepsilon})^{1/2}$. Evaluating this expression (numerically) in the limit as ε goes to zero, we find that the half-derivative of x is (almost)

$$\frac{d^{1/2}}{dx^{1/2}}(x) = 2\sqrt{\frac{x}{\pi}} \tag{4}$$

We can thus write the above as ${}_0D_x^{1/2}(x) = 2\sqrt{x/\pi}$ indicating that differentiation is starting at start point $x=0$. This concept of defining the fractional derivative with backward shift operator gives ${}_aD_x^{\upsilon}[f(x)]$, the forward derivative of function defined in the interval $[a,b]$. We can have backward derivative too as ${}_xD_b^{\upsilon}[f(x)]$, by using forward shift operator; in this case future points of the function is to be known a priory. In a sense therefore forward derivative constructed by back-shift operator is causal.

This is exactly what we would expect based on a straightforward interpolation of the derivatives of a power of x. Recalling that the first few (whole) derivatives of x^m are

$$\frac{d}{dx}(x^m) = mx^{m-1}$$
$$\frac{d^2}{dx^2}(x^m) = m(m-1)x^{m-2}$$
$$\frac{d^3}{dx^3}(x^m) = m(m-1)(m-2)x^{m-3}$$

Thus we expect to find that the general form of the nth derivative of x^m is

$$\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}x^{m-n}$$

Replacing the integer n with the general value υ, and using the gamma function to express the factorial, this suggests that the a fractional derivative of x^n is simply

$$\frac{d^\upsilon}{dx^\upsilon}(x^m) = \frac{m!}{\Gamma(m-\upsilon+1)}x^{m-\upsilon} = \frac{\Gamma(m+1)}{\Gamma(m-\upsilon+1)}x^{m-\upsilon} \tag{5}$$

$$\frac{d^{1/2}}{dx^{1/2}}(x) = \frac{1!}{\Gamma(1-[1/2]+1)}x^{1-(1/2)} = \frac{1}{\Gamma(3/2)} = 2\sqrt{\frac{x}{\pi}}$$

which is exactly the same answer to L'Hopital's question as we got previously, i.e., the half-derivative of x is given by (4). The (5) is Euler formula holds for $m > -1$; that is requires function to be better behaved than x^{-1}.

Now, since analytic functions can be expanded into power series ($f(x) = \sum_k a_k x^k$) we can use equation (5), applying it term by term to determine the fractional derivatives of all such functions. Furthermore, applying this formula with negative values of υ gives a plausible expression for the fractional- integral of a power of x. For example, to find the whole integral of x^3 we set $m = 3$ and $\upsilon = -1$ and then compute

$$\frac{d^{-1}}{dx^{-1}}(x^3) = \frac{3!}{\Gamma(3-[-1]+1)}x^{3-(-1)} = \frac{3!}{\Gamma(5)}x^4 = \frac{6}{24}x^4 = \frac{1}{4}x^4$$

Note that the above integration is valid only if the initial point be zero, else initial value be subtracted.

So, in a sense, equation (5) is an algebraic expression of the fundamental theorem of calculus, i.e., the inverse relationship between the operations of differentiation and integration, since the $n-$th derivative of the $-n$ th derivative (integration) is the identity, (provided the initial values at the start point of the function is zero). The unification of these two operations makes it even less surprising that generalized differentiation is non-local, just as is integration.

9. Half derivative of a constant and other functions-and paradoxical case

Incidentally, the generalized derivative as developed so far gives some slightly surprising results. For example, the half-derivative of any constant function $\mathrm{C}x^0$ is

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}(\mathrm{C}) = \mathrm{C}\frac{0!}{\Gamma(1/2)} = \frac{\mathrm{C}}{\sqrt{\pi x}} \tag{6}$$

Thus, not only is the half-derivative of a constant with (respect to x) non-zero, it is infinite at $x=0$, and decays to zero at $x=\infty$. Nevertheless, equations (3) and (5) are agreeably consistent with each other, giving some confidence in the significance of this generalization of the derivative. Given this equivalence, one might wonder about the value of the elaborate derivation of equation (3) when it seems to be so much easier and more direct to arrive at equation (5). In answer to this there are two points to consider. First, equation (3) applies to fairly arbitrary functions, whereas equation (5) applies only to functions expressible as power series. Still, a very large class of functions can be expressed as power series, so this in itself is not an overriding factor. More important is the fact that equation (5) gives no hint of the non-locality of the generalized derivative, i.e., the dependence on the function over a finite range rather than just at a single point, and the need to specify (implicitly or explicitly) the chosen range. The importance of this can be seen in several different ways. Perhaps the most significant reason for taking care of the derivative interval is brought to light when we try to apply equation (3) or (5) to the simple exponential function. We previously proposed that the general $n-$th derivative of e^{ax} is simply $(a^n)e^{ax}$, and yet if we expand the exponential function e^x into a power series

$$e^{ax} = 1 + \frac{a}{1!}x + \frac{a^2}{2!}x^2 + \frac{a^3}{3!}x^3 + \ldots$$

and apply equation (5) to determine the half-derivative, term by term, we get

$$\frac{d^{1/2}(e^x)}{dx^{1/2}} = \frac{1}{\sqrt{\pi x}}\left(1 + 2x + \frac{4}{3}x^2 + \frac{8}{15}x^3 + \frac{16}{105}x^4 + \ldots\right)$$

A plot of this function, along with e^x, is shown in the figure below

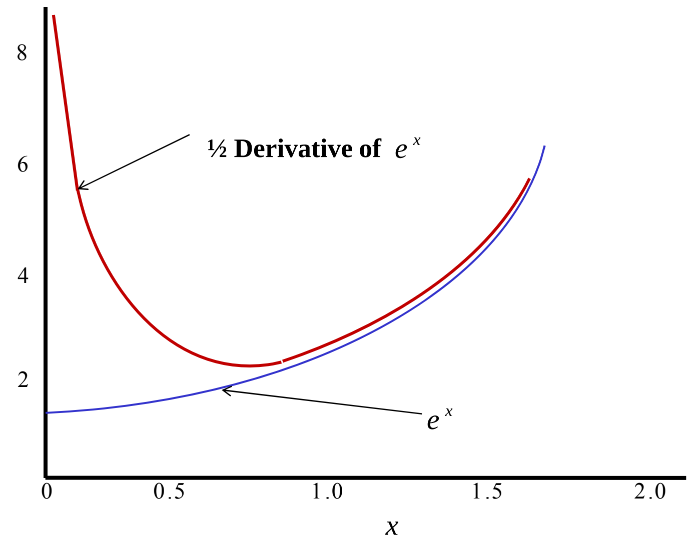

Figure-2: Plot of half-derivative of e^x and e^x

Here we see one of the paradoxes that might have intrigued Leibniz. According to a very reasonable general definition we expect any derivative (including fractional derivatives) of the exponential function to equal itself, and the exponential goes to 1 as x goes to zero, and yet our carefully-derived formulas for the half-derivative of the exponential function goes to infinity at $x = 0$. Clearly something is wrong. Must we abandon the elegant exponential approach, along with its beautiful explanation of the trigonometric derivatives as simple phase shifts, etc? No, we can reconcile our results, provided we recognize that the derivative is non-local, and therefore depends on the chosen range of differentiation; that is like integration it has lower and upper limits.

10. Fractional Derivatives with lower limit to minus infinity

The lower terminal to minus infinity is Liouvelli formulation, and this refers also to steady state systems. In these cases our simplified results of phase shifting of trigonometric functions do hold. Consider the two anti-differentiations (integrations) shown below

$$\int_{x_0}^{x} u^3 \mathrm{d}u = \frac{x^4}{4} - \frac{x_0^4}{4}$$

$$\int_{x_0}^{x} e^u \mathrm{d}u = e^x - e^{x_0}$$

The first integral shows that when we say x^3 is the derivative of $x^4/4$ we are implicitly assuming $x_0 = 0$, which is consistent with our derivation of equation (3). However, the lower integral shows that, by saying e^x is the derivative of e^x, we are implicitly assuming $x_0 = -\infty$.

Thus ranges of integration/differentiation we have tacitly assumed for these two definitions are different. To get agreement between the interpolated binomial expansion method and the definition based on exponential functions we must return to equation (2), and replace the condition $x_0 = 0$ with the condition $x_0 = -\infty$. This is easy to do, because it simply amounts to setting the upper summation limit to infinity, i.e., we take the following formula for our generalized derivative

$$\frac{\mathrm{d}^\upsilon}{\mathrm{d}x^\upsilon} f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\upsilon} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(\upsilon+1)}{j!\Gamma(\upsilon+1-j)} f(x - j\varepsilon) \tag{7}$$

With this, we do indeed find that the υ–th derivative of e^{ax} is simply $(a^\upsilon)e^{ax}$, consistent with the purely exponential approach.

11. Repeated integration approach to get fractional derivative

Still another approach to fractional calculus is to begin with a generalization of the formula for repeated integration. Suppose the function $f(x)$ has the specified values

f_0, f_1, f_2, f_3, f_4, f_5 at equally spaced intervals of width ε. The integral of this function from $x = 0$ to 5ε can be approximated by ε times the cumulative sum of these values, and the integral of this new function is ε times the cumulative sum of those values, and so on. This is illustrated in the table below.

f_0	f_0
f_1	$f_0 + f_1$
f_2	$f_0 + f_1 + f_2$
f_3	$f_0 + f_1 + f_2 + f_3$
f_4	$f_0 + f_1 + f_2 + f_3 + f_4$
f_5	$f_0 + f_1 + f_2 + f_3 + f_4 + f_5$

$$S_1 = \int f(x)\,dx \qquad dx = \varepsilon = 1$$
$$S_1 = f_0 + f_1 + f_2 + f_3 + f_4 + f_5$$

f_0	f_0
f_1	$2f_0 + f_1$
f_2	$3f_0 + 2f_1 + f_2$
f_3	$4f_0 + 3f_1 + 2f_2 + f_3$
f_4	$5f_0 + 4f_1 + 3f_2 + 2f_3 + f_4$
f_5	$6f_0 + 5f_1 + 4f_2 + 3f_3 + 2f_4 + f_5$

$$S_2 = \int\int f(x)\,dx \qquad dx = \varepsilon = 1$$
$$S_2 = (6-0)f_0 + (6-1)f_1 + (6-2)f_2 + (6-3)f_3 + (6-4)f_4 + (6-5)f_5$$

f_0	f_0
f_1	$3f_0 + f_1$
f_2	$6f_0 + 3f_1 + f_2$
f_3	$10f_0 + 6f_1 + 3f_2 + f_3$
f_4	$15f_0 + 10f_1 + 6f_2 + 3f_3 + f_4$
f_5	$21f_0 + 15f_1 + 10f_2 + 6f_3 + 3f_4 + f_5$

$$S_3 = \int\int\int f(x)\,dx \qquad dx = \varepsilon = 1$$
$$S_3 = \frac{(6+1)(6-0)}{2}f_0 + \frac{(6-0)(6-1)}{2}f_1 + \frac{(6-1)(6-2)}{2}f_2 + \frac{(6-2)(6-3)}{2}f_3$$
$$= + \frac{(6-3)(6-4)}{2}f_4 + \frac{(6-4)(6-5)}{2}f_5$$

Table-1: Cumulative sum showing 1st, 2nd, and 3rd integration of function for six points

Thus the 1st, 2nd, and 3rd “integrations” yield the values (the ε is common term not shown)

$$S_1 = f_0 + f_1 + f_2 + f_3 + f_4 + f_5$$
$$S_2 = (6-0)f_0 + (6-1)f_1 + (6-2)f_2 + (6-3)f_3 + (6-4)f_4 + (6-5)f_5$$
$$S_3 = \frac{(6+1)(6-0)}{2}f_0 + \frac{(6-0)(6-1)}{2}f_1 + \frac{(6-1)(6-2)}{2}f_2 + \frac{(6-2)(6-3)}{2}f_3$$
$$= + \frac{(6-3)(6-4)}{2}f_4 + \frac{(6-4)(6-5)}{2}f_5$$

As we divide the overall interval into more and more segments, ε becomes arbitrarily small, and so do the differences between the factors in any given term, so the successive integrations give

$$\frac{d^{-2}}{dx^{-2}}f(x)=\int_0^x\int_0^{u_1}f(u_1)du_2du_1=\int_0^x(x-u)f(u)du$$

$$\frac{d^{-3}}{dx^{-3}}f(x)=\int_0^x\int_0^{u_1}\int_0^{u_2}f(u_1)du_3du_2du_1=\frac{1}{2!}\int_0^x(x-u)^2f(u)du$$

and so on like following.

$$\frac{d^{-n}}{dx^{-n}}f(x)=\underbrace{\int_0^x\int_0^{u_1}\int_0^{u_2}............\int_0^{u_{n-1}}}_{n}f(u_1)du_ndu_{n-1}.....du_1=\frac{1}{(n-1)!}\int_0^x(x-u)^{n-1}f(u)du$$

$$=\frac{1}{\Gamma(n)}\int_0^x(x-u)^{n-1}f(u)du$$

Thus we have Cauchy’s expression for repeated integrals, which is

$$\underbrace{\int_0^x\int_0^{u_1}\int_0^{u_2}............\int_0^{u_{n-1}}}_{n}f(u_1)du_ndu_{n-1}.....du_1=\frac{1}{(n-1)!}\int_0^x(x-u)^{n-1}f(u)du$$

which we can express using the gamma function instead of factorials, for fractional order υ as

$$\frac{d^{-\upsilon}}{dx^{-\upsilon}}f(x)=\frac{1}{\Gamma(\upsilon)}\int_0^x(x-u)^{\upsilon-1}f(u)du \tag{8}$$

The (8) is Riemann fractional integral formula. The convergence properties of this formula are best when υ has a value between 0 and 1.

Let us evaluate double integral of $\sqrt{x}$ take then $\upsilon=2$ and apply (8)

$${}_0D_x^{-2}\left[\sqrt{x}\right]=\frac{d^{-2}\sqrt{x}}{dx^{-2}}=\frac{1}{\Gamma(2)}\int_0^x(x-u)\sqrt{u}du$$

$$=\int_0^x(x\sqrt{u}-\sqrt{u^3})du=\left[\frac{2}{3}x\sqrt{u^3}-\frac{2}{5}\sqrt{u^5}\right]_{u=0}^{u=x}$$

$$=\frac{4}{15}\sqrt{x^5}$$

Now let us do semi integration of $\sqrt{x}$, take then $\upsilon=1/2$ and apply (8)

$$ {}_0D_x^{-1/2} = \frac{d^{-1/2}}{dx^{-1/2}}\sqrt{x} = \frac{1}{\Gamma(1/2)}\int_0^x \frac{\sqrt{u}}{(x-u)^{-\frac{1}{2}+1}}du = \frac{1}{\Gamma(1/2)}\int_0^x \frac{udu}{\sqrt{u}(\sqrt{x-u})} $$

$$ = \frac{1}{\Gamma(1/2)}\int_0^x \frac{u}{\sqrt{xu-u^2}}du = \frac{1}{\Gamma(1/2)}\int_0^x \frac{udu}{\sqrt{\frac{x^2}{4}-u^2-\frac{x^2}{4}+2x\frac{u}{2}}} $$

$$ = \frac{1}{\Gamma(1/2)}\int_0^x \frac{udu}{\sqrt{\frac{x^2}{4}-\left(u-\frac{x}{2}\right)^2}} $$

Put $\quad u = \dfrac{x + x\sin\theta}{2} \qquad du = \dfrac{x}{2}(\cos\theta)d\theta$

$u = 0;\quad \theta = -\pi/2 \qquad u = x;\quad \theta = +\pi/2$

With these substitutions we proceed

$$ {}_0D_x^{-1/2} = [\sqrt{x}] = \frac{1}{\Gamma(1/2)}\int_{-\pi/2}^{+\pi/2} d\theta \frac{\frac{x}{2}(\cos\theta)\left(\frac{x+x\sin\theta}{2}\right)}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}} $$

$$ = \frac{1}{\Gamma(1/2)}\int_{-\pi/2}^{+\pi/2} d\theta \frac{\frac{x}{2}(\cos\theta)\left(\frac{x+x\sin\theta}{2}\right)}{\left(\frac{x}{2}\right)\sqrt{1-\sin^2\theta}} $$

$$ = \frac{1}{\Gamma(1/2)}\int_{-\pi/2}^{+\pi/2} d\theta\left(\frac{x+x\sin\theta}{2}\right) = \frac{1}{\Gamma(1/2)}\left[\frac{x\theta}{2}-\frac{x\cos\theta}{2}\right]_{\theta=-\pi/2}^{\theta=+\pi/2} $$

$$ = \frac{1}{\Gamma(1/2)}\left(\frac{\pi x}{2}\right) $$

$$ = \frac{\sqrt{\pi}}{2}x $$

The same we will get via Euler formula as shown below

$$ {}_0D_x^{\upsilon}[x^m] = \frac{\Gamma(m+1)}{\Gamma(m-\upsilon+1)}x^{m-\upsilon} $$

for semi-integration of $\quad \sqrt{x} \qquad m = \dfrac{1}{2} \qquad \upsilon = -\dfrac{1}{2}$

$$ {}_0D_x^{-1/2}\left[x^{1/2}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+1\right)}x^{\frac{1}{2}+\frac{1}{2}} = \frac{\Gamma(3/2)}{\Gamma(2)}x = \Gamma\left(\frac{3}{2}\right)(x) = \frac{\sqrt{\pi}}{2}x $$

There are two different ways in which this formula might be applied. For example, if we wish to find the (7/3)-rd ($\mu = 7/3$)derivative of a function ($\mathrm{d}^{7/3} f(x)/\mathrm{d}x^{7/3}$), we could begin by differentiating the function three whole times (taking nearest integer say m just greater than μ ; that is $m = 3$), and then apply the above formula with $\upsilon = 2/3$ to "deduct" two thirds i.e. $\upsilon = (m - \mu) = (3 - [7/3]) = 2/3$ of a differentiation

Alternatively we could begin by applying the above formula with $\upsilon = 2/3$ and then differentiate the resulting function three whole times ($m = 3$).

These two alternatives for fractional derivatives are called the Right Hand Definition (Caputo) and the Left Hand Definitions (Riemann-Liouvelli) respectively. Although these two definitions give the same result in many circumstances especially when the start point of the process is at $-\infty$, they are not entirely equivalent, because (for example) the half-derivative of a constant is zero by the Right Hand Definition-Caputo, whereas the Left Hand Definition gives for the half-derivative of a constant the result given previously as equation (6). In general, the Left Hand Definition is more uniformly consistent with the previous methods, but the Right Hand Definition has also found some applications.

Equation (8) highlights (again) the non-local character of fractional operations, because it explicitly involves an integral, which we have stipulated to range from 0 to x . For any whole number of differentiations we don't need to invoke this integral, but for a non-integer number of differentiations we must include the effect of this integral, which implies that the result depends not just on the values of function at x , but over the stipulated range from 0 to x .

To illustrate the use of equation (8), we will (again) determine the half-derivative of $f(x) = x$, as L'Hopital requested. Using the Left Hand Definition, we first apply half of an integration to this function using equation (8) with $\upsilon = 1/2$, giving

$$\frac{d^{-1/2}}{dx^{-1/2}}(x)=\frac{1}{\Gamma(1/2)}\int_0^x (x-u)^{-1/2}(u)du \qquad \text{put} \qquad x-u=z, \quad du=-dz$$

$$=\frac{1}{\Gamma(1/2)}\int_x^0 (-dz)\frac{(x-z)}{\sqrt{z}}=\frac{1}{\Gamma(1/2)}\left[\int_x^0 z^{1/2}dz-\int_x^0 xz^{-1/2}dz\right]$$

$$=\frac{1}{\Gamma(1/2)}\left[-\frac{2}{3}x^{3/2}+2x^{3/2}\right]=\frac{1}{\Gamma(1/2)}\frac{4x^{3/2}}{3}$$

$$=\frac{4}{3\sqrt{\pi}}x^{3/2}$$

Then we apply one whole differentiation to give the net result of a half-derivative

$$\frac{d^{1/2}}{dx^{1/2}}(x)=\frac{d}{dx}\left(\frac{4x^{3/2}}{3\sqrt{\pi}}\right)=2\sqrt{\frac{x}{\pi}}$$

in agreement with equation (6).

In operator sense we have for Riemann-Liouvelli fractional derivative as

$${}_0D_x^{1/2}[f(x)]=D^1\left[{}_0D_x^{-\left(1-\frac{1}{2}\right)}f(x)\right] \qquad \text{here} \qquad m=1 \qquad D^1\equiv\frac{d}{dx}$$

$${}_0D_x^{\upsilon}\left[f(x)\right]=D^m\left[{}_0D_x^{-(m-\upsilon)}f(x)\right] \qquad \upsilon>0 \qquad (m-1)<\upsilon<m, \qquad m\in\mathbb{Z}^+ \qquad D^m\equiv\frac{d^m}{dx^m}$$

In this case the Right Hand Definition (Caputo) gives the same result. Choose $m=1$ then differentiate the function $f(x)=x$ once to have $f^{(1)}(x)=1$, do the semi-integration of this $f^{(1)}(x)=1$ that is ${}_0D_x^{-1/2}[1]=\Gamma(1)x^{1/2}/\Gamma(3/2)=2\sqrt{x/\pi}$. In this process we first differentiate the function m times, then follow up by remainder fraction and integrate it by that fraction. In operator sense the Caputo derivative is as follows (we write C here to distinguish from Riemann-Liouvelli fractional derivative)

$${}_0^CD_x^{1/2}[f(x)]={}_0D_x^{-1/2}\left[D^1f(x)\right] \qquad \text{here} \qquad m=1 \qquad D^1\equiv\frac{d}{dx}$$

$${}_0^CD_x^{\upsilon}\left[f(x)\right]={}_0D_x^{-(m-\upsilon)}\left[D^mf(x)\right] \qquad \upsilon>0 \qquad (m-1)<\upsilon<m, \qquad m\in\mathbb{Z}^+ \qquad D^m\equiv\frac{d^m}{dx^m}$$

Now suppose we apply this method to the exponential function. Since our definition has been based on the range from 0 to x, whereas we've seen that the "exponential approach" to fractional derivatives is essentially based on the range from -∞ to x, we expect to find disagreement, and indeed for the half-derivative of e^x we get (by applying (8) with

$\upsilon = 1/2$ and then differentiating one whole time)

$$\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}}(e^x) = e^x\left[\operatorname{erf}\sqrt{x} + \frac{e^{-x}}{\sqrt{\pi x}}\right]$$

This is identical to the half-derivative of e^x given by equation (5), shown in red in the plot presented previously (when we only had the series expansion of this function). Again, we can reconcile this approach with the "exponential approach" by changing the lower limit on the integration from 0 to -∞. When we make this change, equation (8) gives

$$\frac{\mathrm{d}^{-1/2}}{\mathrm{d}x^{-1/2}}(e^x) = \frac{1}{\Gamma(1/2)}\int_{-\infty}^{x}(x-u)^{-1/2}(e^u)\mathrm{d}u = e^x$$

and of course the whole derivative of this is also e^x, so the half-derivative of e^x by this method is indeed e^x, provided we use a suitable range of differentiation

We note that Riemann-Liouvelli fractional derivative does not require function to be differentiable (needs only to be continuous), whereas in Caputo case differentiability is essential.

12. The Laplace Transform and Fourier Transform of Fractional derivative

The Laplace Transform of fractional derivative-integral of order α operation is

$$\mathcal{L}\left\{{}_0\mathrm{D}_x^{\alpha} f(x)\right\} = s^{\alpha}\mathcal{L}\left\{f(x)\right\} - \sum_{k=0}^{n-1} s^k\left[{}_0\mathrm{D}_x^{\alpha-1-k} f(x)\right]_{\text{at}\quad x=0} \tag{9}$$

Where Laplace Transform defined as

$$\mathcal{L}\left\{f(x)\right\} \stackrel{\text{def}}{=} \int_0^{\infty}\mathrm{d}x\{e^{-sx} f(x)\}$$

In Laplace definition above the order of differ-integration $\alpha \in \mathbb{R}$; and the integer $n \in \mathbb{Z}$ such that $(n-1) < \alpha \le n$. In this expression (9) when $\alpha < 0$, that is operation is fractional integration, the term involving summation becomes zero for any function, $f(x)$ with available Laplace Transform. Also one can have similar to Laplace

Transform of fractional differ-integrals of $f(x)$; a Fourier Transform of fractional differ-integral operation. A function $f(x)$, which is "well-behaved" at $x=-\infty$, we can have

$$\mathcal{F}\left\{{}_{-\infty}\mathrm{D}_x^{\alpha} f(x)\right\}=(\mathrm{i}\omega)^{\alpha}\mathcal{F}\left\{f(x)\right\} \tag{10}$$

and therefore we have fractional derivative/integral operation as inverse Fourier transformed one

$${}_{-\infty}\mathrm{D}_x^{\alpha} f(x)=\mathcal{F}^{-1}\left\{(\mathrm{i}\omega)^{\alpha}\mathcal{F}\{f(x)\}\right\}$$

Where the Fourier and Inverse Fourier Transform is depicted as following

$$\mathcal{F}\left\{f(x)\right\}=F(\omega)\overset{\text{def}}{=}\int_{-\infty}^{+\infty}\mathrm{d}x\{e^{\mathrm{i}\omega x}f(x)\} \qquad f(x)=\mathcal{F}^{-1}\left\{F(\omega)\right\}\overset{\text{def}}{=}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega\{e^{-\mathrm{i}\omega x}F(\omega)\}$$

In some cases (especially for steady state systems with lower terminal of differ-integration $a=-\infty$) the Fourier Transformation method is another way to find fractional derivative/fractional integration of function $f(x)$. That is

(i) Obtain the Fourier Transform of $f(x)$ as $F(\omega)$.

(ii) Then this transformed $F(\omega)$ in frequency ω domain we multiply by $(\mathrm{i}\omega)^{\alpha}$, where $\alpha\in\mathbb{R}$.

(iii) The resulting function $(\mathrm{i}\omega)^{\alpha}F(\omega)$ we inverse Fourier transform, to get ${}_{-\infty}\mathrm{D}_x^{\alpha} f(x)$.

13. Conclusion

What we discussed in tutorial that there is a concept of fractional order differentiation rather the concepts generalizes the notion of differentiation and integration and extends it to arbitrary order (other than the whole number). Hence there exists semi-derivative of function what was queried by L'Hopital way back in 1695. This fractional calculus is gaining popularity to describe natural phenomena. In context of electromagnetic theory the fractional curl operator as we saw gives insight into wave propagation in media with positive and negative refractive index, explains duality and maps the waves in between original and dual solutions of Maxwell's equation, plus explains polarization of the vector field. However, this tutorial is 'tip of ice berg'.

This was tip of an iceberg

End of this tutorial