National Work Shop

On

Application of Fractional Calculus in Engineering

March 9-11; 2012

RAIT Navi-Mumbai

Reality of Fractional Calculus

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The Year of Mathematics 2012

………Hon. Prime Minister Dr. Manmohan Singh

We

salute all the Mathematicians who have gifted a wonderful tool ………….

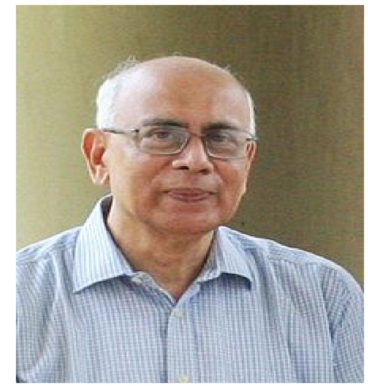
“The Fractional Calculus”

……….

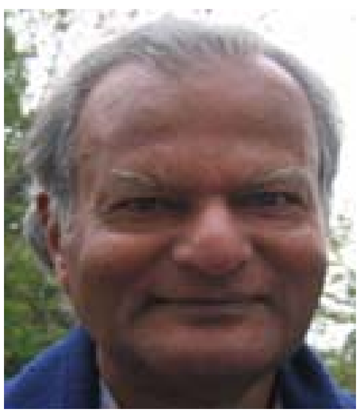
Saluting Indian Mathematicians in the field of Fractional Calculus

H M Srivastava

S Ramanujam



V Balakrishnan



S C Dutta Roy

O.P.Agarwal
Anil.D. Gangal
Kiran .M. Kolwankar
R.K.Saxena
R.K.Raina
Rasajit Kumar Bera
…………………………

Lokenath Debnath

Prelude

I (by profession an Electrical and Electronics Engineer) can state that

***"Mathematics goes far beyond our physical understanding"*.**

Mathematics is what nature understands,

***but which one*?**

We Scientists & Engineers try to find and keep on finding-
***"that particular mathematics"*.**
As an engineer my past one and half decade of work is mainly to ascribe
physical/engineering/geometrical
sense to wonderful *three hundred years old topic* of
fractional calculus-
and make
***product and science out of this subject*.**

Why?

Why?

Because it was for search to make
"Fuel Efficient Control System"
for Nuclear Reactors!!

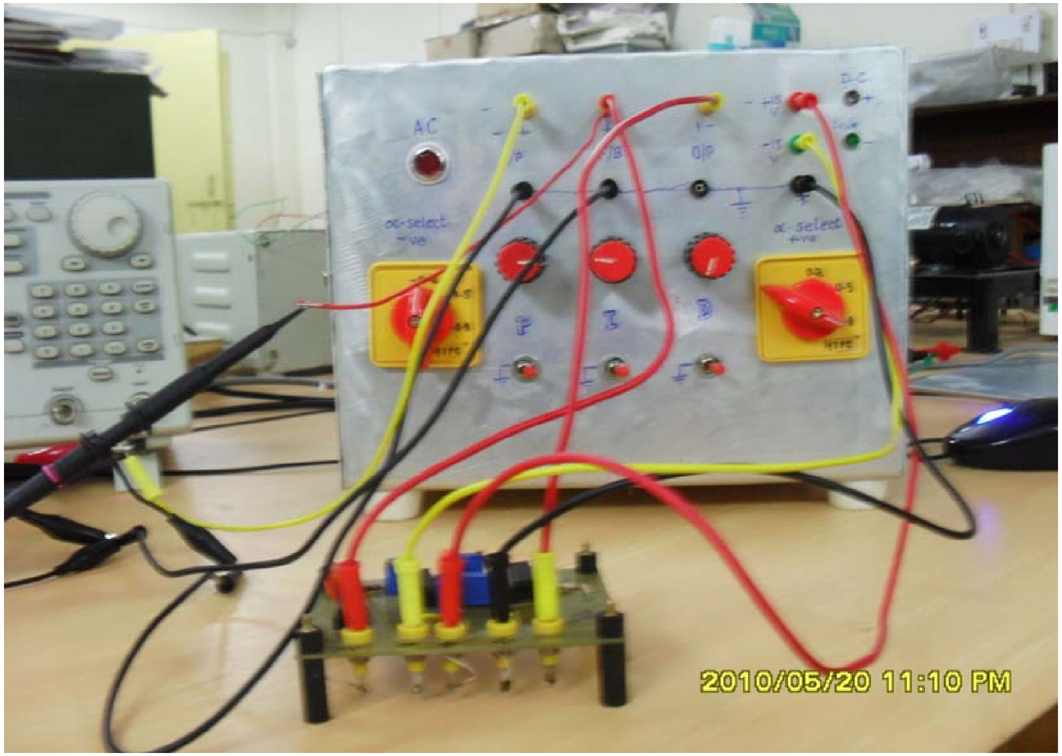
My small contribution is also to relate real life processes and explanation with physical behavior to the wonderful tool of mathematics that is fractional calculus; also to give physical sense to solution of solvable extraordinary differential equations.

Still continuing……… and is unfinished. Still learning Fractional Calculus by fractions!!

It is reality now once considered stupidity

A great gift from mathematicians got translated into a product (circuits) for Fractional Order Controls-for "*Efficient Control System with Robustness*". Here the Fractional Differential Equations are working in electronics circuit.

Under three Patents.



$$u(t) = K_P e(t) + K_I \{ {}_0D_t^{-\alpha} e(t) \} + K_D \{ {}_0D_t^{\beta} e(t) \}$$

$$\alpha ; \beta \in (0,1)$$

$$e(t) = r(t) - c(t)$$

$$\mathrm{PI}^{\alpha}\mathrm{D}^{\beta}$$

What we intend?

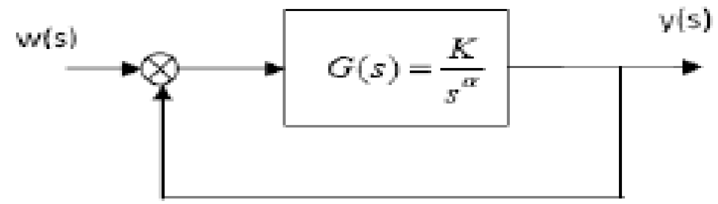


Figure-1 Ideal Bode's loop

ideally is to have flat loop TF

(a Greens function who has its argument frequency independent)

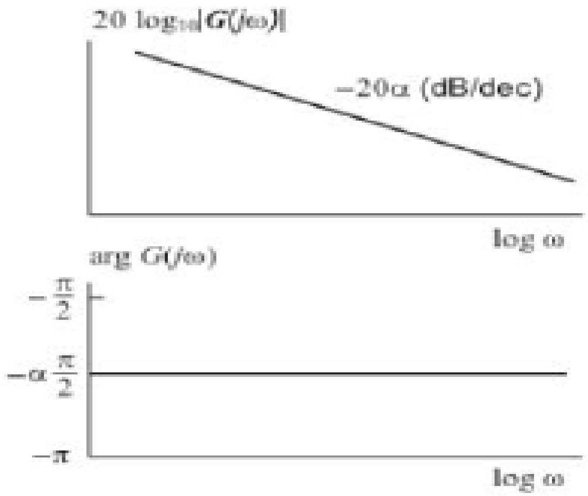


Figure-2 Gain and Phase plots for open loop ideal TF $L(s)$

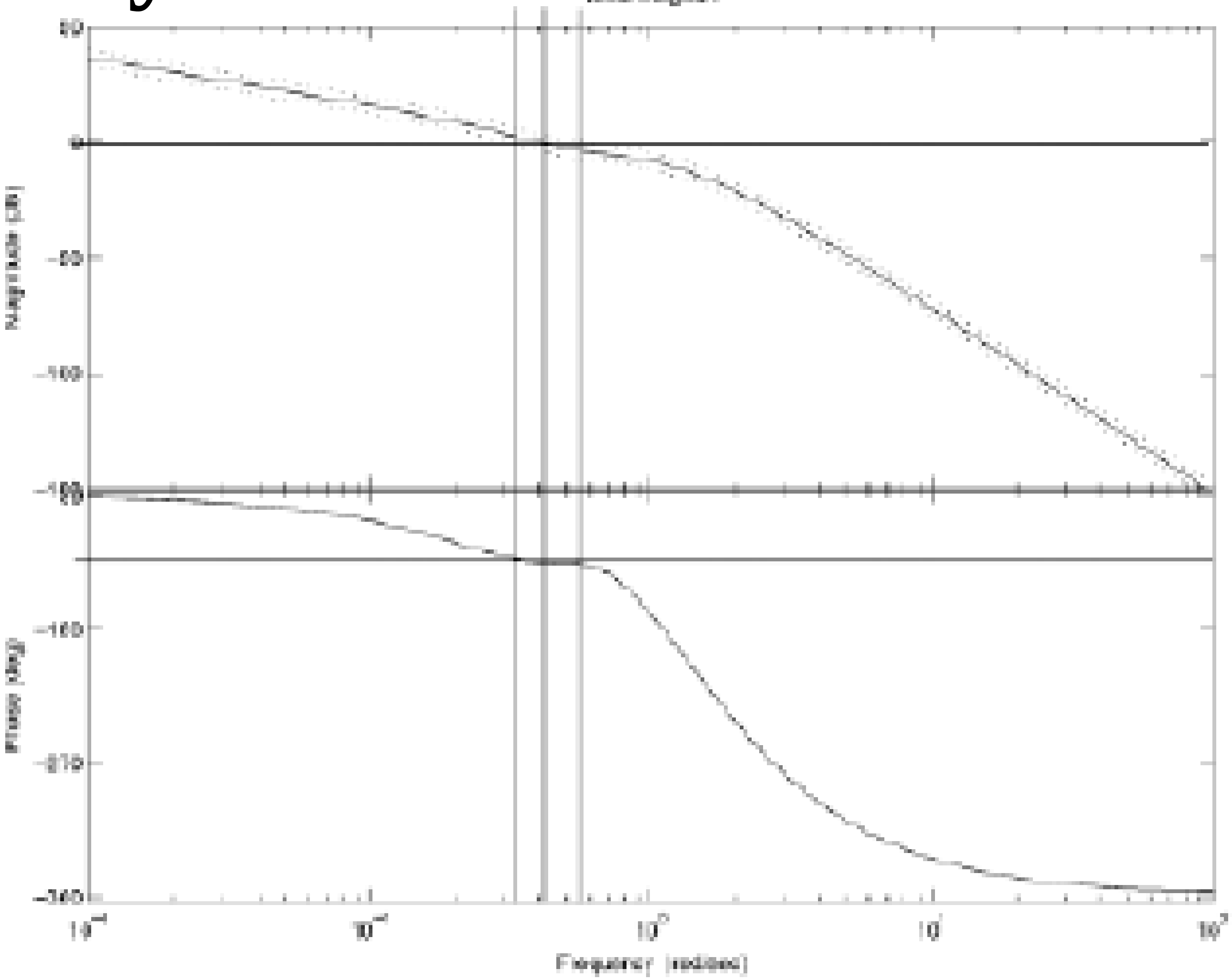
Bode's TF can be used as reference system in the following forms, I get from figure-1

$$G_{OL}(s)=\frac{K}{s^{\alpha}} \quad \textbf{meaning} \quad G_{CL}(s)=\frac{K}{s^{\alpha}+K} \quad \textbf{for non-integer order} \quad (1<\alpha<2)$$

General characteristics of Bode's open loop TF magnitude plot has slope of $-\alpha\, 20\mathrm{dB/decade}$, the gain cross over frequency is depending loop gain K phase angle is constant $-\alpha\pi/2$. The closed loop TF thus has 'ideally' Gain margin $A_m=\infty$, and phase margin $\Phi_m=\pi(1-\alpha/2)$. The closed loop response to a step input is

$$y(t)=Kt^{\alpha}E_{\alpha,\alpha+1}(-Kt^{\alpha})$$

Practically we can do a limited flattening



Suppose the phase Bode's plot of a system is made flat i.e. the phase derivative w.r.t. the frequency is made zero, then the system is robust to gain variations. This property is called iso-damping and the frequency at which the phase derivative becomes zero is called the tangent frequency .

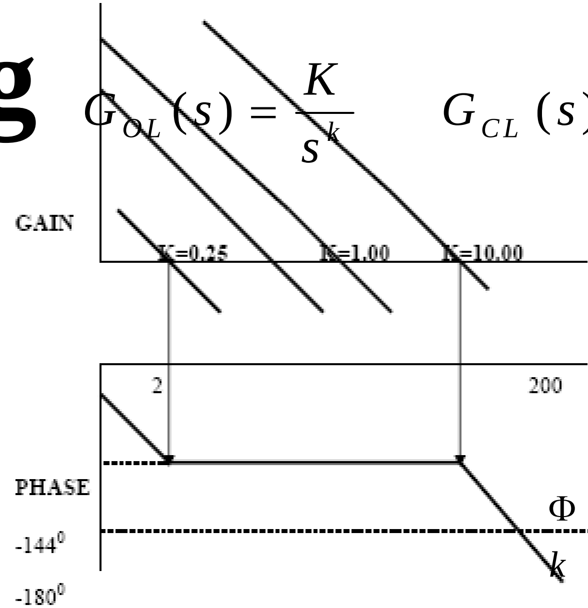
Enlarging

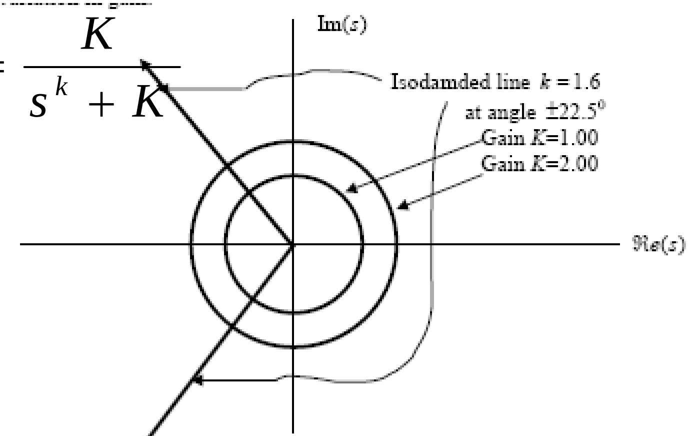
$$G_{OL}(s) = \frac{K}{s^k} \qquad G_{CL}(s) = \frac{K}{s^k + K}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\angle G(s)\Big|_{s=\mathrm{j}\omega_c} = 0$$

gives

$$\angle \frac{\mathrm{d}G(s)}{\mathrm{d}s}\Big|_{s=\mathrm{j}\omega_c} = \angle G(s)\big|_{s=\mathrm{j}\omega_c}$$





$$\Phi_m = 180^0 - 144^0 = 36^0 = \pi(1 - k/2)$$

$$k = 8/5 = 1.6$$

Complex Green' function $G(\mathrm{j}\omega) = x(\omega) + \mathrm{j}y(\omega)$ **has angle as frequency dependence**

$$\angle G(\mathrm{j}\omega) = \tan^{-1}\left[y(\omega)/x(\omega)\right]$$

Doing the arithmetic as following we get

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\angle G(\mathrm{j}\omega) = \frac{1}{1+(y/x)^2}\frac{\mathrm{d}(y/x)}{\mathrm{d}\omega} = \frac{x^2}{x^2+y^2}\left(\frac{(\mathrm{d}y/\mathrm{d}\omega)}{x} - \frac{y(\mathrm{d}x/\mathrm{d}\omega)}{x^2}\right) = \frac{1}{x^2+y^2}\left(x\frac{\mathrm{d}y}{d\omega} - y\frac{\mathrm{d}x}{d\omega}\right) = 0$$

$$x\frac{\mathrm{d}y}{\mathrm{d}\omega} - y\frac{\mathrm{d}x}{\mathrm{d}\omega} = 0 \quad \text{i.e.} \quad \frac{y}{x} = \frac{(\mathrm{d}y/\mathrm{d}\omega)}{(\mathrm{d}x/\mathrm{d}\omega)}$$

where $\tan\angle G(\mathrm{j}\omega) = y/x$ **and**

$$\tan\angle \mathrm{d}G(\mathrm{j}\omega)/\mathrm{d}\omega = (\mathrm{d}y/\mathrm{d}\omega)/(\mathrm{d}x/\mathrm{d}\omega)$$

Gives statement of isodamping

$$\angle \frac{\mathrm{d}G(\mathrm{j}\omega)}{\mathrm{d}s}\Big|_{s=\mathrm{j}\omega_c} = \angle G(\mathrm{j}\omega)\big|_{s=\mathrm{j}\omega_c}$$

Inference!

$$\frac{\mathrm{d}}{\mathrm{d}s}\angle G(s) = 0$$

Frequency independent Green's function's argument returns a same feed-back quantity even if the gain be changed or parameters of the plant be changed; does imply that at any condition the control loop delay be the same, and control actions be the same; thus are likely to record similar response to a particular stimulus at many values of parametric spread of the system (several gain changes gives similar response)

The basic reason of isodamping

Mathematically

Berkhauson's criteria for instability

$$\left|G(s)\right|\angle G(s) = 1\angle 180^0$$

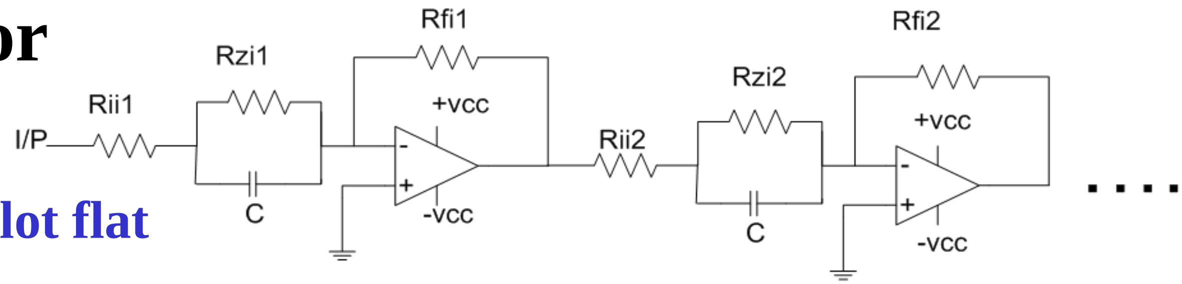
Isodamped case says that at cross over we have $\left|G(s)\right| = 1$ **zero dB.**

The angle $\frac{d}{ds}\angle G(s) = 0$ **is constant say** $\angle G(s) = 144^0$

gives the same amount of feed back quantity always with respect to gain change

resulting in same reaction with wide spread of gain; with out going into oscillations

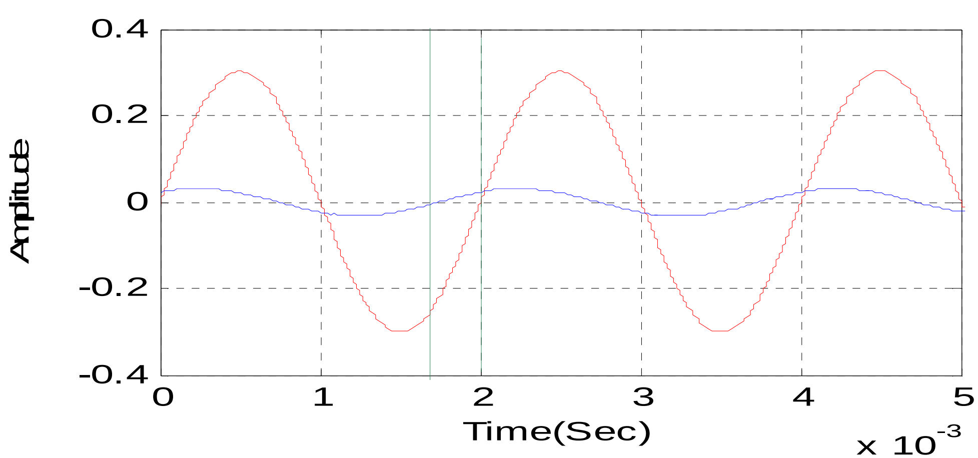
Fractal real poles and real zeros interlaced to give fractional order differentiator

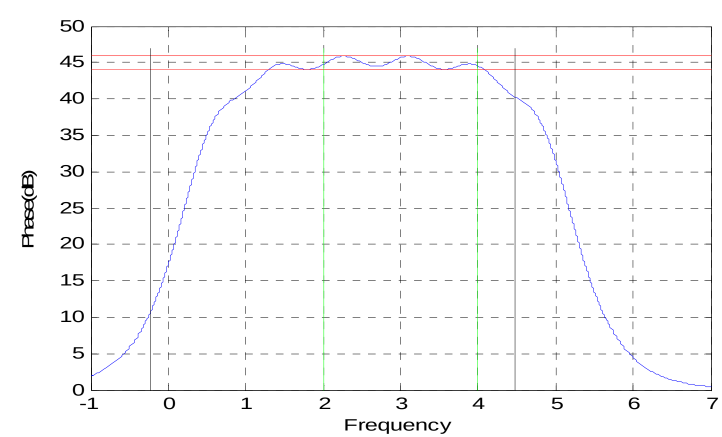


Making the Bode phase plot flat

$$s^{1/2} \cong \frac{(s - z_1)(s - z_2)..}{(s - p_1)(s - p_2)..}$$

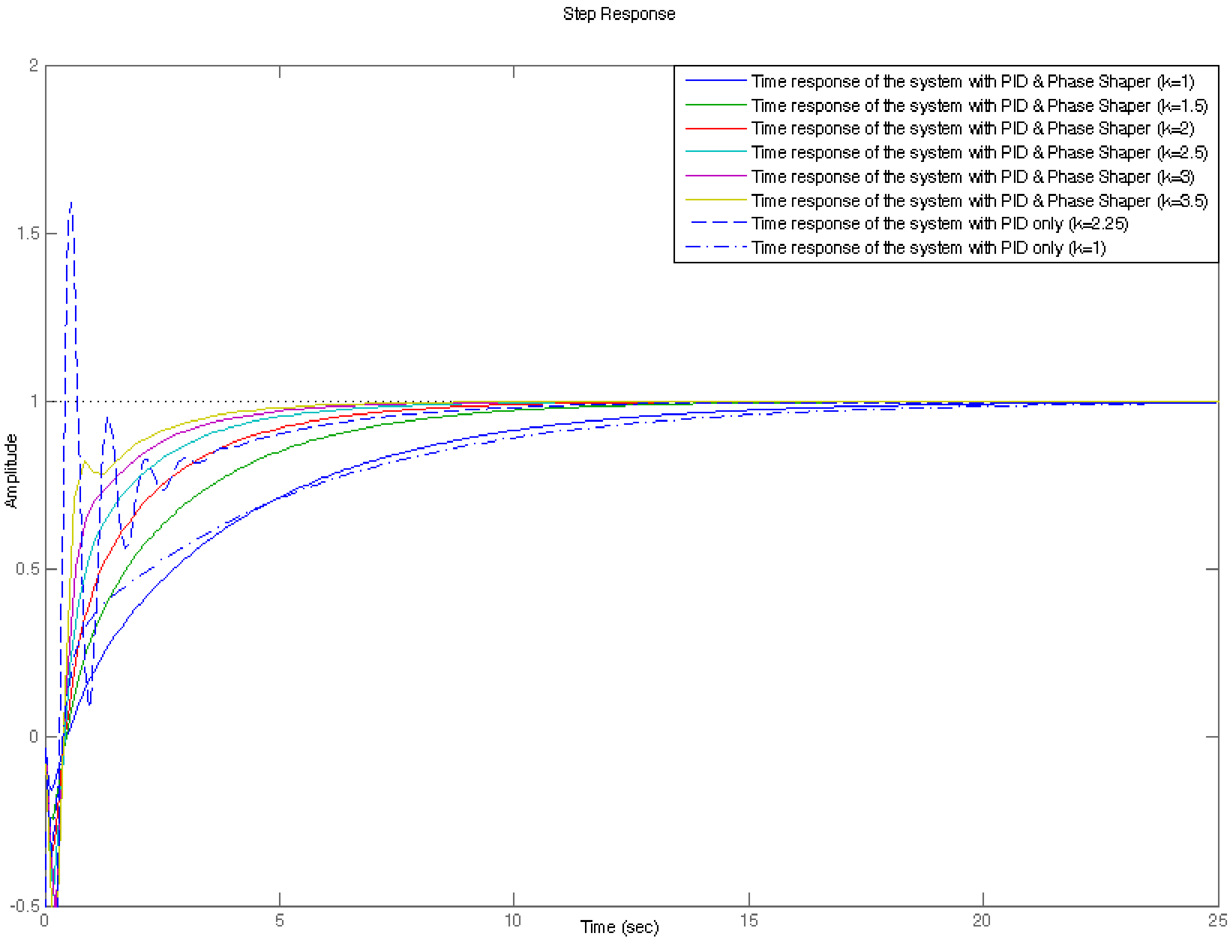
i	Zi	Pi	Ci	Rfi= Rii		Rzi	
				Ω	TP	Ω	TP
1	2.2537	6.0406	1μ	264.07k	500k	443.71k	500k
2	15.955	42.764	1μ	37.30k	50k	62.67k	100k
3	112.95	302.75	680nf	11.21k	20k	18.83k	20k
4	799.65	2143.3	68nF	10.94k	20k	18.39k	20k
5	5661.1	15173	10nF	10.51k	20k	17.64	20k
6	40078	107420	1nF	14.85k	20k	24.95k	50k





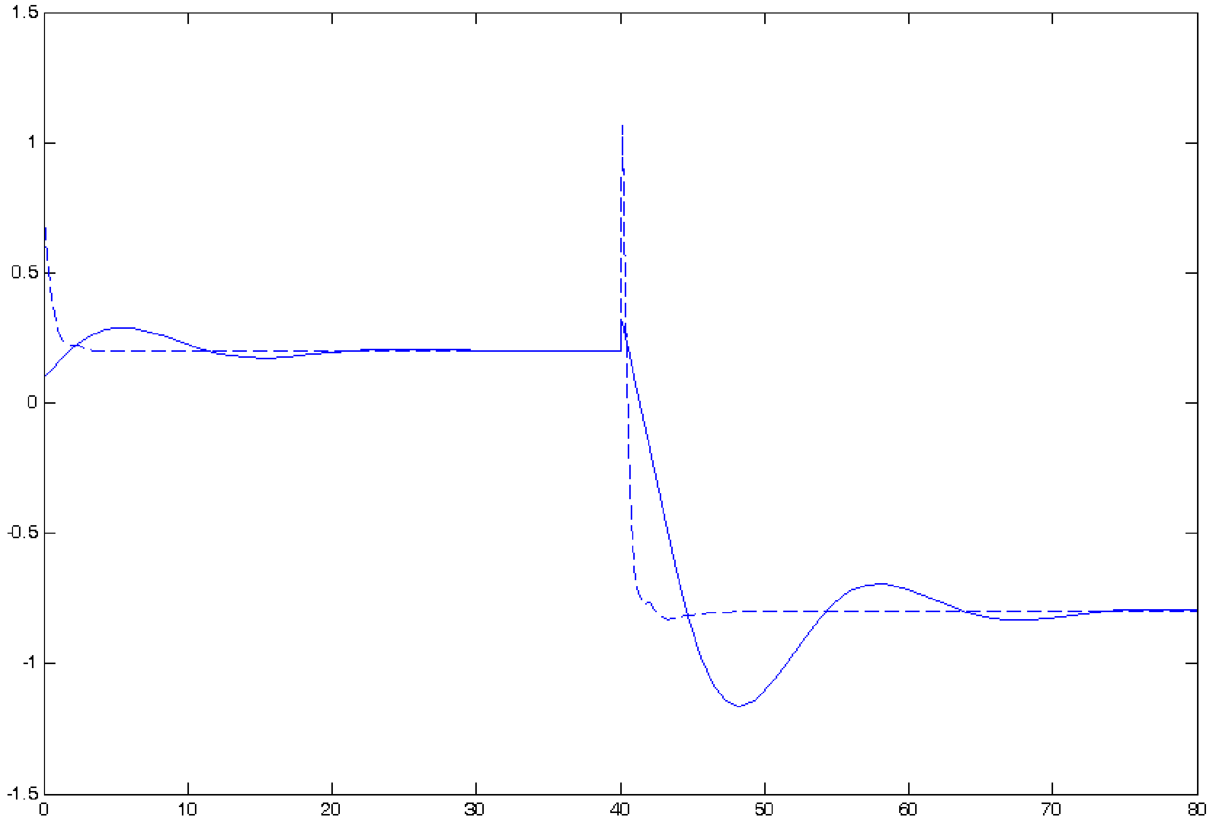
What we get?

Overshoot unchanged while parameter spreads five folds with Fractional order PID

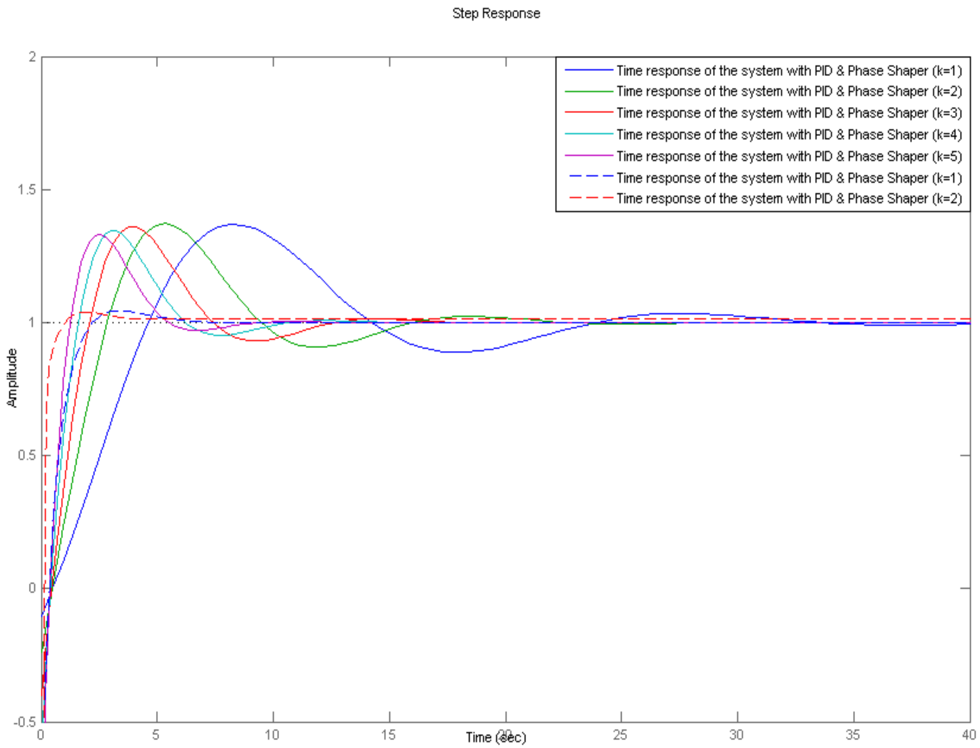


With less effort?

Controller effort is lesser with Fractional order PID



Robustness enhanced!!



Iso-damped response with wide parametric spreads

The scalar gain, as shown in Figure can be varied by 500% keeping the overshoot constant- with Fractional Order PID or Fractional Order Controller

The closed loop system, with the PID controller alone becomes unstable with two fold increase in gain- "parametric spreads"

A reality today product based on Fractional Calculus!!

Hardwire set up to control DC Motor servo position system with FO-PID circuits

Does ‘d / d*t*’ represent accumulation or loss always ?

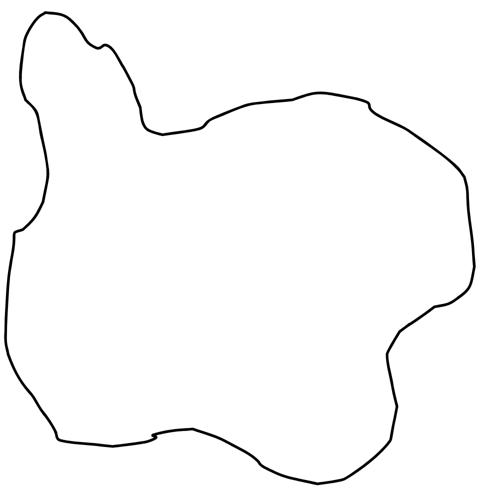
Well if there are temporary traps then?

Well if the boundary is partly reflecting?

Well if the elementary element (area, volume etc) be not a point quantity?

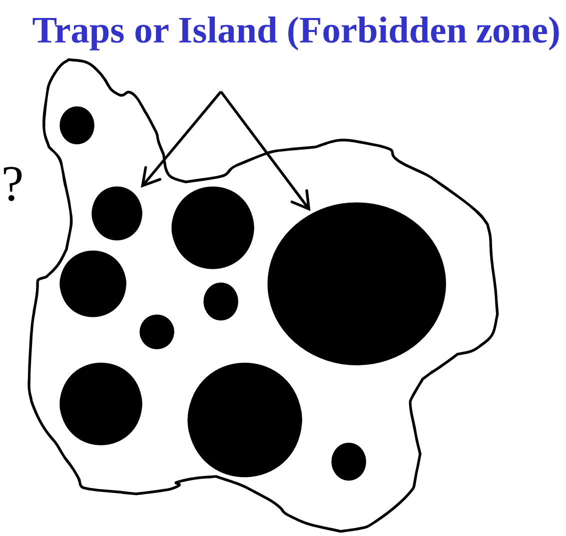
Well if mass of ball is not a point quantity?

Well if spring is not mass-less?



$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi \rangle = \mathrm{GAIN} - \mathrm{LOSS?}$$

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\langle \phi \rangle = \mathrm{GAIN} - \mathrm{LOSS?}$$



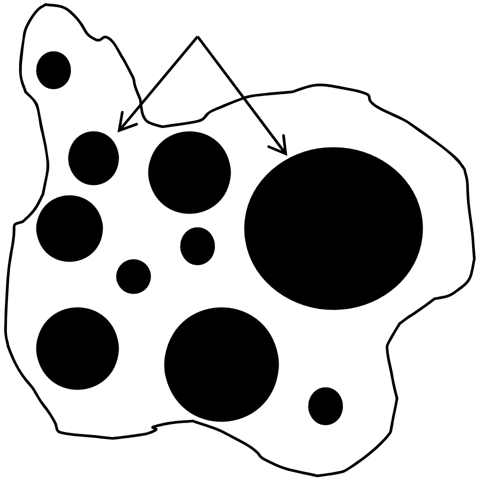
Some are entraps temporarily indicating slow rate of change than d / d*t*

The particles cannot have the island paths indicating fast rate of change than d / d*t*

Spatial Disorder Heterogeneity giving fractional temporal derivative?

Traps or Island (Forbidden zone)

$$\frac{d^{\alpha}}{dt^{\alpha}}\langle\phi\rangle = \mathrm{GAIN} - \mathrm{LOSS}?$$



It is indeed true….

Perin's Brownian Motion & Fick's Diffusion

In discrete time steps of span Δt the walker (particle) is assumed to jump to one nearest neighbour. One dimensional lattice be taken, then pdf (concentration/ number of particles/flux) at position j at time $t + \Delta t$ in dependence on population of position of two adjacent sites $j \pm 1$ at time t , is:

$$N_j(t + \Delta t) = \frac{1}{2} N_{j-1}(t) + \frac{1}{2} N_{j+1}(t) \qquad \text{.............(1)}$$

½ is for isotropic case.

In continuum limit with $\Delta t \to 0$ and $\Delta x \to 0$, we have Taylor's expansion:

$$N_j(t + \Delta t) = N_j(t) + \Delta t \frac{\partial N_j}{\partial t} + R_1\left([\Delta t]^2\right) \qquad \text{........................(2)}$$

$$N_{j\pm 1}(t) = N_j(x, t) \pm \Delta x \frac{\partial N}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 N}{\partial x^2} + R_2\left([\Delta x]^3\right) \qquad \text{........................(3)}$$

Using (2) and (3) and putting in (1) and recognizing that $N(x, t) = N_j(t)$, neglecting $R_1; R_2$ we get:

$$\frac{\partial N(x,t)}{\partial t} = \mathbb{D} \frac{\partial^2 N(x,t)}{\partial x^2}$$

$$\mathbb{D} = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t}$$

is Fick's law

A normal diffusion equation

Start with continuity equation $\frac{\mathrm{d}}{\mathrm{d}t} M(r,t) = -j(r,t)$

where $M(r,t) = \int_0^r \mathrm{d}r P(r,t)$ **and** $j(r,t)$ **is the total probability current at *r* from origin.**

The above equation must be supplemented by constitutive equation relating the current *j* (*r*, *t*) to the probability density function p.d.f *P* (*r*, *t*) i.e.

$$j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r}$$

From constitutive and continuity laws we get Fick's law as $\frac{\partial}{\partial t} P(r,t) = \mathbb{D}_0 \frac{\partial^2}{\partial r^2} P(r,t)$

The plume is Gaussian and MSD is linear with time, for delta function at origin we have

$$P(r,t) = \frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}$$

A normal diffusion equation & its fractional calculus version

The Gaussian plume $P(r,t) = \frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}$

Take Laplace $P(r,s) = \mathcal{L}\{P(r,t)\} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}\right\} = \left(\frac{1}{\sqrt{\mathbb{D}_0 s}}\right) e^{\left(-r\sqrt{\frac{s}{\mathbb{D}_0}}\right)}$ **……1**

Take derivative $\frac{d}{dr} P(r,s) = \frac{-1}{\sqrt{\mathbb{D}_0 s}} \sqrt{\frac{s}{\mathbb{D}_0}} e^{-r\sqrt{s/\mathbb{D}_0}} = -\frac{1}{\mathbb{D}_0} e^{-r\sqrt{s/\mathbb{D}_0}}$ **……2**

Constitutive eqn. $j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r}$ **……………………………………………………3**

Take Laplace use 2 $j(r,s) = -\mathbb{D}_0 \frac{\partial P(r,s)}{\partial r} = -\mathbb{D}_0 \frac{-1}{\mathbb{D}_0} e^{-r\sqrt{s/\mathbb{D}_0}} = e^{-r\sqrt{s/\mathbb{D}_0}}$ **…… ……….4**

Manipulate 4 use 1 $j(r,s) = \sqrt{s\mathbb{D}_0} P(r,s)$ **…… ………………………………………5**

Use $\sqrt{s} \leftrightarrow \mathcal{L}\left\{d^{1/2}/dt^{1/2}\right\}$ **in 5 to get** $j(r,t) = \sqrt{\mathbb{D}_0} \frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}}$ **…… …………………..6**

Equating 6 and 3 we get fractional counter part of integer order Fick's law that is

$\frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0} \frac{\partial P(r,t)}{\partial r}$ **Equivalent to original** $\frac{\partial}{\partial t} P(r,t) = \mathbb{D}_0 \frac{\partial^2}{\partial r^2} P(r,t)$

This is standard Brownian process BM with linear MSD

Anomalous diffusion Fractional Brownian Motion

$$\frac{\partial^{1/2} P_{\mathrm{BM}}(x,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0}\,\frac{\partial P_{\mathrm{BM}}(x,t)}{\partial |x|}$$

The Brownian Motion normal Fickian diffusion

$$P_{\mathrm{BM}}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t}}\, e^{\left(-\frac{x^2}{4\mathbb{D}_0 t}\right)}$$

$$\left\langle x^2(t) \right\rangle \equiv 2\mathbb{D}_0 t$$ **Linear scaling of MSD**

Fractional Brownian Motion (FBM) is simplest mathematical model extension of Gaussian stochastic process (random walk) whose variance (MSD) does not scale linearly with time its p.d.f. is:

FBM is natural generalization of BM,
here with a stretched exponential

$$P_{\mathrm{FBM}}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t^{2/d_w}}}\, e^{\left(-\frac{x^2}{4\mathbb{D}_0 t^{2/d_w}}\right)}$$

$$\left\langle x^2(t) \right\rangle \equiv 2\mathbb{D}_0 t^{2/d_w} \qquad 1 \le d_w < \infty \qquad d_w$$ **Anomalous diffusion exponent**

Brownian Case is with anomalous diffusion exponent as 2 is following:

For non-Brownian case:

$$\frac{\partial^{1/d_w} P_{\mathrm{FBM}}(x,t)}{\partial t^{1/d_w}} = -\mathrm{A}\,\frac{\partial P_{\mathrm{FBM}}(x,t)}{\partial |x|}$$

Anomalous transport in an irregular media-passing comment

Transport phenomena in complex systems such as random fractal structures exhibit anomalous features which are qualitatively different from the standard regular systems.

In the case of fractals such anomalies are due to constraint on the transport process on all lengths scales.

These constraints may be seen as temporal correlations existing on time scales (memory mechanism!!)

$$X^2 \equiv \langle x^2(t) \rangle \cong t^{2/d_w} \qquad \langle P(x,t) \rangle \cong t^{-d_f/d_w} e^{\left[-C\left(\frac{x}{X}\right)^u\right]}$$

$$u = d_w/(d_w - 1)$$

d_f **Fractal dimension.**

d_w **Anomalous diffusion exponent**

A Fractional Brownian Motion Process (FBM), represented with memory

A Fractional Brownian Motion Process (FBM), represented with memory, described as integral transform of Brownian Motion (BM). The convolution with memory kernel that is $K(t)$ **; first proposed by Mandelbrot.**

$$x_{\mathrm{FBM}}(t) = \int_{-\infty}^{t} K_M(t-\tau)\mathrm{d}x_{\mathrm{BM}}(\tau)$$

Where $x_{\mathrm{FBM}}(t)$ **and** $x_{\mathrm{BM}}(t)$ **are the position of the particle undergoing the FBM and BM process respectively.**

$$K_M(t-\tau) \stackrel{\mathrm{def}}{=} \begin{cases} (t-\tau)^{(1/d_w)-(1/2)} - (-\tau)^{(1/d_w)-(1/2)} & ;\tau < 0 \\ (t-\tau)^{(1/d_w)-(1/2)} & ;0 < \tau < t \end{cases}$$

The kernel resembles the singular memory kernel associated with fractional integral (derivatives).

$$I_t^{\phi} f(t) = d_t^{-\phi} f(t) = \frac{1}{\Gamma(\phi)} \int_0^t (t-\tau)^{\phi-1} f(\tau) d\tau$$

With anomalous exponent equal to 2 we get a no-memory case with $K_M(t-\tau) = 1$ **in this case**

$$x_{\mathrm{FBM}}(t) = x_{\mathrm{BM}}(t)$$

A Fractional Brownian Motion Process (FBM), its interpretation

A walker undergoing FBM remembers his past, while a walker undergoing BM does not remember its past. Well a walker can remember its past and have preferences in same direction giving persistence walk, or a walker remembering its past can change its direction giving anti-persistence walks, are the cases of anomalous transport. This gives concept of sub or super diffusion, in FBM context.

Note, the variable $x_{\mathrm{FBM}}(t)$ may not be physical distance. It could be

1. Computer net work delay.
2. Could be price of stock market.
3. Could be random returns of insurance system.
4. Could be infected population by swine flu

In general could be variable for physical systems represented by random stochastic process.

This is beginning of Fractional Order Signal Processing

Discrete difference to continuum limit

Backward difference (construction of up-shift operator)

Backward difference of Stochastic process & Taylor's expansion $X(t) = \sum_{i=1}^{N} \xi_i \quad t = N\tau$

$$X(t+\tau) = X(t) + \tau D X(t) + \frac{\tau^2}{2!} D^2 X(t) + \ldots + \frac{\tau^n}{n!} D^n X(t) + \ldots = [e^{\tau D}] X(t)$$

Derivative operator D **can define 'shift operator as** $E_\tau \equiv E_\tau\left(X\{t\}\right) = X(t+\tau)$

From above Taylor's expansion we can write: $E_\tau X(t) = \left\{e^{\tau D}\right\} X(t)$

Also we can formulate:

$$E_\tau \equiv 1 - \Delta_{(-)} = e^{\tau D}$$

Backward difference operator or Up-shift operator is:

$$\Delta_{(-)} = 1 - E_\tau = 1 - e^{\tau D} = 1 - \left(1 + \tau D + \frac{(\tau D)^2}{2} + \ldots\right) \approx -\tau D$$

In a way we have defined a backward derivative operator as: $\lim_{\tau \to 0}\left(\frac{\Delta_{(-)}}{\tau}\right) = -D$

It is suggestive that D **is time derivative operator and when applied to a continuous function yields:**

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = -\lim_{\tau \to 0} \frac{\Delta_{(-)} X(t)}{\tau}$$

Fractional Brownian Motion

& Fractional Stochastic Difference

For unit interval of time, with down shift operator fractional stochastic process may be modeled as fractional random walk. If ξ_j is random variable used to represent step taken in discrete time j then $0 < \alpha < 1$

$(1 - E^{-1})^{\alpha} X_j = \xi_j$ represent analog of Fractional Brownian Motion.

Whereas BM is $(1 - E^{-1}) X_j = \xi_j$ $\quad X(t) = \sum_{i=1}^{N} \xi_i \quad t = N\tau \quad X(t) - X(t-\tau) = \xi_N$

Shift operator is $E_{\tau}^{-1} \equiv e^{-\tau D} \quad D \equiv \lim_{\tau \to 0} \frac{\mathrm{d}}{\mathrm{d}t} \quad E^{-1} X_j \equiv \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) X_j$

In continuum BM is Langevin equation $\frac{\mathrm{d}}{\mathrm{d}t} x_{BM}(t) = \lambda x_{BM}(t) \quad \lambda = \tau^{-1}$

Inverting this we write:

$$X_j = (1 - E^{-1})^{-\alpha} \xi_j$$

$$(1 - E^{-1})^{-\alpha} = \sum_{k=0}^{\infty} \begin{pmatrix} -\alpha \\ k \end{pmatrix} (-1)^k E^{-k}$$

$$\begin{pmatrix} -\alpha \\ k \end{pmatrix} = \begin{pmatrix} k + \alpha - 1 \\ k \end{pmatrix} = \frac{(k + \alpha - 1)!}{k!(\alpha - 1)!}$$

$$X_j = \sum_{k=0}^{\infty} \frac{(k + \alpha - 1)!}{k!(\alpha - 1)!} E^{-k} \xi_j = \sum_{k=0}^{\infty} \frac{(k + \alpha - 1)!}{k!(\alpha - 1)!} \xi_{j-k}$$

$$X_j = \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha)}{\Gamma(k+1)(\alpha - 1)!} \xi_{j-k}$$

Asymptotic behavior of FBM:

We wish to determine asymptotic form of $X_j = \sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)(\alpha-1)!} \xi_{j-k}$
using Stirling's formula for gamma function ratio

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \approx x^{a-b}$$

So we have: $X_j \approx \sum_{k=0}^{\infty} \frac{k^{\alpha-1}}{(\alpha-1)!} \xi_{j-k}$

Strength of the above decrease asymptotically with increasing time lag k
as an 'inverse power law' as long as $\alpha < 1$

Spectrum of FBM

Of fractional time series $X_j = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!}\xi_{j-k}$ **D.F.T as** $X_\omega = \theta_\omega \xi_\omega$

spectrum is defined as

$$S(\omega) = \left\langle \left| X_\omega \right|^2 \right\rangle$$

Assuming random fluctuations have a 'white-noise' spectrum of constant strength then

$$S(\omega) = \left\langle \theta_\omega^{\ 2} \right\rangle$$

$$\theta_\omega = \sum_{k=0}^{\infty} \frac{(k+\alpha-1)!}{k!(\alpha-1)!} e^{-ik\omega} = \left(1 - e^{-i\omega}\right)^{-\alpha}$$

$$S(\omega) = \frac{1}{\left[2\sin\left(\omega/2\right)\right]^{2\alpha}} \approx \frac{1}{\omega^{2\alpha}}$$

Fractional stochastic process driven by 'white-noise' has inverse power law

Set $\alpha = H - \frac{1}{2}$ **so that the spectrum equation becomes** $S(\omega) = \frac{1}{\omega^{2\alpha}} = \frac{1}{\omega^{2H-1}}$

H **is Hurst exponent (defines measure of self-similarity & irregularity of random graph)**

Network delay-a FBM!!

$\langle \tau_{\text{Delay}}(t) \rangle$

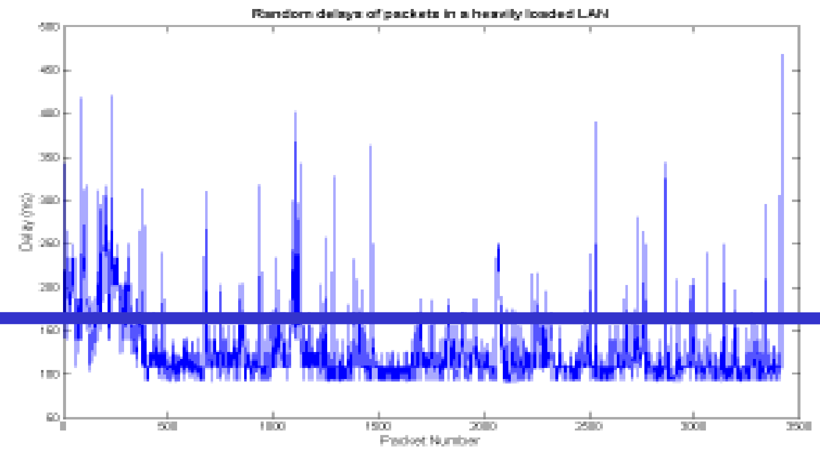


Fig. 1. Time domain presentation of the network induced stochastic delay.

Estimates of Network Delay Irregular graph gives, a Case of FBM

$$H = 0.88$$
$$d_f = 2 - H = 1.2$$
$$\beta = 2H - 1 = 0.76$$

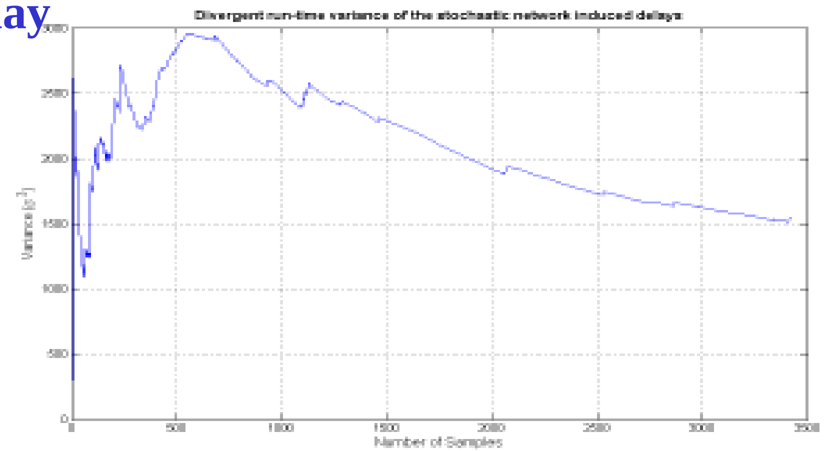


Fig. 2. Diverging run-time variance for the random network induced delay.

A Brownian Case BM Has the irregularity Parameters as

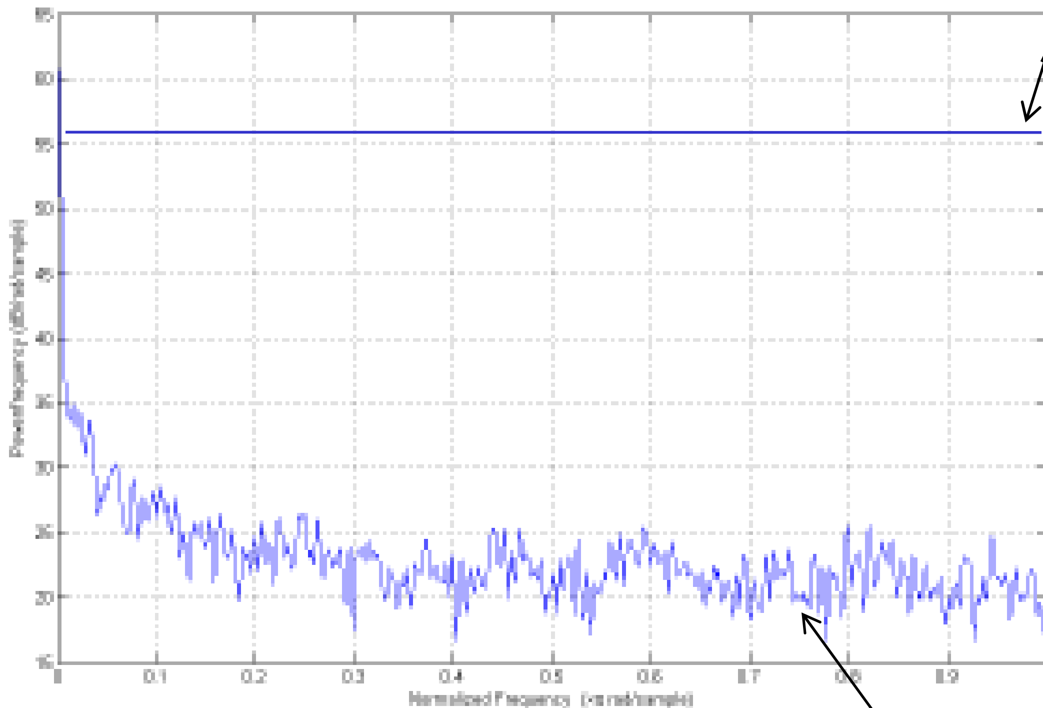
$$H = 0.5$$
$$d_f = 2 - H = 1.5$$
$$\beta = 2H - 1 = 0$$

Delay in LAN stochastic nature, random nature has wide spikiness , statistical study show the behavior is a 'power-law'. Brownian motion (FBN) is to model this stochastic process by "Fractional Langevin equation" driven by 'shot-noise'.

Power spectral density of random walk-(network delay)

White-noise

$$S(\omega) \sim |\omega|^{-\beta} = 1$$



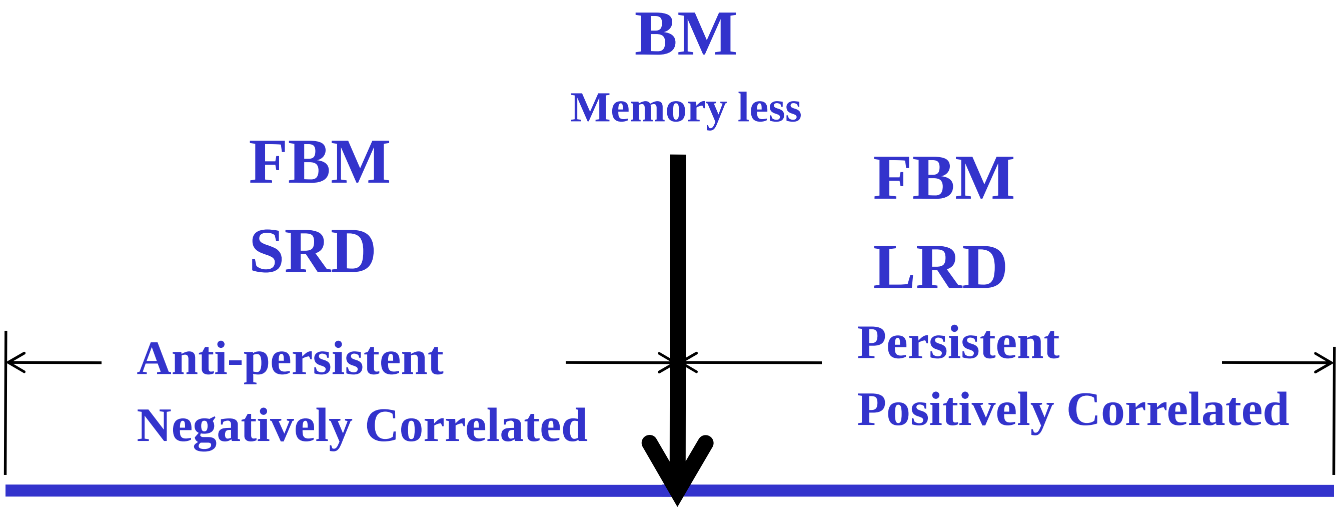
$$S(\omega) \sim |\omega|^{-\beta} = |\omega|^{-0.76}$$

Equation of motion of this delay dynamics could be Fractional Langevin's equation

$$\frac{d^{1-\beta}}{dt^{1-\beta}} x_{FBM} - \lambda x_{FBM} = 0$$

$$\frac{d}{dt} x_{BM} - \lambda x_{BM} = 0$$

Persistent-
Anti-persistent walks



$$H = 0 \qquad d_f = 2 \qquad \beta = -1$$

$$H = 0.5 \qquad d_f = 1.5 \qquad \beta = 0$$

$$H = 1 \qquad d_f = 1 \qquad \beta = 1$$

$$\beta = 2H - 1$$

$$d_f = 2 - H$$

Long Range Dependency (LRD)

The processes where the dependency is LRD, are characterized by Fractional Differential Equations

The correlations linger-does not decay fast like exponentials, or memory less process

The stochastic processes governed by irregularity exponent *H* defines the character of process if they are to be having memory or not.

They are LRD are LONG TAILED responses with MEMORY positively correlated

Phase Table for the Fractional Diffusion Equation

$$\frac{\partial}{\partial t}\langle \phi(x,t)\rangle = \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}(\mathbb{D}_{\alpha,\mu})\frac{\partial^{\mu}}{\partial x^{\mu}}\langle \phi(x,t)\rangle$$

Temporal Fractional Order α	Spatial Fractional Order μ	Type of Walk	Average Waiting Time T	Jump-Length Variance σ^2	Nature of Diffusion
$0<\alpha<1$	$0<\mu<2$	Long-Jump	∞	∞	Non-Markovian
$\alpha \geq 1$	$0<\mu<2$	Long-Jump	$<\infty$	∞	Markovian
$0<\alpha<1$	$\mu \geq 2$	Sub-diffusion	∞	$<\infty$	Non-Markovian
$\alpha \geq 1$	$\mu \geq 2$	Brownian	$<\infty$	$<\infty$	Markovian

We are used to $\alpha = 1, \mu = 2$

The fractional order comes as observation of asymptotic behavior in space time relaxation.

Long jumps & indefinite wait times

Anomalous transport

Clearly we can have argument that massive particle (as neutron) cannot jump infinitely far. For such massive particles finite velocity of propagation exists making instantaneous long jumps impossible. But in reality we have nuclear reactors where the dimensions are large especially for high power reactors. The neutrons do have ‘coupling’ between spatially distributed ‘point’ reactors. These power reactors are having dimensions much larger than the average diffusion lengths of neutron-and are called coupled core reactors. The spatial heterogeneity in small scales do manifest as fractal dimensions in space which makes the anomalous transport of neutrons contrary to belief that it can only reside and ‘walk’ as local Brownian motion. Long-jumps can therefore take place for these ‘massive’ neutrons-in a ‘fractal’ heterogeneous spatial backdrop, along with long wait times, in heterogeneous lattice. The ‘Fractal background’ helps to have walk-through, making long jumps possible!

Fractional Reactor Kinetic Equation & solution

$$\frac{1}{v_c}\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) = \mathbb{D}\nabla\phi(x,t) + \left(\gamma\Sigma_f - \Sigma_a\right)\phi(x,t) + \lambda C(x,t) \qquad 0 < \alpha < 1/2$$

$$\frac{\partial^\alpha}{\partial t^\alpha}C(x,t) = \beta\gamma\Sigma_f\phi(x,t) - \lambda C(x,t) \qquad \phi(x,0) = \phi_0(x) = 1.0$$

$v_c = 220{,}000\text{cm/s}$ $B = \beta\gamma\Sigma_f = 000735\text{cm}^{-1}$ $\mathbb{D} = 0.356\text{cm}^2\text{s}^{-\alpha}$ $\lambda = 0.08\text{s}^{-1}$ $\Sigma = \gamma\Sigma_f - \Sigma_a = 0.005\text{cm}^{-1}$

Time (s)	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
0.00010	1.04535×10^8	6.21599×10^6	3.58370×10^5	2.00988×10^4	1.13743×10^3
0.00039	1.59622×10^8	1.42279×10^7	1.22266×10^6	1.01416×10^5	8.23638×10^3
0.00068	1.89937×10^8	1.99782×10^7	2.02159×10^6	1.97079×10^5	1.86944×10^4
0.00097	2.12264×10^8	2.48265×10^7	2.78898×10^6	3.01545×10^5	3.16326×10^4
0.00126	2.3039×10^8	2.91405×10^7	2.78898×10^6	3.01545×10^5	3.16326×10^4
0.00155	2.45852×10^8	3.3087×10^7	4.26719×10^6	5.29060×10^5	6.34629×10^4
0.00184	2.59449×10^8	3.67595×10^7	4.98668×10^6	6.50071×10^5	8.19238×10^4
0.00213	2.71652×10^8	4.02168×10^7	5.69632×10^6	7.75079×10^5	1.01890×10^5
0.00242	2.8277×10^8	4.34989×10^7	6.39764×10^6	9.03656×10^5	1.23257×10^5
0.00271	2.93014×10^8	4.66344×10^7	7.09176×10^6	1.03547×10^6	1.45939×10^5
0.00300	3.02536×10^8	4.96447×10^7	7.77958×10^6	1.17024×10^6	1.69866×10^5

The neutron flux growths for the lower values of $\alpha < 0.3$ has saturation tendency with time. From observations , the inference is drawn that neutron flux multiplication can be obtained at fractional orders values !! $\alpha > 0.3$

This implies that there is possibility of nuclear reactor achieving the desired neutron multiplication factor or criticality at fractional values indicating concept of 'fractional criticality'. $k_\infty^{(\alpha)}$

Convolution & Evolution of Process Dynamics

$$\frac{d}{dt}\Phi(t) = -\int_0^t dt' K(t-t')\Phi(t)$$

Convolution is rolled up condition. Non-exponential relaxation implies memory that is the underlying fundamental relaxation process are non-Markovian. Natural way to incorporate such memory effect is via fractional calculus, via involved convolution integrals in times; the present state is being influenced by all the states the system has been running through at times 0, 1, 2,…, *t*. The power law kernel defining the fractional expression represents a particular long memory.

$$\frac{d}{dt}\Phi(t) = -K_0\Phi(t)$$

Process in this above case at present condition is just entering via present state and not past states-is memory less system.

Relaxation

Maxwell-Debye exponential relaxation process:

Standard Maxwell Debye relaxation is

$$\tau \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -\Phi(t)$$

Gives pure exponential solution with single relaxation time constant

$$t > 0; \qquad \Phi(0) = \Phi_0$$

$$\Phi(t) = \Phi_0 e^{-t/\tau}$$

Integral representation of Maxwell-Debye relaxation is:

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau}\frac{\mathrm{d}^{-1}}{\mathrm{d}t^{-1}}\Phi(t)$$

The integral equation can be formally extended to Fractional Integral equation by replacing

$$\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}} \to \frac{1}{\tau^{\beta}}\frac{d^{-\beta}}{dt^{-\beta}} \equiv \tau^{-\beta}\,{}_0D_t^{-\beta} \qquad \beta > 0$$

which leads to

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau^{\beta}}\,{}_0D_t^{-\beta}\Phi(t)$$

Relaxation anomalous (weak)

Non Debye non-exponential relaxation process:

Kohlraush Williams Watts (KWW) relaxation law:

$$\Phi(t) = \Phi_0 e^{\left\{\left(-\,t/\tau\right)^{\alpha}\right\}}$$

Nutting Power Law relaxation:

$$\Phi(t) = \Phi_0 \left(1 + t/\tau\right)^{-n} \; ; 0 < n < 1$$

Observed in:

Dielectric relaxation, Stress Relaxation, Strain relaxation, NMR relaxation
Diffusion controlled relaxation, electrical circuits,…….; unlike normal relaxation $\Phi(t) = \Phi_0 e^{-t/\tau}$

Relaxation anomalous (weak) with memory

Memory Integrals: $$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -\int_0^t K(t-\tau)\Phi(\tau)\,\mathrm{d}\tau = -K(t) * \Phi(t)$$

Represents Memory Integral i.e. all instances for $\tau = 0$ to $\tau = t$ contribute to situation at $\tau = t$

1. **Memory breaks down i.e. Markovian Case:** $K(t) = K_0\delta(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t) = -\int_0^t K_0\delta(t-\tau)\Phi(\tau)\,\mathrm{d}\tau = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \exp\left(-K_0 t\right)$$

Maxwell-Debye case

$$\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -\frac{\Phi(t)}{\tau} = -K(t) * \Phi(t)$$

How the process time constant is related to Kernel of Memory integral, good research case?

2. **The opposite case Constant Memory i.e. leading to oscillatory case**

$$K(t) = K_0$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Phi(t) = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \cos(\sqrt{K_0}\,t)$$

Memory Integral (Contd.)

3. **Slowly varying Kernel which for small time behaves as power law gives KWW relaxation process**

$$K(t) \approx K_0 t^{\gamma}$$

$$\Phi(t) = \Phi_0 \exp\left(-K_0 t^{\gamma+2}\right)$$

4. **Relaxation for Fractional Differential/Integral equation & its Memory Kernel**

$$K(t) = K_0 t^{q-2}; \qquad 0 < q \le 2$$

$$\frac{d}{dt}\Phi(t) = -\frac{1}{\tau^q}\left[{}_0D_t^{1-q}\Phi(t)\right]$$

$$\tau^q = \left[K_0 \Gamma(q-1)\right]^{-1}$$

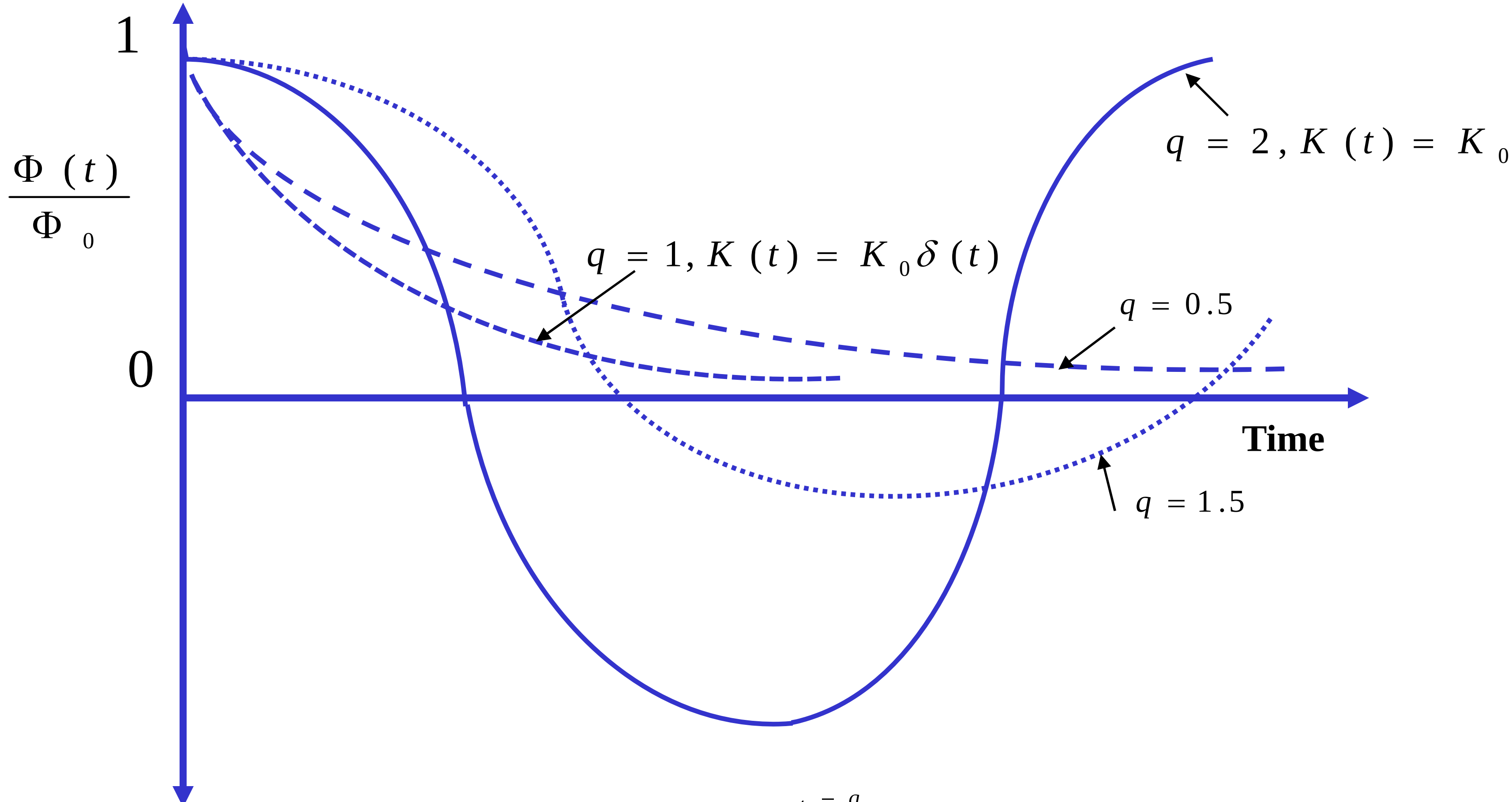
Apply ${}_0D_t^{-1}$ **on both sides to get:** $\Phi(t) - \Phi_0 = -\tau^{-q}\,{}_0D_t^{-q}\Phi(t)$

Apply ${}_0D_t^{q}$ **on both sides to get FDE & using Fractional Derivative of constant C as , non zero, that is** $\Phi_0 t^{-q} / \Gamma(1-q)$

We obtain

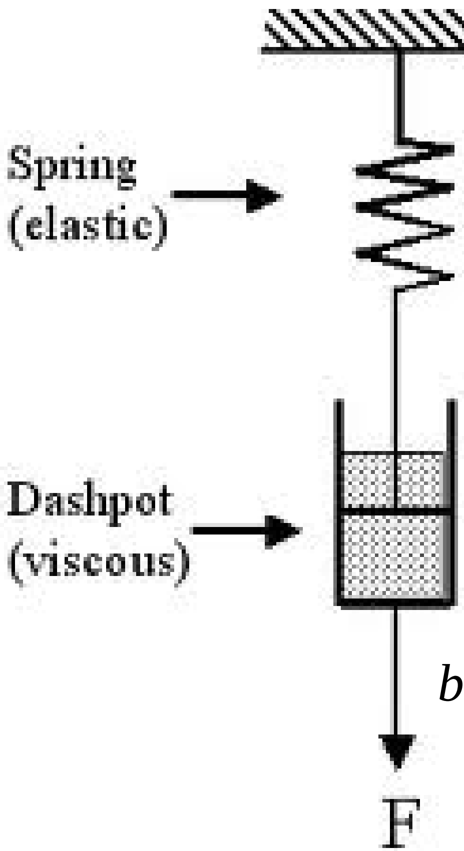
$${}_0D_t^{q}\Phi(t) - \Phi_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\Phi(t)$$

Memory Kernel & Fractional Differential Equation for Relaxation kinetics



$$ {}_0D_t^q \Phi(t) - \Phi_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\Phi(t) $$

Viscoelasticity

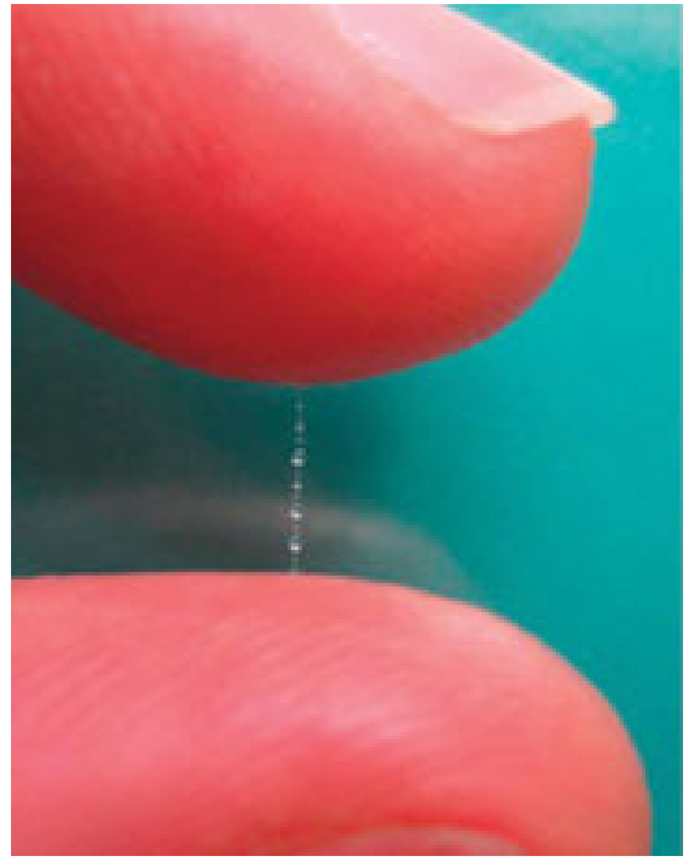


$$F_{spring} \sim Ex$$

$$F_{viscous} \sim b\dot{x}$$

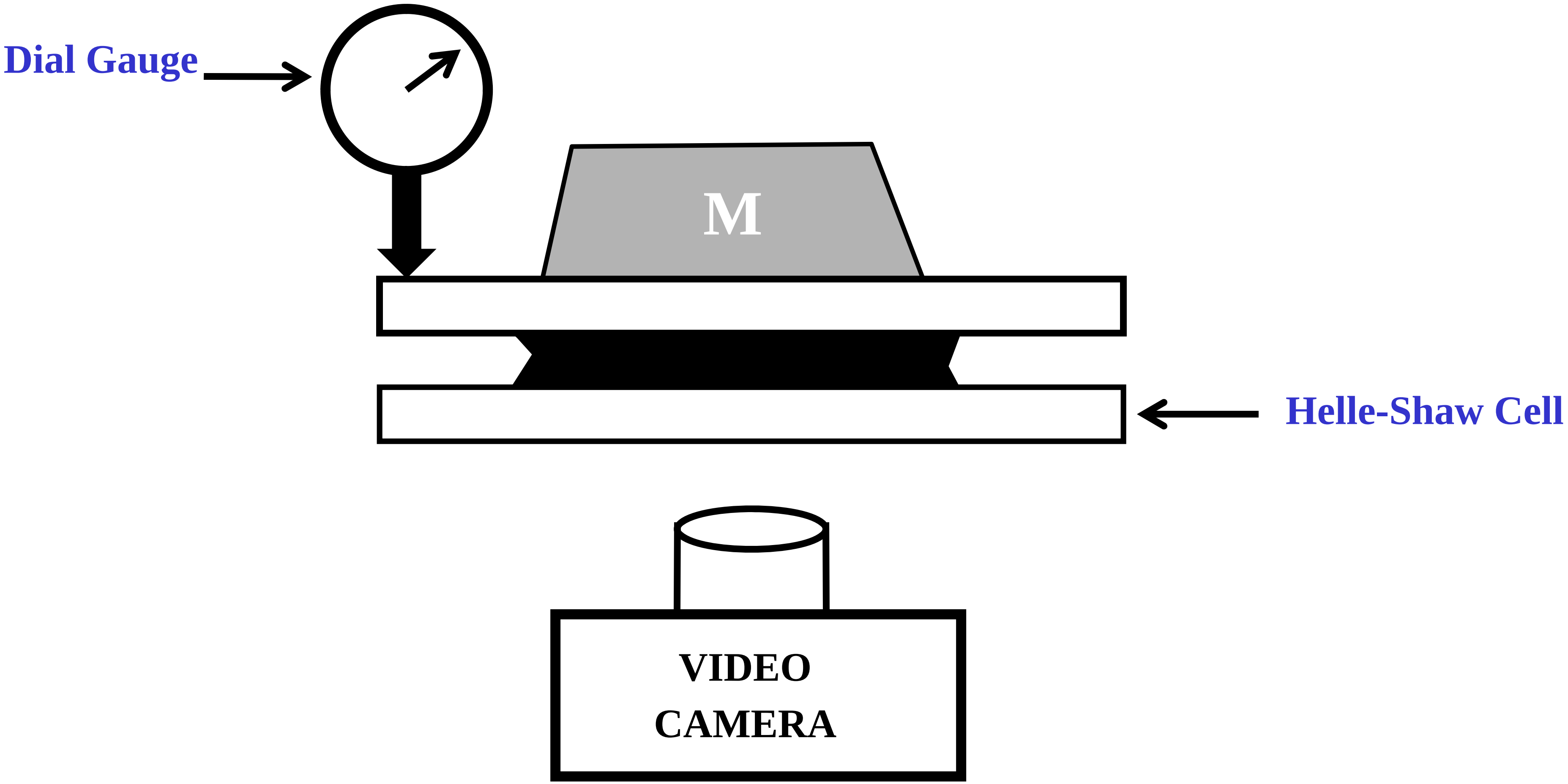
$$b \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon + E\varepsilon = \sigma$$

Maxwell Viscoelastic Model



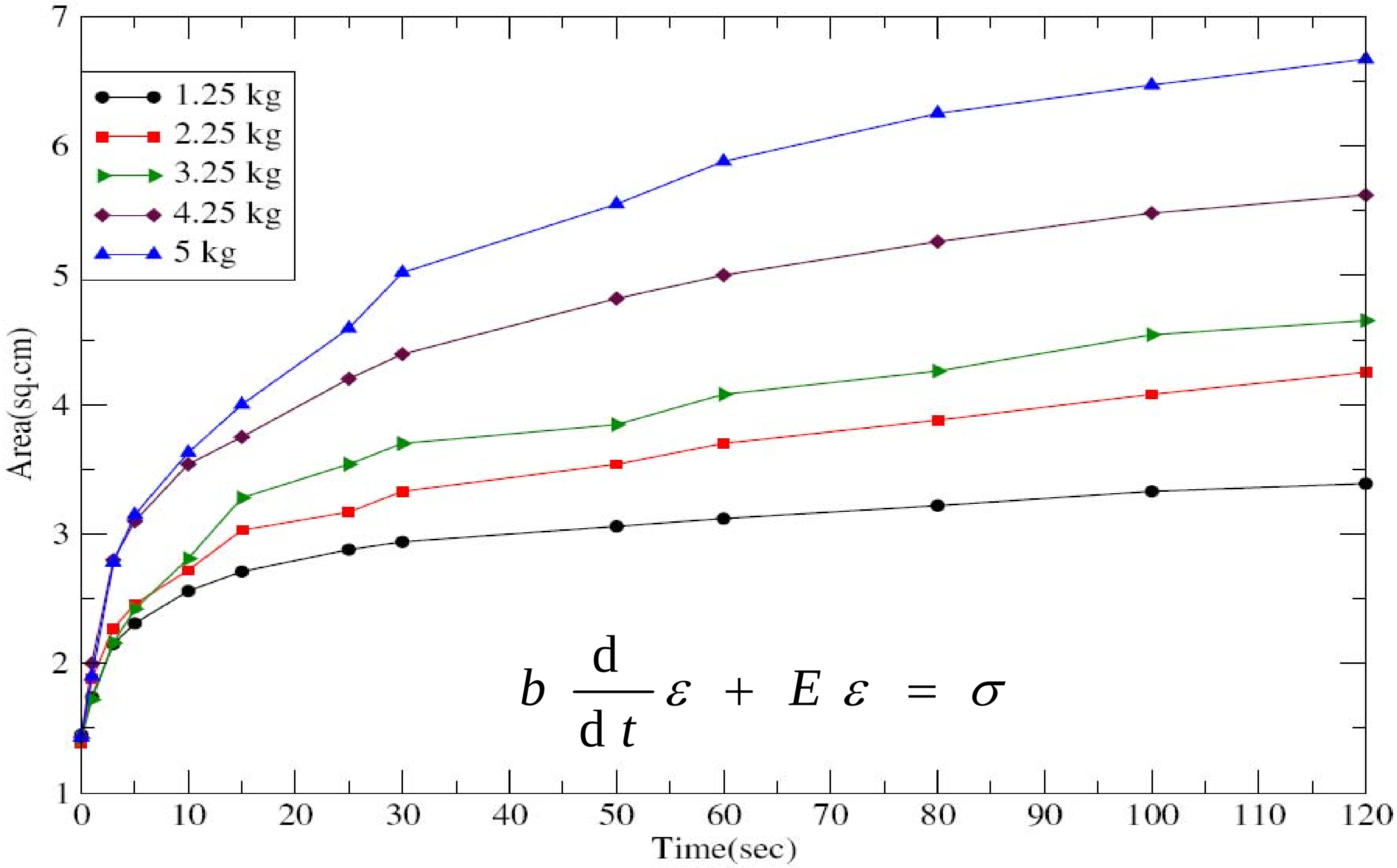
Our Experiment –

forcing a fluid to spread under a load



Normal result

Case of Newtonian Fluid



$$b \frac{d}{dt}\varepsilon + E\varepsilon = \sigma$$

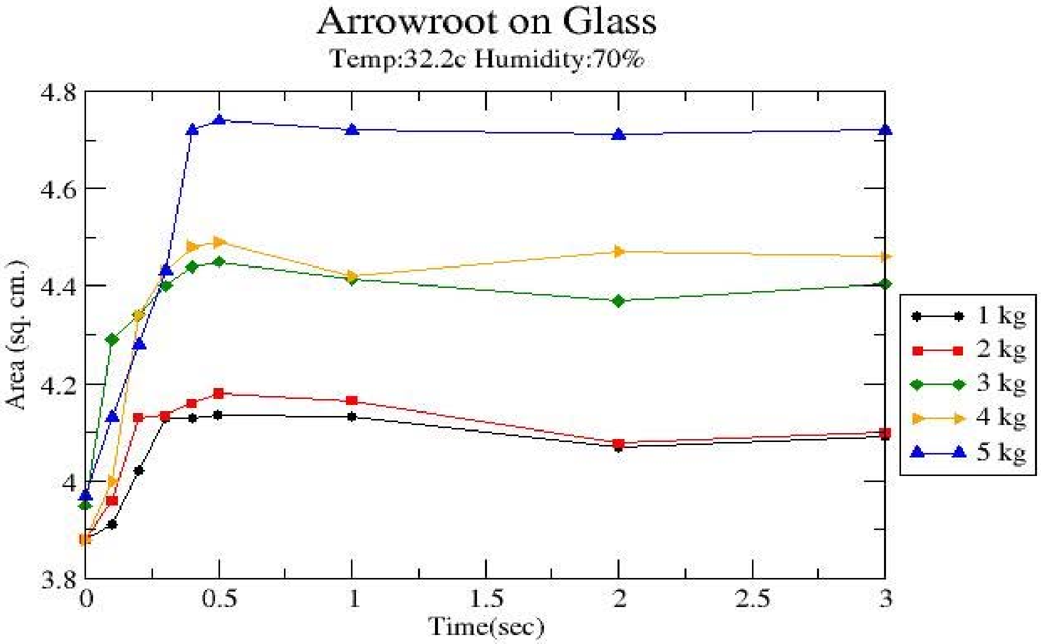
Anomalous result

Non-Newtonian case

Area-Time plot Oscilatory!

$$b \frac{d^q}{dt^q}\varepsilon + E\varepsilon = \sigma$$

$$0 < q < 2$$



Oscillatory spreading

The fractional order *q* encompasses all the anomalous components, including nonlinear viscous constant and Elastic Modulus!

The *b* is different than normal viscosity

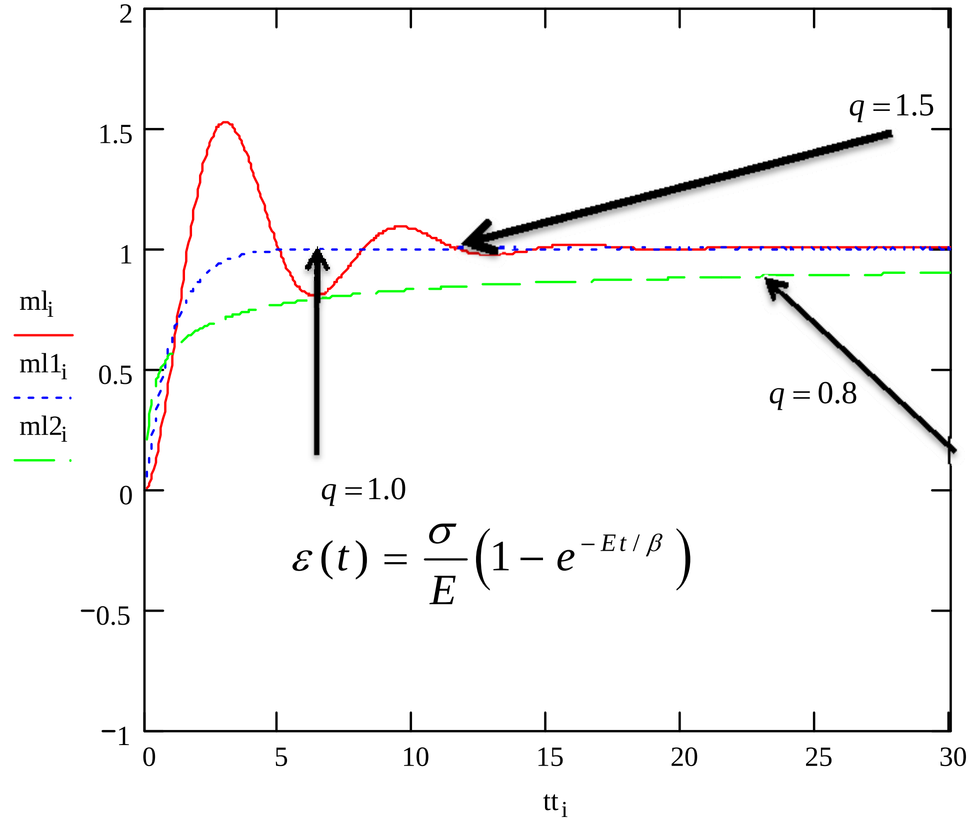
$$b_q \frac{d^q}{dt^q}\varepsilon + E\varepsilon = \sigma$$

$$\sigma < \sigma_c$$

$$\varepsilon(s) = \frac{\sigma}{b_q}\left[\frac{1}{s(s^q + E/b_q)}\right]$$

$$= \frac{\sigma}{E}\left[\frac{1}{s} - \frac{s^{q-1}}{s^q + E/b_q}\right]$$

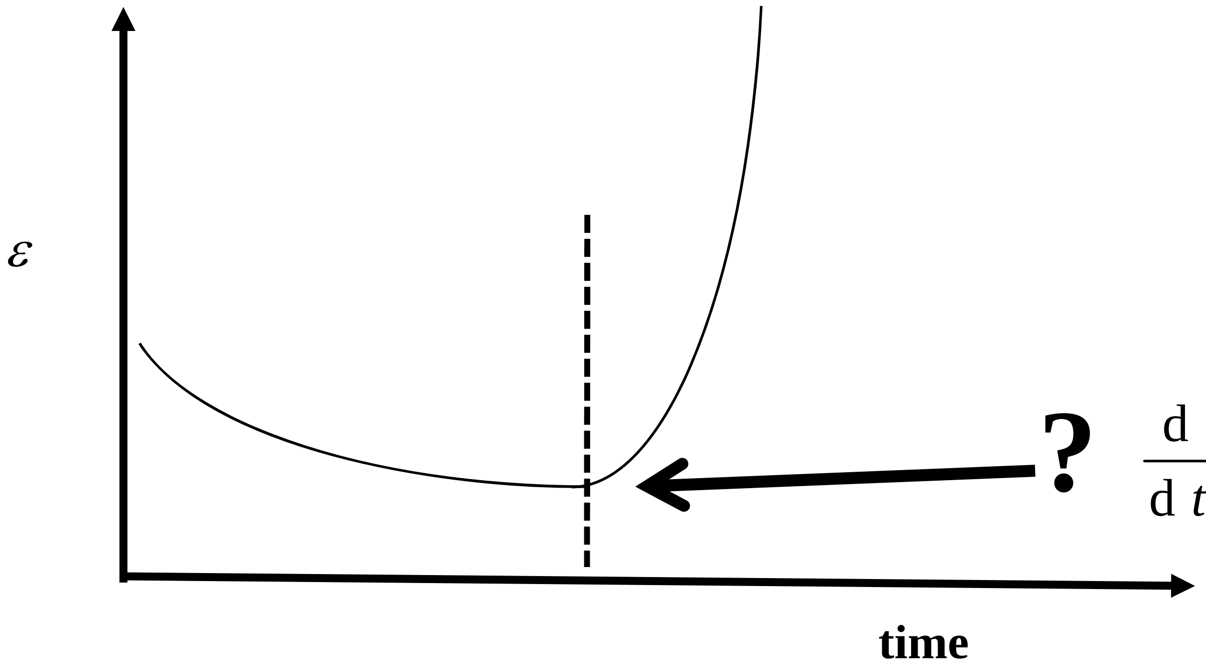
$$\varepsilon(t) = \frac{\sigma}{E}\left[1 - E_q\left(-\frac{Et}{b_q}\right)\right]$$



Motion opposite to force!!

severe anomaly

Abrupt direction change in the dial gauge reading; dial moves opposite to the applied force!!



$$\frac{d}{dt}\varepsilon = \infty$$

Critical point

-power law algebraic fit!

$$\langle S_{12} \rangle = r^{-a}$$

Two point correlation when normal relaxation is governed by a χ distance; beyond which the other points are not aware of happenings! So $\langle S_{12} \rangle = e^{-r/\chi}$

r is the distance from origin; and χ the correlation length.

Near critical point's approach the χ starts growing and becomes ∞ at critical point , every point knows about every other point!!

$$r^{-a} = e^{\ln r^{-a}} = e^{1/(\ln r^{-a})^{-1}} = e^{-r/-r(\ln r^{-a})^{-1}} = e^{-r/-r(-a)^{-1}(\ln r)^{-1}}$$

$$r^{-a} = e^{-r/(r(a \ln r)^{-1})} \qquad \chi = r(a \ln r)^{-1}$$

One sees that in critical case there is no length scale χ , hence correlation function does not decay over any characteristic length scales, they decay algebraic with power law

Local Fractional Derivative

Note the non-local character of the Fractional Derivative defined by RL definition and the non-constant 'fractional-derivative' of non-zero constant. These two features makes extraction of scaling information somewhat difficult. The problem is overcome by 'Local Fractional Derivative' LFD. Sometimes it is desirable to have 'Local-Character' in wide range of applications ranging from structure of 'differentiable' manifolds to various physical models. Secondly, the Fractional Derivative of constant is non-zero, consequently the magnitude of Fractional Derivative changes with addition of a constant to a function. The notion of LFD must address these issues.

The logic is take fractional integral of order $(1-\alpha)$ **of function minus the value of function at point of interest; then Differentiate the function and put limit tending to that point of interest.**

$$I(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} \big(f(t) - f(x_0) \big) \mathrm{d}t$$

$$\mathbf{D}^{\alpha} f(x_0) = \lim_{x \to x_0} I^{/}(x)$$

$$\mathrm{D}^{\alpha} f(x)_{@\ x=x_0} = \lim_{x \to x_0} \frac{\mathrm{d}^{\alpha} \big(f(x) - f(x_0) \big)}{\mathrm{d}(x-x_0)^{\alpha}}$$

Kolwankar-Gangal (KG) Local Fractional Derivative

Kolwankar-Gangal definition:

$$D^{\alpha} f(x)_{@\, x=x_0} = \lim_{x \to x_0} \frac{d^{\alpha}\left(f(x) - f(x_0)\right)}{d(x - x_0)^{\alpha}}$$

$$\frac{d^{\alpha}}{dx^{\alpha}} f(y) = \frac{1}{\Gamma(1-\alpha)} \lim_{x \to y} \frac{d}{dx} \int_y^x (x-t)^{-\alpha} \left(f(t) - f(y)\right) dt$$

In the limit the first derivative of fractional integral $I(x)$ **of order** $(1-\alpha)$

$$I(x) = \frac{1}{\Gamma(1-\alpha)} \int_y^x (x-t)^{-\alpha} \left(f(t) - f(y)\right) dt$$

$$D^{\alpha} f(x)_{@\, x=y} = \lim_{x \to y} I'(x)$$

Local Fractional Derivative (LFD)

Kolwankar-Gangal (KG)

For a function $f : [0,1] \rightarrow \mathbf{R}$ **, the limit**

$$\mathbf{D}^q f(x_0) = \lim_{x \rightarrow x_0} \frac{d^q \left(f(x) - f(x_0) \right)}{\left[d(x - x_0) \right]^q}$$

exists and is finite; then we say LFD of order q **where** $0 < q < 1$ **at** $x = x_0$ **exits**

In this definition the lower limit x_0 **is treated as a constant. The subtraction of** $f(x_0)$ **corrects for the fact that fractional derivative of constant (in RL) is not zero. Where the limit** $x \rightarrow x_0$ **is taken to remove non-local contents. This LFD (removing the non-local contents) allows the study of point wise behavior of** $f(x)$ **.**

$$\mathbf{D}^1 f(0) = \lim_{x \rightarrow 0} \frac{d}{d(x - 0)} [f(x) - f(0)] = \lim_{x \rightarrow 0} \frac{d}{dx} f(x)$$

$$\mathbf{D}^1 f(0) = \lim_{x \rightarrow 0} \frac{d}{dx} f(x)$$ **Slope at origin!**

KG Local Fractional Derivative for fractional order more than one

$$\mathbf{D}^q f(x_0) = \lim_{x\to x_0} \frac{\mathrm{d}^q\left(f(x) - \left[f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}}{2!}(x-x_0)^2 + \ldots \frac{f^{(N)}}{N!}(x-x_0)^N \right]\right)}{\left[\mathrm{d}(x-x_0)\right]^q}$$

$$= \lim_{x\to x_0} \frac{\mathrm{d}^q\left(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}}{\Gamma(n+1)}(x-x_0)^n \right)}{\left[\mathrm{d}(x-x_0)\right]^q}$$

If the limit exists and is finite, where *N* is the largest integer for which *N*-th derivative of function $f(x)$ at point x_0 exists and is finite, then we say LFD of order q $N < q \le N+1$, at x_0 exists.

When q is positive integer, then integer order derivative is recovered.

For $q = 1, N = 0$

$$\mathbf{D}^1 f(x_0) = \lim_{x\to x_0} \frac{d}{d[(x-x_0)]}[f(x) - f(x_0)]$$

Critical Order

Critical order $\alpha(x_0)$

Supremum of q **all Local Fractional Derivative of order less than** q **exists at** x_0 **.**

$$f(x) = a + bx + c|x|^{\beta} \qquad 1 < \beta < 2$$

$$f(0) = a \qquad f^{(1)}(0) = b \qquad f^{(2)}(0) = \infty$$ **Nearest integer** $N = 1$

$$\mathbf{D}^q f(0) = \lim_{x\to 0} \frac{d^q\left(f(x) - \left[f(0) + f^{(1)}(0)(x-0)\right]\right)}{\left[d(x-0)\right]^q} = \lim_{x\to 0} \frac{d^q}{dx^q}\left[a + bx + c|x|^{\beta} - (a+bx)\right]$$

$$= \lim_{x\to 0} c\,\frac{\Gamma(\beta+1)}{\Gamma(\beta-q+1)}|x|^{\beta-q} = \begin{cases} \infty & ; q > \beta \\ 0 & ; q < \beta \end{cases}$$

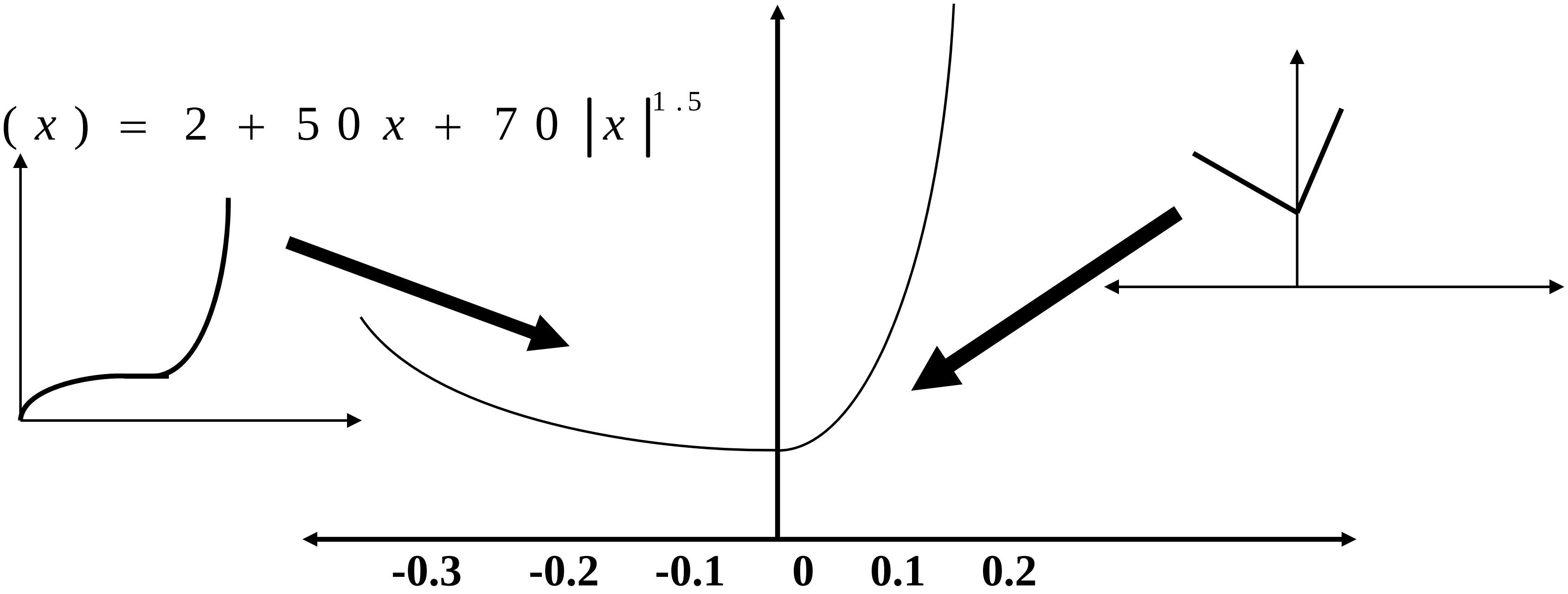
Critical Order of the function $f(x)$ **at** $x = 0$ **is** $\alpha(0) = \beta$

LFD of critical order has value at origin for this function i.e.

$$\mathbf{D}^{\beta} f(0) = c\,\Gamma(\beta+1)$$

Abrupt phase transition to continuous phase transition

$$f(x) = 2 + 50x + 70|x|^{1.5}$$



This is notion to extend Ehrenfest's classification of thermodynamic phase transition, magnetic property at critical point, or yield point (stain) beyond critical stress to continuous transition.

In simplified terms magnify the critical point which takes place abruptly and approximate by polynomial to get Fractional Differentiability at critical point.

Non-differentiability can be magnified and studied near critical points.

Fractional Differentiation of continuous but non-differentiable graph

$$W_{\lambda}(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^{k} x \qquad \lambda > 1 \qquad d_B = s \qquad 1 < s < 2$$

$$W_{\lambda}(0) = 0$$

Use scaling law $\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q}$ **we get** $\frac{d^q}{dx^q} \sin \lambda^k x = (\lambda^k)^q \frac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$

and the fractional derivative of Wierstrauss's function for $0 < q < 1$

$$\frac{d^q}{dx^q} W_{\lambda}(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \frac{d^q \sin \lambda^k x}{dx^q} = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$$

Using $\frac{d^q}{dx^q} \sin x = \frac{d^q}{dx^q} \int_0^x \cos t dt = \frac{d^{q-1}}{dx^{q-1}} \cos x$ **, the FD is**

$$\frac{d^q}{dx^q} W_{\lambda}(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

KG LFD & Fractal Dimmension

The critical order of FD of Wierstrauss's function:

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

The fractional integral $I_x^p \cos \lambda^k x$ **of order** $p = 1 - q$ **is bounded uniformly for all values of** $\lambda^k x$ **. This implies that the RHS will converge for** $s - 2 + q < 0$ **or** $q < 2 - s$ **and diverge for** $q > 2 - s$ **at the point zero (*x*=0)**
The value of fractional integral is zero hence FD at zero of Wierstrauss's is ZERO

This Wierstrauss's function is continuously Fractionally Differentiable locally For orders $q < (2 - s)$ **and not between orders** $(2 - s)$ **to one .**
This implies that this Wierstrauss's function has Critical Order $\alpha = (2 - s)$ **at all points, which is equal roughness exponent and thereby box dimension of the graph!.**

LFD is perhaps a tool to extract local dimension of the irregular rough function

Critical Order LFD and Fractal (Box) dimension

$f : [0,1] \to \mathbf{R}$, be a continuous (real) function

If $\lim_{x \to x_0} \frac{d^q \left(f(x) - f(x_0) \right)}{\left[d(x - x_0) \right]^q} = 0$ for $q < \alpha$

Then $\dim_B f(x_0) \leq 2 - \alpha$

Holder Exponent $\alpha(x_0)$ of a function $f(x)$ defined by this is the largest exponent such that there exists a polynomial $P_n(x)$ that satisfies

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^{\alpha}$$

There is clear change in behavior when q crosses the Critical Order. $\alpha(x_0)$

Study of non-differentiability at critical points

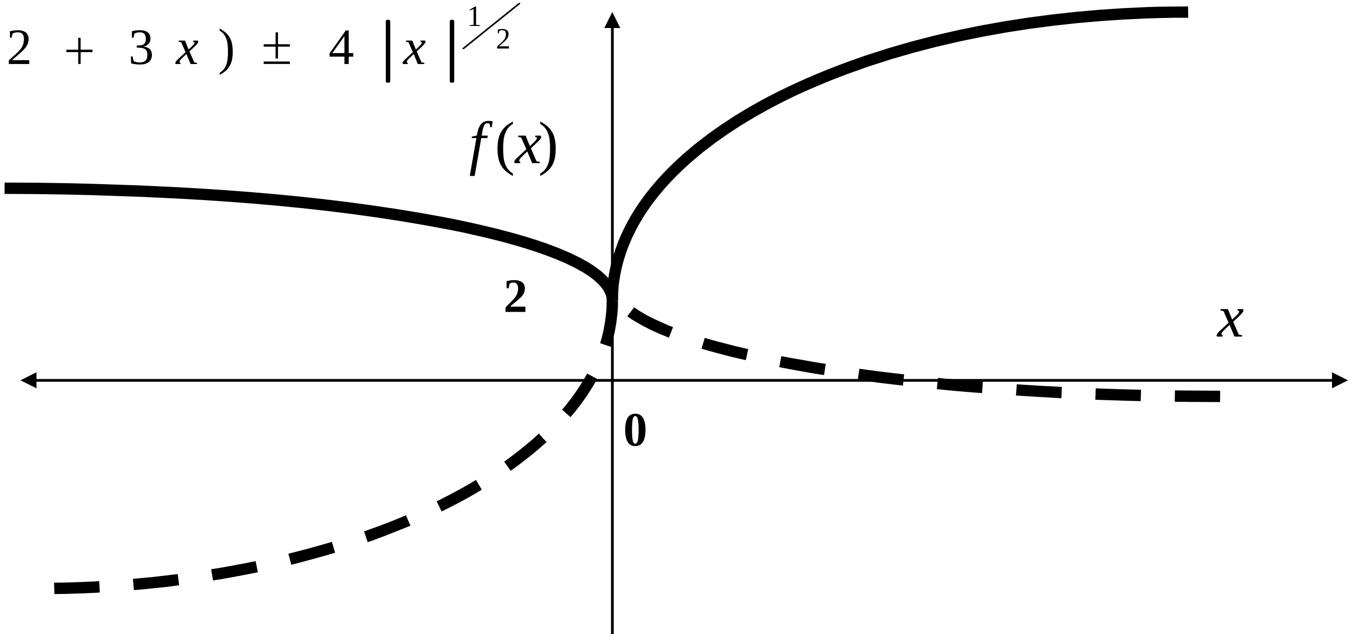
Phase Transition (Non-Differentiability) at critical point:

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^h$$

$$x_0 = 0; P(x) = 2 + 3x$$

Critical Point at zero and the polynomial is linear.

$$f(x) = (2 + 3x) \pm 4 \left| x \right|^{1/2}$$



Non-Differentiability (critical point) at point zero having a value 2 is approximated by a function.

Response function of several processes diverge algebraically near critical point.

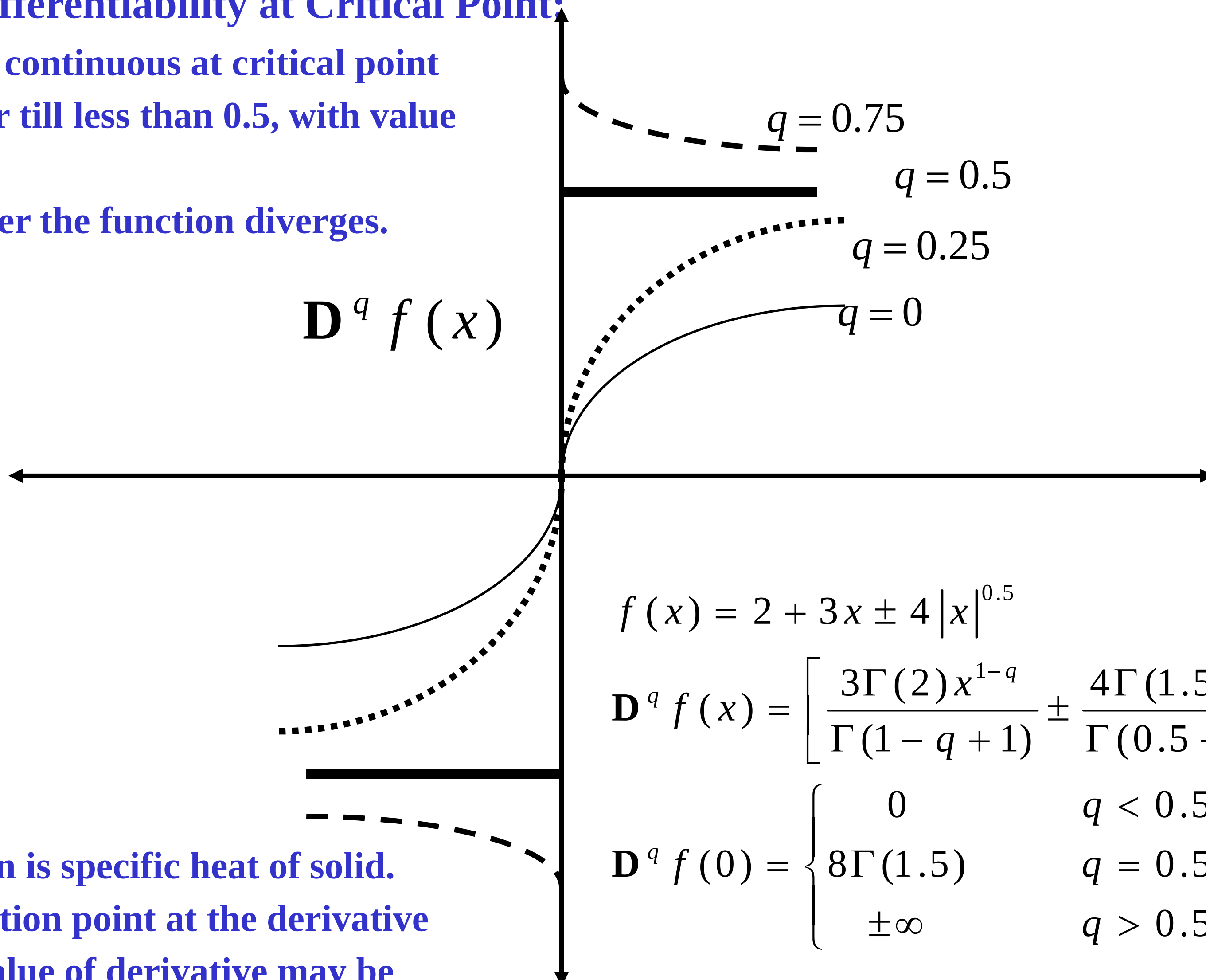
Example Vander wall's equation at critical point, our experiment on Helle-Shaw cell, Curie point.

Magnifying non-differentiability for Critical points

Fractional Differentiability at Critical Point:

The function is continuous at critical point from zero order till less than 0.5, with value zero.

Beyond 0.5 order the function diverges.



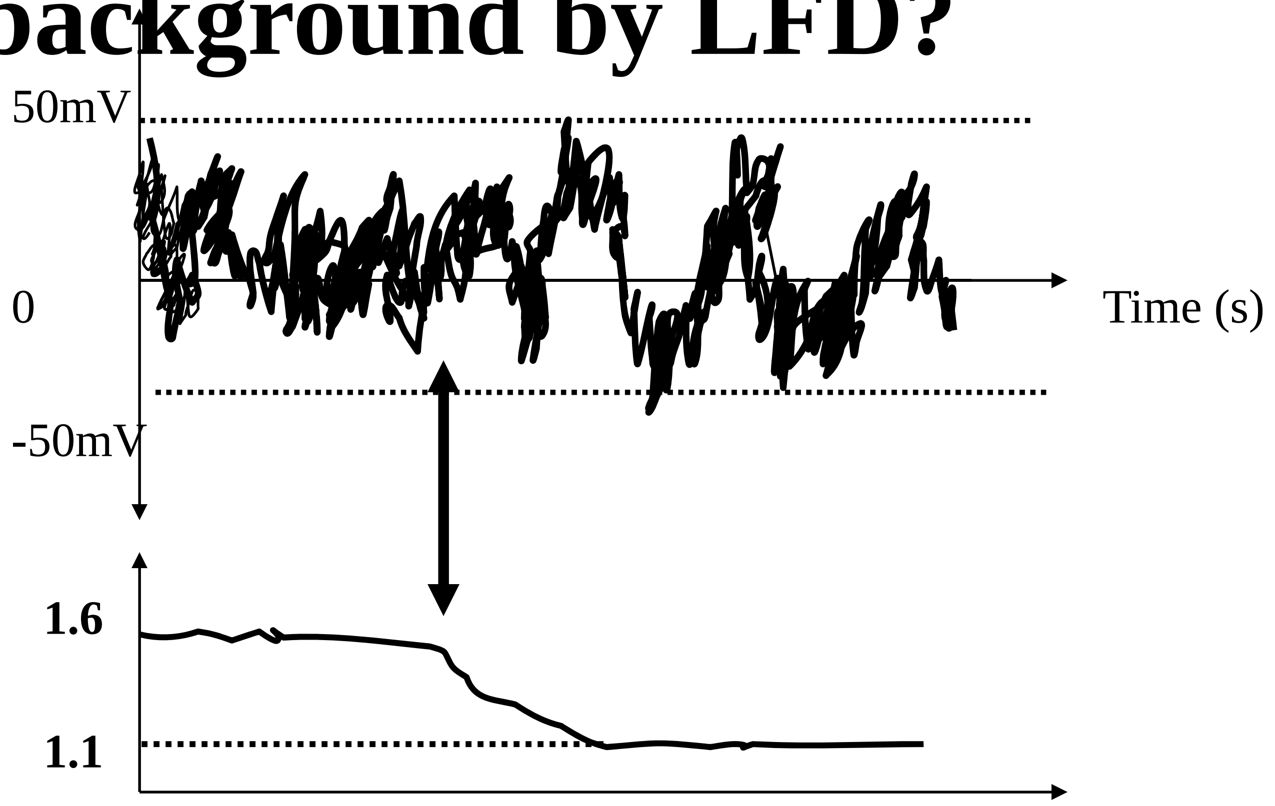
$$f(x) = 2 + 3x \pm 4|x|^{0.5}$$

$$\mathbf{D}^{q} f(x) = \left[\frac{3\Gamma(2)x^{1-q}}{\Gamma(1-q+1)} \pm \frac{4\Gamma(1.5)x^{0.5-q}}{\Gamma(0.5-q+1)} \right]$$

$$\mathbf{D}^{q} f(0) = \begin{Bmatrix} 0 & q < 0.5 \\ 8\Gamma(1.5) & q = 0.5 \\ \pm\infty & q > 0.5 \end{Bmatrix}$$

Say the function is specific heat of solid. At phase transition point at the derivative order 0.5 the value of derivative may be regarded as 'Fractional Latent Heat'

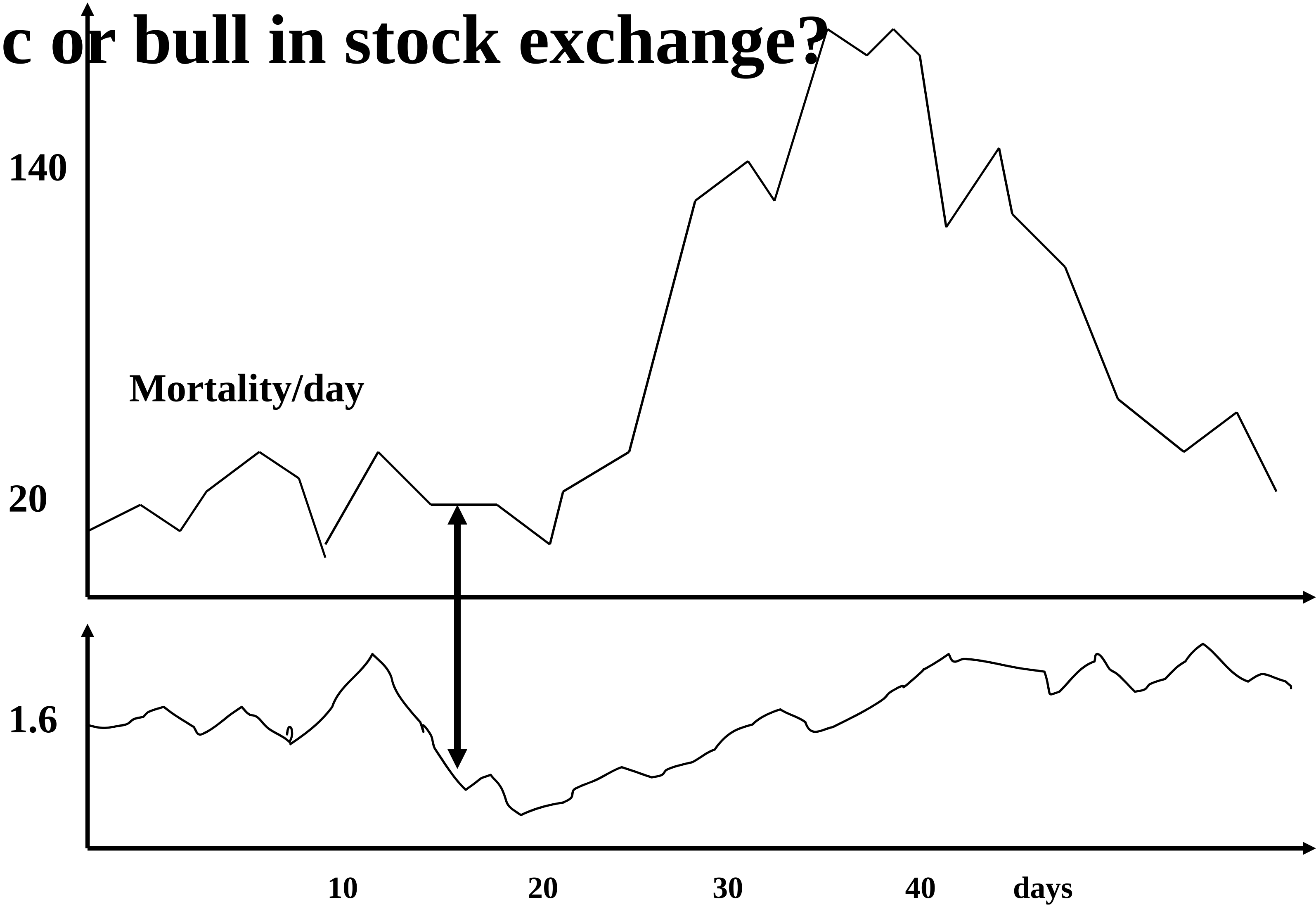
Extraction of information from large noisy background by LFD?



Exactly where the signal starts in high noise background

Signal buried in 85% white noise, the change in dimension indicates the first arrival time of signal.

LFD-Fractal Dimension indicating on-set of Epidemic or bull in stock exchange?



The cause of epidemics exhibited significant change in fractal dimension. Initially behaved as Brownian Motion 1.4-1.6, then dropped to 1.3-1.1 indicating on set of burst between 0-16 day (became regular from irregular) and again raised to 1.4 behaves variably.

Line/surface/volume integrals of Fractal Distribution

Fractal Distribution represented by Fractional Continuous Medium and then we perform the integration.

The fractional Integrals are considered as an approximate integrals on fractals.

This type of new approach is applicable in processes where fractal features of the process or the medium impose the necessity of using non traditional tools in regular smooth physical equations.

Smoothening the microscopic characteristics over physically infinitesimal Volume/surface/line transforms the initial fractal distribution into fractional continuum model.

The order of fractional integration is of fractal dimension.

Fractional Continuum media representing Fractal Distribution!

RL fractional integration is ${}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)\mathrm{d}u$ **Replaced** $\mathrm{d}u$ **by** $\mathrm{d}V_3$

where $\mathrm{d}V_3 = (\mathrm{d}x)(\mathrm{d}y)(\mathrm{d}z)$ **Let us write 3D RL integration as**

$${}_0D_{x,y,z}^{-3\alpha} f(x,y,z) = \frac{1}{\Gamma^3(\alpha)} \int_0^x \int_0^y \int_0^z (x-u)^{\alpha-1}(y-v)^{\alpha-1}(z-w)^{\alpha-1} f(u,v,w)\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

Placing above $d = 3\alpha$ **and rewriting the same as:**

$${}_0D_{x,y,z}^{-d} f(x,y,z) = \frac{1}{[\Gamma(d/3)]^3} \int_0^x \int_0^y \int_0^z (x-u)^{\frac{d}{3}-1}(y-v)^{\frac{d}{3}-1}(z-w)^{\frac{d}{3}-1} f(u,v,w)\mathrm{d}V_3$$

The LHS of above is 'volume' integration on fractional volume V_d **with elemental volume** $\mathrm{d}V_d$ **symbolically** ${}_0D_{x,y,z}^{-d} f(x,y,z) = \int_V f(x,y,z)(\mathrm{d}V_d)$ **. Equating the two with simplification we get:**

$$\int_V f(x,y,z)(\mathrm{d}V_d) = \int_V \frac{1}{\Gamma^3(d/3)}(x-u)^{\frac{d}{3}-1}(y-v)^{\frac{d}{3}-1}(z-w)^{\frac{d}{3}-1} f(u,v,w)\mathrm{d}V_3$$

Approximation from Euclidian media to Fractional Continuum Media

$$\int_V f(x,y,z)(\mathrm{d}V_d) = \int_V \frac{1}{\Gamma^3(d/3)}(x-u)^{\frac{d}{3}-1}(y-v)^{\frac{d}{3}-1}(z-w)^{\frac{d}{3}-1} f(u,v,w)\mathrm{d}V_3$$

We infer that fractional volume element $\mathrm{d}V_d$ **is related to 3D- Euclidian elemental volume element** $\mathrm{d}V_3$ **as** $\mathrm{d}V_d = K_3(x,y,z,d)\mathrm{d}V_3$ **with**

$$K_3(x,y,z,d) = \frac{(xyz)^{\frac{d}{3}-1}}{\Gamma^3(d/3)}$$

Take $\Delta x = \Delta y = \Delta z = \Delta r$ **and** $x \approx y \approx z = r$ **then we have**

$$K_3(r,d) \cong \frac{r^{d-3}}{\Gamma^3(d/3)} \quad \textbf{for} \quad 2 < d < 3$$

Similar method will yield fractional continuum media for surface and line

$$\mathrm{d}S_d = K_2(r,d)\mathrm{d}S_2 \qquad 1 < d < 2 \qquad K_2(r,d) = \frac{r^{d-2}}{\Gamma^2(d/2)}$$

$$\mathrm{d}L_d = K_1(r,d)\mathrm{d}L_1 \qquad 0 < d < 1 \qquad K_1(r,d) = \frac{r^{d-1}}{\Gamma(d)}$$

Integration on fractional continuous media

Let the 'particle' number density be $n(r,t) = n(r)$

Then total number of particles in fractionally continuous volume is $N_d(r) = \int_W n(r,t)\,dV_d$

For 3D volume we write $N(r) = \int_W n(r,t)\,dV_3$ **with** $dV_3 = 4\pi r^2 dr$ **say the density is constant as** $n(r,t) = n_0$ **then in 3D case we get**

$$N(r) = \int_0^R 4\pi r^2 n_0\,dr = \frac{4}{3}\pi R^3 n_0$$

Using the approximation for fractal distributed case to have fractionally continuous Media

$$dV_d = \frac{r^{d-3}}{\Gamma^3(d/3)}dV_3 = \frac{r^{d-3}}{\Gamma^3(d/3)}\left(4\pi r^2\right)dr$$

we get total number of particles as

$$N_d(r) = \int_0^R 4\pi r^2 n_0\,dV_d = \frac{4\pi n_0}{\Gamma^3(d/3)}\int_0^R r^2 r^{d-3}dr$$

$$= \frac{4\pi n_0}{\Gamma^3(d/3)}\frac{R^d}{d} \sim R^d$$

$$2 < d < 3$$

Some laws on Fractal Geometries

Flux through a fractal surface:

A flowing quantity trough a fractal surface be represented as:

$$\phi_{S_d} = \int_S \left(J(r,t) \bullet \mathrm{d}S_d \right) \qquad \mathrm{d}S_d \equiv K_2(r,d)\,\mathrm{d}S_2 \qquad \mathrm{d}S_d = \frac{r^{d-2}}{\Gamma^2(d/2)}\mathrm{d}S_2$$

$$1 < d < 2$$

Gauss's law on Fractal:

$$\int_{\partial W} \left(J(r,t) \bullet \mathrm{d}S_2 \right) = \int_W \mathbf{div}[J(r,t)]\mathrm{d}V_3$$

$$\mathrm{d}S_d = K_2(r,d)\mathrm{d}S_2 \qquad \mathrm{d}V_d = K_3(r,d)\mathrm{d}V_3$$

$$\int_{\partial W} \left(J(r,t) \bullet \mathrm{d}S_d \right) = \int_W (K_3(r,d_3))^{-1}\mathbf{div}[K_2(r,d_2)J(r,t)]\mathrm{d}V_d$$

Stroke's law on Fractal:

$$\int_L \left(E \bullet \mathrm{d}L_1 \right) = \int_S [\mathbf{curl}E]\mathrm{d}S_2$$

$$\mathrm{d}L_d = K_1(r,d)\mathrm{d}L_1 \qquad \mathrm{d}S_d = K_2(r,d)\mathrm{d}S_2$$

$$\int_L \left(E \bullet \mathrm{d}L_d \right) = \int_S \left(K_2(r,d_2) \right)^{-1}[\mathbf{curl}K_1(r,d_1)E]\mathrm{d}S_d$$

Existence of Magnetic charges?

In normal cases of smooth geometry $\mathbf{div}\, B = 0$ **indicating no magnetic charges at point exists . Magnetic mono-pole not possible.**

Fractional generalization however gives: $\mathbf{div}[K_2(r, d_2)B] \neq 0$

$$\mathbf{div}\, B = B.\mathbf{grad}\, K_2(r, d_2)$$

For $d_2 \neq 2$ **;** $\mathbf{grad}\, K_2(r, d_2) \neq 0$ **indicating** $\mathbf{div}\, B \neq 0$
Existence of 'magnetic monopole charges' with magnitude of

$$e_m \approx B.\nabla K_2(r, d_2)$$

For fractal distribution we have thus all sets of conservation laws and set of Maxwell equations and electrodynamics do get modified.

This method perhaps is suitable for dusty plasma cases.

Relaxation through several states (Variety of time constants)

Nonlinear system relaxation through several states

$$\Psi(t) = [\Psi] = [\psi_1(t) \quad \psi_2(t) \quad \psi_3(t) \quad * \quad * \quad \psi_\infty(t)]$$

$$\begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ * \\ \psi_N(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & * & a_{1N} \\ a_{21} & a_{22} & * & a_{2N} \\ * & * & * & * \\ a_{N1} & a_{N2} & * & a_{NN} \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t} \\ e^{-\lambda_2 t} \\ * \\ e^{-\lambda_N t} \end{bmatrix}$$

$$\Psi_1(t) = a_{11}e^{-\lambda_1 t} + a_{21}e^{-\lambda_2 t} + \ldots\ldots + a_{NN}e^{-\lambda_N t}$$

$$\Psi_2(t) = a_{21}e^{-\lambda_1 t} + a_{22}e^{-\lambda_2 t} + \ldots\ldots + a_{NN}e^{-\lambda_N t}$$

....

$$\Psi_N(t) = a_{N1}e^{-\lambda_1 t} + a_{N2}e^{-\lambda_2 t} + \ldots\ldots + a_{NN}e^{-\lambda_N t}$$

$$\Psi(t) = \left\| [\Psi] \right\| = \sqrt{[\Psi]^T [\Psi]}$$

Disordered Relaxation

Ordered Relaxation: (Intense & Strong)

Standard Maxwell Debye relaxation

Gives pure exponential solution with single relaxation time constant; this is strong relaxation (without-memory).

$$\tau \frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) = -\Psi(t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) + \lambda\Psi(t) = \delta(t)$$

$$\tau^{-1} = \lambda; \qquad \Psi(0) = 1; \qquad \Psi(0^-) = 0$$

$$\Psi(t) = e^{-t/\tau} = e^{-\lambda t}$$

Disordered Relaxation: (Intermittent & weak)

For complex dissipating process we (may) have several time constants and let us have this ‘disorder’ in a power law representation so, the PDE is, (this is weak relaxation, obtained by modifying above)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t) + \lambda\Psi(t) = \delta(t)$$

$$\frac{\partial}{\partial t}\Psi(\lambda, t) + (\lambda)^{1/\phi}\Psi(\lambda, t) = \delta(t)$$

$$0 < \phi < 1$$

Power law is scale free with preferential ‘statistics’ ;why a random walker prefers to have his/her state maintained; also the law states that ‘rich becoming richer’ thus preferential

Origination of Fractional Differential Equation in complex 'disordered' relaxation

$$\frac{\partial}{\partial t}\Psi(\lambda,t)+(\lambda)^{1/\phi}\Psi(\lambda,t)=\delta(t)$$

The solution is $\Psi(\lambda,t)=e^{\left(-\lambda^{\{1/\phi\}}t\right)}$ **'impulse response function'** $h(\lambda,t)$

On integrating this $h(\lambda,t)$ **w.r.t.** λ **for all** $0,\infty$ **, we obtain the function in time.**

$$g(t)=\int_0^\infty h(\lambda,t)\,d\lambda=\int_0^\infty e^{\left\{-\lambda^{(1/\phi)}t\right\}}d\lambda$$

With change of variable and recasting with definition of Gamma function

as $\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}dx$ **we get:**

'Impulse response' of Linear Constant Coefficient System starting from rest:

$$g(t)=\frac{\Gamma(1+\phi)}{t^{\phi}}$$

Observed is 'power-law' (long-tailed) decay (lingering memory!!)

Response of relaxation to arbitrary forcing function-origin of $D_t^{\phi} f(t)$

$$\frac{\partial}{\partial t}\Psi(\lambda,t) + (\lambda)^{1/\phi}\Psi(\lambda,t) = \dot{f}(t) \qquad g(t) = \frac{\Gamma(1+\phi)}{t^{\phi}}$$

Response to the arbitrary forcing function is:

$$r(t) = g(t) * \dot{f}(t)$$

$$= \int_0^t d\tau\, g(t-\tau)\,\dot{f}(t)$$

Here put the value of Green's function $g(t)$

$$= \Gamma(1+\phi)\int_0^t \frac{\dot{f}(t)}{(t-\tau)^{\phi}}d\tau = \Gamma(1+\phi)\int_0^t \frac{\dot{f}(t-\tau)}{\tau^{\phi}}d\tau$$

$$= \Gamma(1+\phi)\Gamma(1-\phi)\int_0^t \frac{\tau^{-\phi}}{\Gamma(1-\phi)}\dot{f}(t-\tau)\,d\tau$$

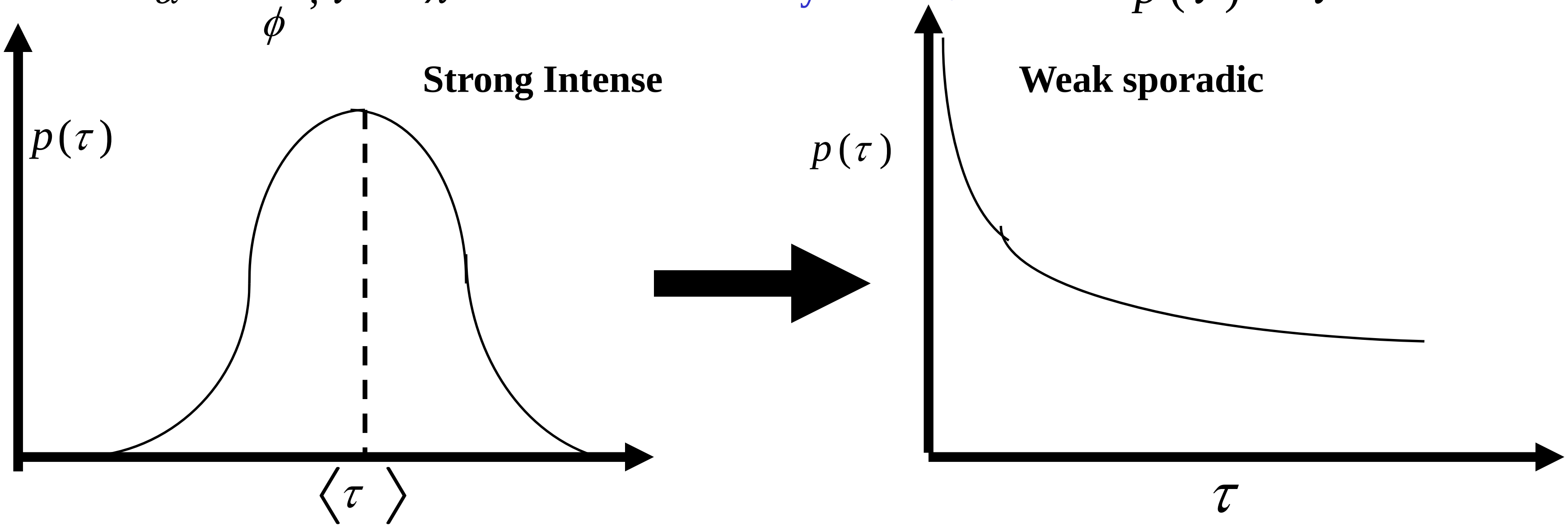
$$= \Gamma(1+\phi)\Gamma(1-\phi)\, I_t^{\phi-1}\left\{\dot{f}(t)\right\}$$

$$= \Gamma(1+\phi)\Gamma(1-\phi)\, D_t^{\phi} f(t)$$

Intermittency of relaxation random normal, and scale-free power law:

$$(\lambda)^{1/\phi}\ ;\ 0<\phi<1$$

Let $\alpha = \dfrac{1}{\phi};\ \tau = \lambda^{-1}$ Then Probability Distribution is $p(\tau) \approx \tau^{-\alpha}$



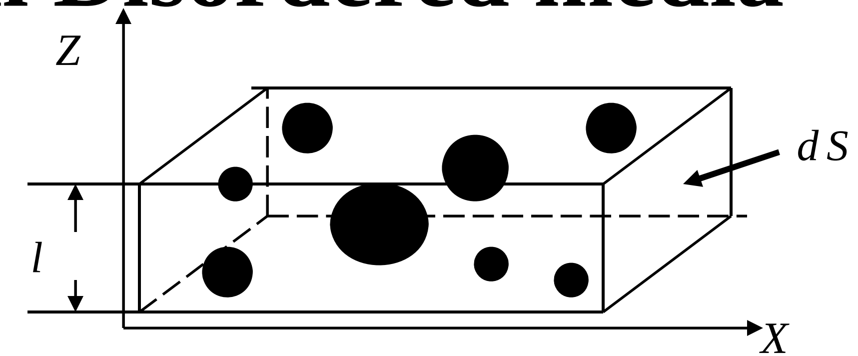
A power law distribution $p(x) \approx x^{-\alpha}$ implies that is asymptotically scale-invariant. In general it is convenient to write from a minimum value as: $p(x) = \dfrac{\alpha-1}{x_{\min}}\left(\dfrac{x}{x_{\min}}\right)^{-\alpha};\ \alpha > 1$

Moments of power law distribution

$$\langle x^m \rangle = \int_{x_{\min}}^{\infty} x^m p(x)\,dx = \frac{\alpha-1}{\alpha-1-m}\, x_{\min}^m$$

Which is well defined for $m < \alpha - 1$. That means for $m \geq \alpha - 1$ all moments diverge. Can have no defined average or variance!! It is preferential 'rich-getting –richer'.

Transport in Spatial Disordered media



Let the largest diameter of the porous media be $a_0 ; l >> a_0$ **in elementary** $\mathrm{d}V$

The volume of grains $\mathrm{d}V_{gr}$ **in** $\mathrm{d}V$.

$\mathrm{d}V_{gr} = (\mathrm{d}V)K(x)$ **Where** $K(x)$ **is volume of grain per unit volume** $\mathrm{d}V$ **near** x .

Grain free volume is thus: $\mathrm{d}V_{free} = \mathrm{d}V - \mathrm{d}V_{gr} = \mathrm{d}V[1 - K(x)] = \varepsilon(x)\mathrm{d}V$

Porosity of the medium at x **is:** $\varepsilon(x) = 1 - K(x) = \frac{\mathrm{d}V_{free}}{\mathrm{d}V}$

Let $\mathrm{d}S_{gr}(z)$ **be the area common to the plane & to the grain. The volume of grains cutting the plane is:** $dV_{gr} = \int_0^l dS_{gr}(z)dz$ **and the grain free area in this plane is:** $dS_{free} = dS - dS_{gr}(z)$

Averaging over length the grain free area is:

$$\mathrm{d}S_{free} = \frac{1}{l}\int_0^l \mathrm{d}S_{free}(z)\mathrm{d}z = \frac{1}{l}\int_0^l [\mathrm{d}S - \mathrm{d}S_{gr}(z)]\mathrm{d}z = \mathrm{d}S - \frac{\mathrm{d}V_{gr}}{l}$$

$$\mathrm{d}V_{gr} = K(x)l\mathrm{d}S$$

Diffusion in porous media:

Mass balance in the grain free area/volume $u(x,t)$ **is concentration &** j **is particle flux**

$$\frac{\partial}{\partial t}\int_{V_{free}} u \, \mathrm{d}V_{free} = -\int_{S_{free}} j \, \mathrm{d}S_{free}$$

$$\mathrm{d}S_{free} = \varepsilon(x)\mathrm{d}S; \qquad \mathrm{d}V_{free} = \varepsilon(x)\mathrm{d}V$$

$$\frac{\partial}{\partial t}(u\varepsilon) + \mathrm{div}(j\varepsilon) = 0$$

$$j = -\mathbb{D}_0 \nabla u$$

$$\frac{\partial}{\partial t}u\varepsilon - \mathbb{D}_0 \nabla . \varepsilon \nabla u = 0$$

Porosity a stochastic process

Let porosity be a random process with $\varepsilon_p(x)$ having zero average which takes the value between 0 and 1. $\varepsilon(x) = e^{-\varepsilon_p(x)}$. Substituting this in and expanding

$$\frac{\partial}{\partial t} u \varepsilon - \mathbb{D}_0 \nabla . \varepsilon \nabla u = 0$$

We get $$\frac{\partial}{\partial t} u - \mathbb{D}_0 \nabla^2 u = - \mathbb{D}_0 (\nabla \varepsilon_p) . \nabla u$$

A "Stochastic Differential Equation"-with a source term !!

The stochastic disorder process has returned a 'self energy term'- the idea is to find equation for average $< u >$ for all possible realizations of ε_p for standard initial condition $u(x,0) = \delta(x)$ with natural boundary condition

The *u* (*x*, *t*) is 'functional' of the porosity. The averaging < *u* (*x*, *t*) > with standard techniques of Stochastic Differential Equation yield a 'different' differential equation for the average. Use of perturbative technique says that disorder free greens function Propagates through several realizations of porosity, to return (with approximation) the temporal Fractional Derivative-as 'self energy term'

Evolution of fractional derivative in time due to spatial disorder

The *u* (*x*, *t*) is 'functional' of the porosity. The averaging < *u* (*x*, *t*) > with standard techniques of Stochastic Differential Equation yield a 'different' differential equation for the average. Use of perturbative technique says that disorder free greens function propagates through several realizations of porosity, to return (with approximation) the temporal Fractional Derivative-as 'self energy term'. Use of Feynman diagram and Complex analysis transforms, the obtained stochastic differential equation

$$\frac{\partial}{\partial t}u(x,t)-\mathbb{D}_0\frac{\partial^2}{\partial x^2}u(x,t)=-\mathbb{D}_0\left(\frac{\partial}{\partial x}\varepsilon_p(x)\right)\left(\frac{\partial}{\partial x}u(x,t)\right)$$

to

$$\frac{\partial}{\partial t}\left\langle u(x,t)\right\rangle-\mathbb{D}_0\frac{\partial^2}{\partial x^2}\left\langle u(x,t)\right\rangle$$

$$=\frac{32\pi a^3 l\mathbb{D}_0^{1/2}}{L^4}\frac{\partial^{1/2}}{\partial t^{1/2}}\left\langle u(x,t)\right\rangle-\frac{160\pi l^3a^3}{\mathbb{D}_0^{1/2}L^6}\frac{\partial^{3/2}}{\partial t^{3/2}}\left\langle u(x,t)\right\rangle+\frac{32\pi a^3l^5}{\mathbb{D}_0^{3/2}L^8}\frac{\partial^{5/2}}{\partial t^{5/2}}\left\langle u(x,t)\right\rangle$$

Let us extend our thought

A lumped system, spring dash pot, or RL circuit is ODE, system without memory!

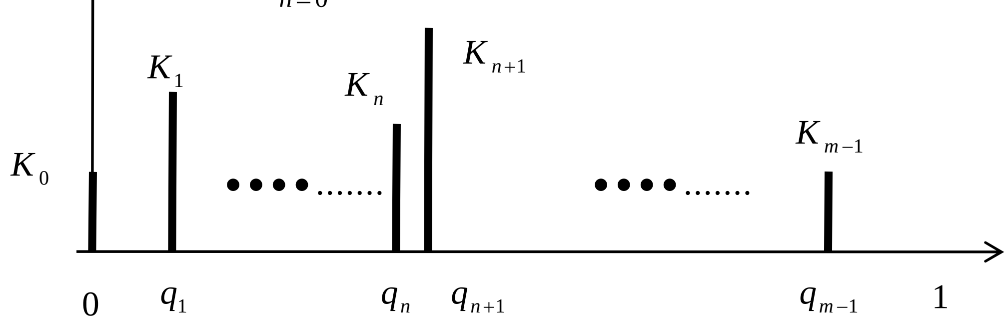
$$K_1 \frac{\mathrm{d}}{\mathrm{d}t} f(t) + K_0 f(t) = g(t)$$

A fractional generalization gives a system with memory

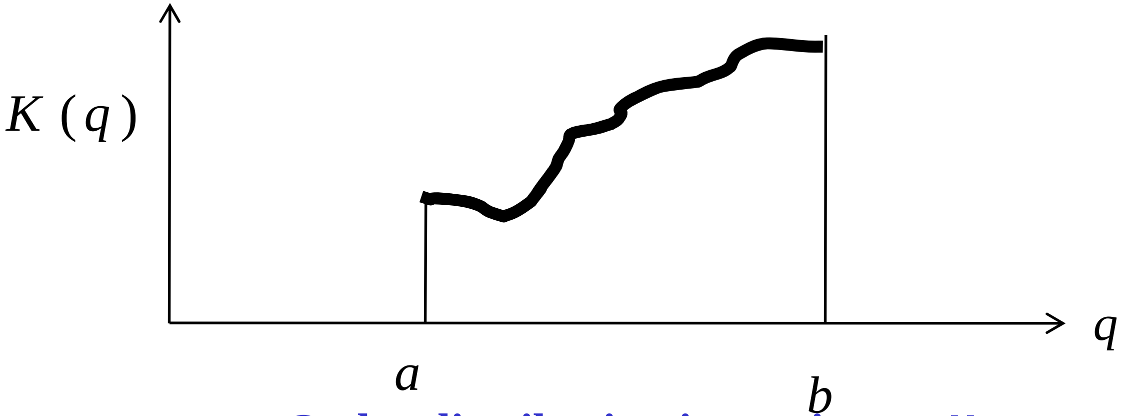
$$K_1 \frac{\mathrm{d}^q}{\mathrm{d}t^q} f(t) + K_0 f(t) = g(t)$$

A fractional order system with several fractionally damped element

$$\sum_{n=0}^{m=1} K_n \frac{\mathrm{d}^{q_n}}{\mathrm{d}t^{q_n}} f(t) = g(t)$$



What happens?



Order distribution is continuous!!

$$\sum_{n=0}^{m=1} K_n \frac{\mathrm{d}^{q_n}}{\mathrm{d}t^{q_n}} f(t) = g(t)$$

The summation $\sum$ is changed to $\int$

We get Continuous Order Differential Equation
(infinite memory types!!)

$$\int_a^b \left(K(q) \frac{\mathrm{d}^q}{\mathrm{d}t^q} f(t) \right) dq = g(t)$$

At the end of the day

one has to solve

Fractional (Extraordinary)

Differential Equations