National Work Shop

Fractional Calculus: Theory & Application

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Department of Mathematics University of Pune

Reality of Fractional Calculus

in six physical applications

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In this year of mathematics-2012

Saluting the mathematicians who gave us wonderful world of

Fractional Calculus

Proud to be standing at the venue that has given us

“Local Fractional Derivative”

and

“Existence uniqueness solution of Fractional Differential Equation (with Caputo derivative)”

That is Univ. of Pune

Some theory & applications in this deliberation-six of them

1. Concept of non-canonical sources (mass, charge).

2. Fractional Cross Product & Fractional Curl.

3. Dissipation mechanism and Fractional Order Differ-ential equation.

4. Spatial Roughness manifesting to Fractional Order Differential equation-(super-capacitor)

5. Gramian for Fractional Order State Space system

6. Generalization of Fractional Calculus and utility

Revision of few salient concepts

Useful revision-delta function

In many applications we would like to be able to handle quantities as

1. The density $\rho(\mathrm{r})$ of point mass at r_0 or say charge
2. The probability density function of a discrete distribution
3. The force $F(t)$ associated with an instantaneous collision
4. The derivative of a function with a step discontinuity

We may regard each of these as result of a limiting process in which a quantity gets More and more concentrated in space or time.

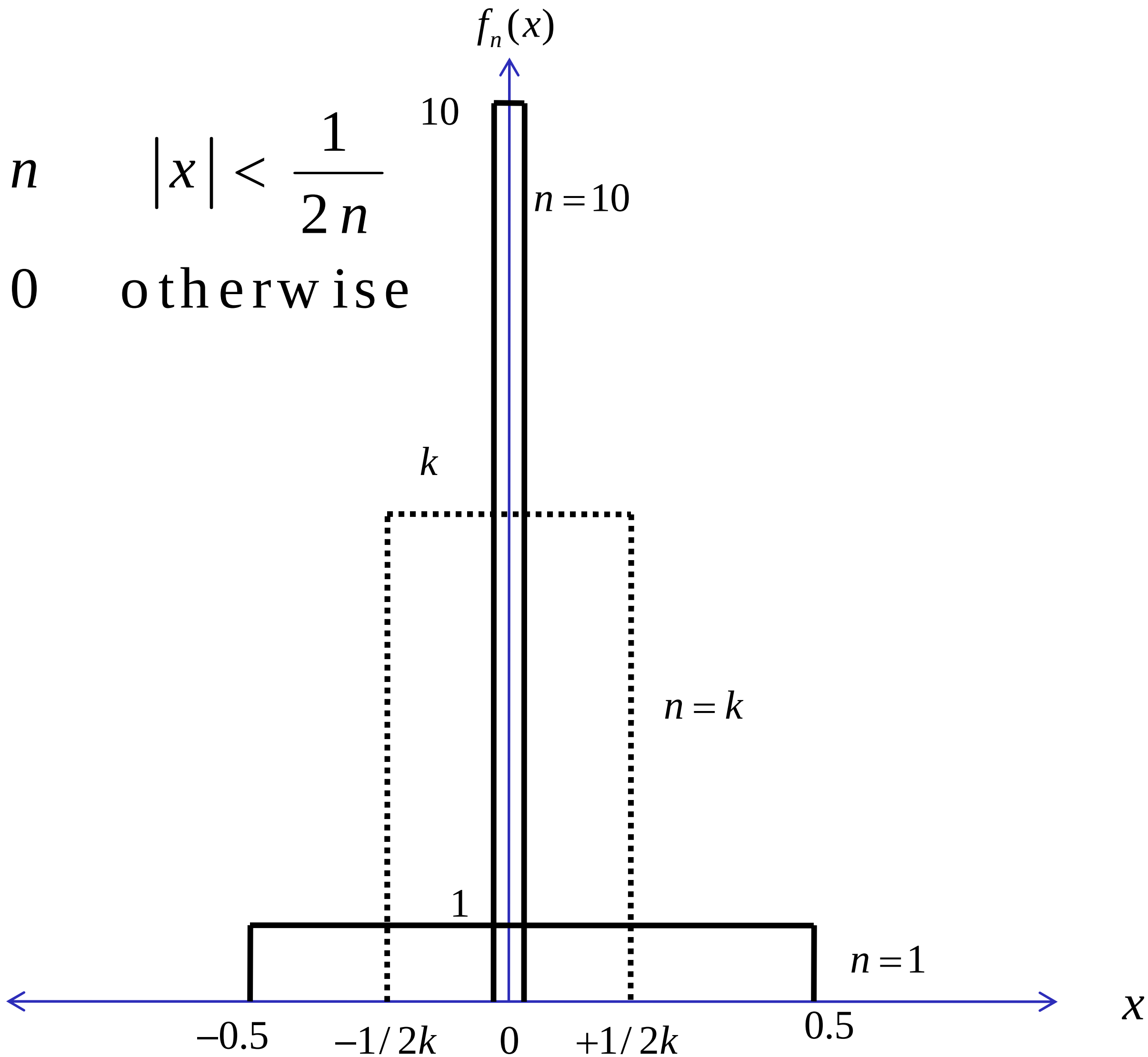
Consider the sequence of functions

$$f_n(t) = \begin{cases} n & |t| < \dfrac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

These functions get narrower and taller as $n \to \infty$. The area in any interval enclosing the origin tends to one for n large.

Sequence of function

$$f_n(x) = \begin{cases} n & |x| < \dfrac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$



Sampling window for window width zero

The (Dirac) delta or impulse function

These sequence of functions do not tend point wise to any limit as *n* tends to infinite, but physically it is useful to think of "point" sources such as point charges point masses etc. We would like to invent a "limit" for this sequence of functions which should have certain properties. It is called the "(Dirac) delta function" or "unit impulse" and is denoted by $\delta(x)$

$$f_n(x) = \begin{cases} n & |x| < \dfrac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b \delta(x)\mathrm{d}x = \begin{cases} 1 & \text{if} \quad 0 \in (a,b) \\ 0 & \text{if} \quad 0 \notin (a,b) \end{cases}$$

For any *g*(*x*) continuous at *x*=0, we have that $\int_{-\infty}^{+\infty} f_n(x)\, g(x) = g(0)$

So we would like delta function to have property

$$\int_{-\infty}^{+\infty} \delta(x)\, g(x) = g(0)$$

Not ordinary function but are "distributions"

Test function $g(x) \in \mathcal{D}$ → **Dirac delta** → $\int_{-\infty}^{+\infty} \delta(x)\, g(x) = g(0)$

Test function → **Functional** → **Number**

Test function belongs to $\mathcal{D}$ class of functions infinitely differentiable and vanishes outside a finite interval

The delta distribution is $\langle \delta, g \rangle = \int_{-\infty}^{+\infty} \delta(x)\, g(x)\mathrm{d}x = g(0)$ **a singular distribution , i.e. not a regular distribution. The regular distributions are induced by locally integrable functions.**

Differentiation of distribution

For test function

$$\phi(t) \in \mathcal{D} \qquad \left\langle f'(t), \phi(t) \right\rangle = \int_{-\infty}^{+\infty} f'(t)\phi(t)\mathrm{d}t$$

$$= \left[f(t)\phi(t) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t)\phi'(t)\mathrm{d}t$$

$$= -\left\langle f(t), \phi'(t) \right\rangle$$

where the boundary terms from the integration by parts is set to zero since test functions vanish outside a bounded set

Derivative in distributional sense

Step function $H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$ **undefined at** $t = 0$ **with no derivative existing**

is also locally integrable and thus induces regular distribution

$$\langle H(t), \phi(t) \rangle = \int_{-\infty}^{+\infty} H(t)\phi(t)\,\mathrm{d}t$$

$$= \int_{0}^{\infty} \phi(t)\,\mathrm{d}t$$

$$\langle H'(t), \phi(t) \rangle = -\langle H(t), \phi'(t) \rangle$$

$$= -\int_{0}^{\infty} \phi'(t)\,\mathrm{d}t = -\phi(\infty) + \phi(0)$$

$$= \phi(0) = \langle \delta(t), \phi(t) \rangle$$

therefore $H'(t) = \delta(t)$

in distributional sense; this holds despite the fact $H(t)$ **is not differentiable in ordinary sense!**

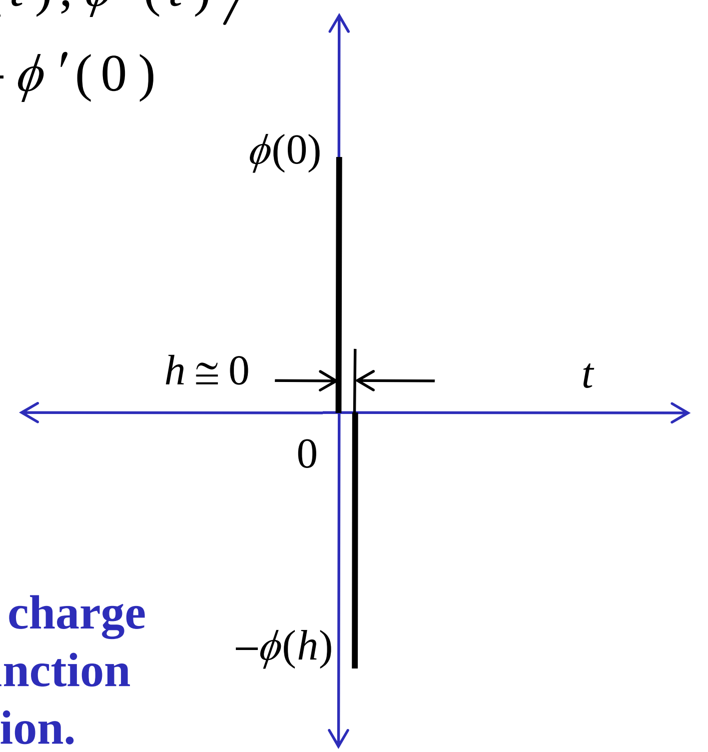
Derivative in distributional sense for (Dirac) delta function

$\phi(t) \in \mathcal{D}$ **the action of δ' on test function**

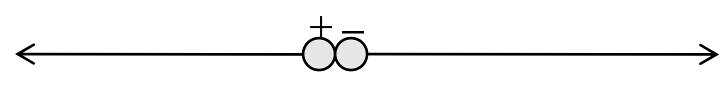
$$\langle \delta'(t), \phi(t) \rangle = -\langle \delta(t), \phi'(t) \rangle$$
$$= -\phi'(0)$$

Formally $\int_{-\infty}^{+\infty} \delta'(t)\phi(t)\,dt = -\phi'(0)$

$$-\phi'(0) = -\lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h}$$



This is called dipole or doublet distribution. Just as charge density for a point charge can be written as delta function the charge distribution of dipole by dipole distribution.



Integration in distributional sense for (Dirac) delta function

$\phi(t) \in \mathcal{D}$

$$\langle \delta, \phi \rangle = \int_{-\infty}^{+\infty} \delta(t)\phi(t)\mathrm{d}t = \phi(0)$$

$$\phi(t) = 1$$

$$\langle \delta, 1 \rangle = \int_{-\infty}^{+\infty} \delta(t)\mathrm{d}t = \phi(0) = 1$$

Fractional differ-integrations are

…in between one full integration and one full differentiation

…generalization of normal differentiation and integration

…having causal, $_{-\infty}D_t^{\alpha}$, $_{0}D_t^{\alpha}$ **non-causal structures** $_{t}D_{\infty}^{\alpha}$, $_{t}D_b^{\alpha}$

…having past information contents-thus have memory for causal case, and future in non-causal case.

…having Riemann-Liouvelli (RL), Caputo types-the most popular ones. These two derivative are related via value of function at the start point. Well if the function value at the start point is zero then RL-Caputo are same; or if the start point of differ-integration is at time immemorial then also these RL-Caputo are same. Else they are different.

…having fractional order initial states for initializing the Laplace of RL-differ-integral; integer order initial states for initializing the Laplace of Caputo type

…having Fractional order Local Derivatives are derived from basic RL derivative -called KG-LFD; useful for getting fractional derivative of non differentiable points, fractional Taylor expansions, fractional differential geometry, fractal study etc.

The fundamental theorem

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

$$x \in [a,b] \qquad f'(x) \in AC[a,b]$$

$$f(x) = D_x(I_x f(x)) = f(a) + \int_a^x f'(t)\,dt$$

$$f(x) = D(If(x)) = f(a) + I(Df(x))$$

$$f(x) = D_x^n(I_x^n f(x)) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt$$

$$f(x) = D^n I^n f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1)}(x-a)^k + \frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt$$

Differentiation of integration is-constant(s) at start point plus integration of differentiation
This is also RL, Caputo fractional derivative; a very important generalization.

$$D^n I^{n-\alpha} f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt$$

$$D_x^\alpha f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} + {}^C D_x^\alpha f(x)$$

$$0 < \alpha < 1 \qquad D_x^\alpha f(x) = \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + {}^C D_x^\alpha f(x)$$

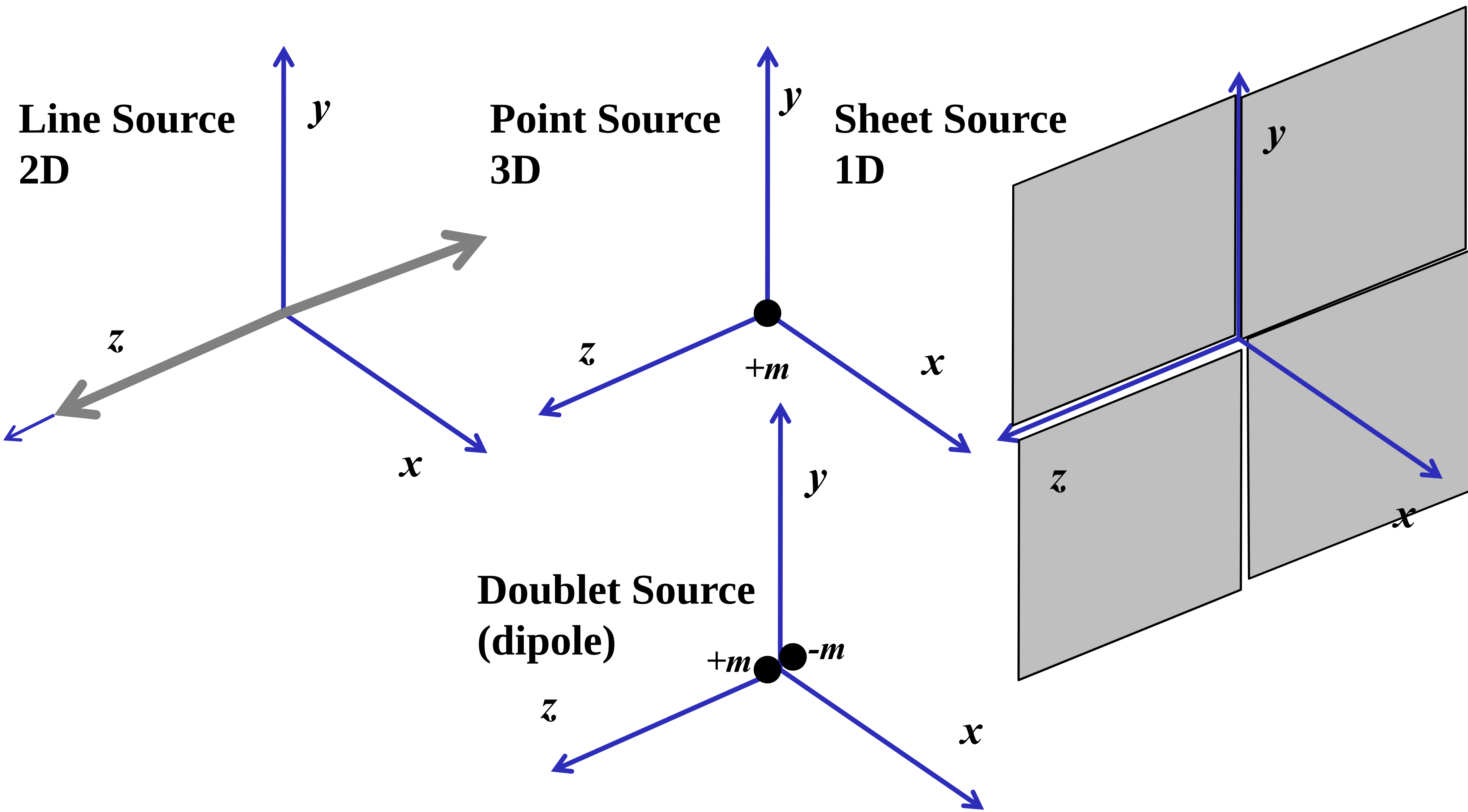
The order α of differ-integrals in D_t^{α} are

$\alpha \in \mathbb{Z}$	**Integer order differ-integrals**
$\alpha \in \mathbb{Q}$	**Fractional (rational) order differ-integrals**
$\alpha \in \mathbb{R}$	**Fractional (real) order differ-integrals**
$\alpha \sim n\beta\,; \quad n \in \mathbb{N}$	**Fractional (sequential) order differ-integrals**
$\alpha \in \mathbb{C}$	**Fractional (complex) order differ-integrals**
$\alpha \sim \sum_{k=0}^{N} q_k\,; \quad q \in \mathbb{R}$	**Fractional (distributed) order differ-integrals**
$\alpha \sim \int_{-\infty}^{+\infty} k(q)\,\mathrm{d}q$	**Fractional (continuous) order differ-integrals**

End of part-revision

Concept of non-canonical sources (mass, charge).

Canonical sources (mass, charge).



Use of canonical sources

To get potential functions

Due to point source 3D

$$\Phi = \iiint_V \frac{\rho}{r} \mathrm{d}V \qquad \rho \equiv \mathrm{m} / \mathrm{cm}^3$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Due to line source 2D

$$\Phi = \iiint_L \frac{\lambda}{r} \mathrm{d}L \qquad \lambda \equiv \mathrm{m} / \mathrm{cm}$$

$$r = \sqrt{x^2 + y^2}$$

Due to sheet source 1D

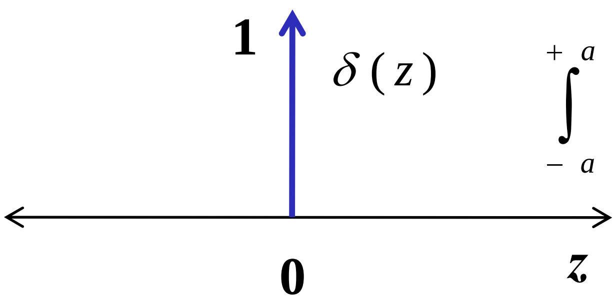
$$\Phi = \iint_S \frac{\sigma}{r} \mathrm{d}S \qquad \sigma \equiv \mathrm{m} / \mathrm{cm}^2$$

$$r = \sqrt{x^2} = x$$

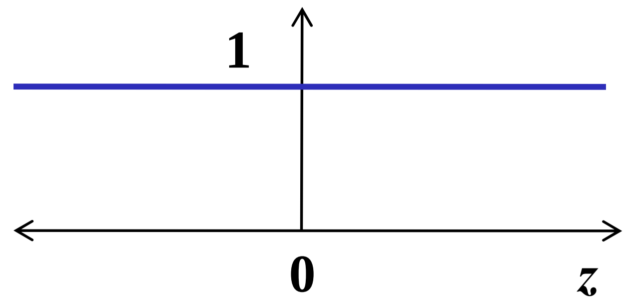
Integration of delta function

$$\delta(r) = \delta(x)\delta(y)\delta(z)$$

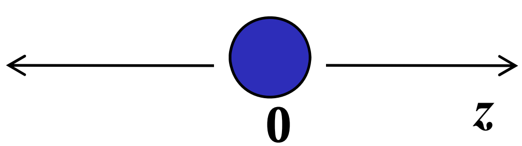
Integrate in *z*-direction



$$\int_{-a}^{+a} \delta(z)\,dz = 1$$



$$\int_{-a}^{+a} \delta(r)\,dz = \int_{-\infty}^{+\infty} \delta(x)\delta(y)\delta(z)\,dz = \delta(x)\delta(y)(1)$$

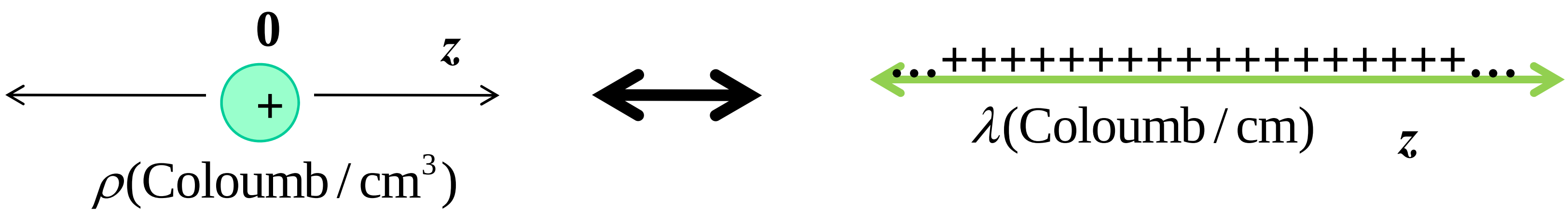


$\rho(\text{gm}/\text{cm}^3)$

$\lambda(\text{gm}/\text{cm})$

Integration of delta function is stretching!

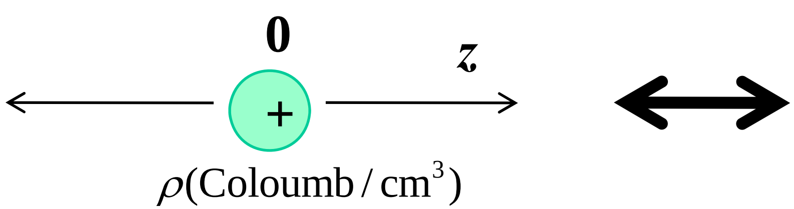
Gives impression that point source is pulled (stretched) in the *z* direction, to get uniform line source of mass charge etc…

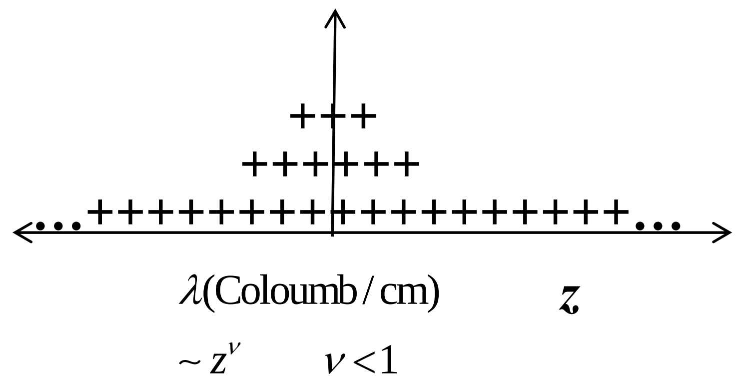


What happens in between-partially stretched point source?-a case of Fractional Integration

Fractional integration of delta function

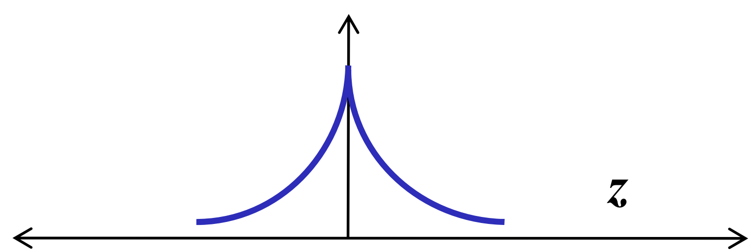
Partially stretched case we get in between Point and Line Source





$$_{-\infty}D_z^{-\alpha}\,\delta(z) = \frac{1}{\Gamma(\alpha)}\,z^{\alpha-1}$$

$$0 < \alpha < 1$$

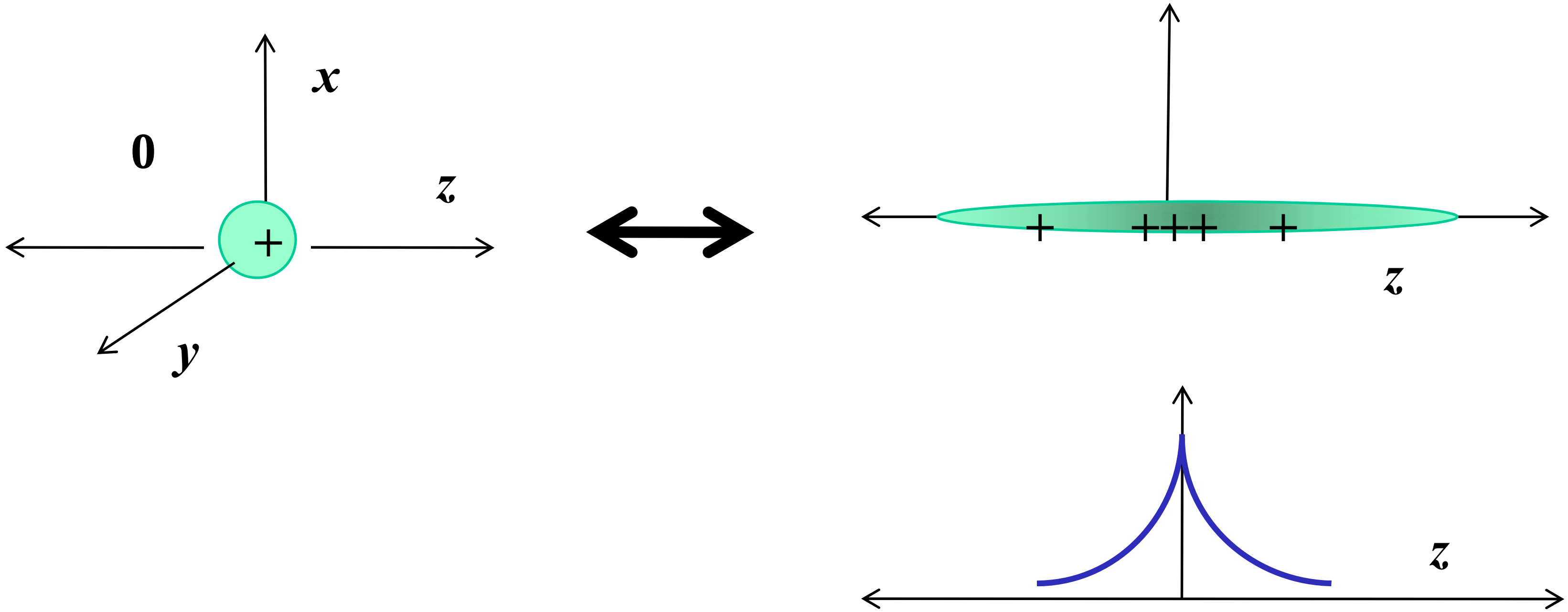


In between point and line source

$${}_{-\infty}D_z^{-\alpha}\delta(r) = {}_{-\infty}D_z^{-\alpha}\delta(x)\delta(y)\delta(z) = \delta(x)\delta(y)\frac{z^{\alpha-1}}{\Gamma(\alpha)}$$

$$\alpha \in \mathbb{R}^{+} \qquad 0 < \alpha < 1$$

$$S_\alpha(x,y,z) = \frac{1}{2}\left[{}_{-\infty}D_z^{-\alpha}\delta(x)\delta(y)\delta(z) + {}_{-\infty}D_{-z}^{-\alpha}\delta(x)\delta(y)\delta(z)\right] = \frac{\delta(x)\delta(y)\left|z\right|^{\alpha-1}}{2\Gamma(\alpha)}$$



Similarly stretching the line source in *y* direction-in between line and sheet source!

$$ {}_{-\infty}D_y^{-\beta}\delta(x)\delta(y) = \frac{\delta(x)y^{\beta-1}}{\Gamma(\alpha)} $$

$$ S_\beta(x,y) = \frac{1}{2}\left[{}_{-\infty}D_z^{-\beta}\delta(x)\delta(y) + {}_{-\infty}D_{-z}^{-\beta}\delta(x)\delta(y)\right] = \frac{\delta(x)\left|y\right|^{\alpha-1}}{2\Gamma(\alpha)} $$

$$ \beta \in \mathbb{R}^+ \qquad 0 < \beta < 1 $$

We obtain in between line and sheet source, that is between 2D and 1D case

Sources from canonical to non canonical

Point 3D source $$S_{3D} = \delta(r) = \delta(x)\delta(y)\delta(z)$$

In-between Point-Line (3D-2D) source $$S_{\alpha} = \frac{\delta(x)\delta(y)\left|z\right|^{\alpha-1}}{2\Gamma(\alpha)}; 0 < \alpha < 1$$

Line 2D Source $$S_{2D} = \delta(r) = \delta(x)\delta(y)$$

In-between Line-Sheet (2D-1D) source $$S_{\beta} = \frac{\delta(x)\left|y\right|^{\beta-1}}{2\Gamma(\beta)}; 0 < \beta < 1$$

Sheet 1D source $$S_{1D} = \delta(r) = \delta(x)$$

Waves from canonical sources

Green's function of the Helmholtz equation gives waves propagation

$$\nabla^2 G(\vec{r},\vec{r}_0;k) + k^2 G = -\delta(\vec{r}-\vec{r}_0)$$

For unbounded space i.e. with b.c. of vanishing field at infinity, and for constant *k* and sources at origin r_0=0

1D case giving plane wave for uniform plane sheet located at *y-z* plane

$$\delta_1(\vec{r}) = \delta(x) \qquad G_1(\vec{r},\vec{r}_0;k) = G_1(|x|;k) = i\frac{e^{ik|x|}}{2k}$$

2D case giving cylindrical wave for uniform line-source located along *z*- axis

$$\delta_2(\vec{r}) = \delta(x)\delta(y) \qquad G_2(\vec{r},\vec{r}_0;k) = G_1(\sqrt{x^2+y^2};k) = \frac{i}{4}H_0^{(1)}(k\sqrt{x^2+y^2})$$

3D case giving spherical wave for point-source located at origin

$$\delta_3(\vec{r}) = \delta(x)\delta(y)\delta(z)$$

$$G_3(\vec{r},\vec{r}_0;k) = G_3(\sqrt{x^2+y^2+z^2};k) = \frac{e^{ik\sqrt{x^2+y^2+z^2}}}{4\pi\sqrt{x^2+y^2+z^2}}$$

Henkel's function $H_0^{(1)}$

Some interpretation physically

Plane wave 1D

$$\left|G_1(|x|;k)\right| \underset{k|x|\gg 1}{\sim} \left(\frac{1}{2k}\right)$$

magnitude is independent of x

Cylindrical wave 2D

$$\left|G_2(\sqrt{x^2+y^2};k)\right| \underset{k\sqrt{x^2+y^2}\gg 1}{\sim} \frac{1}{4}\sqrt{\frac{2}{\pi k\sqrt{x^2+y^2}}} \propto (x^2+y^2)^{-1/4} \equiv \rho^{-1/2}$$

magnitude drops as square root of distance of observation from the line source

Spherical wave 3D

$$\left|G_3(\sqrt{x^2+y^2+z^2};k)\right| \underset{k\sqrt{x^2+y^2+z^2}\gg 1}{\sim} \frac{1}{4\pi}\frac{1}{\sqrt{x^2+y^2+z^2}} \equiv R^{-1}$$

magnitude drops as distance of observation from origin

Waves from non-canonical source

If we sit along *x* axis in the far zone *kx>>1* and measure *G*, we will be observing that for 1D delta function source the magnitude of *G* does not change with *x*, where as for 2D source the variation is $x^{-1/2}$.

Would it be possible to have solution to Helmholtz equation with different source where the magnitude of *G* drops slower than $x^{-1/2}$ but also not independent of *x* .

Could it drop as $x^{-(1-f)/2}; \quad 1 < f < 2$?

Obviating Fractional Waves!

Intermediate solutions as fractional waves due intermediate non canonical source

Intermediate 1-2 source is: $_{e}S_{f}(x,y)=\dfrac{\delta(x)\,|y|^{1-f}}{2\Gamma(2-f)};\quad 1<f<2$

Waves observation in *x-z* plane

$$_{e}G_{f}(x,y=0;k)=\frac{-i\Gamma[(f-1)/2]\cos(f\pi/2)}{4\sqrt{\pi}}\left(\frac{x}{2k}\right)^{(2-f)/2}H^{(1)}_{(f-2)/2}(kx),\quad x>0$$

$$\cong\frac{\Gamma[(f-1)/2]\cos(f\pi/2)}{4\pi\,2^{(1-f)/2}k^{(3-f)/2}}x^{(1-f)/2}$$

Waves observation outside *x-z* plane

$\rho\equiv\sqrt{x^2+y^2}\qquad \varphi=\tan^{-1}(y/x)\qquad k\rho\gg 1\qquad \varphi>0$

$$_{e}G_{f}(x,y;k)\cong\frac{-i}{4\pi}\cos(f\pi/2)(k\sin\varphi)^{f-2}\sqrt{\frac{2\pi}{k\rho}}e^{ik\rho-i\pi/4}+\frac{i}{4k^{2-f}\Gamma(2-f)}\frac{e^{ik|x|}}{(k|y|)^{f-1}}$$

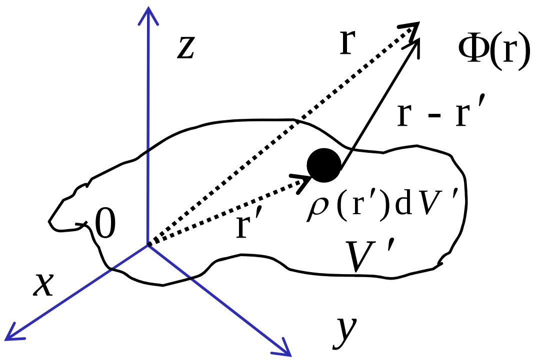
Cylindrical part varying as $\rho^{-1/2}$ plus non-uniform plane wave part Propagating in *x* but amplitude varying with $|y|^{1-f}$

Practicality

The idea of slowly varying localized wave carrying energy in space-time has been subject of interest to many researchers and solutions such as focus wave modes, localized wave transmission, electromagnetic missiles have been suggested in literatures. Here we find totally different approach to find solution of the Helmholtz equation with the distributed source defined as fractional integration of higher dimensional Dirac delta function.

So effectively 2D-1D Green's function of the Helmholtz equation are smoothly connected by varying the order of the fractional integration of 2D source term-and got the "Intermediate Waves" or "Fractional Waves"

Multi-pole expansion of fields



Scalar potential

$$\Phi(\mathrm{r}) = \int_{V'} \rho(\mathrm{r}') \frac{e^{ik|\mathrm{r}-\mathrm{r}'|}}{4\pi\varepsilon\,|\mathrm{r}-\mathrm{r}'|} \mathrm{d}V'; \quad k = \sqrt{\varepsilon\mu}$$

Taylor expansion in 3D for kernel

$$\frac{e^{ik|\mathrm{r}-\mathrm{r}'|}}{|\mathrm{r}-\mathrm{r}'|} = \sum_{0}^{\infty} \frac{1}{n!}(-\mathrm{r}'.\nabla)^n \frac{e^{ikr}}{r}; \quad r = |\mathrm{r}|$$

We get the potential

$$\Phi(\mathrm{r}) = \frac{1}{4\pi\varepsilon}\left(q\frac{e^{ikr}}{r} - \mathrm{p}\bullet\nabla\frac{e^{ikr}}{r} + \frac{1}{2!}\mathrm{Q}\bullet\nabla\nabla\frac{e^{ikr}}{r} + \ldots \right)$$

Where $q \equiv \int_{V'} \rho(\mathrm{r}')\mathrm{d}V'$ $\quad \mathrm{p} \equiv \int_{V'} \rho(\mathrm{r}')\mathrm{r}'\mathrm{d}V'$ $\quad \mathrm{Q} \equiv \int_{V'} \rho(\mathrm{r}')\mathrm{r}'\mathrm{r}'\mathrm{d}V'$

Thus source density at non origin is represented as sum of charge at origin, dipole at origin plus quadra-pole at origin….

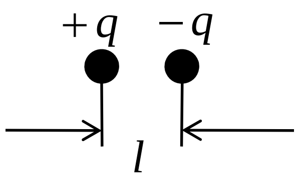
$$\rho(\mathrm{r}') = q\delta(\mathrm{r}) - \mathrm{p}.\nabla\delta(\mathrm{r}) + (1/2!)\mathrm{Q}.\nabla\nabla\delta(\mathrm{r}) + \ldots$$

Derivative of monopole is dipole

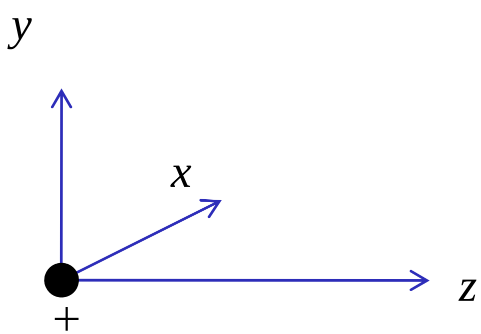
Charge distribution of point monopole is $\rho_1(\mathrm{r}) = q\delta(\mathrm{r})$

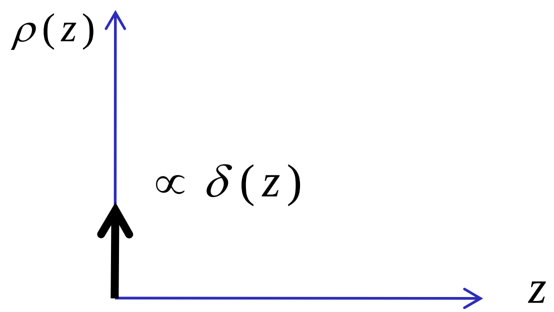
Charge distribution of point dipole is $\rho_2(\mathrm{r}) = -p.\nabla\,\delta(\mathrm{r})$

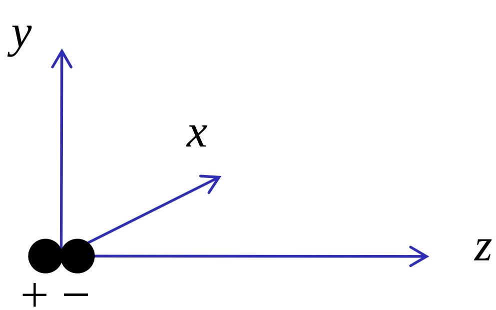
Dipole

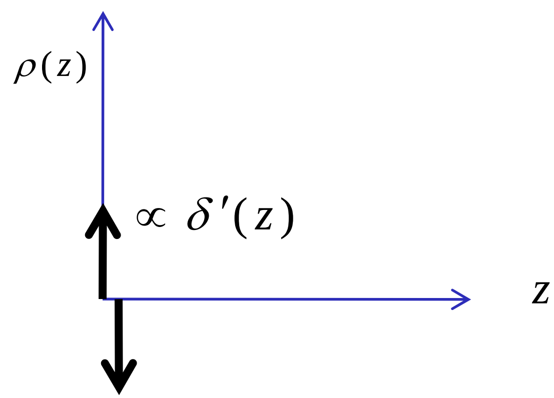


Dipole moment $p = ql\,; \quad l \to 0$



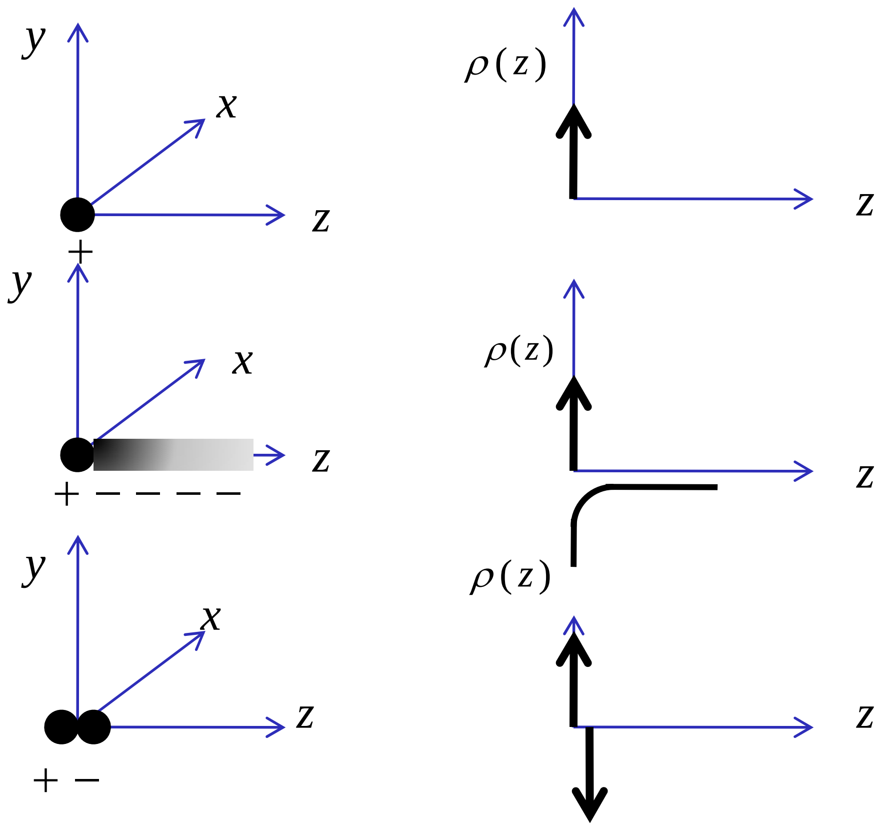




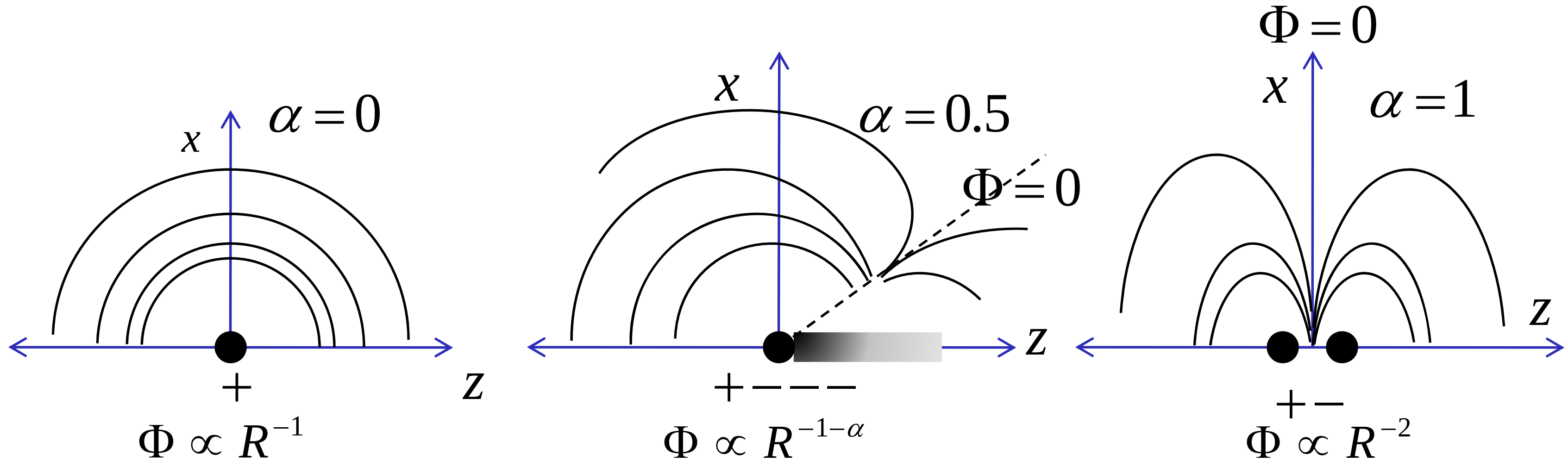


Fractional derivative of monopole to get fractional poles!

$$\rho_{2^\alpha,z}(\mathrm{r}) = q l^\alpha \; {}_{-\infty}D_z^\alpha \delta(\mathrm{r}) = q l^\alpha \; {}_{-\infty}D_z^\alpha \delta(x)\delta(y)\delta(z) = q l^\alpha \delta(x)\delta(y)\frac{z^{-1-\alpha}}{\Gamma(-\alpha)}$$



Potential of fractional pole



Generalized charge density mapping monopole to dipole

$$\rho_{2^\alpha,z}(\mathrm{r}) = ql^\alpha \, {}_{-\infty}D_z^\alpha \delta(\mathrm{r}) = ql^\alpha \, {}_{-\infty}D_z^\alpha \delta(x)\delta(y)\delta(z) = ql^\alpha \delta(x)\delta(y)\frac{z^{-1-\alpha}}{\Gamma(-\alpha)}$$

Respective potential functions are

$$R = \sqrt{x^2 + z^2} \qquad \tan\theta = x/z$$

$$\alpha = 0 \qquad \Phi_{2^0,z}(x,y,z) = \frac{q}{4\pi\varepsilon R} \qquad \text{monopole}$$

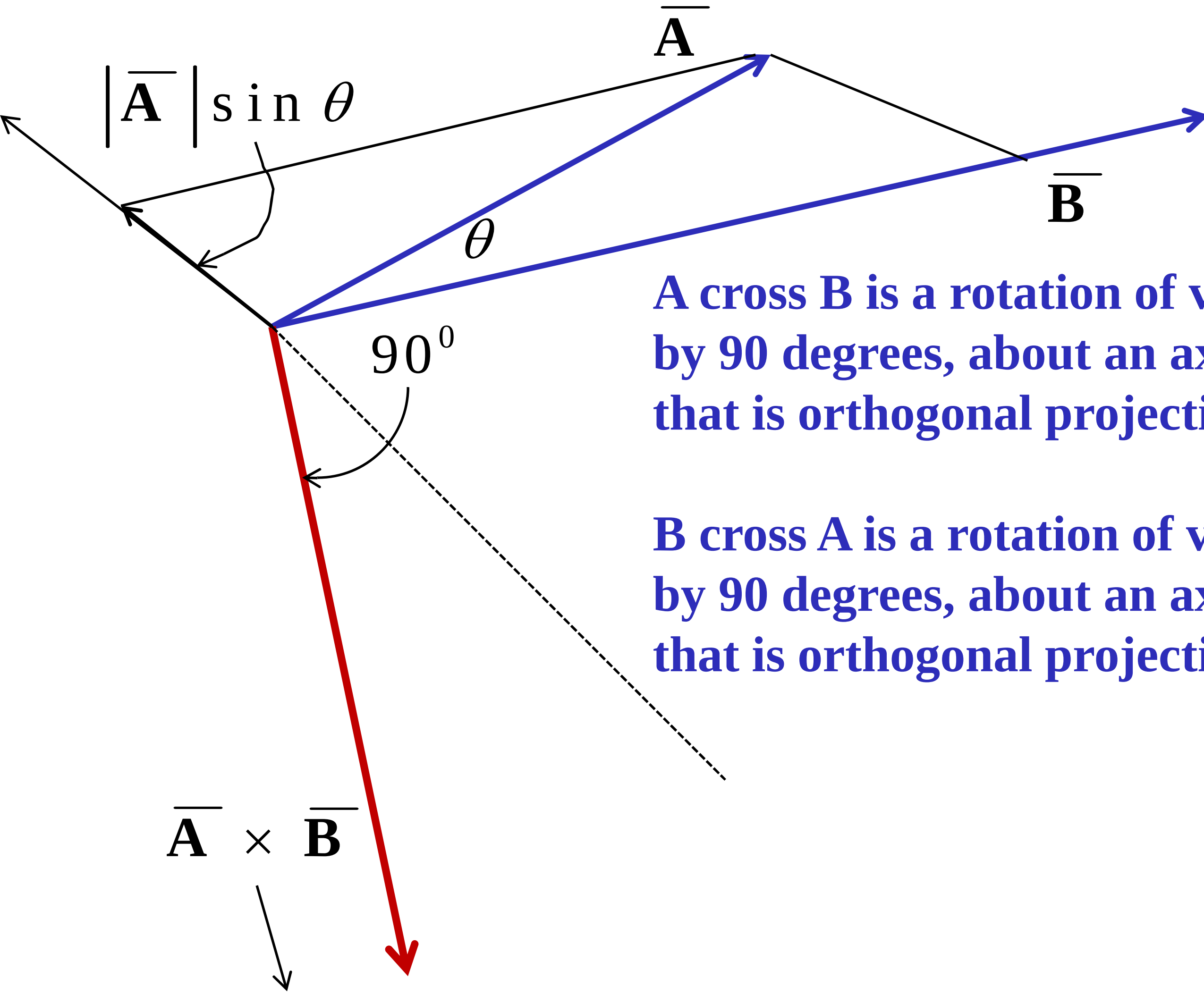
$$\alpha = 1 \qquad \Phi_{2^1,z}(x,y,z) = \frac{ql}{4\pi\varepsilon R^2} P_1(-\cos\theta) \qquad \text{dipole}$$

$$0 < \alpha < 1 \qquad \Phi_{2^\alpha,z}(x,y,z) = \frac{ql^\alpha \Gamma(\alpha+1)}{4\pi\varepsilon R^{1+\alpha}} P_\alpha(-\cos\theta) \qquad \text{fractional pole}$$

End of part-1

Fractional Cross Product & Fractional Curl.

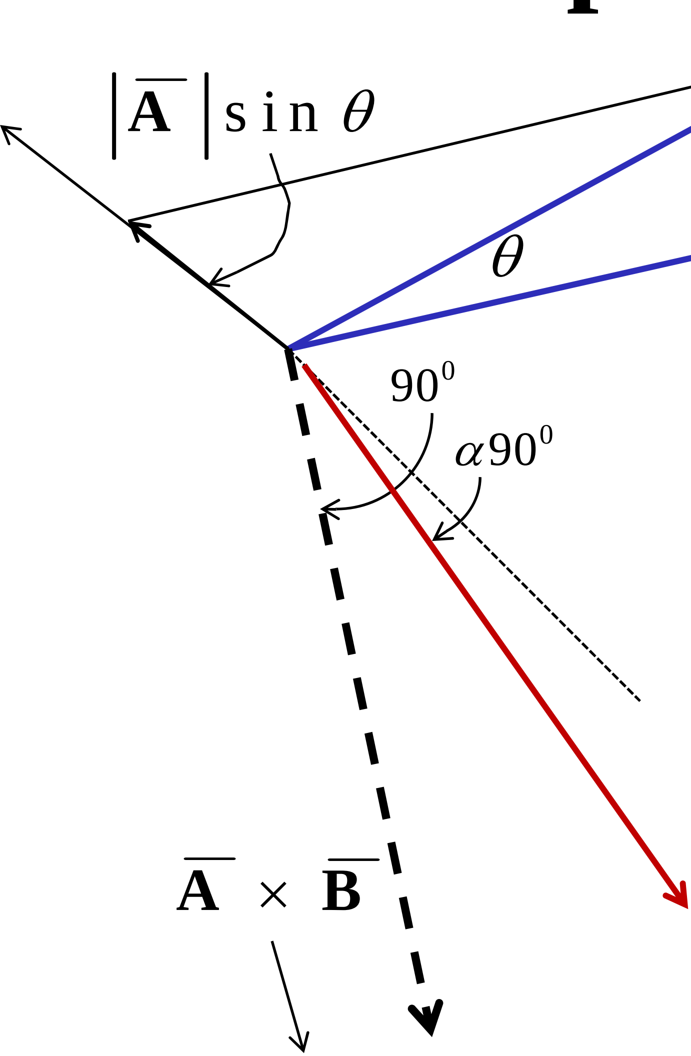
Cross-product of two vectors



A cross B is a rotation of vector B anticlockwise by 90 degrees, about an axis given by $\mathbf{A}\ \sin\theta$ that is orthogonal projection component of A on B

B cross A is a rotation of vector A anticlockwise by 90 degrees, about an axis given by $\mathbf{B}\ \sin\theta$ that is orthogonal projection component B on A.

Fractional cross-product of two vectors



A cross B is a rotation of vector B anticlockwise by fraction of 90 degrees, about an axis given by $\mathbf{A}\ \sin\theta$ **that is orthogonal projection component of A on B**

……… A Fractional Cross Product!!

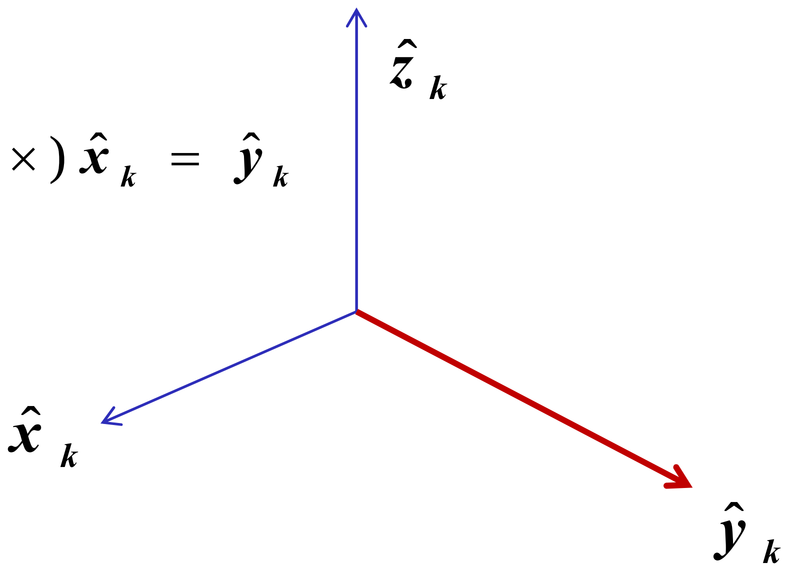
We cannot have rotation in quantum of 90 degrees always!

$(\overline{\mathbf{A}}\times)^{\alpha}\ \overline{\mathbf{B}}$ **Fractionally rotated by** $\alpha 90^0$

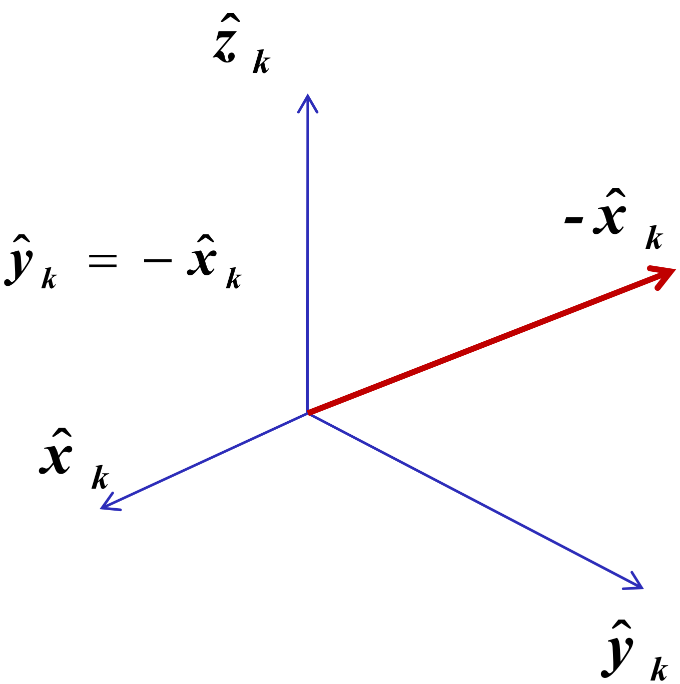
$0 < \alpha < 1$

Cross product of orthogonal unit vectors

$$(\hat{z}_k \times)\,\hat{x}_k = \hat{y}_k$$



$$(\hat{z}_k \times)\,\hat{y}_k = -\hat{x}_k$$

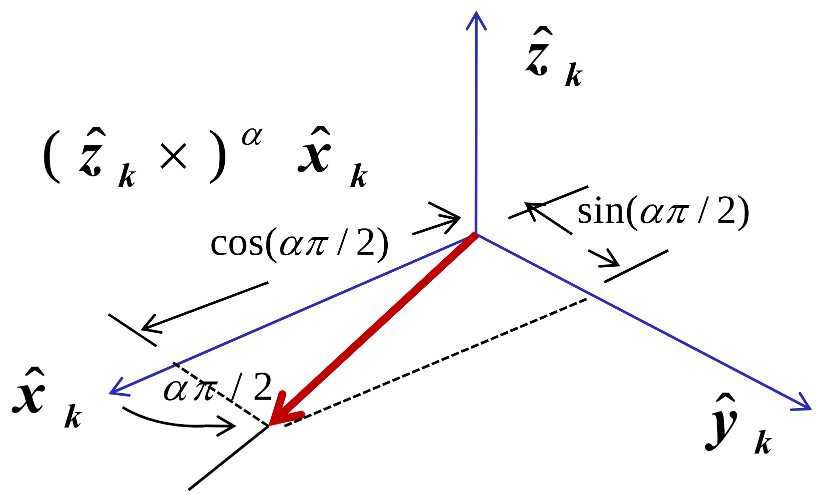


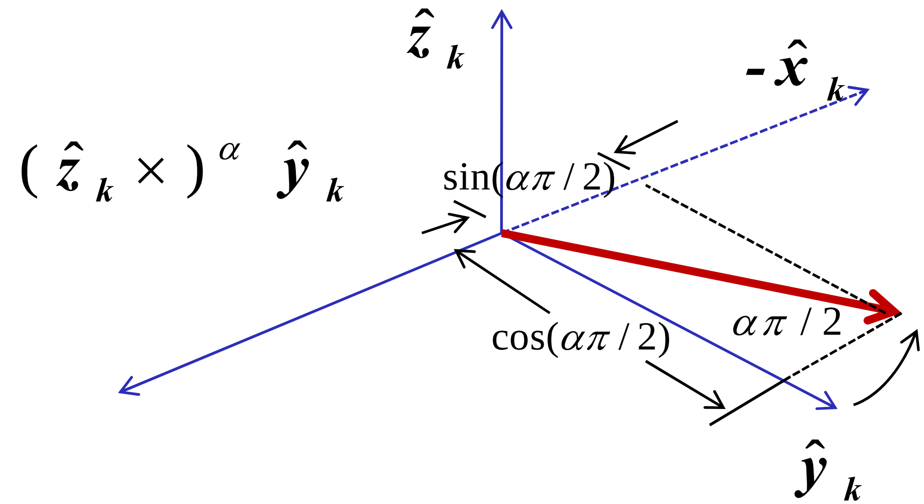
$$(\hat{z}_k \times)\hat{x}_k = \hat{y}_k \qquad (\hat{z}_k \times)\hat{y}_k = -\hat{x}_k \qquad (\hat{z}_k \times)\hat{z}_k = 0$$

$$|\hat{x}_k| = |\hat{y}_k| = |\hat{z}_k| = 1$$

Anticlockwise rotation of $\hat{x}_k$ about $\hat{z}_k$ **Anticlockwise rotation of $\hat{y}_k$ about $\hat{z}_k$**

Fractional cross product of orthogonal unit vectors





Anticlockwise rotation of $\hat{x}_k$ about $\hat{z}_k$

$$(\hat{z}_k \times)^{\alpha}\, \hat{x}_k = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k$$

Anticlockwise rotation of $\hat{y}_k$ about $\hat{z}_k$

$$(\hat{z}_k \times)^{\alpha}\, \hat{y}_k = \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k$$

Several combinations of fractional cross products

The fractional cross product obtained

$$(\hat{z}_k \times)^\alpha \, \hat{x}_k = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k$$

$$(\hat{z}_k \times)^\alpha \, \hat{y}_k = \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k$$

$$(\hat{z}_k \times)^\alpha \, \hat{z}_k = 0$$

The other combinations are similarly

$$(\hat{x}_k \times)^\alpha \, \hat{y}_k = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{z}_k$$

$$(\hat{x}_k \times)^\alpha \, \hat{z}_k = \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{z}_k$$

$$(\hat{y}_k \times)^\alpha \, \hat{z}_k = \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{z}_k$$

$$(\hat{y}_k \times)^\alpha \, \hat{x}_k = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{z}_k$$

Fractional cross product of any vector

We learnt that the fractional cross product is rotation by a fractional angle $\alpha\pi/2$ of the vector on which the cross product operation is carried on, and the rotation about the axis of the vector which is doing the cross-product operation. The rotation operation being a linear operation we can use superposition to get the general fractional cross product's expression. Let us take an example of a vector

$$\bar{\mathbf{A}} = (1)\hat{\boldsymbol{x}}_k + (1)\hat{\boldsymbol{y}}_k + (1)\hat{\boldsymbol{z}}_k = (1,1,1) \qquad (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\bar{\mathbf{A}} = (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\hat{\boldsymbol{x}}_k + (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,y_k + (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\hat{\boldsymbol{z}}_k$$

By superposition of obtained earlier result of $(\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\hat{\boldsymbol{x}}_k \quad (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\hat{\boldsymbol{y}}_k \quad (\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\hat{\boldsymbol{z}}_k$

$$(\hat{\boldsymbol{z}}_k \times)^{\alpha}\,\bar{\mathbf{A}} = \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_k + \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_k + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_k + (0)\hat{\boldsymbol{z}}_k$$

$$= \left[\cos\left(\frac{\alpha\pi}{2}\right) - \sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{x}}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right) + \cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{\boldsymbol{y}}_k + (0)\hat{\boldsymbol{z}}_k$$

Checking the result

$$\bar{\mathbf{A}} = (1)\hat{x}_k + (1)\hat{y}_k + (1)\hat{z}_k = (1,1,1)$$

$$(\hat{z}_k\times)^{\alpha}\bar{\mathbf{A}} = \left[\cos\left(\frac{\alpha\pi}{2}\right) - \sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right) + \cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k + (0)\hat{z}_k$$

Put above $\alpha = 1$ **we get** $(\hat{z}_k\times)^{1}\bar{\mathbf{A}} = (-1,1,0)$

which is same as normal cross product , demonstrated below

$$(\hat{z}_k\times)\bar{\mathbf{A}} = (-1)\hat{x}_k + (1)\hat{y}_k + (0)\hat{z}_k = (-1,1,0) = \det\begin{bmatrix}\hat{x}_k & \hat{y}_k & \hat{z}_k \\ 0 & 0 & 1 \\ 1 & 1 & 1\end{bmatrix}$$

Interestingly the expression is not returning identity operator when we put $\alpha = 0$

Modify as

$$(\hat{z}_k\times)^{\alpha}\bar{\mathbf{A}} = \left[\cos\left(\frac{\alpha\pi}{2}\right) - \sin\left(\frac{\alpha\pi}{2}\right)\right]\hat{x}_k + \left[\sin\left(\frac{\alpha\pi}{2}\right) + \cos\left(\frac{\alpha\pi}{2}\right)\right]\hat{y}_k + (\delta_{\alpha})\hat{z}_k$$

with $\delta_{\alpha} = 1$ for $\alpha = 0$ else for $\alpha \neq 0$ $\delta_{\alpha} = 0$

Matrix representation for fractional cross product

For $\alpha \neq 0$

$$\bar{\mathbf{A}} = (1,1,1) \qquad (\hat{z}_k \times)^{\alpha} \bar{\mathbf{A}} = \begin{bmatrix} \hat{x}_k \left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{x}_k \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{x}_k[0] \\ \hat{y}_k \left[\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{y}_k \left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{y}_k[0] \\ \hat{z}_k[0] & \hat{z}_k[0] & \hat{z}_k[0] \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\bar{\mathbf{F}}_k = (F_{xk}, F_{yk}, F_{zk}) = F_{xk}\hat{x}_k + F_{yk}\hat{y}_k + F_{zk}\hat{z}_k$ **for** $\alpha \neq 0$

$$(\hat{z}_k \times)^{\alpha} \bar{\mathbf{F}}_k = \begin{bmatrix} \hat{x}_k \left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{x}_k \left[-\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{x}_k[0] \\ \hat{y}_k \left[\sin\left(\frac{\alpha\pi}{2}\right)\right] & \hat{y}_k \left[\cos\left(\frac{\alpha\pi}{2}\right)\right] & \hat{y}_k[0] \\ \hat{z}_k[0] & \hat{z}_k[0] & \hat{z}_k[0] \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix}$$

Superposing other combination general expression for fractional cross product

For $\alpha \neq 0$

$$((x_k, y_k, z_k)\times)^{\alpha}(F_{xk}, F_{yk}, F_{zk})$$

$$= \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0] + \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[F_{yk}\cos\left(\frac{\alpha\pi}{2}\right) - F_{zk}\sin\left(\frac{\alpha\pi}{2}\right)\right] + \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[F_{yk}\sin\left(\frac{\alpha\pi}{2}\right) + F_{zk}\cos\left(\frac{\alpha\pi}{2}\right)\right]$$

$$= \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[F_{xk}\cos\left(\frac{\alpha\pi}{2}\right) + F_{zk}\sin\left(\frac{\alpha\pi}{2}\right)\right] + \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0] + \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[-F_{xk}\sin\left(\frac{\alpha\pi}{2}\right) + F_{zk}\cos\left(\frac{\alpha\pi}{2}\right)\right]$$

$$= \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[F_{xk}\cos\left(\frac{\alpha\pi}{2}\right) - F_{yk}\sin\left(\frac{\alpha\pi}{2}\right)\right] + \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[F_{xk}\sin\left(\frac{\alpha\pi}{2}\right) + F_{yk}\cos\left(\frac{\alpha\pi}{2}\right)\right] + \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0]$$

$$((\hat{x}_k, \hat{y}_k, \hat{z}_k)\times)^{\alpha}(F_{xk}, F_{yk}, F_{zk})$$

$$= \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[2F_{xk}\cos\left(\frac{\alpha\pi}{2}\right) + (F_{zk} - F_{yk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

$$+ \hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[2F_{yk}\cos\left(\frac{\alpha\pi}{2}\right) + (F_{xk} - F_{zk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

$$+ \hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[2F_{zk}\cos\left(\frac{\alpha\pi}{2}\right) + (F_{yk} - F_{xk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

Verification

$$((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)^{\alpha}(F_{xk},F_{yk},F_{zk})$$

$$= \hat{\boldsymbol{x}}_{\boldsymbol{k}}\left[2F_{xk}\cos\left(\frac{\alpha\pi}{2}\right)+(F_{zk}-F_{yk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

$$+\hat{\boldsymbol{y}}_{\boldsymbol{k}}\left[2F_{yk}\cos\left(\frac{\alpha\pi}{2}\right)+(F_{xk}-F_{zk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

$$+\hat{\boldsymbol{z}}_{\boldsymbol{k}}\left[2F_{zk}\cos\left(\frac{\alpha\pi}{2}\right)+(F_{yk}-F_{xk})\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

Putting $\alpha = 1$

$$((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)\overline{\mathbf{F}}_k = \hat{\boldsymbol{x}}_{\boldsymbol{k}}(F_{zk}-F_{yk})+\hat{\boldsymbol{y}}_{\boldsymbol{k}}(F_{xk}-F_{zk})+\hat{\boldsymbol{z}}_{\boldsymbol{k}}(F_{yk}-F_{xk})$$

$$((\hat{x}_k,\hat{y}_k,\hat{z}_k)\times)(F_{xk},F_{yk},F_{zk}) =$$

$$\det\begin{bmatrix}\hat{x}_k & \hat{y}_k & \hat{z}_k\\ 1 & 1 & 1\\ F_{xk} & F_{yk} & F_{zk}\end{bmatrix} = \hat{\boldsymbol{x}}_{\boldsymbol{k}}(F_{zk}-F_{yk})+\hat{\boldsymbol{y}}_{\boldsymbol{k}}(F_{xk}-F_{zk})+\hat{\boldsymbol{z}}_{\boldsymbol{k}}(F_{yk}-F_{xk})$$

Which is normal cross product operation

Fractional cross product to fractional curl

A Fourier transform of fractional derivative of f (..)

$$\mathcal{F}\left\{{}_{-\infty}D_x^{\alpha} f(t)\right\} = (\mathrm{i}\omega)^{\alpha}\mathcal{F}\left\{f(t)\right\} \qquad {}_{-\infty}D_x^{\alpha} f(t) = \mathcal{F}^{-1}\left\{(\mathrm{i}\omega)^{\alpha}\mathcal{F}\{f(t)\}\right\}$$

$$\mathcal{F}\left\{{}_{-\infty}D_x^{\alpha} f(x)\right\} = (\mathrm{ik})^{\alpha}\mathcal{F}\left\{f(x)\right\} \qquad {}_{-\infty}D_x^{\alpha} f(x) = \mathcal{F}^{-1}\left\{(\mathrm{ik})^{\alpha}\mathcal{F}\{f(x)\}\right\}$$

Fourier transforming vector

$$\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}}) \equiv (f_x, f_y, f_z) = f_x\hat{\boldsymbol{x}} + f_y\hat{\boldsymbol{y}} + f_z\hat{\boldsymbol{z}} \qquad \overline{\mathrm{k}} \equiv (k_x, k_y, k_z) = k_x\hat{\boldsymbol{x}}_{\boldsymbol{k}} + k_y\hat{\boldsymbol{y}}_{\boldsymbol{k}} + k_z\hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

$$\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}}) = \mathcal{F}\left\{\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\}$$

$$\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}}) = \mathcal{F}\left\{\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}\hat{\boldsymbol{x}}\,\mathrm{d}\hat{\boldsymbol{y}}\,\mathrm{d}\hat{\boldsymbol{z}}\{\overline{\mathbf{F}}(\hat{x},\hat{y},\hat{z})e^{-\mathrm{i}k_x\hat{\boldsymbol{x}}-\mathrm{i}k_y\hat{\boldsymbol{y}}-\mathrm{i}k_z\hat{\boldsymbol{z}}}\}$$

$$\mathcal{F}\left\{\nabla\times\overline{\mathbf{F}}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})\right\} = (\mathrm{i}\overline{\mathrm{k}})\times\overline{\mathbf{F}}_k(\hat{\boldsymbol{x}}_{\boldsymbol{k}},\hat{\boldsymbol{y}}_{\boldsymbol{k}},\hat{\boldsymbol{z}}_{\boldsymbol{k}})$$

So in order to fractionalize the curl operation what we need is obtain fractional cross product $(\mathrm{i}\overline{\mathrm{k}}\times)^{\alpha}\overline{\mathbf{F}}_k$ **in** $\overline{\mathrm{k}}$ **domain, then Fourier invert the result back to domain** $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ **to get** $(\nabla\times)^{\alpha}\overline{\mathbf{F}}$

Fractional curl for vector in k domain

Let us have a vector field which has variation in only $\hat{z}$ direction can be described as Fourier Transformed vector field as

$$\overline{\mathbf{F}}_k(z) = F_x \hat{\boldsymbol{x}}_{\boldsymbol{k}} + F_y \hat{\boldsymbol{y}}_{\boldsymbol{k}} + F_z \hat{\boldsymbol{z}}_{\boldsymbol{k}}$$

We have to find gradient in z-direction, we need to first take fractional cross product of the vector field with $\hat{\boldsymbol{z}}_{\boldsymbol{k}}$ axis, or find $(\boldsymbol{z}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k$

We know $(\boldsymbol{z}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k$ and thus multiply simply by $(\mathrm{i}k)^{\alpha}$ to get:

$$(\mathrm{i}k\hat{\boldsymbol{z}}_{\boldsymbol{k}}\times)^{\alpha}\overline{\mathbf{F}}_k = \begin{bmatrix} \hat{\boldsymbol{x}}_k\left[\cos\left(\frac{\alpha\pi}{2}\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{x}}_k\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{x}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} \\ \hat{\boldsymbol{y}}_k\left[\sin\left(\frac{\alpha\pi}{2}\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{y}}_k\left[\cos\left(\frac{\alpha\pi}{2}\right)\right](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{y}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} \\ \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} & \hat{\boldsymbol{z}}_{\boldsymbol{k}}[0](\mathrm{i}k)^{\alpha} \end{bmatrix} \begin{bmatrix} F_{xk} \\ F_{yk} \\ F_{zk} \end{bmatrix} \qquad \alpha \neq 0$$

Fractional curl for vector in (x, y, z) domain

On Fourier inverting the k-domain result we get:

$$(\nabla_z \hat{z}\times)^{\alpha}\overline{\mathbf{F}} = \begin{bmatrix} \hat{x}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{x}\left[-\sin\left(\frac{\alpha\pi}{2}\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{x}[0] \\ \hat{y}\left[\sin\left(\frac{\alpha\pi}{2}\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{y}\left[\cos\left(\frac{\alpha\pi}{2}\right)\right]({}_{-\infty}D_z^{\alpha}) & \hat{y}[0] \\ \hat{z}[0] & \hat{z}[0] & \hat{z}[0] \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad \alpha \neq 0$$

$$\overline{\mathbf{F}} \equiv \hat{x} f_x + \hat{y} f_y + \hat{z} f_z$$

Putting $\alpha = 1$

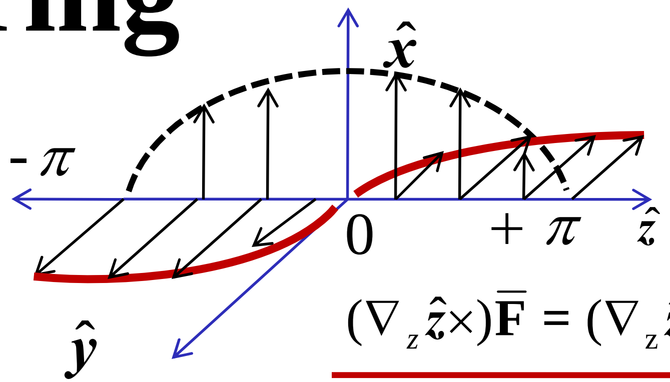
$$(\nabla_z \hat{z}\times)\overline{\mathbf{F}} = \begin{bmatrix} 0 & \hat{x}(-{}_{-\infty}D_z^1) & 0 \\ \hat{y}({}_{-\infty}D_z^1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \hat{x}(-{}_{-\infty}D_z^1 f_y) + \hat{y}({}_{-\infty}D_z^1 f_x)$$

$$(\nabla_z \hat{z}\times)\overline{\mathbf{F}} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & {}_{-\infty}D_z^1 \\ f_x & f_y & f_z \end{bmatrix} = \hat{x}(-{}_{-\infty}D_z^1 f_y) + \hat{y}({}_{-\infty}D_z^1 f_x)$$

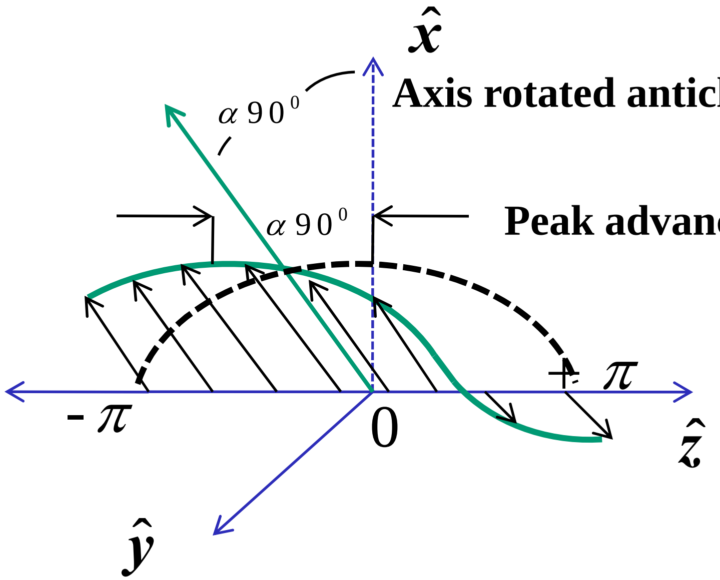
Which is normal curl operation

Fractional curl of vibrating string

$$\overline{\mathbf{F}}(z) = \hat{x}(\cos z) + \hat{y}(0) + \hat{z}(0)$$



$$(\nabla_z \hat{z}\times)\overline{\mathbf{F}} = (\nabla_z \hat{z}\times)\hat{x}\cos z = \hat{y}(\sin z) = \hat{y}\cos(z + [\pi/2])$$



$$(\nabla_z \hat{z}\times)^{\alpha}\overline{\mathbf{F}} = \left[\hat{x}\cos\left(\frac{\alpha\pi}{2}\right) + \hat{y}\sin\left(\frac{\alpha\pi}{2}\right)\right]\{{}_{-\infty}D_z^{\alpha} f_x(z)\} = \left[\hat{x}\cos\left(\frac{\alpha\pi}{2}\right) + \hat{y}\sin\left(\frac{\alpha\pi}{2}\right)\right]\cos\left(z + \frac{\alpha\pi}{2}\right)$$

Other components of fractional curl

$$(\nabla_z \times)^\alpha \overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left[\cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_z^\alpha f_x) - \sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_z^\alpha f_y)\right] + \hat{\boldsymbol{y}}\left[\sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_z^\alpha f_x) + \cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_z^\alpha f_y)\right] + \hat{\boldsymbol{z}}(0)({}_{-\infty}D_z^\alpha f_z)$$

$$(\nabla_y \times)^\alpha \overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left[\cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_y^\alpha f_x) + \sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_y^\alpha f_z)\right] + \hat{\boldsymbol{y}}(0)({}_{-\infty}D_y^\alpha f_y) + \hat{\boldsymbol{z}}\left[-\sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_y^\alpha f_x) + \cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_y^\alpha f_z)\right]$$

$$(\nabla_x \times)^\alpha \overline{\mathbf{F}} = \hat{\boldsymbol{x}}(0)({}_{-\infty}D_x^\alpha f_x) + \hat{\boldsymbol{y}}\left[\cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_x^\alpha f_y) - \sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_x^\alpha f_z)\right] + \hat{\boldsymbol{z}}\left[\sin\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_x^\alpha f_y) + \cos\left(\frac{\alpha\pi}{2}\right)({}_{-\infty}D_x^\alpha f_z)\right]$$

$$\overline{\mathbf{F}} \equiv \hat{\boldsymbol{x}} f_x + \hat{\boldsymbol{y}} f_y + \hat{\boldsymbol{z}} f_z \qquad \nabla \equiv \hat{\boldsymbol{x}}\nabla_x + \hat{\boldsymbol{y}}\nabla_y + \hat{\boldsymbol{z}}\nabla_z$$

Complete expression of fractional curl

Adding all components

$$(\nabla\times)^{\alpha}\,\overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left[\left(\frac{\partial^{\alpha} f_x}{\partial y^{\alpha}}+\frac{\partial^{\alpha} f_x}{\partial z^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_z}{\partial y^{\alpha}}-\frac{\partial^{\alpha} f_y}{\partial z^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right]$$
$$+\hat{\boldsymbol{y}}\left[\left(\frac{\partial^{\alpha} f_y}{\partial x^{\alpha}}+\frac{\partial^{\alpha} f_y}{\partial z^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_x}{\partial z^{\alpha}}-\frac{\partial^{\alpha} f_z}{\partial x^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right]$$
$$+\hat{\boldsymbol{z}}\left[\left(\frac{\partial^{\alpha} f_z}{\partial x^{\alpha}}+\frac{\partial^{\alpha} f_z}{\partial y^{\alpha}}\right)\cos\left(\frac{\alpha\pi}{2}\right)+\left(\frac{\partial^{\alpha} f_y}{\partial x^{\alpha}}-\frac{\partial^{\alpha} f_x}{\partial y^{\alpha}}\right)\sin\left(\frac{\alpha\pi}{2}\right)\right]$$

Putting $\alpha = 1$

$$(\nabla\times)\overline{\mathbf{F}} = \hat{\boldsymbol{x}}\left(\frac{\partial f_z}{\partial y}-\frac{\partial f_y}{\partial z}\right)+\hat{\boldsymbol{y}}\left(\frac{\partial f_x}{\partial z}-\frac{\partial f_z}{\partial x}\right)+\hat{\boldsymbol{z}}\left(\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)$$

$$(\nabla\times)\overline{\mathbf{F}} = \begin{bmatrix} 0 & -\hat{\boldsymbol{x}}\frac{\partial}{\partial z} & \hat{\boldsymbol{x}}\frac{\partial}{\partial y} \\ \hat{\boldsymbol{y}}\frac{\partial}{\partial z} & 0 & -\hat{\boldsymbol{y}}\frac{\partial}{\partial x} \\ -\hat{\boldsymbol{z}}\frac{\partial}{\partial y} & \hat{\boldsymbol{z}}\frac{\partial}{\partial x} & 0 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{bmatrix}$$

That is normal curl

Complete expression of fractional curl-compact form

$\alpha \neq 0$

$$(\nabla\times)^{\alpha}\,\overline{\mathbf{F}} = \left[\sin\left(\frac{\alpha\pi}{2}\right)\right]\left(\det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial^{\alpha}}{\partial x^{\alpha}} & \frac{\partial^{\alpha}}{\partial y^{\alpha}} & \frac{\partial^{\alpha}}{\partial z^{\alpha}} \\ f_x & f_y & f_z \end{bmatrix}\right) + \left[\cos\left(\frac{\alpha\pi}{2}\right)\right]\left(\begin{bmatrix} \nabla_{yz}^{\alpha} & 0 & 0 \\ 0 & \nabla_{xz}^{\alpha} & 0 \\ 0 & 0 & \nabla_{xy}^{\alpha} \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{x}} f_x \\ \hat{\boldsymbol{y}} f_y \\ \hat{\boldsymbol{z}} f_z \end{bmatrix}\right)$$

where $\nabla_{yz}^{\alpha} \equiv \frac{\partial^{\alpha}}{\partial y^{\alpha}} + \frac{\partial^{\alpha}}{\partial z^{\alpha}}$ $\quad \nabla_{xz}^{\alpha} \equiv \frac{\partial^{\alpha}}{\partial x^{\alpha}} + \frac{\partial^{\alpha}}{\partial z^{\alpha}}$ $\quad \nabla_{xy}^{\alpha} \equiv \frac{\partial^{\alpha}}{\partial x^{\alpha}} + \frac{\partial^{\alpha}}{\partial y^{\alpha}}$

Example using expression of fractional curl-compact form

A vector field of travelling wave pointed *x*, travelling in *z*

$$\overline{\mathbf{F}} = \hat{\boldsymbol{x}}(Ae^{ik_0 z}) \qquad f_x = Ae^{ik_0 z} \qquad f_y = 0 \qquad f_z = 0$$

For this field $\partial^\alpha F_x / \partial x^\alpha = \partial^\alpha F_x / \partial y^\alpha = 0$ **and thus** $\nabla^\alpha_{xy} \overline{\mathbf{F}} = 0$

For this field $\nabla^\alpha_{yz} \equiv \partial^\alpha / \partial z^\alpha$ **and** $\nabla^\alpha_{xz} \equiv \partial^\alpha / \partial z^\alpha$

So we use fractional curl expression to write

$$(\nabla\times)^\alpha \overline{\mathbf{F}} = \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \frac{\partial^\alpha}{\partial z^\alpha} \\ f_x & 0 & 0 \end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix} \frac{\partial^\alpha}{\partial z^\alpha} & 0 & 0 \\ 0 & \frac{\partial^\alpha}{\partial z^\alpha} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{x}} f_x \\ 0 \\ 0 \end{bmatrix}$$

$$= \hat{\boldsymbol{y}}\sin\left(\frac{\alpha\pi}{2}\right)\left[\frac{\partial^\alpha}{\partial z^\alpha} f_x\right] + \hat{\boldsymbol{x}}\cos\left(\frac{\alpha\pi}{2}\right)\left[\frac{\partial^\alpha}{\partial z^\alpha} f_x\right]$$

Using (Liouvelli) Fractional Derivative of exponential function as

$${}_{-\infty}D^\alpha_t e^{\lambda t} = \partial^\alpha e^{\lambda t} / \partial^\alpha t = \lambda^\alpha e^{\lambda t}$$ **we arrive at final result**

$$(\nabla\times)^\alpha \{\hat{\boldsymbol{x}}(Ae^{ik_0 z})\} = (ik_0)^\alpha A\left[\hat{\boldsymbol{x}}\cos\frac{\alpha\pi}{2} + \hat{\boldsymbol{y}}\sin\frac{\alpha\pi}{2}\right]e^{ik_0 z}$$

..a tilted traveling wave *x-y* plane, travelling in *z*

Circular Polarized vector field & its fractional curl

Right circular field in *x-y* plane travelling in *z* with circular radius E_0

$$\overline{\mathbf{F}} = E_0(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})e^{ik\hat{z}} \qquad f_z = 0 \qquad f_x = E_0 e^{ikz} \qquad f_y = \mathrm{i}E_0 e^{ikz}$$

So we use fractional curl expression to write

$$(\nabla\times)^{\alpha}\overline{\mathbf{F}} = \left(\sin\frac{\alpha\pi}{2}\right)\det\begin{bmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & \frac{\partial^{\alpha}}{\partial z^{\alpha}} \\ E_0 e^{ikz} & \mathrm{i}E_0 e^{ikz} & 0 \end{bmatrix} + \left(\cos\frac{\alpha\pi}{2}\right)\begin{bmatrix} \frac{\partial^{\alpha}}{\partial z^{\alpha}} & 0 & 0 \\ 0 & \frac{\partial^{\alpha}}{\partial z^{\alpha}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\boldsymbol{x}}E_0 e^{ikz} \\ \hat{\boldsymbol{y}}\mathrm{i}E_0 e^{ikz} \\ 0 \end{bmatrix}$$

$$= \hat{\boldsymbol{x}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^{\alpha}}{\partial z^{\alpha}}\mathrm{i}E_0 e^{ikz}\right] - \hat{\boldsymbol{y}}\left(\sin\frac{\alpha\pi}{2}\right)\left[-\frac{\partial^{\alpha}}{\partial z^{\alpha}}E_0 e^{ikz}\right]$$

$$+\hat{\boldsymbol{x}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^{\alpha}}{\partial z^{\alpha}}E_0 e^{ikz}\right] + \hat{\boldsymbol{y}}\left(\cos\frac{\alpha\pi}{2}\right)\left[\frac{\partial^{\alpha}}{\partial z^{\alpha}}\mathrm{i}E_0 e^{ikz}\right]$$

Use in above the following expressions for final result

$${}_{-\infty}D_t^{\alpha}e^{\lambda t} = \partial^{\alpha}e^{\lambda t}/\partial^{\alpha}t = \lambda^{\alpha}e^{\lambda t}; \quad \pm\mathrm{i} = e^{\pm\mathrm{i}\pi/2}; \quad \cos\theta \pm \mathrm{i}\sin\theta = e^{\pm\mathrm{i}\theta}$$

Interesting result of fractional curl of circular polarized vector

$$(\nabla\times)^{\alpha}\bar{\mathbf{F}} = E_0\{\hat{\boldsymbol{x}}[-\mathrm{i}\sin(\alpha\pi/2)] + \hat{\boldsymbol{y}}\sin(\alpha\pi/2) + \hat{\boldsymbol{x}}\cos(\alpha\pi/2) + \hat{\boldsymbol{y}}[i\cos(\alpha\pi/2)]\}(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$= \left[\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\sin\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{x}} + \left(\sin\frac{\alpha\pi}{2} + \mathrm{i}\cos\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$= \left[\left(\cos\frac{\alpha\pi}{2} - \mathrm{i}\sin\frac{\alpha\pi}{2}\right)\hat{\boldsymbol{x}} + \left(\cos\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\} + \mathrm{i}\sin\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$= \left[\left(e^{-\mathrm{i}\frac{\alpha\pi}{2}}\right)\hat{\boldsymbol{x}} + \left(e^{\mathrm{i}\left\{\frac{\pi}{2} - \frac{\alpha\pi}{2}\right\}}\right)\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$= \left[\left\{e^{\left(-\mathrm{i}\frac{\pi}{2}\right)}\right\}^{\alpha}\hat{\boldsymbol{x}} + \left\{e^{i\frac{\pi}{2}}\right\}\left\{e^{\left(-\mathrm{i}\frac{\pi}{2}\right)}\right\}^{\alpha}\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz} = \left[(-\mathrm{i})^{\alpha}\hat{\boldsymbol{x}} + (\mathrm{i})(-\mathrm{i})^{\alpha}\hat{\boldsymbol{y}}\right]E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$= (-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$\mathrm{curl}^{\alpha}\left\{(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} + \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

$$\mathrm{curl}^{\alpha}\left\{(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (+\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}} - \mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$$

Complex order curl

From result $\mathrm{curl}^{\alpha}\left\{(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0 e^{\mathrm{i}kz}\right\}=(-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{\alpha}e^{\mathrm{i}kz}$

Put $\mathrm{i}\beta$ **in place of** α **and simplify as**

$$\mathrm{curl}^{\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0 e^{\mathrm{i}kz}\right\}=(-\mathrm{i})^{\mathrm{i}\beta}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz}$$

$$=\left(e^{-i\frac{\pi}{2}}\right)^{i\beta}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz}=(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\beta\pi/2}E_0\right\}(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz}$$

From above two results we write

$$\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0 e^{\mathrm{i}kz}\right\}=(-\mathrm{i})^{\mathrm{i}\beta}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})E_0(\mathrm{i}k)^{i\beta}e^{\mathrm{i}kz}$$

$$=(-\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\beta\pi/2}E_0\right\}(\mathrm{i}k)^{\alpha+\mathrm{i}\beta}e^{\mathrm{i}kz}$$

The fractional field

In all our examples we have seen by virtue of fractional differentiation for obtaining fractional curl, we get term $(\mathrm{i}k)^{\alpha}$ therefore, after getting the fractional curl operation the dimensions of the vector quantity is altered.
In order to maintain same dimensions of the original vector field, the obtained result be divided by $(\mathrm{i}k)^{\alpha}$

Therefore we may write the 'fractional' operation as

$$\overline{\mathbf{F}}_f = (\mathrm{i}k)^{-\alpha}\left\{(\nabla\times)^{\alpha}\,\overline{\mathbf{F}}\right\}$$

Therefore the fractional field of for right and left circularly polarized vector field is

$$\overline{\mathbf{F}}_{f(\pm)} = (\mathrm{i}k)^{-(\alpha+i\beta)}\,\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}}\pm\mathrm{i}\hat{\boldsymbol{y}})E_0e^{\mathrm{i}kz}\right\} = (\mp\mathrm{i})^{\alpha}(\hat{\boldsymbol{x}}\pm\mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}kz}$$

$$= \left\{e^{\mp\mathrm{i}(\pi/2)}\right\}^{\alpha}(\hat{\boldsymbol{x}}\pm\mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}kz} = (\hat{\boldsymbol{x}}\pm\mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}\left(kz\mp\frac{\alpha\pi}{2}\right)}$$

Physical interpretation

Complex order curl of a circularly polarized vector field gives following (complex) fractional field

$$\overline{\mathbf{F}}_{f(\pm)} = (\mathrm{i}k)^{-(\alpha+i\beta)}\,\mathrm{curl}^{\alpha+\mathrm{i}\beta}\left\{(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})E_0 e^{\mathrm{i}kz}\right\}$$

$$= \left\{e^{\mp\mathrm{i}(\pi/2)}\right\}^{\alpha}(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}kz} = (\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\left\{e^{\pm\beta\pi/2}E_0\right\}e^{\mathrm{i}\left(kz\mp\frac{\alpha\pi}{2}\right)}$$

The fractional field obtained by complex order curl operation (after normalizing) gives interpretation as following. The real part of the fractional order $\alpha > 0$ **gives a spatial phase lead in case of right circularly polarized vector field, gives a spatial phase lag for left circularly polarized vector. In other words the real part of fractional operator modifies the phase of the vector field. Whereas the** $\beta > 0$ **imaginary part of the operator modifies the amplitude. For the right circularly polarized vector field it increases the amplitude (in this case radius in** $x - y$ **plane); and for left circularly polarized vector field the fractional field reduces the radius of the vector field's amplitude .**

Other fractional fields in reality

Defining fractional field as explained above, we get in electromagnetic theory, with $(\bar{\mathbf{E}}, \eta\bar{\mathbf{H}})$ as electric field and magnetic fields solution to the Maxwell equations; then we get fractional fields as in

$$\bar{\mathbf{E}}_f = (ik)^{-\alpha}\left[(\nabla\times)^{\alpha}\bar{\mathbf{E}}\right] \qquad (\eta\bar{\mathbf{H}}_f) = (ik)^{-\alpha}\left[(\nabla\times)^{\alpha}(\eta\bar{\mathbf{H}})\right]$$

$$\alpha = 0 \qquad \bar{\mathbf{E}}_f = \bar{\mathbf{E}} \qquad \bar{\mathbf{H}}_f = \bar{\mathbf{H}}$$

Original solution

$$k = \omega\sqrt{\mu\varepsilon} \qquad \eta = \sqrt{\varepsilon/\mu} = |\bar{\mathbf{E}}|/|\bar{\mathbf{H}}|$$

$$\mathrm{curl}(\eta\bar{\mathbf{H}}) = -(ik)\bar{\mathbf{E}} \qquad \mathrm{curl}(\bar{\mathbf{E}}) = (ik)\eta\bar{\mathbf{H}} \qquad \mathrm{div.}(\eta\bar{\mathbf{H}}) = 0 \qquad \mathrm{div.}(\bar{\mathbf{E}}) = 0$$

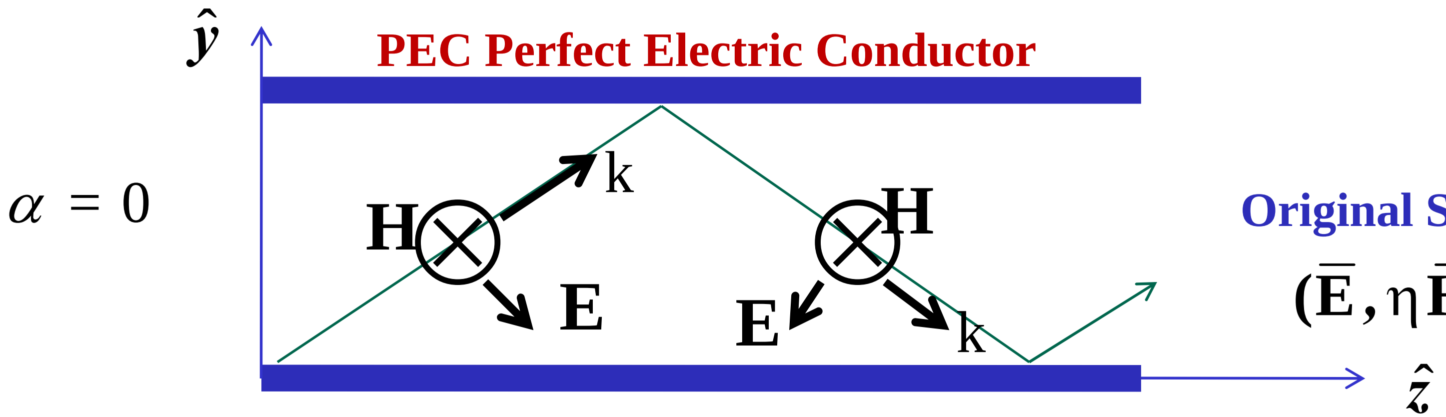
$$\alpha = 1 \qquad \bar{\mathbf{E}}_f = \eta\bar{\mathbf{H}} \qquad \bar{\mathbf{H}}_f = -\bar{\mathbf{E}}$$

Dual to the solution

Duality means if $(\bar{\mathbf{E}}, \eta\bar{\mathbf{H}})$ is solution to Maxwell's equation, then is $(\eta\bar{\mathbf{H}}, -\bar{\mathbf{E}})$ its dual solution.

$\bar{\mathbf{E}}_f, (\eta\bar{\mathbf{H}}_f)$ therefore gives in between original and dual to original field

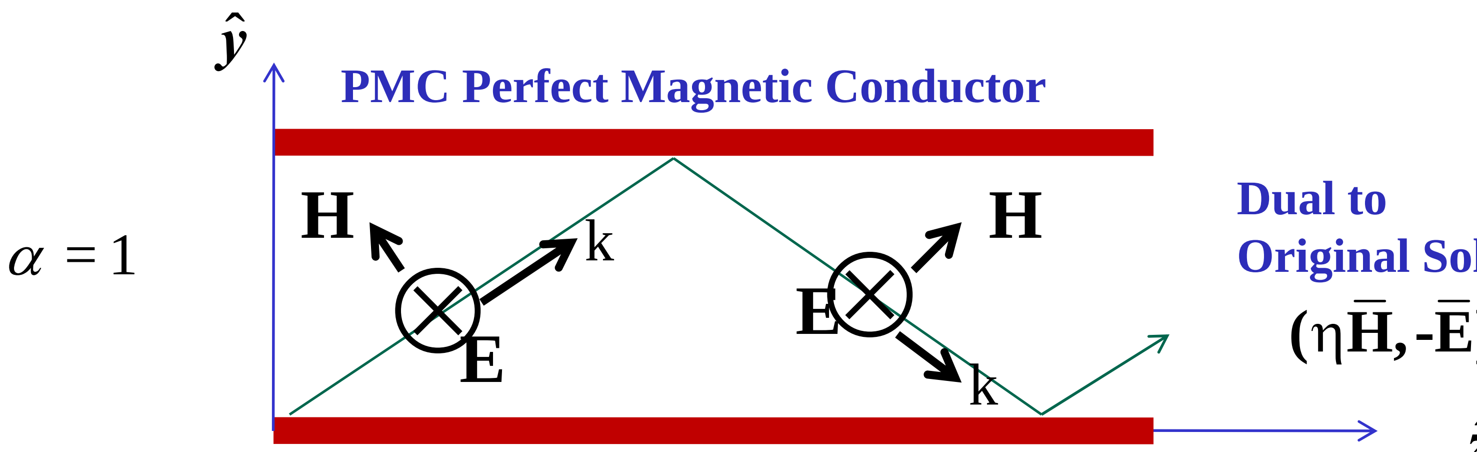
Dual solutions of EM field inside wave -guide



Original Solution

$(\bar{\mathbf{E}}, \eta \bar{\mathbf{H}})$

Transverse Magnetic Mode (TM) propagating between two PEC boundaries



Dual to Original Solution

$(\eta \bar{\mathbf{H}}, -\bar{\mathbf{E}})$

Transverse Electric Mode (TE) propagating between two PMC boundaries

Fractional fields inside wave-guide practical reality

We have the wave guide walls not with PEC or PMC

We have material which are in between PEC and PMC

This gives propagation inside a wave guide which is in-between original and dual to original solutions, that is in between TM and TE

Case of fractional field

End of part-2

Dissipation mechanism and Fractional Order Differential equation……..with memory

Wave equation-partial differential equation

$$\frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} u(x,t) = 0 \qquad u(0,0) = 1$$

$$u(x,t) = \mathrm{Re}\left\{e^{\mathrm{i}(\omega_1 t - k_1 x)}\right\} \qquad \omega_1 \in \mathbb{R}^+ \qquad k_1 \in \mathbb{R}^+$$

$$k_1 = \omega_1 / v \qquad \omega_1 = 2\pi\upsilon_1$$

A pure oscillatory loss less system may be regarded as a system whose memory does not decay with time or space . Meaning that an excitation once put into a system is always memorized and it remains in the system; as though system has ‘constant memory . While the system having dissipation, has memory which decays with time and space . This mechanism of dissipation with constant memory and with decaying memory is having relation with fractional order derivative . Thus a dissipating wave mechanism is well characterized by fractional derivative. This we will elaborate in following slides

Wave equation-partial differential equation-with dissipation

The dissipation of wave is a complex phenomena, and is studied classically by adding few first order derivatives terms in PDE

Depending on the complexity of the dissipation mechanism we add the several of these extra terms as

$$\sum_{n=1}^{N} c_n (\partial u(x,t)/\partial x) + \sum_{n=1}^{M} d_n (\partial u(x,t)/\partial t) \qquad c_n ; d_n \in \mathbb{R}$$

$$\frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} u(x,t) - c_1 \frac{\partial}{\partial x} u(x,t) - d_1 \frac{\partial}{\partial t} u(x,t) = 0$$

Fractional wave equation!

$$\frac{\partial^2}{\partial x^2}u(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) = 0 \qquad u(0,0) = 1$$

What we show here that wave dissipation can be simplified by fractional derivatives in the wave equation and writing the equation as

$$\frac{1}{\varsigma^{2(1-\beta)}}\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - \frac{1}{v^2}\frac{1}{\xi^{2(1-\alpha)}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = 0$$

$$0 < \alpha \leq 1 \qquad 0 < \beta \leq 1$$

Fractional derivative defined in Caputo sense

$$\frac{\partial^q}{\partial t^q}f(t) = \frac{1}{\Gamma(n-q)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}}\mathrm{d}\tau \qquad (n-1) \leq q < n \qquad n \in \mathbb{Z}^+; q \in \mathbb{R}^+$$

Generalization of wave equation

In generalizing wave-equation with the fractional derivative we have introduced two parameters ς ξ

$$\frac{1}{\varsigma^{2(1-\beta)}}\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}}-\frac{1}{v^2}\frac{1}{\xi^{2(1-\alpha)}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t)=0$$

Space parameter ς which has dimensions of length; indicates the attached immediate derivative operator is fractional derivative with respect to space; and parameter ξ having dimension of time indicates that immediately attached derivative operator is fractional derivative with respect to time.

Also note that the powers of these new parameters are adjusted so that in above we are subtracting quantities with same dimensions (unit as per length square).

Separation of variables

Assume $u(x,t) = f(t)g(x)$ **with** $\beta = 1$

that is with only the time fractional derivative part the equation is:

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{1}{\xi^{2(1-\alpha)}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = 0$$

The equation will be having solution of the form $u(x,t) = u_0 e^{-ik_1 x}.f(t)$

Substituting this in above equation we get the following

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} f(t) + v^2 k_1{}^2 \xi^{2(1-\alpha)} f(t) = 0 \qquad \frac{\mathrm{d}^{2\alpha}}{\mathrm{d}t^{2\alpha}} f(t) + \omega_\alpha^2 f(t) = 0 \qquad \omega_\alpha^2 = v^2 k_1^2 \xi^{2(1-\alpha)}$$

Similarly consider $\alpha = 1$

and write the equation with as

$$\frac{1}{\varsigma^{2(1-\beta)}}\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) = 0$$

A particular solution to equation is $u(x,t) = \tilde{u}_0 e^{\mathrm{i}\omega_1 t}.g(x)$

Substituting this we have the following equation

$$\frac{\partial^{2\beta}}{\partial x^{2\beta}} g(x) + \frac{\omega_1^2}{v^2}\varsigma^{2(1-\beta)} g(x) = 0 \qquad \frac{\mathrm{d}^{2\beta}}{\mathrm{d}x^{2\beta}} g(x) + k_\beta^2 g(x) = 0 \qquad k_\beta^2 = \frac{\omega_1^2}{v^2}\varsigma^{2(1-\beta)}$$

Comment on generalization

The wave equation $$\frac{\partial^2}{\partial x^2}u(x,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) = 0 \qquad u(0,0) = 1$$

The wave equation generalized $$\frac{1}{\varsigma^{2(1-\beta)}}\frac{\partial^{2\beta}u(x,t)}{\partial x^{2\beta}} - \frac{1}{v^2}\frac{1}{\xi^{2(1-\alpha)}}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u(x,t) = 0$$

The solution is in form $$u(x,t) = u_0 e^{-ik_1 x}.f(t) \qquad u(x,t) = \tilde{u}_0 e^{i\omega_1 t}.g(x)$$

and

$$\frac{d^{2\alpha}}{dt^{2\alpha}}f(t) + \omega_\alpha^2 f(t) = 0 \qquad \omega_\alpha^2 = v^2 k_1^2 \xi^{2(1-\alpha)} \qquad \frac{d^{2\beta}}{dx^{2\beta}}g(x) + k_\beta^2 g(x) = 0 \qquad k_\beta^2 = \frac{\omega_1^2}{v^2}\varsigma^{2(1-\beta)}$$

The expressions are generalized from standard mechanism of solution to partial differential wave equation by separation of variables methods as being done for normal case to get spatial and temporal parts.

The fractional orders α, β **is indicative of complex dissipative mechanism how a wave relaxes into a media, and has relation to memory in space time**

Temporal part's solution

The FDE is $$\frac{d^{2\alpha}}{dt^{2\alpha}} f(t) + \omega_\alpha^2 f(t) = 0 \qquad 0 < \alpha \leq 1$$

The solution for FDE in terms of Mittag-Leffler function is

$$f(t) = E_{2\alpha}(-\omega_\alpha{}^2 t^{2\alpha}) \qquad E_{2\alpha}(-\omega_\alpha{}^2 t^{2\alpha}) = \sum_{m=0}^{\infty} \frac{(-\omega_\alpha{}^2 t^{2\alpha})^m}{\Gamma(2m\alpha + 1)}$$

For $\alpha = 1$ $$\frac{d^2 f(t)}{dt^2} + \omega_1^2 f(t) = 0 \qquad E_2(-\omega_1^2 t^2) = \cosh(\sqrt{-\omega_1^2 t^2}) = \cosh(i\omega_1 t) = \cos\omega_1 t$$

SHO case with $\omega_1^2 = k_1^2 v^2$ $$\frac{d^2}{dt^2} f(t) + k_1^2 v^2 f(t) = 0 \qquad f(t) = \cos\omega_1 t$$

is constant memory case

For $\alpha = 0.5$ $$\frac{d}{dt} f(t) + \omega_{0.5}{}^2 f(t) = 0 \qquad f(t) = E_1(-\omega_{0.5}{}^2 t) = e^{-\omega_{0.5}{}^2 t}$$

is no memory case

In terms of EM parameters of travelling plane wave we also have

$$\omega_\alpha = \tau^{-\alpha} = \frac{k_1 c}{\sqrt{\varepsilon\mu}\,\xi^{(\alpha-1)}} = k_1 v \xi^{1-\alpha} = \omega_1 \xi^{1-\alpha}$$

Spatial part's solution

The FDE $$\frac{d^{2\beta}}{dx^{2\beta}} g(x) + k_\beta^2 g(x) = 0 \qquad 0 < \beta \le 1$$

In terms of EM parameters of travelling plane wave we also have

$$k_\beta^2 = \frac{\omega_1^2}{v^2} \varsigma^{2(1-\beta)} = k_1^2 \varsigma^{2(1-\beta)}$$

The solution to FDE is $$g(x) = E_{2\beta}(-k_\beta^2 x^{2\beta}) = \sum_{m=0}^{\infty} \frac{(-k_\beta^2 x^{2\beta})^m}{\Gamma(2m\beta + 1)}$$

For $\beta = 1$ **SHO case with constant memory**

$$\frac{d^2}{dx^2} g(x) + k_1^2 g(x) = 0 \qquad g(x) = E_2(-k_1^2 x^2) = \cosh(\sqrt{-k_1^2 x^2}) = \cosh(ik_1 x) = \cos(k_1 x)$$

For $\beta = 0.5$ **SHO case with no memory exponential decay case**

$$\frac{d}{dx} g(x) + k_{0.5}^2 g(x) = 0 \qquad g(x) = E_1(-k_{0.5}^2 x) = e^{-k_{0.5}^2 x}$$

Reviewing relaxation

Standard Maxwell Debye relaxation is

$$\tau \frac{d}{dt}\Phi(t) = -\Phi(t)$$

$$t > 0;\ \Phi(0) = \Phi_0$$

$$\Phi(t) = \Phi_0 e^{-t/\tau}$$

Gives pure exponential solution with single relaxation time constant

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}}\Phi(t)$$

Also the Integral representation of Maxwell-Debye relaxation is:

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}}\Phi(t)$$

The integral equation can be formally extended to Fractional Integral equation by replacing $\frac{1}{\tau}\frac{d^{-1}}{dt^{-1}} \rightarrow \frac{1}{\tau^{\beta}}\frac{d^{-\beta}}{dt^{-\beta}}$ which leads to

$$\Phi(t) - \Phi_0 = -\frac{1}{\tau^{\beta}}\, {}_0D_t^{-\beta}\Phi(t)$$

Non Debye non-exponential relaxation process

Kohlrausch Williams Watts (KWW) relaxation law:

$$\Phi(t) = \Phi_0 e^{\left\{\left(-\,t/\tau\right)^{\alpha}\right\}}$$

Nutting Power Law relaxation:

$$\Phi(t) = \Phi_0 \left(1 + t/\tau\right)^{-n} ;\ 0 < n < 1$$

Observed in:

Dielectric relaxation, Stress Relaxation, Strain relaxation, NMR relaxation
Diffusion controlled relaxation, electrical circuits,…….; unlike normal relaxation $\Phi(t) = \Phi_0 e^{-t/\tau}$

Memory vis-à-vis fractional calculus

1. Non-exponential relaxation implies MEMORY i.e. the underlying fundamental relaxation process are NON-MARKOVIAN

2. Natural way to incorporate memory effect is fractional calculus via the involved Convolution integral in time (space). The present state is being influenced by all the states, the system has been running through at the times $t' = 0, 1, \ldots\ldots\ldots t$

3. The power-law Kernel defines the fractional expression represents a particular long memory.

Fractional Integration is

$$ {}_0D_t^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau = \mathrm{P}_\beta(t) * f(t) $$

$$ \mathrm{P}_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)} \qquad \mathrm{P}_{\beta\to 1}(t) = H(t) \qquad \mathrm{P}_{\beta\to 0}(t) = \delta(t) $$

Memory integrals

$$\frac{d\Phi(t)}{dt} = -\int_0^t K(t-\tau)\Phi(\tau)d\tau = -K(t)*\Phi(t)$$

Represents Memory Integral i.e. all instances for $\tau = 0$ **to** $\tau = t$ **contribute to situation at** $\tau = t$

1. **Memory breaks down i.e. Markovian Case:**

$$K(t) = K_0\delta(t)$$

$$\frac{d}{dt}\Phi(t) = -\int_0^t K_0\delta(t-\tau)\Phi(\tau)d\tau = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \exp\{-K_0 t\}$$

$$\frac{d\Phi(t)}{dt} = -\frac{\Phi(t)}{\tau} = -K(t)*\Phi(t)$$

2. **The opposite case Constant Memory i.e. leading to oscillatory case**

$$K(t) = K_0$$

$$\frac{d^2}{dt^2}\Phi(t) = -K_0\Phi(t)$$

$$\Phi(t) = \Phi_0 \cos(\sqrt{K_0}\, t)$$

Slowly varying memory

3. Slowly varying Kernel which for small time behaves as power law gives KWW relaxation process

$$K(t) \approx K_0 t^{\gamma} \qquad \Phi(t) = \Phi_0 \exp\left\{-K_0 t^{\gamma+2}\right\}$$

4. **Relaxation for Fractional Differential/Integral equation & its Memory Kernel**

$$K(t) = K_0 t^{q-2};\ 0 < q \le 2$$

$$\frac{d}{dt}\Phi(t) = -\frac{1}{\tau^q}\left[{}_0D_t^{1-q}\Phi(t)\right]; \qquad \tau^q = \left[K_0\Gamma(q-1)\right]^{-1}$$

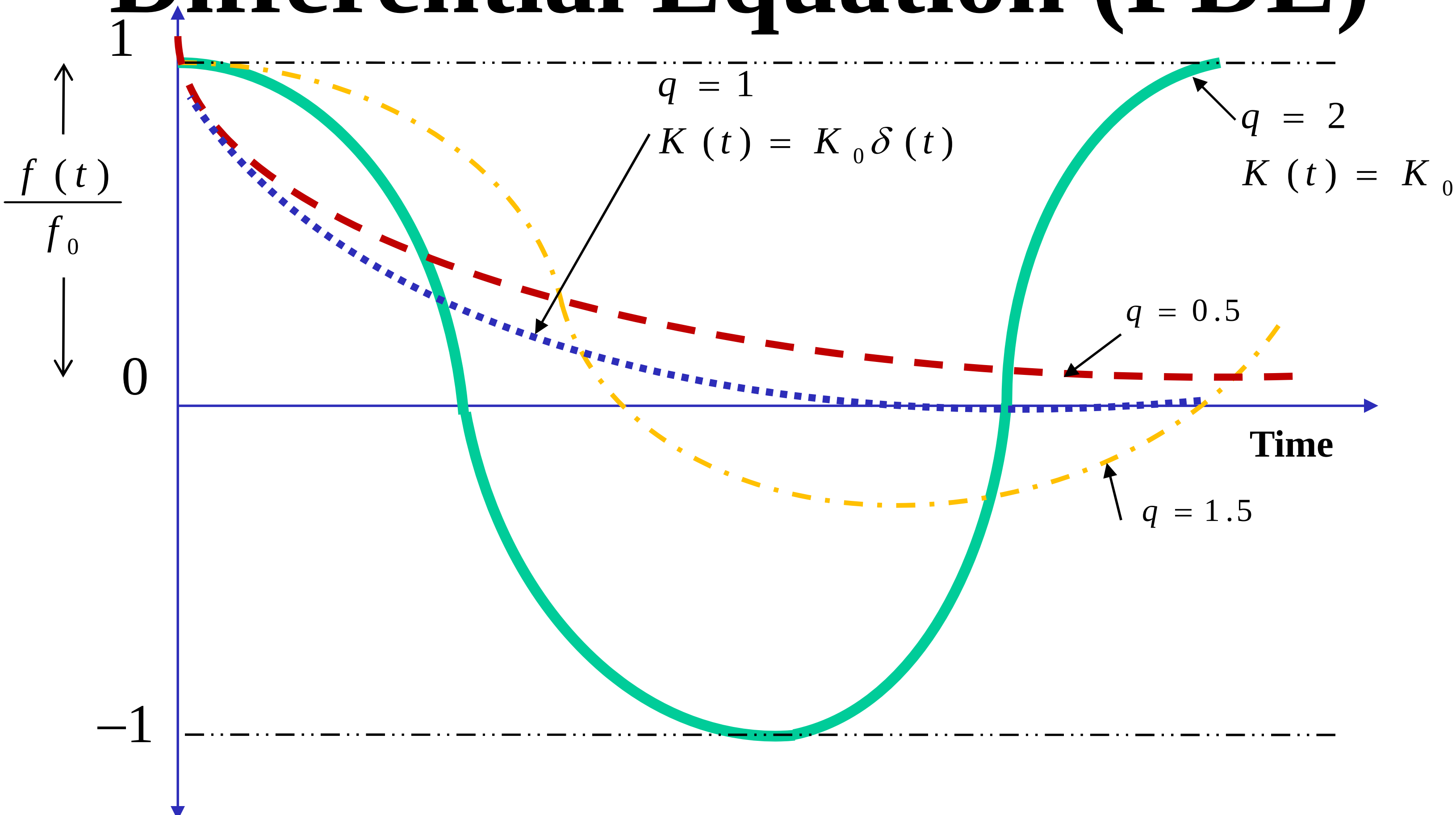
Apply ${}_0D_t^{-1}$ **on both sides to get:** $\Phi(t) - \Phi_0 = -\tau^{-q}\,{}_0D_t^{-q}\Phi(t)$

Apply ${}_0D_t^{q}$ **on both sides to get FDE**

$${}_0D_t^{q}\Phi(t) - \Phi_0\frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\Phi(t)$$

Using Fractional Derivative of constant *C* as , non zero, that is $C\,t^{-q}/\Gamma(1-q)$ **we get this**

Memory Kernel & Fractional Differential Equation (FDE)



$$\frac{d^q}{dt^q} f(t) + \tau^{-q} f(t) = 0; \quad q \in \mathbb{R}^+$$

$$1 \le q \le 2$$

$$f(0) = f_0$$

Normalized time scale solution

$$\frac{d^{2\alpha}}{dt^{2\alpha}} f(t) + \omega_\alpha^2 f(t) = 0; \qquad \omega_\alpha^2 = \omega_1^2 \xi^{2(1-\alpha)} = 4\pi^2 \upsilon_1^2 \frac{\xi^2}{\xi^{2\alpha}}$$ **with** $\alpha = 1 \quad T_1 = 1/\upsilon_1 \quad \upsilon_1 = \omega_1 / 2\pi$

We get $$\frac{d^{2\alpha}}{dt^{2\alpha}} f(t) + 4\pi^2 \frac{\upsilon_1^2 \xi^2}{\xi^{2\alpha}} f(t) = 0$$

Putting $\alpha = \upsilon_1^2 \xi^2;$ $$\frac{d^{2\alpha}}{dt^{2\alpha}} f(t) + 4\pi^2 \frac{\alpha}{\xi^{2\alpha}} f(t) = 0$$

$\xi \equiv$ time (s)

$\upsilon_1 \equiv$ frequency $(s)^{-1}$

$\alpha \equiv$ no dimmension

For verification put $\alpha = 1$ **we get** $$\frac{d^2}{dt^2} f(t) + 4\pi^2 \frac{1}{\xi^2} f(t) = 0$$

where $4\pi^2 \xi^{-2} = \omega_1^2 = (2\pi\upsilon_1)^2$ **the integer order angular frequency**

Define normalized time as $\tilde{t} = t / T_1$ **then** $\alpha = \upsilon_1^2 \xi^2 = \frac{\xi^2}{T_1^2} \qquad 0 < \xi < T_1 \qquad T_1 = \upsilon_1^{-1}$

Temporal solution in terms of normalized time

$$f(t) = E_{2\alpha}(-\omega_\alpha^2 t^{2\alpha}) = E_{2\alpha}(-\omega_1^2 \xi^{2(1-\alpha)} t^{2\alpha})$$

$$= E_{2\alpha}(-4\pi^2 \upsilon_1^2 \xi^2 \xi^{-2\alpha} t^{2\alpha}) = E_{2\alpha}\left(-4\pi^2 \frac{1}{T_1^2} \frac{\xi^2}{\xi^{2\alpha}} t^{2\alpha}\right)$$

$$= E_{2\alpha}\left(-4\pi^2 \frac{1}{T_1^2} \frac{\xi^2}{\xi^{2\alpha}} t^{2\alpha} \frac{T_1^{2\alpha}}{T_1^{2\alpha}}\right) = E_{2\alpha}\left(-4\pi^2 \frac{\xi^{2(1-\alpha)}}{T_1^{2(1-\alpha)}} \left(\frac{t}{T_1}\right)^{2\alpha}\right)$$

$$= E_{2\alpha}\left(-4\pi^2 \left(\frac{\xi}{T_1}\right)^{2(1-\alpha)} \left(\frac{t}{T_1}\right)^{2\alpha}\right) = E_{2\alpha}\left(-4\pi^2 \left(\left(\frac{\xi}{T_1}\right)^2\right)^{(1-\alpha)} \left(\frac{t}{T_1}\right)^{2\alpha}\right)$$

$$= E_{2\alpha}\left(-4\pi^2 \alpha^{(1-\alpha)} \{\tilde{t}\}^{2\alpha}\right)$$

Normalized length scale solution

Similarly we have FDE spatial equation

$$\frac{d^{2\beta}}{dx^{2\beta}} g(x) + k_\beta^2 g(x) = 0 \qquad k_\beta^2 = \frac{\omega_1^{\,2}}{v^2} \varsigma^{2(1-\beta)} = k_1^2 \varsigma^{2(1-\beta)} = \frac{4\pi^2}{\lambda_1^2} \varsigma^{2(1-\beta)}$$

$$k_1 = 2\pi / \lambda_1 = \omega_1 / v = 2\pi \upsilon / v$$

Define $\beta = \frac{\varsigma^2}{\lambda_1^{\,2}} = \frac{\varsigma^2 k_1^2}{4\pi^2}$ $\qquad 0 < \varsigma \le \lambda_1$

$\zeta \equiv \text{length}$

$\lambda \equiv \text{length}^{-1}$

$\beta \equiv \text{no dim ension}$

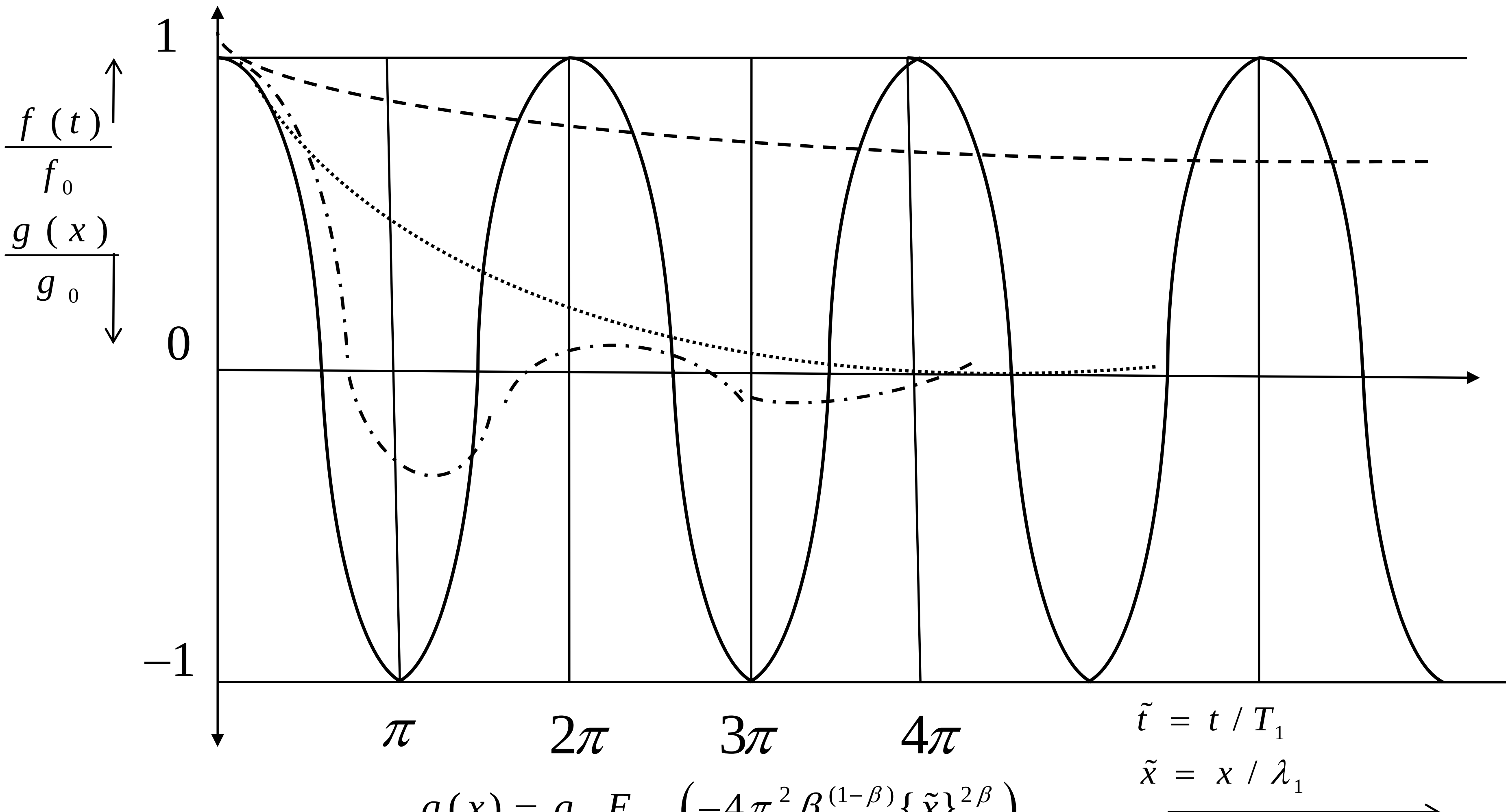
$$\frac{d^{2\beta}}{dx^{2\beta}} g(x) + 4\pi^2 \frac{\varsigma^2}{\lambda_1^2} \varsigma^{-2\beta} g(x) = 0 \qquad \frac{d^{2\beta}}{dx^{2\beta}} g(x) + 4\pi^2 \frac{\beta}{\varsigma^{2\beta}} g(x) = 0$$

Define normalized length as $\tilde{x} = x / \lambda_1$

Spatial solution in terms of normalized length scale is

$$g(x) = E_{2\beta}\left(-4\pi^2 \beta^{(1-\beta)} \{\tilde{x}\}^{2\beta}\right)$$

Plot



$$g(x) = g_0 . E_{2\beta}\left(-4\pi^2 \beta^{(1-\beta)}\{\tilde{x}\}^{2\beta}\right)$$

$$f(t) = f_0 . E_{2\alpha}\left(-4\pi^2 \alpha^{(1-\alpha)}\{\tilde{t}\}^{2\alpha}\right)$$

Interpretations

Therefore a general normalized solution to fractional wave equation with complex dissipation mechanism and memory with initial value $u(0,0) = 1$

$$u(x,t) = E_{2\alpha}\left(-\alpha^{(1-\alpha)}\tilde{t}^{2\alpha}\right)E_{2\beta}\left(-\beta^{(1-\beta)}\tilde{x}^{2\beta}\right) \qquad \tilde{t} = t/T_1;\ \tilde{x} = x/\lambda_1$$

The solutions are plotted in figure, in normalized scales of $\tilde{t}$; $\tilde{x}$. The case of constant memory is depicted when $\alpha = 1 = \beta$ gives a non decaying period of $\tilde{t}_{\text{period}};\tilde{x}_{\text{period}} = 2\pi$

$\alpha = \beta = 0.75$
we obtain the oscillatory but decaying relaxation (discharge). Here the wave dissipation has decaying memory kernel

$\alpha = \beta = 0.5$
we are getting non oscillatory discharge, without memory a pure exponential dissipation

$\alpha = \beta = 0.25$

we obtain a lingering tailed discharge with slowly decaying memory kernel.

Interestingly the fractional orders α , β in this case are fractions of full oscillatory non dissipative wave's period and wave length that is T_1 λ_1

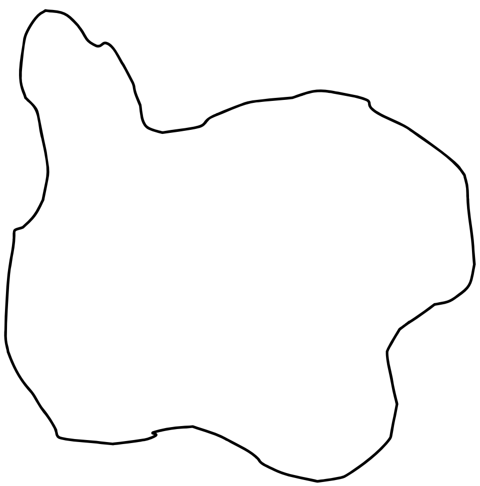
$$\alpha = \upsilon_1^2\xi^2 = \frac{\xi^2}{T_1^2} \qquad \beta = \frac{\varsigma^2}{\lambda_1^2} = \frac{\varsigma^2 k_1^2}{4\pi^2}$$

End of part-3

Spatial Roughness manifesting to Fractional Order Differential equation-(super-capacitor)

Does ‘d / dt’ represent accumulation or loss always ?

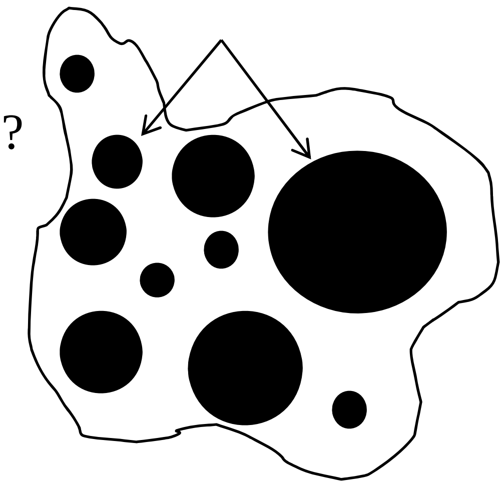
Well if there are temporary traps then?
Well if the boundary is partly reflecting?
Well if the elementary element (area, volume etc) be not a point quantity?
Well if mass of ball is not a point quantity?
Well if spring is not mass-less?



$$\frac{d}{dt}\langle\phi\rangle = \mathrm{GAIN} - \mathrm{LOSS?}$$

$$\frac{d^{\alpha}}{dt^{\alpha}}\langle\phi\rangle = \mathrm{GAIN} - \mathrm{LOSS?}$$

Traps or Island (Forbidden zone)



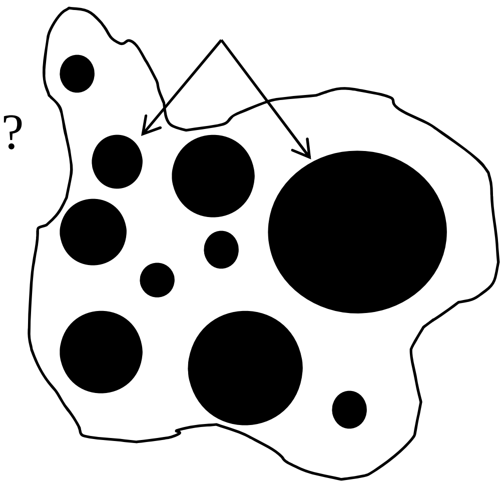
Some are entraps temporarily indicating slow rate of change than d / dt

The particles cannot have the island paths indicating fast rate of change than d / dt

Spatial disorder heterogeneity giving fractional temporal derivative?

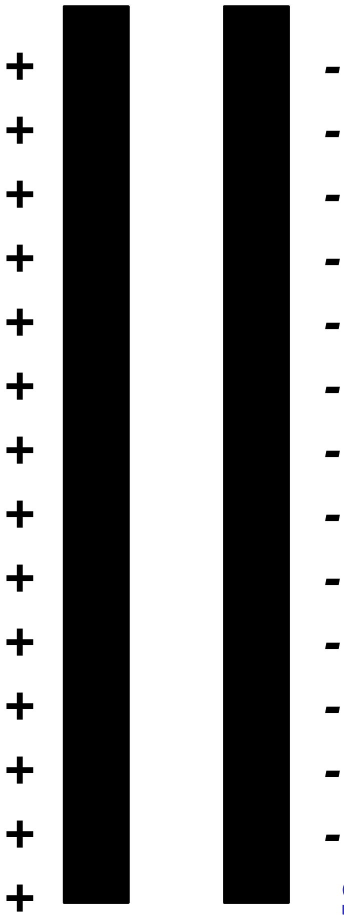
Traps or Island (Forbidden zone)

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\langle\phi\rangle = \mathrm{GAIN} - \mathrm{LOSS?}$$

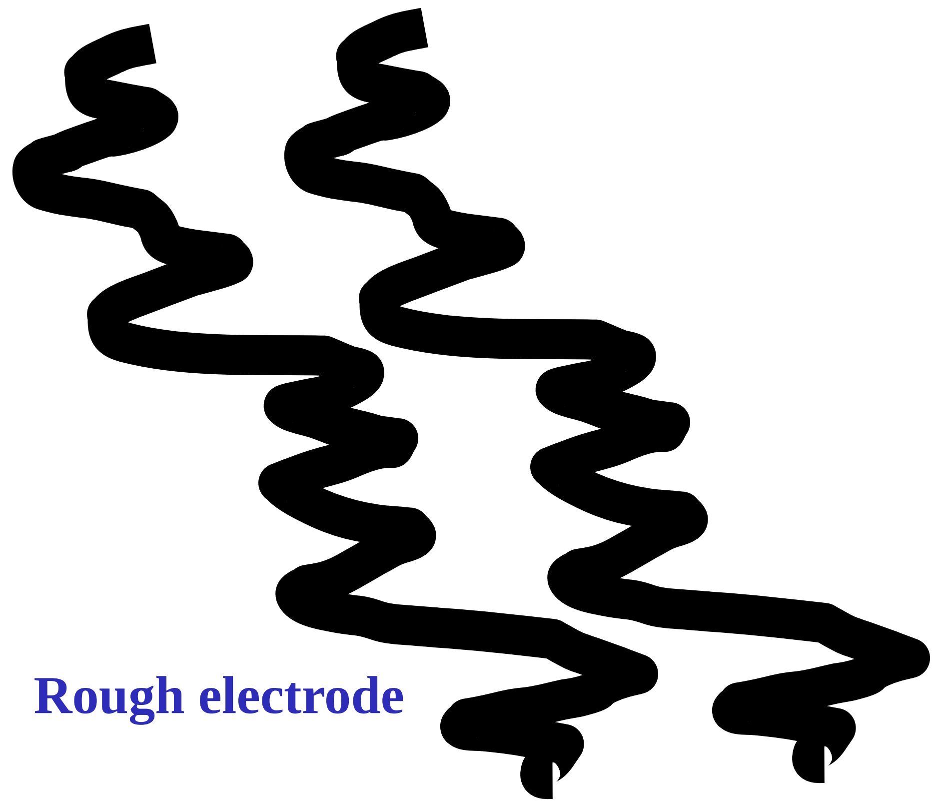


It is indeed true….

We take example of super-capacitor



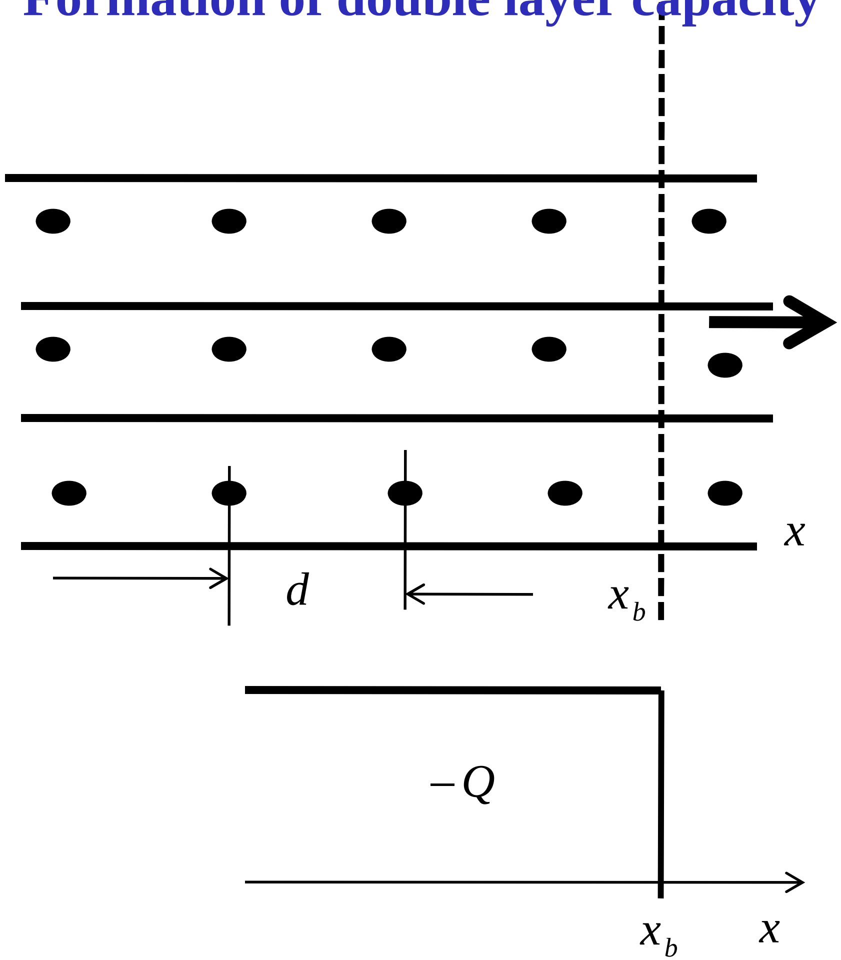
Smooth electrode



Rough electrode

Charge distribution at cleavage of electrode crystal

Formation of double layer capacity



Electrode material we consider as simple case made of positive nuclei on fixed 'regular' grid points inside a sea of homogeneous distribution of negative charge. By cleaving the electrode one obtains two halves which can be considered as electrodes; the cleaving is at x_b location, as depicted. Let us assume that cleavage has made interface of metal (electrode) and organic (electrolyte), and immediate picture of negative charge sea.

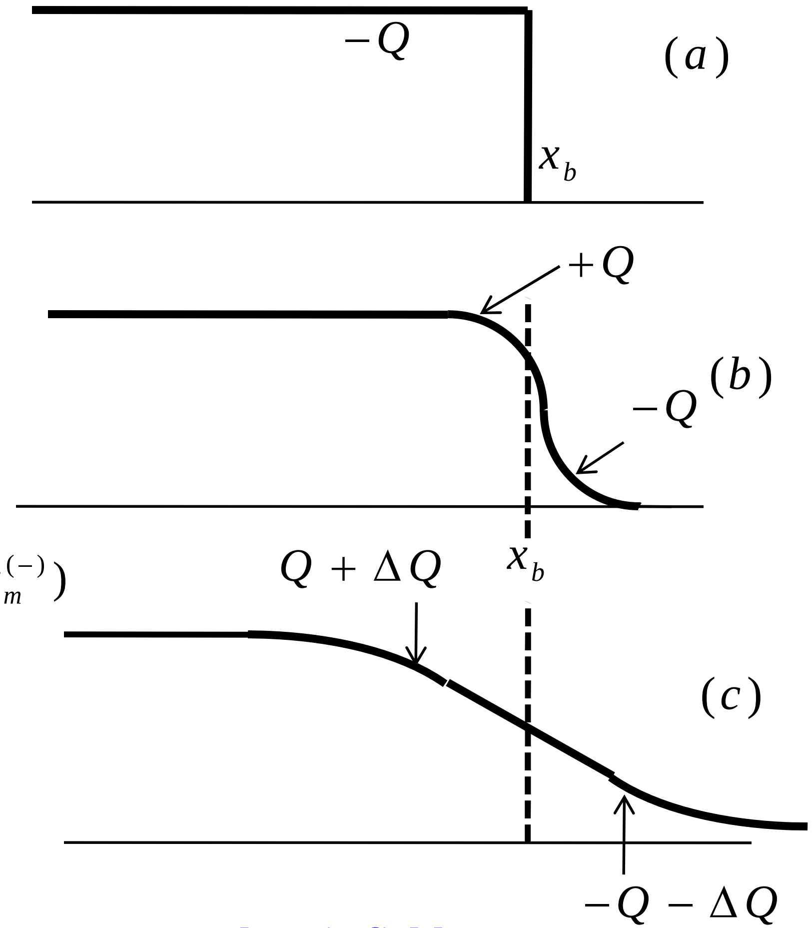
Double layer capacity

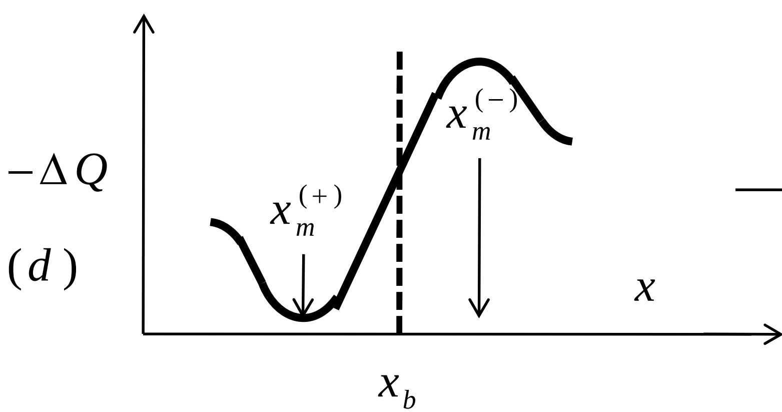
This special charge separation forms "capacity"; the metal-electrolyte capacity-and formation of double layer capacity C_m

$$x_m^{(+)} = \frac{\int_{-\infty}^{x_b} [\Delta Q(x)](x_b - x)\mathrm{d}x}{\int_{-\infty}^{x_b} \Delta Q(x)\mathrm{d}x}$$

$$x_m^{(-)} = \frac{\int_{x_b}^{\infty} [\Delta Q(x)](x - x_b)\mathrm{d}x}{\int_{x_b}^{\infty} \Delta Q(x)\mathrm{d}x}$$

$$C_m = \frac{1}{4\pi}(x_m^{(+)} - x_m^{(-)})$$



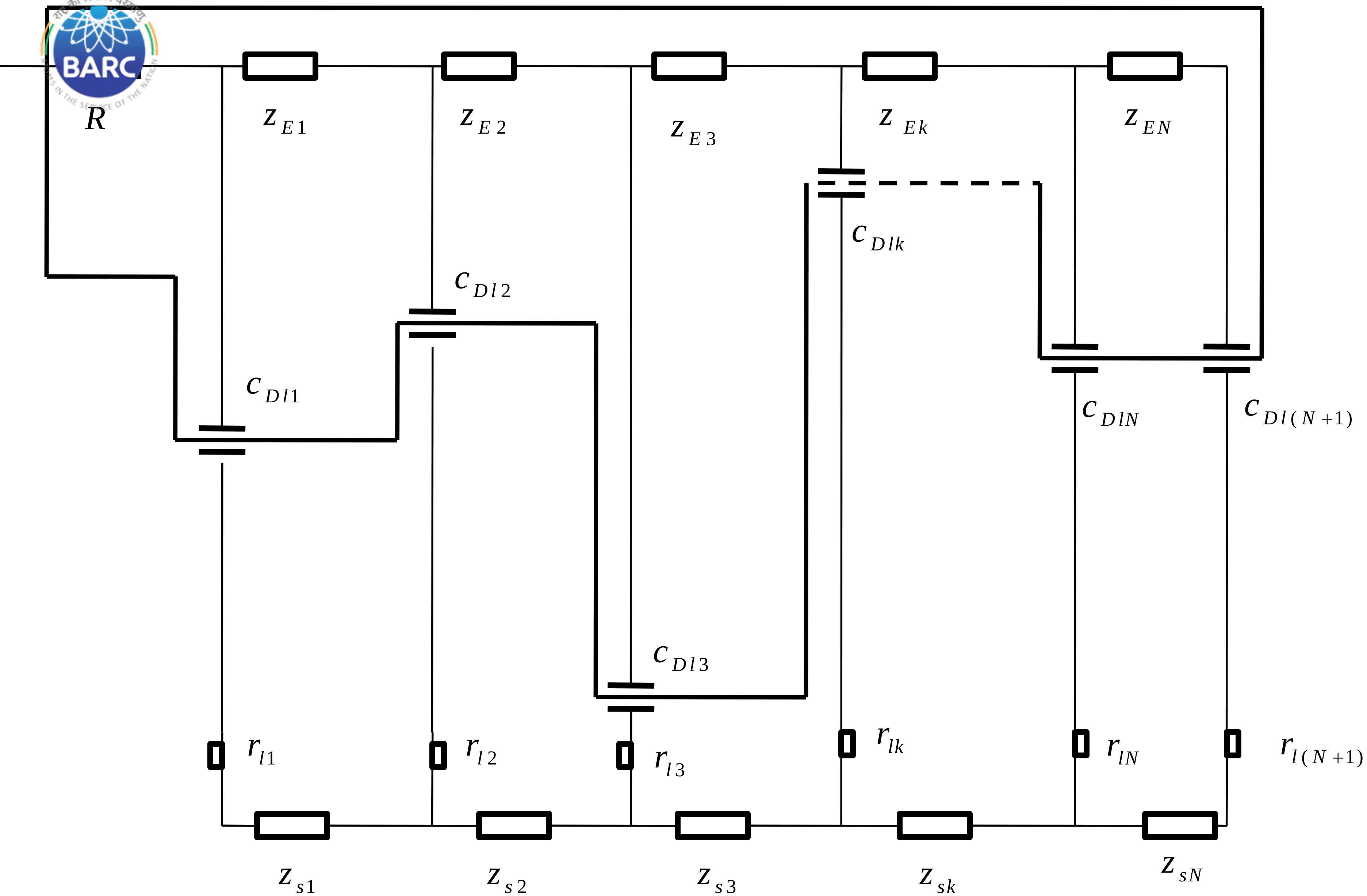


Electric field Perpendicular to electrode

Disordered system

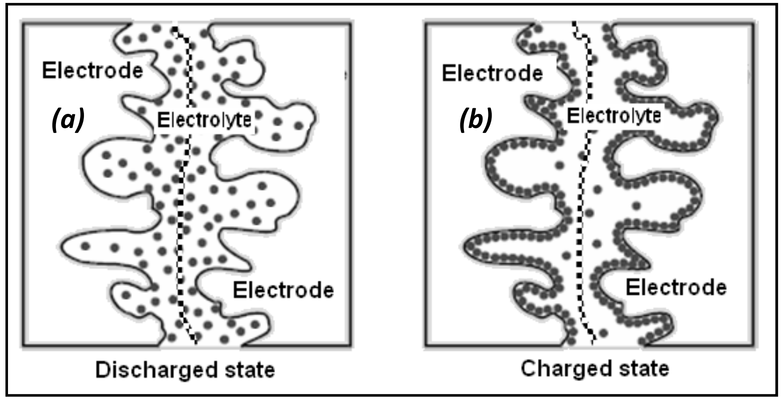
The distribution function is different for different cleavage. Some cleavage may be symmetrical, as ideal as shown some may have different nuclei distribution near cleavage, with different numbers as per crystal face cleavage of electrode. This distribution thus gives rise to several time-constant system, better described by fractional calculus, with unique disorder parameter .

Now if the above obtained capacity is same, assuming electrode surface is uniformly smooth, then we do not have problem to model; the capacitor system. However, due to rough nature the charge distribution function at each of the cleavage is different- the distribution does not and need not be a normal, Gaussian type. This fractal charge distribution can lead to a capacity of ‘rough’ electrode other than normal or Gaussian distribution-leading to power law distribution too! This is ‘disordered’ system.

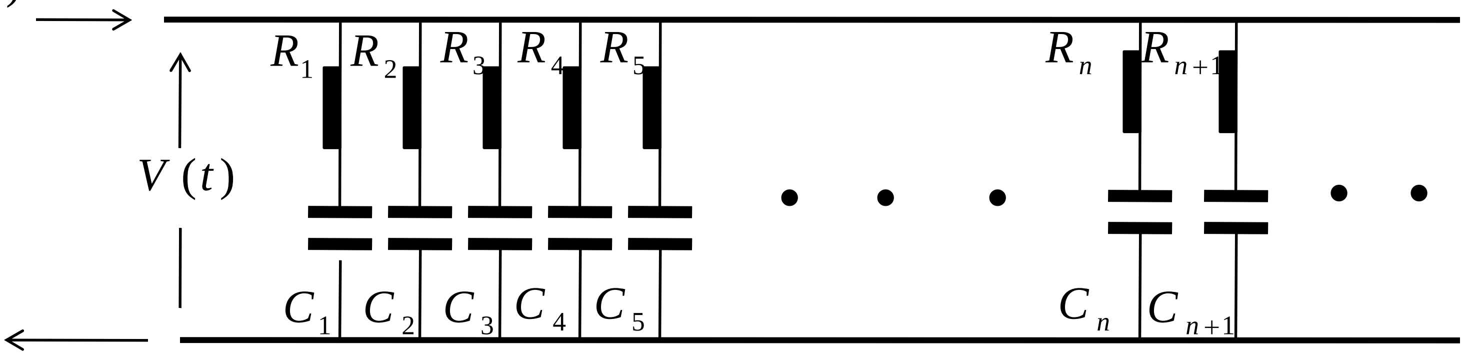


R
Z_{E1}
Z_{E2}
Z_{E3}
Z_{Ek}
Z_{EN}
C_{Dlk}
C_{Dl2}
C_{Dl1}
C_{DlN}
$C_{Dl(N+1)}$
C_{Dl3}
r_{l1}
r_{l2}
r_{l3}
r_{lk}
r_{lN}
$r_{l(N+1)}$
Z_{s1}
Z_{s2}
Z_{s3}
Z_{sk}
Z_{sN}

Rough electrode carbon - aerogel



Disordered relaxation with several time-constans



Fractional differential equation for rough electrodes

$$I(t) = K \frac{d^{\gamma}}{dt^{\gamma}} V(t)$$

Impedance of rough electrode

$$Z(s) = \frac{1}{Ks^{\gamma}}; \quad 0 < \gamma < 1$$

A several time-constant system-no average

Consider a partial differential equation (PDE)

$$\frac{\partial}{\partial t} u(\lambda, t) + (\lambda)^{1/\alpha} u(\lambda, t) = \delta(t) \qquad \alpha > 0$$

The above PDE is having free parameter λ

Now if for the free parameter $\alpha = 1$ then we have single time constant system $\lambda = \tau^{-1}$

with solution as $u(\lambda, t) = \exp(-\lambda t)$ with initial condition $u(\lambda, 0) = 1$

The several time constants (discharge rate) is taken as power law distribution as $(\lambda)^q$

$q = 1/\alpha \qquad 0 < \alpha < 1$

The strong-discharge or exponential discharge with one time constant follow a normal distribution with well defined average that represents average time constant or discharge rate, and that normal distribution has well defined standard deviation. Unlike the normal distribution the ‘power-law’ distribution has no defined average or moments (standard deviation); and is representation of system which has variety. The heterogeneity or the disordered system thus has varieties of ways by which dissipation mechanism takes place.

"Impulse response function" to "Impulse response"

$$\frac{\partial}{\partial t} u(\lambda,t) + (\lambda)^{1/\alpha} u(\lambda,t) = \delta(t)$$

$$u(\lambda,t) = h(\lambda,t) = \exp\left(-\lambda^{1/\alpha} t\right)$$

$h(\lambda,t)$ **denotes 'impulse response function'. On integrating this 'impulse response function' for free variable we get the function of time and that is called 'impulse response' Green's function**

$$g(t) = \int_0^\infty h(\lambda,t)d\lambda = \int_0^\infty \exp\left(-\lambda^{1/\alpha} t\right)\mathrm{d}\lambda = \frac{\Gamma(1+\alpha)}{t^\alpha}$$

To get above substitute $\lambda^{1/\alpha} = x \qquad \lambda = (x/t)^\alpha \qquad \mathrm{d}\lambda = \lambda^{1-(1/\alpha)}(\alpha/t)\mathrm{d}x$

Steps are

$$g(t) = \int_0^\infty e^{-x}\left(\frac{\alpha}{t}\right)\lambda\lambda^{-(1/\alpha)}\mathrm{d}x = \int_0^\infty e^{-x}\left(\frac{x}{t}\right)^\alpha\left(\frac{x}{t}\right)^{-1}\mathrm{d}x$$

$$= \left(\frac{\alpha}{t}\right)\int_0^\infty e^{-x}\frac{x^{\alpha-1}}{t^{\alpha-1}}\mathrm{d}x = \left(\frac{\alpha}{t^\alpha}\right)\int_0^\infty e^{-x}x^{\alpha-1}\mathrm{d}x = \frac{\alpha\Gamma(\alpha)}{t^\alpha}$$

$$= \frac{\Gamma(1+\alpha)}{t^\alpha}$$

Using the Gamma definition

$$\Gamma(\alpha) \triangleq \int_0^\infty e^{-y} y^{\alpha-1}\mathrm{d}y$$

$$\alpha\Gamma(\alpha) = \Gamma(1+\alpha)$$

Fractional derivative operator -in disordered system

$$\frac{\partial}{\partial t}u(\lambda,t) + (\lambda)^{1/\alpha}\, u(\lambda,t) = f'(t)$$

Then the response to this new excitation is convolution of Green's function obtained above with the forcing function that is:

$$r(t) = g(t) * f'(t) = \int_0^t g(\tau) f'(t-\tau)\,\mathrm{d}\tau = \Gamma(1+\alpha)\int_0^t \frac{f'(t-\tau)}{\tau^{\alpha}}\mathrm{d}\tau \qquad 0 < \alpha < 1$$

Multiplying and dividing the above expression with $\Gamma(1-\alpha)$ **Using fractional integral**

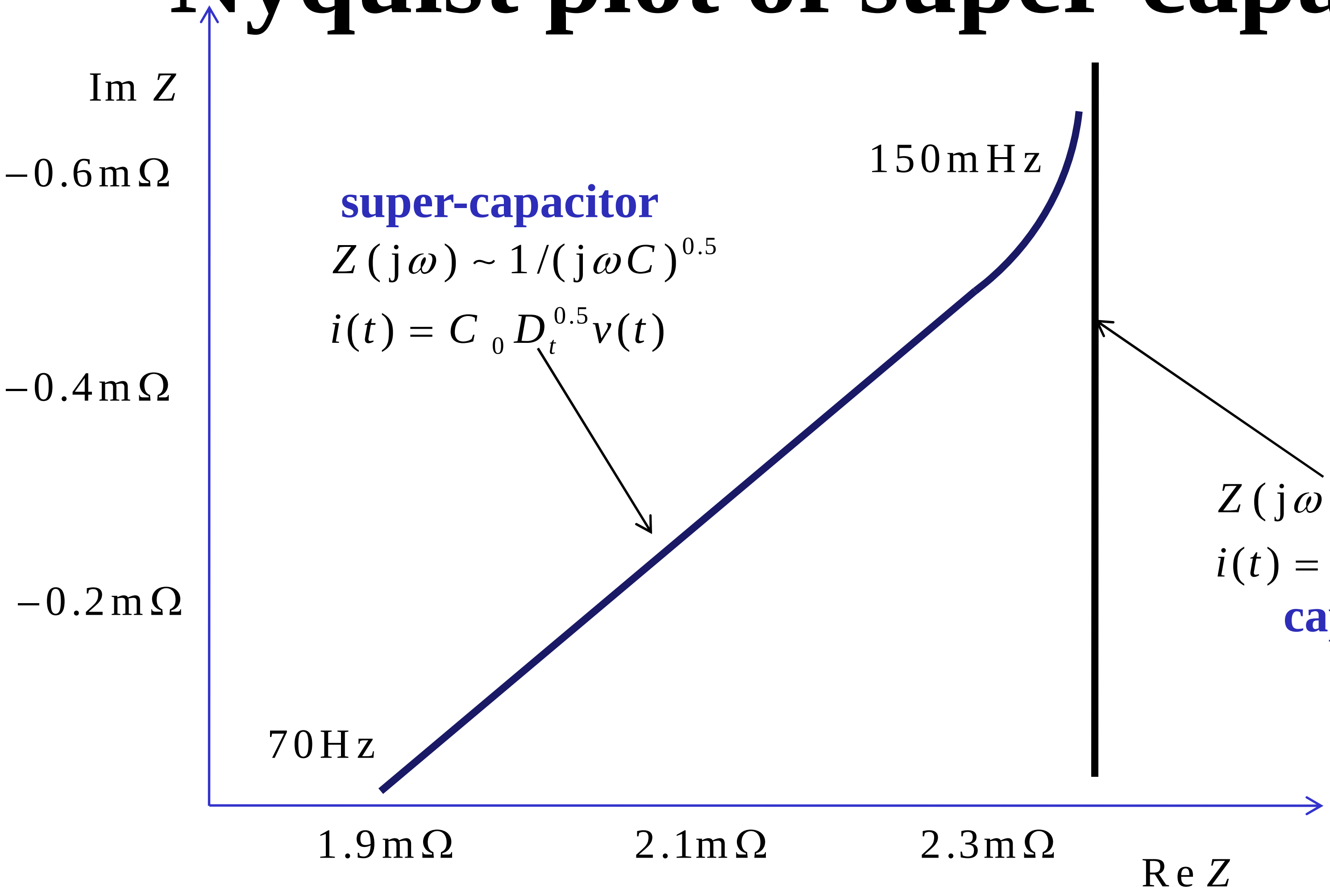
$$r(t) = \Gamma(1+\alpha)\Gamma(1-\alpha)\int_0^t \frac{(t-\tau)^{(-\alpha)}}{\Gamma(1-\alpha)} f'(\tau)\,\mathrm{d}\tau$$

$$\Gamma(1+\alpha)\Gamma(1-\alpha)\, D_t^{-(1-\alpha)}\left[f'(t)\right]$$

$$= \Gamma(1+\alpha)\Gamma(1-\alpha)\, D_t^{\alpha} f(t)$$

Implying the appearance of fractional derivative for cases where several time-constants define a relaxation process. Therefore a disordered relaxation (response) may well be formulated by fractional differential equation, the order giving the 'intermittency' of relaxation disordered process!

Impedance spectroscopy
Nyquist plot of super-capacitor



End of part-4

Gramian for Fractional Order State Space system

Gramian

Gramian matrix of functions is

$$Q(t_0, t) \triangleq \int_{t_0}^{t} \mathrm{F}(t)\mathrm{F}^*(t)\,\mathrm{d}t$$

F^* **denoting transpose of** F **matrix**

The matrix $\mathrm{F}(t)$ **is** $m \times n$ **with functions** $f_i(t)$ $i = 1, 2, 3, \ldots m$

Sequential fractional derivative Miller-Ross ${}_a\mathscr{D}_x^{k\alpha}$

Sequential linear fractional differential equation of order $n\alpha\,; \quad n \in \mathbb{N}$

$$b_0(x)y(x) + b_1(x)\left[{}_a\mathscr{D}_x^{\alpha}y(x)\right] + b_2(x)\left[{}_a\mathscr{D}_x^{2\alpha}y(x)\right] + \ldots. + b_n(x)\left[{}_a\mathscr{D}_x^{n\alpha}y(x)\right] = f(x)$$

${}_a\mathscr{D}_x^{k\alpha}$ **is fractional sequential derivative operator, of commensurate order** α

${}_a\mathscr{D}_x^{\alpha}y(x) = {}_a^{*}D_x^{\alpha}y(x)$ **where** ${}_a^{*}D_x^{\alpha}$ **is RL or Caputo operator, and**

$${}_a\mathscr{D}_x^{k\alpha}y(x) = {}_a\mathscr{D}_x^{\alpha}\,{}_a\mathscr{D}_x^{(k-1)\alpha}y(x)\,; \quad k = 2, 3, \ldots$$

For $k = 2\,; \quad 0 < \alpha < 1/2$ **we have**

$${}_a\mathscr{D}_x^{2\alpha}y(x) = {}_aD_x^{2\alpha}y(x) - {}_aI_x^{1-\alpha}y(a)\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}$$

Comes from the fact that for $\alpha > 0, \beta > 0$ **generally** $D^{\alpha}D^{\beta}f(x) \neq D^{\alpha+\beta}f(x)$

Representing in matrix form

$$\mathscr{D}^{n\alpha} y(x) + \sum_{k=0}^{(n-1)} a_k(x) \mathscr{D}^{k\alpha} y(x) = f(x); \quad n \in \mathbb{N}$$

reduces to $^{*}D^{\alpha}\,\mathrm{Y}(x) = \mathrm{A}(x)\mathrm{Y}(x) + \mathrm{B}(x)$

by changing the variables $y_1(x) = y(x); \quad {}^{*}D^{\alpha} y_j(x) = y_{j+1}(x); \quad j = 1, 2, \ldots, (n-1)$

where $\mathrm{A}(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -a_0 & -a_1 & \ldots & \ldots & -a_{n-1} \end{pmatrix}, \quad \mathrm{B}(x) = \begin{pmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ f(x) \end{pmatrix}; \quad \mathrm{Y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \ldots \\ \ldots \\ y_n(x) \end{pmatrix}$

As an example

$y''(t) + 3\left({}_0D_t^{3/2} y(t)\right) + y(t) = f(t)$ **set** $y''(t) = {}_0\mathscr{D}_t^{4\frac{1}{2}} y(t)$ **gives SLFDE as**

${}_0\mathscr{D}_t^{4\alpha} y(t) + 3\,{}_0\mathscr{D}_t^{3\alpha} y(t) + y(t) = f(t); \quad \alpha = 1/2$ **reduced to** ${}_0D_t^{\alpha}\mathrm{Y}(t) = \mathrm{A}\mathrm{Y}(x) + \mathrm{B}(t)$

with $\mathrm{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -3 \end{pmatrix}; \quad \mathrm{B}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{pmatrix}; \quad \mathrm{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix}$

Fractional order systems (FOS) requiring convolute control

Let us take a fractional differential equation in state variable form as the following

$$\begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} x(t) = A x(t) + B u(t) \\ y(t) = C x(t) \end{cases} \qquad x(0) = x_0 \qquad x_0^{*} = [0, 0, \ldots 0]^{*}$$

A, B, C **are constant matrices Linear time invariant' (LTI) systems**

put $\alpha = 1/2$

$$\frac{d^{1/2}}{dt^{1/2}} x(t) = Ax(t) + Bu(t) \qquad \frac{d}{dt} x(t) = A^{2} x(t) + \left(ABu(t) + B \frac{d^{1/2}}{dt^{1/2}} u(t) \right)$$

The above equivalence states exhibits a non instantaneous effect of control $u(t)$ **on integer order dynamics of state** $x(t)$ **at time** t **through the convolution from initial time** $t = 0$ **to present time** t **via semi derivative of** $u(t)$

Modified performance index required in Fractional Order Systems (FOS)

$u(t)$ **getting modified as** $\sim \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau$

$\int_0^t |u(\tau)|^2 d\tau$ **getting modified as** $\sim \int_0^t |(t-\tau)^{\alpha-1} u(\tau)|^2 d\tau$

Alpha-exponential functions

Two parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad \alpha > 0 \qquad \beta > 0 \qquad z \in \mathbb{C}$$

is extended for matrix case

$$E_{\alpha,\beta}(\mathrm{A}t^{\alpha}) = \sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+\beta)}$$

Put $\beta = \alpha$ **and we define alpha-exponential function-1 as**

$$e_{\alpha}^{\mathrm{A}t} = (t^{\alpha-1})E_{\alpha,\alpha}(\mathrm{A}t^{\alpha}) = (t^{\alpha-1})\sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{k\alpha}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$$

$$\mathcal{L}\left\{e_{\alpha}^{\lambda t}\right\} = (s^{\alpha}-\lambda)^{-1}$$ **is also called as Robotnov-Hartley function**

Put $\beta = 1$ **and we define alpha-exponential function-2 as**

$$\tilde{e}_{\alpha}^{\mathrm{A}t} = E_{\alpha,1}(\mathrm{A}t^{\alpha}) = \sum_{k=0}^{\infty} \mathrm{A}^k \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}$$

$$\mathcal{L}\left\{\tilde{e}_{\alpha}^{\lambda t}\right\} = s^{\alpha-1}(s^{\alpha}-\lambda)^{-1}$$

Alpha-exponential functions as eigenvectors of RL, & Caputo derivatives

$$ {}_{t_0+}^{\;C}D_t^{\alpha}(t-t_0)^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-t_0)^{\beta-\alpha} & \beta \neq 0 \\ 0 & \beta = 0 \end{cases} $$

$$ {}_{t_0+}^{\;C}D_t^{\alpha}\left[\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)}\right] = {}_{t_0+}^{\;C}D_t^{\alpha}\{E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha})\} = {}_{t_0+}^{\;C}D_t^{\alpha}\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{\alpha k}}{\Gamma(k\alpha+1)} $$

$$ = \sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{(k-1)\alpha}}{\Gamma[(k-1)\alpha+1]} = \mathrm{A}E_{\alpha}(\mathrm{A}(t-t_0)^{\alpha}) = \mathrm{A}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} $$

we have useful relation $ {}_{t_0+}^{\;C}D_t^{\alpha}\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{A}\,\tilde{e}_{\alpha}^{\mathrm{A}(t-t_0)} $ **that solves Caputo FDE**

Similarly we get $ {}_{t_0+}D_t^{\alpha}e_{\alpha}^{\mathrm{A}(t-t_0)} = {}_{t_0+}D_t^{\alpha}\sum_{k=0}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} $

$$ = \sum_{k=1}^{\infty}\mathrm{A}^k\frac{(t-t_0)^{k\alpha-1}}{\Gamma(k\alpha)} = \mathrm{A}\,e_{\alpha}^{\mathrm{A}(t-t_0)} $$

we have useful property $ {}_{t_0+}D_t^{\alpha}e_{\alpha}^{\mathrm{A}(t-t_0)} = \mathrm{A}e_{\alpha}^{\mathrm{A}(t-t_0)} $ **that solves R-L FDE**

Relation between the two alpha-exponential functions

$$\int_{t_0}^{t} \mathrm{A} e_\alpha^{\mathrm{A}(t-\tau)} \mathrm{d}\tau = \int_{t_0}^{t} \sum_{k=0}^{\infty} \mathrm{A}^{k+1} \frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \mathrm{d}\tau$$

$$= \sum_{k=1}^{\infty} \mathrm{A}^{k} \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha+1)} = E_\alpha(\mathrm{A}(t-t_0)^\alpha) - \mathrm{I} = \tilde{e}_\alpha^{\mathrm{A}(t-t_0)} - \mathrm{I}$$

we have useful relation $\tilde{e}_\alpha^{\mathrm{A}(t-t_0)} = \mathrm{I} + \int_{t_0}^{t} \mathrm{A} e_\alpha^{\mathrm{A}(t-\tau)} \mathrm{d}\tau$ $\quad \tilde{\Phi}_\alpha(t-t_0) = \mathrm{I} + \int_{t_0}^{t} \Phi_\alpha(t-\tau) \mathrm{A}\, \mathrm{d}\tau$

$\Phi_\alpha(t) = e_\alpha^{\mathrm{A}t}$ **and** $\tilde{\Phi}_\alpha(t) = \tilde{e}_\alpha^{\mathrm{A}t}$

as state transition matrices or the Green's functions for homogeneous fractional multivariate dynamics with Riemann-Liouvelli and Caputo derivative formulations respectively

Solution of differential equation

Let us take general differential equation $\frac{d}{dt}x(t)+a(t)x(t)=bu(t) \qquad x(t_0)=x_0$

multiply both the sides by $e^{\int a(t)dt}$ **and get** $\frac{d}{dt}\left[e^{\int a(t)dt}x(t)\right]=e^{\int a(t)dt}bu(t)$ **and**

integrate $\int_{t_0}^{t}\frac{d}{d\tau}\left[e^{\int a(\tau)d\tau}x(\tau)\right]d\tau=\int_{t_0}^{t}e^{\int a(\tau)d\tau}bu(\tau)d\tau$ **Using** $\Phi(t)=e^{\int a(t)dt}$ **we write**

$$\int_{t_0}^{t}\frac{d}{d\tau}\left[\Phi(\tau)x(\tau)\right]d\tau=\int_{t_0}^{t}\Phi(\tau)bu(\tau)d\tau$$

$$\Phi(t)x(t)-\Phi(t_0)x(t_0)=\int_{t_0}^{t}\Phi(\tau)bu(\tau)d\tau$$

$$x(t)=[\Phi(t)]^{-1}\Phi(t_0)x_0+[\Phi(t)]^{-1}\int_{t_0}^{t}\Phi(\tau)bu(\tau)d\tau$$

For a constant $a(t)=a$ **we have** $\Phi(t)=e^{at}$ **and** $\Phi(t_0)=e^{at_0}$ **and thus we have**

$$x(t)=e^{-at}e^{at_0}x_0+e^{-at}\int_{t_0}^{t}e^{a\tau}bu(\tau)d\tau=e^{a(t_0-t)}x_0+\int_{t_0}^{t}e^{a(\tau-t)}bu(\tau)d\tau$$

$$=\Phi(t_0-t)x_0+\int_{t_0}^{t}\Phi(\tau-t)bu(\tau)d\tau=\Phi(t,t_0)x_0+\int_{t_0}^{t}\Phi(t,\tau)bu(\tau)d\tau$$

Integer order system-solution classical multivariate

$$\Omega: \quad (D_t x)(t) = \mathrm{A}(t)x(t) + \mathrm{B}(t)u(t) \qquad y(t) = \mathrm{C}(t)x(t) + \mathrm{D}(t)u(t)$$

$$x(t) \in \mathbb{R}^{n\times 1}; \quad u(t) \in \mathbb{R}^{p\times 1}; \quad y(t) \in \mathbb{R}^{q\times 1}; \quad \mathrm{A} \in \mathbb{R}^{n\times n}; \quad \mathrm{B} \in \mathbb{R}^{n\times p}; \quad \mathrm{C} \in \mathbb{R}^{q\times n}; \quad \mathrm{D} \in \mathbb{R}^{q\times p}$$

whose entries are continuous function of time

The solution

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)\mathrm{B}(\tau)u(\tau)\,\mathrm{d}\tau = \Phi(t,t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\,\mathrm{d}\tau\right]$$

$$y(t) = \mathrm{C}(t)\Phi(t,t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)\mathrm{B}(\tau)u(\tau)\,\mathrm{d}\tau\right] + \mathrm{D}(t)u(t)$$

$\Phi(t_0,t) = e^{\int_{t_0}^{t}\mathrm{A}(\tau)\mathrm{d}\tau}$ **is state the transition matrix (Green's function) of homogeneous system**

$$(D_t x)(t) = \mathrm{A}(t)x(t)$$

For (LTI) system $\Phi(t_0,t) = \Phi(t-t_0) = e^{\mathrm{A}(t-t_0)} = G(t-t_0)$

The homogeneous solution is given by $G(t-t_0) = \Phi(t-t_0) = e^{\mathrm{A}(t-t_0)}$

& particular solution is convolution

$$x_p(t) = (G \otimes \mathrm{B}u)(t) = \int_{t_0}^{t} G(t-\tau)\mathrm{B}u(\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\,\mathrm{d}\tau$$

We get state trajectory and output trajectory as

$$x(t) = e^{\mathrm{A}t}x_0 + \int_{0}^{t} e^{\mathrm{A}(t-\tau)}\mathrm{B}u(\tau)\,\mathrm{d}\tau = e^{\mathrm{A}t}\left[x_0 + \int_{0}^{t} e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\,\mathrm{d}\tau\right]$$

$$y(t) = \mathrm{C}e^{\mathrm{A}t}\left[x_0 + \int_{0}^{t} e^{-\mathrm{A}\tau}\mathrm{B}u(\tau)\,\mathrm{d}\tau\right] + \mathrm{D}u(t)$$

Solution of FOS with R-L derivative

A non-homogeneous fractional differential equation with RL fractional derivative and with

$$0 < \alpha \leq 1 \qquad (D^{\alpha} \mathrm{Y})(t) = \mathrm{A}\,\mathrm{Y}(t) + \mathrm{B}(t) \qquad \mathrm{Y}_0 = \mathrm{Y}(t_0)$$

General Solution

$$\mathrm{Y}(t) = e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Y}_0 + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\,\mathrm{d}\tau$$

$$= \Phi_{\alpha}(t - t_0) + \int_{t_0}^{t} \Phi_{\alpha}(t-\tau)\mathrm{B}(\tau)\,\mathrm{d}\tau$$

Where

$$G_{\alpha}(t-\tau) = \Phi_{\alpha}(t-\tau) = e_{\alpha}^{\mathrm{A}(t-\tau)}$$ **is Green's function for RL-derivative**

$$\mathrm{Y}_p(t) = (G_{\alpha} \otimes \mathrm{B})(t) = \int_{t_0}^{t} G_{\alpha}(t-\tau)\mathrm{B}(\tau)\,\mathrm{d}\tau$$ **is particular solution**

Solution of FOS with Caputo derivative

Fractional differential equation with Caputo derivative for $0 < \alpha \leq 1$

$({}_{t_0+}^{\;C}D_t^{\alpha}\mathrm{Y})(t) = \mathrm{A}\mathrm{Y}(t)$ **with** $\mathrm{Y}(t_0) = b$ **has general solution**

$$\mathrm{Y}(t) = b + \int_{t_0}^{t}\left(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,b\right)\mathrm{d}\tau = b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,\mathrm{d}\tau\right)$$

$({}_{t_0}^{C}D_t^{\alpha} f)(t) = {}_{t_0}D_t^{\alpha}[f(t) - f(t_0)]$ **is RL-Caputo relation for** $0 < \alpha \leq 1$

Using this we get $({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y})(t) = \mathrm{A}\mathrm{Y}(t)$ $\quad {}_{t_0}D_t^{\alpha}[\mathrm{Y}(t) - b] = \mathrm{A}\mathrm{Y}(t)$

Put $\mathrm{Y}(t) = \mathrm{Z}(t) + b$ **thus we have** $\mathrm{Z}(t_0) = 0 = \mathrm{Z}_0$

Thus the equivalent equation in RL derivative based fractional differential equation is;

$$({}_{t_0}D_t^{\alpha}\mathrm{Z})(t) = \mathrm{A}[\mathrm{Z}(t) + b] = \mathrm{A}\,\mathrm{Z}(t) + \mathrm{A}\,b$$

whose solution we know from just previous derivation with RL, and we thus write the solution as

$$\mathrm{Z}(t) = e_{\alpha}^{\mathrm{A}(t-t_0)}\mathrm{Z}_0 + \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,b)\mathrm{d}\tau = \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,b)\mathrm{d}\tau = \mathrm{Y}(t) - b$$

We have thus the result now that is; $\mathrm{Y}(t) = b + \int_{t_0}^{t}(e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,b)\mathrm{d}\tau = b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,\mathrm{d}\tau\right)$

FOS with Caputo requires-two Green's functions in solution

Similarly for system $\left({}_{t_0}^{C}D_t^{\alpha}\mathrm{Y}\right)(t) = \mathrm{A}\mathrm{Y}(t) + \mathrm{B}(t)$ with $\mathrm{Y}(t_0) = b$

as constant the solution we write as;

$$\mathrm{Y}(t) = b + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\left[\mathrm{B}(\tau) + \mathrm{A}b\right]\mathrm{d}\tau = b\left(\mathrm{I} + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{A}\,\mathrm{d}\tau\right) + \int_{t_0}^{t} e_{\alpha}^{\mathrm{A}(t-\tau)}\mathrm{B}(\tau)\mathrm{d}\tau$$

which is also

$$\mathrm{Y}(t) = b\tilde{\Phi}_{\alpha}(t) + \int_{t_0}^{t}\Phi_{\alpha}(t-\tau)\mathrm{B}(\tau)\mathrm{d}\tau$$

Where state transition matrices are $\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t}$ and $\tilde{\Phi}_{\alpha}(t) = \tilde{e}_{\alpha}^{\mathrm{A}t}$

The solution with Caputo's formulation requiring thus two Green's functions (state transition matrices)!

The Gramian of control for classical integer order system

If it is possible to transfer any initial state $x(t_0)$ in state space of system to any other state $x(t_1)$ by proper choice of $u(t)$ in finite time then system is controllable "reachable"

The Gramian matrix of controllability is defined as $Q(t_0, t_1) \triangleq \int_{t_0}^{t_1} \Phi(t_0, t)\mathrm{B}(t)\mathrm{B}^*(t)\Phi^*(t_0, t)\mathrm{d}t$

A^* denotes transpose of matrix A

The Gramian matrix is non-singular, that is Gramian matrix has full rank; for controllability of the system with A , B pair

For a LTI system, with $t_0 = 0$ the controllability Gramian is

$$Q(t) = \int_0^t e^{-\mathrm{A}\tau}\mathrm{B}\mathrm{B}^* e^{-\mathrm{A}^*\tau}\mathrm{d}\tau = \int_0^t e^{\mathrm{A}(t-\tau)}\mathrm{B}\mathrm{B}^* e^{\mathrm{A}^*(t-\tau)}\mathrm{d}\tau$$

The claim here is that the input control vector function

$$u(t) = -\mathrm{B}^*(t)\Phi^*(t_0, t)Q^{-1}(t_0, t_1)\left[x_0 - \Phi(t_0, t_1)x_1\right]$$

transfers x_0 to x_1 at $t = t_1$

with minimal energy control, given by $J_{(t_0 - t)} = \int_{t_0}^{t} u^*(t)u(t)\mathrm{d}t$

Calculating Φ

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \qquad \mathrm{A} = \begin{pmatrix} 1 & e^{-t} \\ 0 & -1 \end{pmatrix} \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x(0) = x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Find control action from $t \in [0, 1]$

Linear Time Variant (LTV) system, is demonstrated as follows from the two sets of coupled differential equations for states are;

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1 = x_1 + e^{-t} x_2 \qquad \frac{\mathrm{d}}{\mathrm{d}t} x_2 = -x_2 + u(t)$$

In matrix form the solution of homogeneous system as we obtained for the two state variables is expressed as following;

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{(t-t_0)} & \frac{1}{3} e^{(t-2t_0)} - \frac{1}{3} e^{(t_0-2t)} \\ 0 & e^{-(t-t_0)} \end{pmatrix} \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \Phi(t, t_0) \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

Thus from the homogeneous system's solution we obtain the state transition matrix as

$$\Phi(t, \tau) = \begin{pmatrix} e^{t-\tau} & \frac{1}{3}\left(e^{t-2\tau} - e^{-2t+\tau}\right) \\ 0 & e^{-t+\tau} \end{pmatrix} \qquad \Phi(0, \tau) = \begin{pmatrix} e^{-\tau} & \frac{1}{3}\left(e^{-2\tau} - e^{\tau}\right) \\ 0 & e^{\tau} \end{pmatrix}$$

Calculating Gramian etc.

$$Q(0,1)=\int_0^1 \Phi(0,\tau)BB^*\Phi^*(0,\tau)\,d\tau$$

$$=\int_0^1 \begin{pmatrix} \frac{1}{9}\left(e^{-4\tau}-2e^{-\tau}+e^{2\tau}\right) & \frac{1}{3}\left(e^{-\tau}-e^{2\tau}\right) \\ \frac{1}{3}\left(e^{-\tau}-e^{2\tau}\right) & e^{2\tau} \end{pmatrix} d\tau = \begin{pmatrix} 0.2417 & -0.8541 \\ -0.8541 & 3.1945 \end{pmatrix}$$

$$Q^{-1}(0,1)=\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}$$

$$\det Q(0,1)=0.04262\neq 0$$

thus system of this example is completely controllable.

The minimal energy input control vector is

$$\bar{u}(t)=-\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ \frac{1}{3}\left(e^{-2t}-e^{t}\right) & e^{t} \end{pmatrix}\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}\left[\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix} e^{-1} & \frac{1}{3}\left(e^{-2}-e^{1}\right) \\ 0 & e^{1} \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right]$$

$$=-\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ \frac{1}{3}\left(e^{-2t}-e^{t}\right) & e^{t} \end{pmatrix}\begin{pmatrix} 74.9531 & 20.0399 \\ 20.0399 & 5.6710 \end{pmatrix}\begin{bmatrix} 0.4931 \\ -2.7183 \end{bmatrix}=5.8384e^{-2t}-0.3026e^{t}$$

The state trajectory due to above obtained control input is obtained by

$$\bar{x}(t)=\begin{pmatrix} e^{t} & \frac{1}{3}\left(e^{t}-e^{-2t}\right) \\ 0 & e^{-t} \end{pmatrix}\left[\int_0^t \begin{pmatrix} \frac{1}{3}\left(e^{-2\tau}-e^{\tau}\right) \\ e^{\tau} \end{pmatrix}\left(5.8384e^{-2t}-0.3026e^{t}\right)d\tau\right]=\begin{pmatrix} 0.3856e^{t}-0.1513-1.9966e^{-2t}+1.4596e^{-2t} \\ -0.1513e^{t}+5.9897e^{-t}-5.8384e^{-2t} \end{pmatrix}$$

Calculating Φ via Laplace

$\Phi(t) = e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$ **is solution of** $D_t x(t) = Ax(t)$ **for LTI**

As an example $A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}$

$$\Phi(t) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} s+3 & 2 \\ -1 & s \end{pmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s(s+3)+2}\begin{pmatrix} s & -2 \\ 1 & s+3 \end{pmatrix}\right]$$

$$= \mathcal{L}^{-1}\begin{pmatrix} \frac{2}{s+2} - \frac{1}{s+1} & \frac{2}{s+2} - \frac{2}{s+1} \\ -\frac{1}{s+2} + \frac{1}{s+1} & -\frac{1}{s+2} + \frac{2}{s+1} \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-2t} - e^{-t} & 2e^{-2t} - 2e^{-t} \\ -e^{-2t} + e^{-t} & -e^{-2t} + 2e^{-t} \end{pmatrix}$$

$\Phi_\alpha(t) = \mathcal{L}^{-1}\left\{(s^\alpha I - A)^{-1}\right\}; \quad \alpha \in [0,1)$ **is solution of** $D_t^\alpha x(t) = Ax(t)$

This way we can find state transition matrix of LTI-FOS too.

Calculating Φ_α and $x(t)$ for a simple FOS

FOS is Λ : $\quad {}_{0+}^{\;C}D_t^{\alpha} x(t) = u(t); \quad \alpha \in (0,1]; \quad x(0) = a \in \mathbb{R}; \quad x(T) = b \in \mathbb{R}; \quad T > 0$

In terms of system matrix equation ${}^{C}D^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t)$ **in this case** $\mathrm{A}=0; \quad \mathrm{B}=1$

Here

$$\Phi_{\alpha}(t) = e_{\alpha}^{\mathrm{A}t} = t^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\mathrm{A}^{k}t^{k\alpha}}{\Gamma[(k+1)\alpha]}\right) = t^{\alpha-1}\left(\frac{1}{\Gamma(\alpha)} + \mathrm{A}\frac{t^{\alpha}}{\Gamma(2\alpha)} + \ldots\right) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \qquad ;\mathrm{A}=0$$

The integral $\tilde{\Phi}_{\alpha}(t) = \mathrm{I} + \int_0^t e_{\alpha}^{\mathrm{A}\tau}\mathrm{A}\,d\tau = \mathrm{I} = 1$ **for** $\mathrm{A}=0$

Therefore the state trajectory of system-the solution is

$$x(t)\Big|_0^T = x(0)\tilde{\Phi}_{\alpha}(t) + \int_0^T \Phi_{\alpha}(t-\tau)u(\tau)\mathrm{d}\tau = a + \frac{1}{\Gamma(\alpha)}\int_0^T (T-t)^{\alpha-1}u(t)\mathrm{d}t$$

Gramian for Fractional Order System (FOS)

Our integer order Gramian of control is $Q(t_0,t_1) \triangleq \int_{t_0}^{t_1} \Phi(t_0,t)B(t)B^*(t)\Phi^*(t_0,t)\,dt$

or $Q(t) = \int_0^t e^{-A\tau}BB^* e^{-A^*\tau}d\tau = \int_0^t e^{A(t-\tau)}BB^* e^{A^*(t-\tau)}d\tau$ **replacing** e^{At} **by** e_α^{At} **we get**

$$\int_0^t e_\alpha^{A(t-\tau)}BB^* e_\alpha^{A^*(t-\tau)}d\tau = \int_0^T (T-\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(A(T-\tau)^\alpha)^k}{\Gamma((1+k)\alpha)}BB^*(T-\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(A^*(T-\tau)^\alpha)^k}{\Gamma((1+k)\alpha)}d\tau$$

this integral as similar control Gramian. Surely this diverges as $\tau \to T$

In order to have converging integral we multiply the above integral by $(T-\tau)^{2(1-\alpha)}$

and define the new control Gramian (for FOS) as:

$$Q_\alpha(0,T) = \int_0^T (T-t)^{2(1-\alpha)} e_\alpha^{A(T-t)}BB^* e_\alpha^{A^*(T-t)}dt$$

$$= \int_0^T \left[(T-t)^{2(1-\alpha)}\right]\Phi_\alpha(T-t)BB^*\Phi_\alpha^*(T-t)\,dt$$

The control law for FOS

$$\Sigma: \quad {}_{0+}^{C}D_t^{\alpha} x(t) = \mathrm{A}x(t) + \mathrm{B}u(t) \qquad y(t) = \mathrm{C}x(t); \quad 0 < \alpha \le 1; \quad x(0) = a$$

$$x(t) \in \mathbb{R}^{n\times 1}; \quad u(t) \in \mathbb{R}^{m\times 1}; \quad \mathrm{A} \in \mathbb{R}^{n\times n}; \quad \mathrm{B} \in \mathbb{R}^{n\times m}; \quad \mathrm{C} \in \mathbb{R}^{p\times n}$$

$$Q_\alpha(0,T) = \int_0^T (T-t)^{2(1-\alpha)} e_\alpha^{\mathrm{A}(T-t)} \mathrm{B}\mathrm{B}^* e_\alpha^{\mathrm{A}^*(T-t)} \mathrm{d}t = \int_0^T \left[(T-t)^{2(1-\alpha)} \right] \Phi_\alpha(T-t)\mathrm{B}\mathrm{B}^*\Phi_\alpha^*(T-t)\mathrm{d}t$$

For any states $a, b \in \mathbb{R}^{n\times 1}$ **the control law**

$$\overline{u}(t) = -(T-t)^{2(1-\alpha)} \mathrm{B}^* \Phi_\alpha^*(T-t) Q_\alpha^{-1} f_T(a,b); \qquad t \in [0, T)$$

Where $f_T(a,b) = \left(\mathrm{I} + \int_0^T \Phi_\alpha(t)\mathrm{A}\,dt \right) a - b = -b + \tilde{\Phi}_\alpha(T)a$ **and** $\overline{u}(T) = 0$

drives point $a \rightarrow b$

This is similar to integer order system which is $u(t) = -\mathrm{B}^*(t)\Phi^*(t_0,t)Q^{-1}(t_0,t_1)\left[x_0 - \Phi(t_0,t_1)x_1\right]$

with extra factor as $(T-t)^{2(1-\alpha)}$ **for neutralizing singularity at** T.

The control energy for FOS

Among all possible controls driving $a \to b$ **in time** T **the control** $\overline{u}(t)$

$$Q_{\alpha}(0,T) = \int_0^T \left[(T-t)^{2(1-\alpha)} \right] \Phi_{\alpha}(T-t) \mathrm{B}\mathrm{B}^{*} \Phi_{\alpha}^{*}(T-t) \mathrm{d}t$$

minimizes the integral

$$J_{m(0-T)} = \int_0^T \left| (T-t)^{\alpha-1} u(t) \right|^2 \mathrm{d}t$$

This is similar to integer order control effort which $J_{(t_0-t)} = \int_{t_0}^{t} u^{*}(t) u(t) \mathrm{d}t$

with extra term $(T-t)^{2(1-\alpha)}$

as we discussed at length that from integer order multivariate control the instantaneous control $u(t)$ **gets as convoluted action in case of fractional order system** $\sim (T-t)^{(1-\alpha)} u(t)$

Application to FOS

The Fractional Order System is:

$$\Sigma 1: \begin{cases} {}^{C}D_{0+}^{0.5} x_1(t) = x_2(t) \\ {}^{C}D_{0+}^{0.5} x_2(t) = u(t) \end{cases} \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad a = (1,0)^*; \quad b = (0,0)^*$$

Using Laplace we get State Transition Matrix

$$\Phi_{\alpha}(t) = \mathcal{L}^{-1}\left\{(s^{0.5}\mathrm{I} - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\left(\begin{bmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{s} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{pmatrix} \sqrt{s} & -1 \\ 0 & \sqrt{s} \end{pmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\begin{pmatrix} \sqrt{s} & 1 \\ 0 & \sqrt{s} \end{pmatrix}\right\} = \mathcal{L}^{-1}\begin{pmatrix} \frac{1}{\sqrt{s}} & \frac{1}{s} \\ 0 & \frac{1}{\sqrt{s}} \end{pmatrix}$$

Using $\mathcal{L}^{-1}\{s^{-\alpha}\} = t^{\alpha-1}/\Gamma(\alpha);\quad \mathcal{L}^{-1}\{s^{-0.5}\} = 1/\sqrt{\pi t}$ **and** $\mathcal{L}\left\{\begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}\right\} = \frac{1}{s}$ **we got** Φ_{α}

The two Green's function or state transition matrices for the FOS are thus:

$$\Phi_{\alpha}(t) = \begin{pmatrix} \frac{1}{\sqrt{\pi t}} & 1 \\ 0 & \frac{1}{\sqrt{\pi t}} \end{pmatrix}$$ **and** $$\tilde{\Phi}_{\alpha}(t) = \mathrm{I} + \int_0^t \Phi_{\alpha}(\tau)\mathrm{A}d\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{1}{\sqrt{\pi\tau}} & 1 \\ 0 & \frac{1}{\sqrt{\pi\tau}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} d\tau = \begin{pmatrix} 1 & 2\sqrt{\frac{t}{\pi}} \\ 0 & 1 \end{pmatrix}$$

State trajectory-soln. to FOS

$$\sum 1: \quad \begin{cases} {}^{C}D_{0+}^{0.5} x_1(t) = x_2(t) \\ {}^{C}D_{0+}^{0.5} x_2(t) = u(t) \end{cases} \qquad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad a = (1,0)^{*}; \quad b = (0,0)^{*}$$

$$x(t) = \begin{pmatrix} 1 & \dfrac{2\sqrt{t}}{\sqrt{\pi}} \\ 0 & 1 \end{pmatrix} a + \int_0^t \begin{pmatrix} \dfrac{1}{\sqrt{\pi(t-\tau)}} & 1 \\ 0 & \dfrac{1}{\sqrt{\pi(t-\tau)}} \end{pmatrix} \mathrm{B}\, u(\tau)\, \mathrm{d}\tau$$

Taking $u(t) \equiv 1$ **then for a given** a

$$x(t) = \begin{pmatrix} 1 + t & 2\sqrt{\dfrac{t}{\pi}} \end{pmatrix}^{*}$$

Meaning for a constant $u(.) \equiv 1$ **for** $t > 0$ **we cannot steer the given initial condition to the origin**

Gramian, control law & effort for the FOS

Let $f_T(a,b) = \tilde{\Phi}_\alpha(T)a - b = a$

The control Gramian form is

$$Q_\alpha = \int_0^T \left[(T-t)^{2(1-\alpha)} \right] \Phi_\alpha(T-t) \mathrm{B}\mathrm{B}^* \Phi_\alpha^*(T-t) \mathrm{d}t = \begin{pmatrix} \frac{T^2}{2} & \frac{2T^{3/2}}{3\sqrt{\pi}} \\ \frac{2T^{3/2}}{3\sqrt{\pi}} & \frac{T}{\pi} \end{pmatrix}$$

And the control input is:

$$\overline{u}(t) = -(T-t)^{2(1-\alpha)} \mathrm{B}^* \Phi_\alpha^*(T-t) Q_\alpha^{-1} f_T(a,b) = -\frac{18(T-t)}{T^2} + \frac{12\sqrt{T-t}}{T^{3/2}}$$

Drives the point $a \to b$ **with modified energy**

$$J_{m(0-T)} = \int_0^T \left| (T-t)^{-0.5} \overline{u}(t) \right|^2 \mathrm{d}t = \frac{18}{T^2}$$

End of part-5

Generalization of Fractional Calculus and utility

Why Generalization of Fractional Calculus?

To place all the fractional derivatives/integral formulations unified:
Riemann-Liouvelli,
Caputo,
Hadamard,
Erdely-Kober
Riesz,
and many new formulations as required.

Agarwal's formulation of Fractional Calculus of Variations

Agarwal's way for solving extraordinary integral equations

Fractional integral of function with respect to another function

With respect to *z*(*x*) and with weight *w*(*x*), causal integration is thus defined as

$$ {}_{a+}I^{\alpha}_{x,[z(x),w(x)]} f(x) = \frac{1}{[w(x)]\Gamma(\alpha)} \int_a^x [z(x)-z(t)]^{\alpha-1} w(t) z'(t) f(t) \mathrm{d}t \qquad \alpha > 0 $$

With respect to *z*(*x*) and with weight *w*(*x*), non-causal integration is thus defined as

$$ {}_{x}I^{\alpha}_{b-,[z(x),w(x)]} f(x) = \frac{[w(x)]}{\Gamma(\alpha)} \int_x^b [z(t)-z(x)]^{\alpha-1} [w(t)]^{-1} z'(t) f(t) \mathrm{d}t \qquad \alpha > 0 $$

Fractional derivative of function with respect to another function

With respect to *z*(*x*) and with weight *w*(*x*), causal derivative in RL sense is

$$ {}_{a+}D^{\alpha}_{x,[z(x),w(x)]} f(x) = [w(x)]^{-1} \left(\frac{1}{z'(x)} D_x \right)^m [w(x)] \left({}_{a+}I^{m-\alpha}_{x,[z(x),w(x)]} \right) $$

$$ \alpha > 0 \qquad m-1 < \alpha < m $$

With respect to *z*(*x*) and with weight *w*(*x*), non-causal derivative in RL sense is

$$ {}_{x}D^{\alpha}_{b-,[z(x),w(x)]} f(x) = [w(x)] \left(\frac{-1}{z'(x)} D_x \right)^m [w(x)]^{-1} \left({}_{x}I^{m-\alpha}_{b-,[z(x),w(x)]} f(x) \right) $$

$$ \alpha > 0 \qquad m-1 < \alpha < m $$

Similarly one can have Caputo type formulation ${}^{C}_{a+}D^{\alpha}_{x,[z(x),w(x)]}$; ${}^{C}_{x}D^{\alpha}_{b-,[z(x),w(x)]}$

Caputo-RL relation in generalized formulation

From generalization of fundamental theorem we have RL Caputo relation as

$$D_x^{\alpha} f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} + {}^{C}D_x^{\alpha} f(x)$$

Further generalization is

$$_{a+}D^{\alpha}_{x,[z,w]} f(x) = \frac{w(a)}{w(x)} \sum_{k=0}^{m-1} \frac{(D^{(k)}_{x,[z,w]} f)(a)}{\Gamma(k+1-\alpha)} [z(x)-z(a)]^{k-\alpha} + {}_{a+}^{C}D^{\alpha}_{x,[z,w]} f(x)$$

Where causal *D* operator is

$$(D^{(1)}_{x,[z,w]} f)(x) \equiv [w(x)]^{-1} \left[\left(\frac{1}{z'(x)} D_x \right) (w(x) f(x)) \right](x)$$

Reduce to RL or Caputo type

choose $z(t) = t \qquad w(t) = 1 \qquad z'(t) = 1$

$$ {}_{a+}I^{\alpha}_{x,[z,w]} f(x) = \frac{1}{[w(x)]\Gamma(\alpha)} \int_a^x [z(x) - z(t)]^{\alpha-1} w(t) z'(t) f(t) \mathrm{d}t \qquad \alpha > 0 $$

$$ {}_{a+}I^{\alpha}_{x,[t,1]} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \mathrm{d}t = {}_aI^{\alpha}_x f(x) $$

$$ {}_{a+}D^{\alpha}_{x,[z,w]} f(x) = [w(x)]^{-1} \left(\frac{1}{z'(x)} D_x \right)^m [w(x)] \left({}_{a+}I^{m-\alpha}_{x,[z(x),w(x)]} f(x) \right) \qquad \alpha > 0 \qquad m-1 < \alpha < m $$

$$ {}_{a+}D^{\alpha}_{x,[t,1]} f(x) = D^m_x \, {}_{a+}I^{m-\alpha}_{x,[t,1]} f(x) = \frac{d^m}{dx^m} \left(\frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f(t) \mathrm{d}t \right) = {}_aD^{\alpha}_x f(x) $$

similarly the other RL and Caputo are obtained

$$ {}_xI^{\alpha}_{b-,[t,1]} f(x) = {}^C_xI^{\alpha}_b f(x); \qquad {}_xD^{\alpha}_{b-,[t,1]} f(x) = {}_xD^{\alpha}_b f(x) $$

$$ {}^C_{a+}D^{\alpha}_{x,[t,1]} f(x) = {}^C_aD^{\alpha}_x f(x); \qquad {}^C_xD^{\alpha}_{b-,[t,1]} f(x) = {}^C_xD^{\alpha}_b f(x) $$

Reduce to Hadamard (*H*) type

choose $z(t) = \ln t \qquad w(t) = 1 \qquad z'(t) = (1/t)$

$$ {}_{a+}I^{\alpha}_{x,[z,w]} f(x) = \frac{1}{[w(x)]\Gamma(\alpha)} \int_a^x [z(x) - z(t)]^{\alpha-1} w(t) z'(t) f(t) \mathrm{d}t \qquad \alpha > 0 $$

$$ {}_{a+}I^{\alpha}_{x,[\ln t,1]} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\frac{\ln x}{\ln t} \right)^{\alpha-1} \left(\frac{f(t)}{t} \right) \mathrm{d}t = {}^{H}_{a}I^{\alpha}_{x} f(x) $$

$$ {}_{a+}D^{\alpha}_{x,[z,w]} f(x) = [w(x)]^{-1} \left(\frac{1}{z'(x)} D_x \right)^m [w(x)] \left({}_{a+}I^{m-\alpha}_{x,[z(x),w(x)]} f(x) \right) \qquad \alpha > 0 \qquad m-1 < \alpha < m $$

$$ {}_{a+}D^{\alpha}_{x,[\ln t,1]} f(x) = D^m_x \, {}_{a+}I^{m-\alpha}_{x,[\ln t,1]} f(x) = \frac{1}{\Gamma(m-\alpha)} \left(x \frac{d}{dx} \right)^m \left(\int_a^x \left(\frac{\ln t}{\ln x} \right)^{m-\alpha-1} \left(\frac{f(t)}{t} \right) \mathrm{d}t \right) = {}^{H}_{a}D^{\alpha}_{x} f(x) $$

similarly the other are obtained

$$ {}_{x}I^{\alpha}_{b-,[\ln t,1]} f(x) = {}^{H}_{x}I^{\alpha}_{b} f(x); \qquad {}_{x}D^{\alpha}_{b-,[\ln t,1]} f(x) = {}^{H}_{x}D^{\alpha}_{b} f(x) $$

$$ {}^{C}_{a+}D^{\alpha}_{x,[\ln t,1]} f(x) = {}^{HC}_{a}D^{\alpha}_{x} f(x); \qquad {}^{C}_{x}D^{\alpha}_{b-,[\ln t,1]} f(x) = {}^{HC}_{x}D^{\alpha}_{b} f(x) $$

Reduce to Modified Erdelyi-Kober (*MEK*) type

choose $z(t) = t^{\sigma}$ $w(t) = t^{\sigma\eta}$ $z'(t) = \sigma t^{\sigma-1}$

$$ {}_{a+}I^{\alpha}_{x,[z,w]} f(x) = \frac{1}{[w(x)]\Gamma(\alpha)} \int_a^x [z(x) - z(t)]^{\alpha-1} w(t) z'(t) f(t) \mathrm{d}t \qquad \alpha > 0 $$

$$ {}_{a+}I^{\alpha}_{x,[t^{\sigma},t^{\sigma\eta}]} f(x) = \frac{\sigma t^{-\sigma(\eta)}}{\Gamma(\alpha)} \int_a^x (x^{\sigma} - t^{\sigma})^{\alpha-1} t^{\sigma(1-\eta)-1} f(t)\mathrm{d}t = {}^{MEK}_{\ a}I^{\alpha}_{x} f(x) $$

$$ {}_{a+}D^{\alpha}_{x,[z,w]} f(x) = [w(x)]^{-1} \left(\frac{1}{z'(x)} D_x \right)^m [w(x)] \left({}_{a+}I^{m-\alpha}_{x,[z(x),w(x)]} f(x) \right) \qquad \alpha > 0 \qquad m-1 < \alpha < m $$

$$ {}_{a+}D^{\alpha}_{x,[t^{\sigma},t^{\sigma\eta}]} f(x) = D^m_x \, {}_{a+}I^{m-\alpha}_{x,[t^{\sigma},t^{\sigma\eta}]} f(x) = x^{-\sigma\eta} \left(\frac{1}{\sigma x^{\sigma-1}} \frac{d}{dx} \right)^m x^{\sigma n} \left(\frac{\sigma t^{-\sigma(\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma(1-\eta)-1} f(t)\mathrm{d}t}{(x^{\sigma} - t^{\sigma})^{1-\alpha}} \right) = {}^{MEK}_{\ a}D^{\alpha}_{x} f(x) $$

similarly the other are obtained

$$ {}_{x}I^{\alpha}_{b-,[t^{\sigma},t^{\sigma\eta}]} f(x) = {}^{MEK}_{\ x}I^{\alpha}_{b} f(x); \qquad {}_{x}D^{\alpha}_{b-,[t^{\sigma},t^{\sigma\eta}]} f(x) = {}^{MEK}_{\ x}D^{\alpha}_{b} f(x) $$

$$ {}^{C}_{a+}D^{\alpha}_{x,[t^{\sigma},t^{\sigma\eta}]} f(x) = {}^{MEK-C}_{\ a}D^{\alpha}_{x} f(x); \qquad {}^{C}_{x}D^{\alpha}_{b-,[t^{\sigma},t^{\sigma\eta}]} f(x) = {}^{MEK-C}_{\ x}D^{\alpha}_{b} f(x) $$

Generalized fractional integration& derivative operator

Generalized fractional integration with a kernel k_α in interval [*a*,*b*] is

$$\mathcal{I}_P^{\alpha} f(t) = p\int_a^t k_\alpha(t,\tau) f(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t) f(\tau)\mathrm{d}\tau, \quad \alpha \in (0,1); \quad t \in [a,b]$$

$$P = \{a, t, b, p, q\}; \quad P* = \{a, t, q, p\}$$

Choice of kernel and P-set gives RL fractional causal and non-causal (dual) integration

$$k_\alpha(t-\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}; \quad P = \{a,t,b,1,0\}; \quad \mathcal{I}_P^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau = {}_aI_t^{\alpha} f(t)$$

$$k_\alpha(t-\tau) = \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}; \quad P* = \{a,t,b,0,1\}; \quad \mathcal{I}_{P*}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)\mathrm{d}\tau = {}_tI_b^{\alpha} f(t)$$

Generalized derivative operator can be constructed as

$$\mathfrak{D}_P^{\alpha} = D \circ \mathcal{I}_P^{1-\alpha}; \quad D = \mathrm{d}/\mathrm{d}t$$

$${}^{C}\mathfrak{D}_P^{\alpha} = \mathcal{I}_P^{1-\alpha} \circ D$$

These are RL and Caputo type operator got from basic definition of generalized fractional integration

Integration by parts for generalized fractional integration operator

Using definition of generalized integration and Fubini's rule to change order of integration we have the following derivation

$$\int_a^b g(t)(\mathcal{I}_P^{\alpha} f(t))\mathrm{d}t = \int_a^b \mathrm{d}t g(t)\left[p\int_a^t f(\tau)k_\alpha(t,\tau)\mathrm{d}\tau + q\int_t^b f(\tau)k_\alpha(\tau,t)\mathrm{d}\tau \right]$$

$$= p\int_a^b g(t)\mathrm{d}t\int_a^t f(\tau)k_\alpha(t,\tau)\mathrm{d}\tau + q\int_a^b g(t)\mathrm{d}t\int_t^b f(\tau)k_\alpha(\tau,t)\mathrm{d}\tau$$

$$= p\int_a^b f(\tau)\mathrm{d}\tau\int_\tau^b g(t)k_\alpha(t,\tau)\mathrm{d}t + q\int_a^b f(\tau)d\tau\int_a^\tau g(t)k_\alpha(\tau,t)\mathrm{d}t$$

$$= \int_a^b \mathrm{d}\tau f(\tau)\left[p\int_\tau^b g(t)k_\alpha(t,\tau)\mathrm{d}t + q\int_a^\tau g(t)k_\alpha(\tau,t)\mathrm{d}t \right]$$

$$= \int_a^b \mathrm{d}\tau f(\tau)\left[q\int_a^\tau g(t)k_\alpha(\tau,t)\mathrm{d}t + p\int_\tau^b g(t)k_\alpha(t,\tau)\mathrm{d}t \right]$$

$$= \int_a^b f(\tau)(\mathcal{I}_{P*}^{\alpha} g(\tau))\mathrm{d}\tau$$

Integration by parts formulas

Using the derived expression involving generalized integration and its dual that is

$$\int_a^b g(t)(\mathcal{I}_P^\alpha f(t))\mathrm{d}t = \int_a^b f(\tau)(\mathcal{I}_{P*}^\alpha g(\tau))\mathrm{d}\tau$$

We can write the following expressions

$$\int_a^b g(t)\left({}_a I_t^\alpha f(t)\right)\mathrm{d}t = \int_a^b f(\tau)\left({}_t I_b^\alpha g(\tau)\right)\mathrm{d}\tau$$

$$\int_a^b g(t)\left({}_a D_t^\alpha f(t)\right)\mathrm{d}t = \int_a^b f(\tau)\left({}_t D_b^\alpha g(\tau)\right)\mathrm{d}\tau$$

$$\int_a^b z'(x) f(x)({}_{a+} I_{x,[z,w]}^\alpha g(x))\mathrm{d}x = \int_a^b z'(x) g(x)({}_x I_{b-,[z,w]}^\alpha f(x))\mathrm{d}x$$

$$\int_a^b z'(x) f(x)({}_{a+} D_{x,[z,w]}^\alpha g(x))\mathrm{d}x = \int_a^b z'(x) g(x)({}_x D_{b-,[z,w]}^\alpha f(x))\mathrm{d}x$$

Integration by parts expansion for fractional derivative

For $\alpha \in (0,1), \quad t \in [0,1], \quad P = \{a,t,b,1,0\}; \quad P* = \{a,t,b,0,1\}$

Using $\int_a^b g(t)(\mathcal{I}_P^\alpha f(t))\mathrm{d}t = \int_a^b f(\tau)(\mathcal{I}_{P*}^\alpha g(\tau))\mathrm{d}\tau;$ and $\int (I)(II)\mathrm{d}x = I\int (II)\mathrm{d}x - \int (I'\int II\mathrm{d}x)\mathrm{d}x$

We get the following expansion

$$\int_a^b g(t)(\mathfrak{D}_P^\alpha f(t))\mathrm{d}t = g(t)\int_a^b \mathfrak{D}_P^\alpha f(t)\mathrm{d}t - \int_a^b \mathrm{d}t\{(Dg(t))\int \mathfrak{D}_P^\alpha f(t)\mathrm{d}t\}$$

$$= g(t)\int_a^b D\mathcal{I}_P^{1-\alpha} f(t)\mathrm{d}t - \int_a^b \mathrm{d}t\{(Dg(t)\int D\mathcal{I}_P^{1-\alpha} f(t)\mathrm{d}t\}$$

$$= g(t)\mathcal{I}_P^{1-\alpha} f(t)\Big|_a^b - \int_a^b Dg(t)\mathcal{I}_P^{1-\alpha} f(t)\mathrm{d}t$$

$$= g(t)\mathcal{I}_P^{1-\alpha} f(t)\Big|_a^b - \int_a^b f(t)\mathcal{I}_{P*}^{1-\alpha} Dg(t)\mathrm{d}t$$

$$= g(t)\mathcal{I}_P^{1-\alpha} f(t)\Big|_a^b - \int_a^b f(t)({}^C\mathfrak{D}_{P*}^\alpha g(t))\mathrm{d}t$$

$$\int_a^b g(t)({}_aD_t^\alpha f(t))\mathrm{d}t = g(t)[{}_aI_t^{1-\alpha} f(t)]\Big|_a^b - \int_a^b f(t)({}_b^CD_t^\alpha g(t))\mathrm{d}t$$

Similarly we get

$$\int_a^b g(t)({}_a^CD_t^\alpha f(t))\mathrm{d}t = g(t)[{}_aI_t^{1-\alpha} f(t)]\Big|_a^b - \int_a^b f(t)({}_bD_t^\alpha g(t))\mathrm{d}t$$

useful in derivations of Fractional Calculus of Variation principles

Classical Calculus of Variation

Let $J[y]$ **be the "functional" called performance index also.**

$$J[y] = \int_a^b F(t, y(t), y'(t))\mathrm{d}t\,; \quad J[y] = \int_a^b F(x_1, x_2, x_3)\mathrm{d}t$$

$$y(a) = y_a\,, \quad y(b) = y_b$$

Let y **be an extremizer of** J

Then y **satisfies Euler Lagrange equation**

$$\frac{\partial}{\partial x_2} F(t, y(t), y'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial x_3} F(t, y(t), y'(t)) = 0$$

$$\frac{\partial}{\partial y} F(t, y(t), y'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial y'} F(t, y(t), y'(t)) = 0$$

Fractional Calculus of Variation

First obtained by Riew, one of the earliest application of fractional calculus was to construct a complete mechanical description of the non conservative systems including Lagrangians and Hamiltonian mechanics. Fractional calculus provide necessary tools to apply variation principles to system characterized by dissipative forces even possible to deduce "non conservative extremals"

Let $J[y]$ **be the "functional" called performance index also.**

$$J[y] = \int_a^b F(t, y(t), y'(t), p\,{}_a^C D_t^\alpha y(t) + q\,{}_t^C D_b^\alpha y(t))\mathrm{d}t\,; \quad J[y] = \int_a^b F(x_1, x_2, x_3, x_4)\mathrm{d}t$$

$$p, q \in \mathbb{R}, \quad y(a) = y_a, \quad y(b) = y_b$$

Let y **be an extremizer of** J

Then y **satisfies Euler Lagrange equation**

$$\left(p\,{}_t D_b^\alpha \frac{\partial F}{\partial x_4} + q\,{}_a D_t^\alpha \frac{\partial F}{\partial x_4} \right) + \frac{\partial F}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial x_3} = 0$$

Fractional Calculus of Variation with generalized formulation

Agarwal's Fractional Calculus of Variation

Let $J[y]$ **be the "functional" called performance index also.**

$$J[y] = \int_a^b F(t, y(t), {}_a^C D_{t,[z,w]}^{\alpha} y(t))\mathrm{d}t \qquad 0 < \alpha < 1 \qquad y(a) = y_a, \quad y(b) = y_b$$

Let y **be an extremizer of** J

Then y **satisfies Euler Lagrange equation**

$$\frac{\partial F}{\partial y} + {}_t D_{b,[z,w]}^{\alpha} \frac{\partial F}{\partial [{}_a^C D_{t,[z,w]}^{\alpha} y]} = 0$$

When one boundary is not given take that as free boundary and thus we get

$$\left. {}_t D_{b,[z,w]}^{\alpha-1} \frac{\partial F}{\partial [{}_a D_{t,[z,w]}^{\alpha} y]} \right|_{t=b} = 0$$

Example
Fractional Calculus of Variation

Let $J[y]$ be the "functional" called performance index also in domain [0,1] to be minimized

$$J[y]=\int_0^1\left[\frac{1}{2}\left({}_0^C D_{t,[z,w]}^{\alpha}\,y(t)\right)^2-c(t)\,y(t)\right]z'(t)\,\mathrm{d}t \qquad 0<\alpha<1$$

$$F\left(t,y,{}_0^C D_{t,[z,w]}^{\alpha}y\right)=\left[\frac{1}{2}\left({}_0^C D_{t,[z,w]}^{\alpha}y\right)^2-c(t)\,y\right]$$

Euler Lagrange equation's application to above, gives following exterior differential eqn.

$$\frac{\partial F}{\partial y}+{}_t D_{b,[z,w]}^{\alpha}\frac{\partial F}{\partial\left[{}_a^C D_{t,[z,w]}^{\alpha}y\right]}={}_t D_{1-,[z,w]}^{\alpha}\left({}_{0+}^{C} D_{t,[z,w]}^{\alpha}y(t)\right)-c(t)=0$$

When one boundary is not given take that as free boundary and thus we get

$${}_t D_{b,[z,w]}^{\alpha-1}\left.\frac{\partial F}{\partial\left[{}_a D_{t,[z,w]}^{\alpha}y\right]}\right|_{t=b}={}_t I_{1-,[z,w]}^{1-\alpha}\left({}_a^C D_{t,[z,w]}^{\alpha}y\right)(1)=0$$

Solving this exterior differential equations we obtain $y(t)$ extremizing the required functional.
Similarly as in integer order calculus of variation, isoperimetric problems, Hamiltonian can be translated into fractional calculus of variation

Generalized operator of fractional calculus to solve integral equations

To solve the integral equation $\int_a^x \frac{e^{\lambda(x-t)}}{\sqrt{x-t}} y(t)\,\mathrm{d}t = f(x)$

use $w(t) = e^{-\lambda t}, \quad z(t) = t, \quad z'(t) = 1, \quad \alpha = 1/2, \quad m = 1$

$$ {}_{a+}I^{\alpha}_{x,[z(x),w(x)]} f(x) = \frac{1}{[w(x)]\Gamma(\alpha)} \int_a^x [z(x) - z(t)]^{\alpha-1} w(t) z'(t) f(t)\mathrm{d}t \qquad \alpha > 0 $$

Recast the problem as

$$ \Gamma(1/2)\left({}_a I^{1/2}_{x,[t,e^{-\lambda t}]} y(x) \right) = f(x) $$

Take ${}_a D^{1/2}_{x,[t,e^{-\lambda t}]}$ **of both sides and we get the solution in form generalized derivative**

$$ y(x) = \frac{1}{\Gamma(1/2)} {}_a D^{1/2}_{x,[t,e^{-\lambda t}]} f(x) $$

use

$$ {}_{a+}D^{\alpha}_{x,[z(x),w(x)]} f(x) = [w(x)]^{-1} \left(\frac{1}{z'(x)} D_x \right)^m [w(x)]\left({}_{a+}I^{m-\alpha}_{x,[z(x),w(x)]} \right) \qquad \alpha > 0 \qquad m - 1 < \alpha < m $$

to get compact solution $y(x) = \frac{1}{\pi} e^{\lambda x} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{e^{-\lambda t}}{\sqrt{x-t}} f(t)\,\mathrm{d}t$

End of part-6

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I am again at the very beginning

will try again to ‘fractionally’ learn the beautiful subject

the Fractional Calculus what nature understands the best

Thanking you all