Interdisciplinary Problems in Non-Linear Dynamics

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Non-Linear Dynamics with Fractional Calculus

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One of the greatest discovery that Newton made was that:

“The nature follows Mathematics”

The intriguing question for all of us even today is

Which Mathematics?

That is why we are here discussing Non-Linear Dynamics

Prologue to Non-Linear Dynamics

Representing real life systems as “Non-Linear” dynamics systems, “chaotic” systems, “irregular” systems is of increasingly of interest in interdisciplinary subjects; engineering, physical-chemical science, biological science, economics, medicine, geo-physics and so on.

The ‘non-integer’ representation of dimensions or ‘fractal’ dimensions is one possible parameter to characterize a chaotic, non-linear, continuous but non-differentiable systems.

A non-differential system, irregular system, erratic system, chaotic system, rough system, noisy system, has measure parameter as fractal dimension and can be ‘fractionally differentiated’ (otherwise difficult)-they are “uncertain”. These non-differentiability leads to uncertainty, can be studied via making them differentiable functions with ‘mean-function’ approximation at neighborhood.

It is about fractional integration fractional differentiation and fractal dimensions all are generalizations of normal integer order calculus and normal Euclidian dimensions, which enable us to possibly extract information out of these irregular behavior.

Riemann Liouvelli (RL) fractional integration-antiderivative

Repeated n-fold integration generalization to arbitrary order (real/complex)

$$I_t^1 f(t) = d_t^{-1} f(t) = \int_0^t f(\tau)d\tau$$

$$I_t^2 f(t) = d_t^{-2} f(t) = \int_0^t \int_0^t f(\tau)d\tau d\tau = \int_0^t (t-\tau) f(\tau)d\tau$$

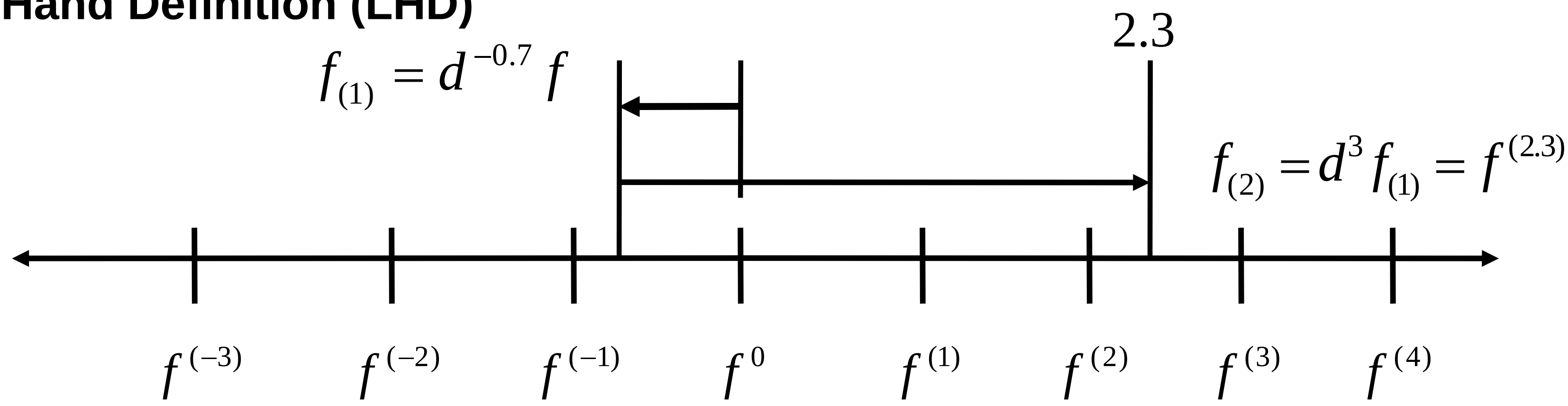
$$I_t^3 f(t) = d_t^{-3} f(t) = \int_0^t \int_0^t \int_0^t f(\tau)d\tau d\tau d\tau = \frac{1}{2}\int_0^t (t-\tau)^2 f(\tau)d\tau$$

$$I_t^n f(t) = d_t^{-n} f(t) = \int_0^t \int_0^t \underbrace{\cdots\cdots\int_0^t f(\tau)d\tau}_{n} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

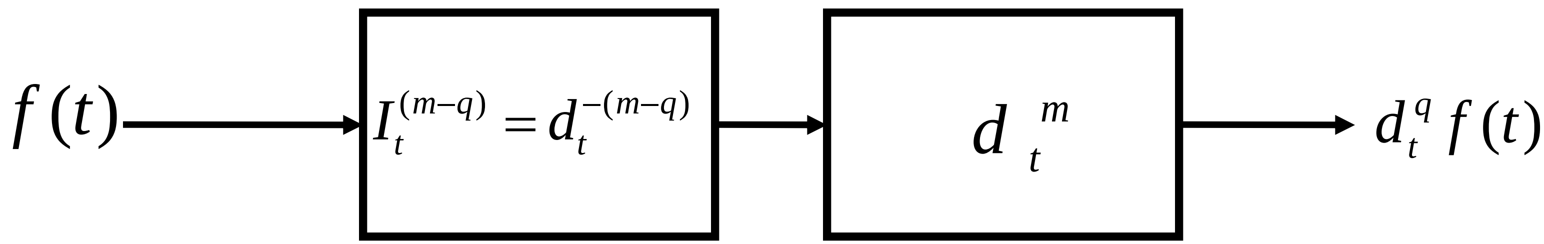
$$I_t^q f(t) = d_t^{-q} f(t) = \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1} f(\tau)d\tau$$

Riemann Liouvelli (RL) Fractional derivative

Left Hand Definition (LHD)



Here 'm' is the integer just greater than fractional order of derivative



$$d_t^q f(t) = d_t^m I_t^{m-q} f(t)$$

$$d_t^q f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-q)}\int_0^t (t-\tau)^{-q-1+m} f(\tau)\,d\tau\right]$$

NOTE: Fractional derivative is non-local property, and to obtain value at point you should know the character not only of past but of future too!! There exist Caputo RHD definition

Fractional derivative the Euler (1730) formula for monomial

$$\frac{d^n f(x)}{dx^n} = \underbrace{\frac{d}{dx}\frac{d}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\frac{d}{dx}}_{n} f(x)$$

$$\frac{d^n}{dx^n}\left\{x^m\right\} = m(m-1)(m-2)\ldots\ldots\ldots\ldots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\ldots\ldots\ldots(m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n}{dx^n}\left\{x^m\right\} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

$$\frac{d^{0.5}}{dx^{0.5}}\left\{x\right\} = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)}x^{1-0.5} = \frac{\sqrt{x}}{\Gamma(1+0.5)} = \frac{\sqrt{x}}{0.5\Gamma(0.5)} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

$$\frac{d^{0.5}}{dx^{0.5}}\left\{C\right\} = \frac{C}{\Gamma(0.5)}x^{-0.5}$$

For positive index the process is differentiation
For negative index the process is integration
Fractional derivative of constant is not ZERO!!
Fractional derivative of constant give SINGULARITY!

Irregularity & its identification

Normal Condition Beats: 90/70/90/70/90/70/90/70/90/70/90/70/90/70/90/70

Abnormal Condition Beats:90/70/70/90/90/90/70/70/90/90/70/90/70/70/90/70

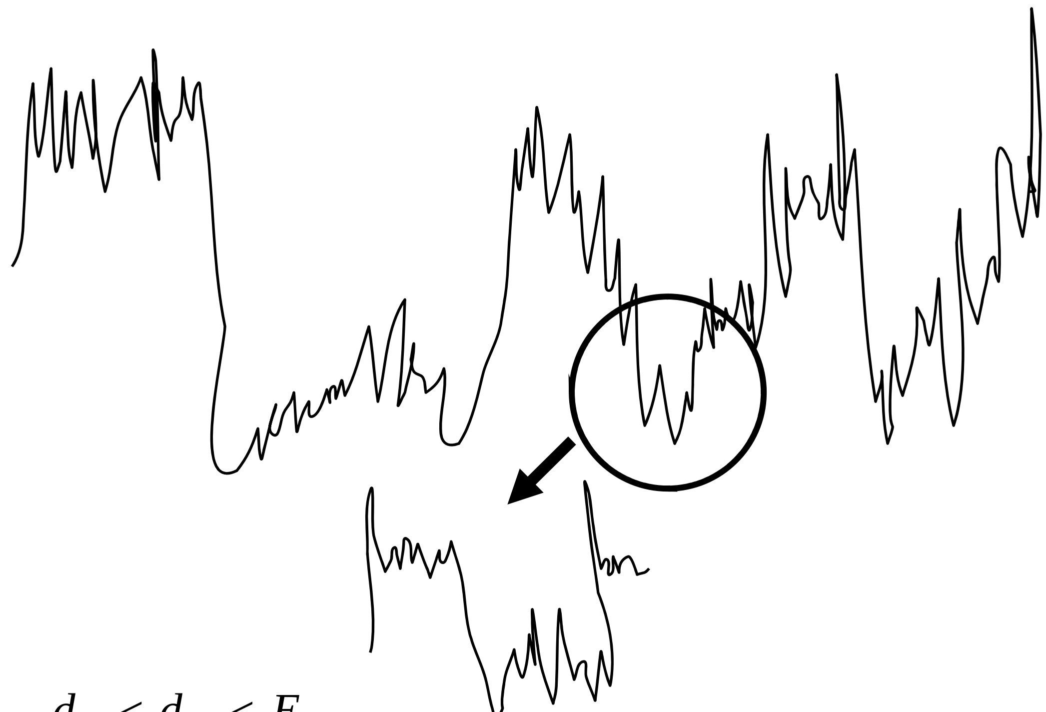
These two series have same mean, median, and variance and the two values (90 or 70) have the same probability of occurring ½ . Statistics fail to distinguish this; yet they are different. In first, one finds the next outcome with absolute certainty. In the second series we only know that next outcome will be either 90 or 70, but our guess will be wrong in 50% cases.

Fractals and multi-fractal functions and corresponding curves or surfaces are found in numerous places in non-linear and non-equilibrium phenomena. Like turbulence, Brownian paths, attractors of some dynamics, economics, seismology records are for examples of occurrence of continuous but highly irregular (non-differentiable) curves and surfaces.

Random functions have defined mean and variance (standard deviation). The non-random (power-law) functions have non definite mean or variance. Graphs are fractal set (self-similar) with no smallest scale. Connection between ‘local’ scaling behavior (fractal dimension) and order of Fractional Derivative is interesting to ‘instrument’ the irregularity of non-linear dynamic systems.

Irregular Rough Functions with fractal dimension to identify

Uni-fractals have same fractal dimensions whereas the multifractals have several fractal dimensions.
Fractal dimension is measure of roughness or irregularity or chaotic curves
They are UNREACHABLE as at a point you cannot draw tangent and reach to neighbourhood.
The first discovery of a continuous and nowhere differentiable function is thought to be by B. Bolzano in 1830 (unpublished till 1890), and the first published function is by K. Weierstrass (published in 1872)
Note this type of graph is visible in CRO for noise signal; (white-noise).
Try and plot a Brownian path, we will get this type of graph



$$d_T < d_F < E$$

$$C \in R^E$$

$$1 < d_F < 2$$

Complex-Weistrauss, defined in 1830!!

$$W_\alpha(x) = \sum_{n=1}^{\infty} (p)^{-\alpha n} \exp\left(i p^n x\right)$$

Cosine-Weistrauss

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi n)$$

$$0 < a < 1, b > 0, ab > 1 + \frac{3}{2}\pi$$

$$d_F = \frac{\log a}{\log b + 2}$$

Movements are Uncertain Irregular and Unreachable

Feynman and Hibbs have shown generic trajectories of quantum particles are continuous and nowhere differentiable, where their quadratic velocity exists and is given by following limit

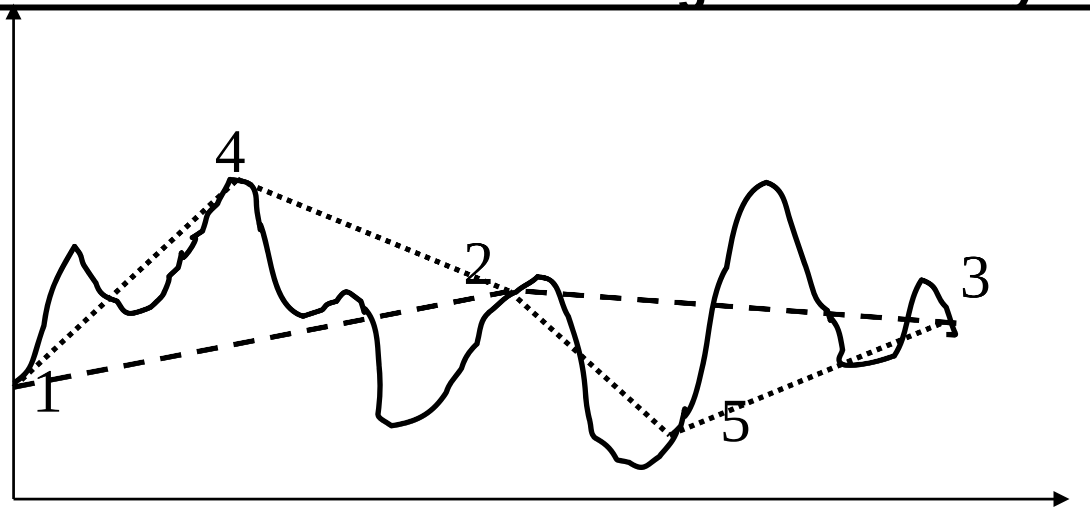
$$\lim_{t \to t'} \frac{\left(f(t) - f(t') \right)^2}{(t - t')} < \infty$$

Moreover for the Brownian motion, Einstein showed the following scaling

$$f(t+\tau) - f(t) \approx \sqrt{\tau} \qquad \tau > 0$$

The position function, which indicates the non-differentiability of the trajectory; above are similar in nature. The set of continuous functions with non zero and finite quadratic velocity is included $C^0\left(\left[a,b\right]\right)$ **the set of continuous and nowhere differentiable functions.**

Measuring the fractal dimension of irregular curve by divider step:



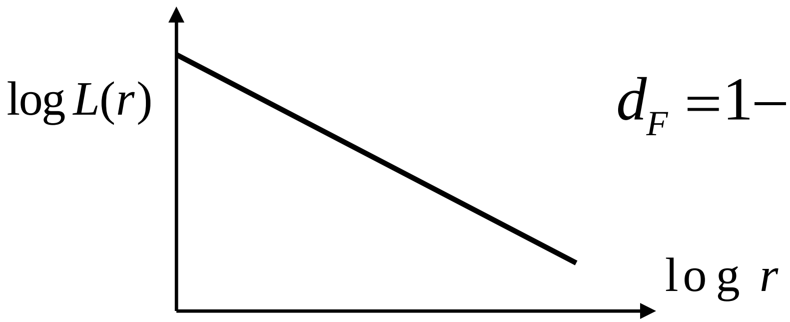
Basis of this is to measure the length of the curve by approximating it with number of straight-line segments called steps. The calculated length of the curve is the product of the number of steps and length of one step. As the step size is decreased the line segments can follow the curve more clearly; smaller scale-structures become more significant and the calculated length of curve increases. If the data (curve) follow a fractal model we have, a relation between the curve length and the step length

$$L(r) \propto r^{1-d_F} \qquad\qquad d_T < d_F < E$$

This is also called “Structured Walk Technique”

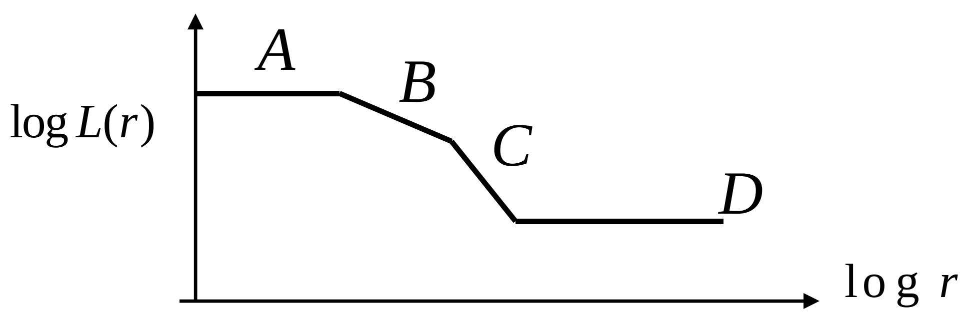
Mandelbrot-Richardson Plot:

Plot of Logarithm of Length and Log of step size is Mandelbrot-Richardson Plot. The slope of that curve is *S* then fractal dimension is (1-*S*) . The S of The curve is equal to or less than zero.



$$d_F = 1 - \lim_{r \to 0} \frac{\log L(r)}{\log r}$$

If the calculations of the length of the curve is performed with a step size that is too long, the main structure of the line described which gives a flat section (D), when the step size is too small, much less than the sampling interval, we are not able to recognize any new structure in the curve again results in flat section (A). The region (B) and (C) indicate two uncorrelated parts in the signal.



Roughness Exponent:

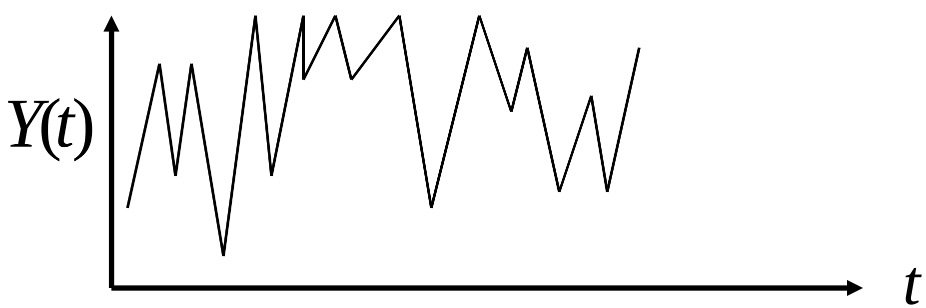
For a function $f(x)$ if there exists a polynomial $P_n(x)$ of degree $n < h$ and a constant C such that

$$\left| f(x) - P_n(x - x_0) \right| \leq C \left| x - x_0 \right|^h$$

The supremum of all exponents $h(x_0) \in (n, n+1)$ is Holder Exponent, which characterizes singularity strength. Holder Exponent describes the 'local' regularity (roughness) of function $f(x)$ at $x = x_0$.

Higher the exponent, more regular is local behavior.

$\left|\Delta Y_i\right| = \left(\Delta t_i\right)^{1/2}$ **is 'square-root' scaling of Brownian Motion, with** $h = 1/2$

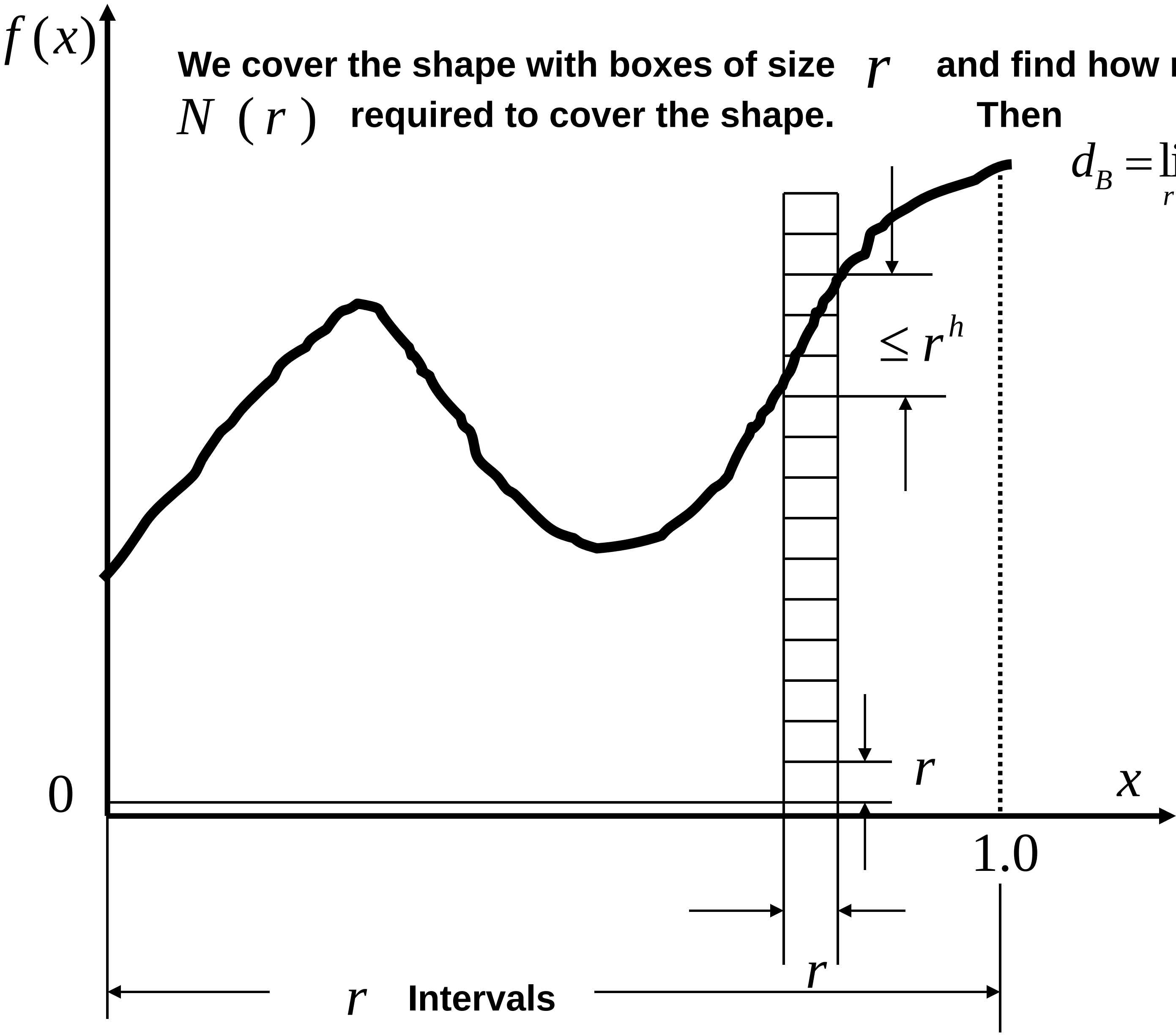


Roughness of Graph is: $h = \log\left|\Delta Y_i\right| / \log\left|\Delta t_i\right|$

Holder Exponent & (box) dimension of graph (function)

We cover the shape with boxes of size r and find how many boxes $N(r)$ required to cover the shape. Then

$$d_B = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)}$$



Box-dimension of irregular graph and Holder's exponent

1. **The function** $f(x)$ **defined in** $[0,1]$
2. **Divide** $0 \leq x \leq 1$ **into** r **equal intervals or almost equal intervals.**
3. **Above each interval make column of width** r **.**
4. **In the situation of scaling condition with Holder Exponent as:** $\Delta y = (\Delta x)^h$

means in each of the column of graph of $f(x)$ **passes through a height** r^h

5. **So the number of boxes needed to cover the part of the graph in that column is about (height of graph)** $\div$ **(height of box)=** $r^h \div r = r^{h-1}$
6. **The number of columns is** $1/r$ **in the length 1.00.**
7. **The number of these boxes of side** r **needed to cover the entire graph is**

$$(r^{h-1}) \times (1/r) = r^{h-2} = N(r)$$

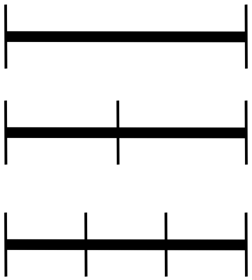
8.

$$d_B = \lim_{r \to 0} \frac{\log N(r)}{\log(1/r)} = \frac{\log r^{h-2}}{\log(1/r)} = 2 - h$$

The Holder exponent of the Brownian path is ½ , the scaling; hence the box, or the fractal dimension of Brownian motion is 1.5. The white noise is also scaled by ½ and has box or fractal dimensions 1.5.

Fractal Dimensions of self-similar figures

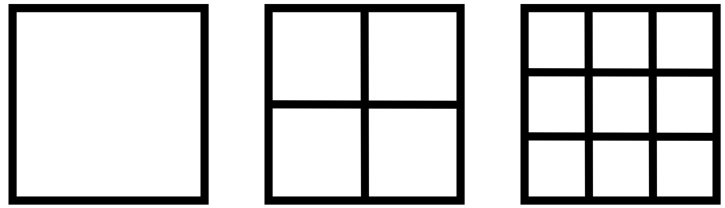
$$d_F = \lim_{\varepsilon \to 0} \frac{\log N}{\log\left(\frac{1}{\varepsilon}\right)}$$



$N = 2$ $\quad r=2, \varepsilon=1/2$

$N = 3$ $\quad r=3, \varepsilon=1/3$

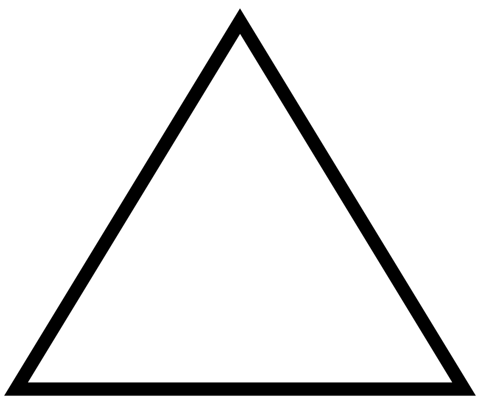
$$d_F = 1$$

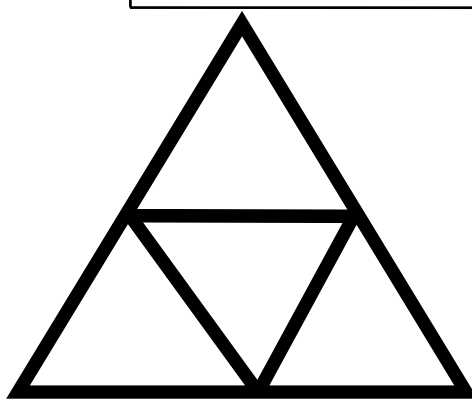


$N=4$ $\quad N=9$

$r=2, \varepsilon=1/2$ $\quad r=3, \varepsilon=1/3$

$$d_F = 2$$





$$N = 3$$

$$r = 2, \varepsilon = 1/2$$

$$d_F = \frac{\log 3}{\log\left(\frac{1}{1/2}\right)} = \log 3/\log 2 = 1.585$$

'Box' could be a circle of radius '*r*' or square of side '*r*' a sphere of radius '*r*' or triangle of side '*r*' or segment of side '*r*'. Infinitesimally all are same!!

Equivalence of Mandelbrot-Richardson fractal dimension and the Box-dimension

$$d_F = 1 - \frac{\log L(r)}{\log r} = 1 + \frac{\log L(r)}{\log(1/r)}$$

$$d_F - 1 = \frac{\log L(r)}{\log(1/r)}$$

$$L(r) = (1/r)^{d_F - 1} = r^{1-d_F}$$

$$d_B = \frac{\log N(r)}{\log(1/r)}$$

$$N(r) = r^{-d_B}$$

$$N(r) = L(r) \times (1/r)$$

$$L(r) = r \times r^{-d_B} = r^{1-d_B}$$

$$d_B = d_F$$

Both the methods give fractal dimension of irregular graphs called fractal dimension or Hausdroff's dimension. The efficient method of estimation is good R&D.

Scaling:

$$\frac{d^q f(\lambda x)}{dx^q} = \lambda^q \frac{d^q f(\lambda x)}{d(\lambda x)^q}$$

This property makes suitable for study of scaling. Local scaling behavior is it related to order of fractional derivative? However, this scaling property of fractional derivative implies study of 'self-similar' objects, distributions etc. Let $y(t) = t^\beta$ **then** $y(\lambda t) = (\lambda t)^\beta = \lambda^\beta y(t)$ **. A scaling in the time axis results in simple scaling of ordinate. The law** $y(t) = t^\beta$ **is not altered. In log-log plot this means a shift of** $\log \lambda$ **on the** $\log t$ **axis and shift** $\beta \log \lambda$ **in the ordinate. This scale-invariance is the cause of stability of system following power-law. Power-law relaxation is observed in various physical systems. Thus one often encounters algebraic relaxation** $\phi(t) \approx t^{-\alpha}$ **with** $0 < \alpha < 1$ **Therefore one often encounters fractional relations in frequency domain** $\omega^{-\alpha} F(\omega)$ **for example** $\omega^{-\alpha} \exp(-\omega \tau)$ **. Fractional relations of this self-similar (fractal) form can arise quite naturally in spectral domain. in time domain fractal form is** $g(t) \approx (t/\tau)^{-\alpha} \gamma^*(\beta, t/\tau)$ **; this form has late time power decay law.**

Local behavior at point:

Note the non-local character of the Fractional Derivative defined by RL definition and the non-constant 'fractional-derivative' of non-zero constant. These two features makes extraction of scaling information somewhat difficult. The problem is overcome by 'Local Fractional Derivative' LFD. Sometimes it is desirable to have 'Local-Character' in wide range of applications ranging from structure of 'differentiable' manifolds to various physical models. Secondly, the Fractional Derivative of constant is non-zero, consequently the magnitude of Fractional Derivative changes with addition of a constant to a function. The notion of LFD must address these issues. The logic is take fractional integral of order $(1-\alpha)$ of function minus the value of function at point of interest; then Differentiate the function and put limit tending to that point of interest.

$$I(x)=\frac{1}{\Gamma(1-\alpha)}\int_{x_0}^{x}(x-t)^{-\alpha}\left(f(t)-f(x_0)\right)dt$$

$$\mathbf{D}^{\alpha}f(x_0)=\lim_{x\to x_0}I^{/}(x)$$

Local Fractional Derivative (LFD) Kolwankar-Gangal (KG)

For a function $f:[0,1]\to\mathbf{R}$ **, the limit**

$$\mathbf{D}^{q} f(x_0) = \lim_{x\to x_0} \frac{d^{q}\left(f(x) - f(x_0)\right)}{\left[d(x - x_0)\right]^{q}}$$

exists and is finite then we say LFD of order q **where** $0 < q < 1$
at $x = x_0$ **exits**
In this definition the lower limit x_0 **is treated as a constant. The subtraction of** $f(x_0)$ **corrects for the fact that fractional derivative of constant (in RL) is not zero. Where the limit** $x\to x_0$ **is taken to remove non-local contents. This LFD (removing the non-local contents) allows the study of point wise behavior of** $f(x)$**.**

$$\mathbf{D}^{1} f(0) = \lim_{x\to 0} \frac{d}{dx} f(x)$$

Slope at origin!

KG Local Fractional Derivative for fractional order more than one:

$$\mathbf{D}^q f(x_0) = \lim_{x\to x_0} \frac{d^q\left(f(x) - \left[f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}}{2!}(x-x_0)^2 + \ldots \frac{f^{(N)}}{N!}(x-x_0)^N \right]\right)}{\left[d(x-x_0)\right]^q}$$

$$= \lim_{x\to x_0} \frac{d^q\left(f(x) - \sum_{n=0}^{N} \frac{f^{(n)}}{\Gamma(n+1)}(x-x_0)^n \right)}{\left[d(x-x_0)\right]^q}$$

Limit exists and is finite, where *N* is the largest integer for which *N*-th derivative of function $f(x)$ at point x_0 exists and is finite, then we say LFD of order q $N < q \le N+1$, at x_0 exists.

When q is positive integer, then integer order derivative is recovered.

For $q = 1, N = 0$

$$\mathbf{D}^1 f(x_0) = \lim_{x\to x_0} \frac{d}{d[(x-x_0)]}[f(x) - f(x_0)]$$

Critical Order:

Critical order $\alpha(x_0)$
Supremum of q **all Local Fractional Derivative of order less than** q **exists at** x_0 **.**

$$f(x) = a + bx + c|x|^{\beta} \qquad 1 < \beta < 2$$

$$f(0) = a \qquad f^{(1)}(0) = b \qquad f^{(2)}(0) = \infty$$

Nearest $N = 1$

$$\mathbf{D}^q f(0) = \lim_{x\to 0} \frac{d^q\left(f(x) - \left[f(0) + f^{(1)}(0)(x-0)\right]\right)}{\left[d(x-0)\right]^q} = \lim_{x\to 0}\frac{d^q}{dx^q}\left[a + bx + c|x|^{\beta} - (a+bx)\right]$$

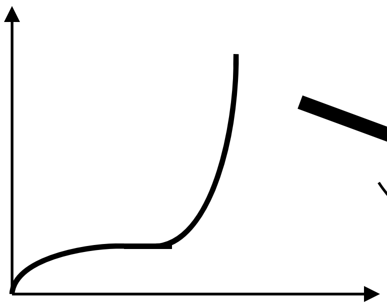
$$= \lim_{x\to 0} c\,\frac{\Gamma(\beta+1)}{\Gamma(\beta-q+1)}|x|^{\beta-q} = \begin{cases} \infty & ; q > \beta \\ 0 & ; q < \beta \end{cases}$$

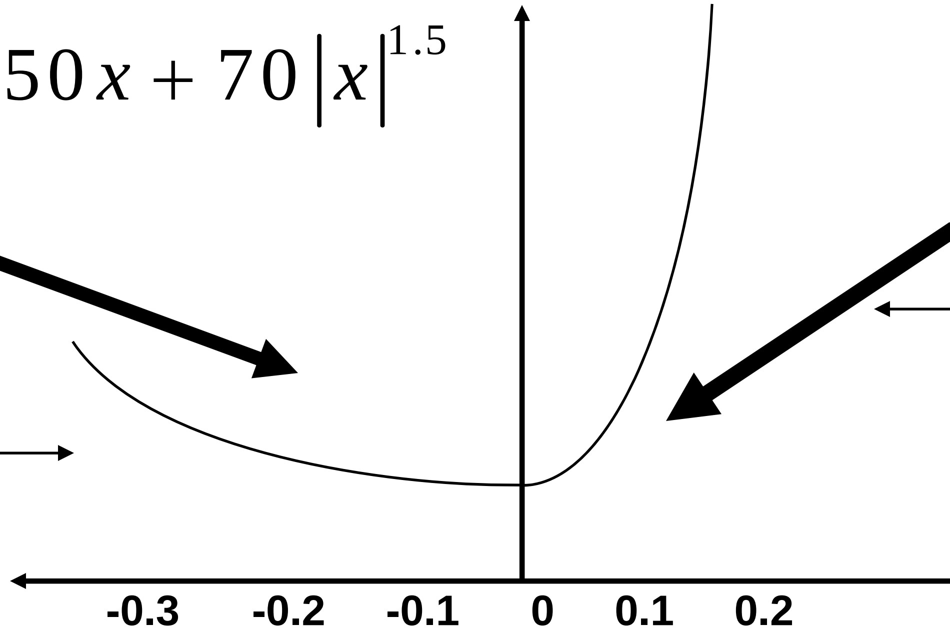
Critical Order of the function $f(x)$ **at** $x = 0$ **is** $\alpha(0) = \beta$

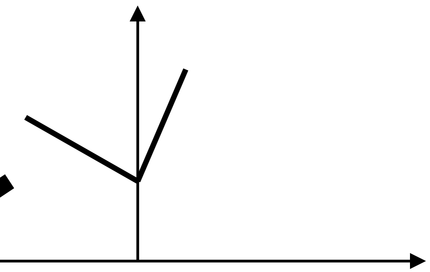
$$\mathbf{D}^{\beta} f(0) = c\Gamma(\beta+1)$$

Abrupt phase transition to continuous phase transition:

$$f(x) = 2 + 50x + 70|x|^{1.5}$$







This is notion to extend Ehrenfest's classification of thermodynamic phase transition, magnetic property at critical point, or yield point (stain) beyond critical stress to continuous transition. In simplified terms magnify the critical point which takes place abruptly and approximate by polynomial to get Fractional Differentiability at critical point. Non-differentiability can be magnified and studied near critical points.

Fractional Differentiation of continuous but non-differentiable graph & its relation with 'Fractal-Dimension'

$$W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin \lambda^k x \qquad \lambda > 1 \qquad d_B = s \qquad 1 < s < 2$$

$$W_\lambda(0) = 0$$

Use scaling law $\dfrac{d^q f(\beta x)}{dx^q} = \beta^q \dfrac{d^q f(\beta x)}{d(\beta x)^q}$ **we get** $\dfrac{d^q}{dx^q} \sin \lambda^k x = (\lambda^k)^q \dfrac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$

and the fractional derivative of Wierstrauss's function for $0 < q < 1$

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \frac{d^q \sin \lambda^k x}{dx^q} = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^q \sin \lambda^k x}{d(\lambda^k x)^q}$$

Using $\dfrac{d^q}{dx^q} \sin x = \dfrac{d^q}{dx^q} \int_0^x \cos t\,dt = \dfrac{d^{q-1}}{dx^{q-1}} \cos x$ **, the FD is**

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

The critical order of FD of Wierstrauss's function:

$$\frac{d^q}{dx^q} W_\lambda(x) = \sum_{k=1}^{\infty} \lambda^{(s-2+q)k} \frac{d^{q-1} \cos \lambda^k x}{d(\lambda^k x)^{q-1}}$$

The fractional integral $I_x^p \cos \lambda^k x$ of order $p = 1 - q$ is bounded uniformly for all values of $\lambda^k x$. This implies that the RHS will converge for $s - 2 + q < 0$ or $q < 2 - s$ and diverge for $q > 2 - s$ at the point Zero
The value of fractional integral is zero hence FD at zero of Wierstrauss's is ZERO

This Wierstrauss's function is continuously Fractionally Differentiable locally For orders $q < (2 - s)$ and not between orders $(2 - s)$ to one .
This implies that this Wierstrauss's function has Critical Order $\alpha = (2 - s)$ at all points, which is equal roughness exponent and thereby box dimension Of the graph.

LFD is perhaps a tool to extract local dimension of the irregular rough function

Critical Order LFD and Fractal (Box) dimension

$f:[0,1]\rightarrow \mathbf{R}$ **, be a continuous (real) function**

If $\lim_{x\rightarrow x_0} \frac{d^q\left(f(x)-f(x_0)\right)}{\left[d(x-x_0)\right]^q}=0$ **for** $q < \alpha$

Then

$$\dim_B f(x_0)\leq 2-\alpha$$

Holder Exponent $\alpha(x_0)$ **of a function** $f(x)$ **defined by this is the largest Exponent such that there exists a polynomial** $P_n(x)$ **that satisfies**

$$\left|f(x)-P_n(x-x_0)\right|=C\left|x-x_0\right|^{\alpha}$$

There is clear change in behavior when q **crosses the Critical Order.** $\alpha(x_0)$

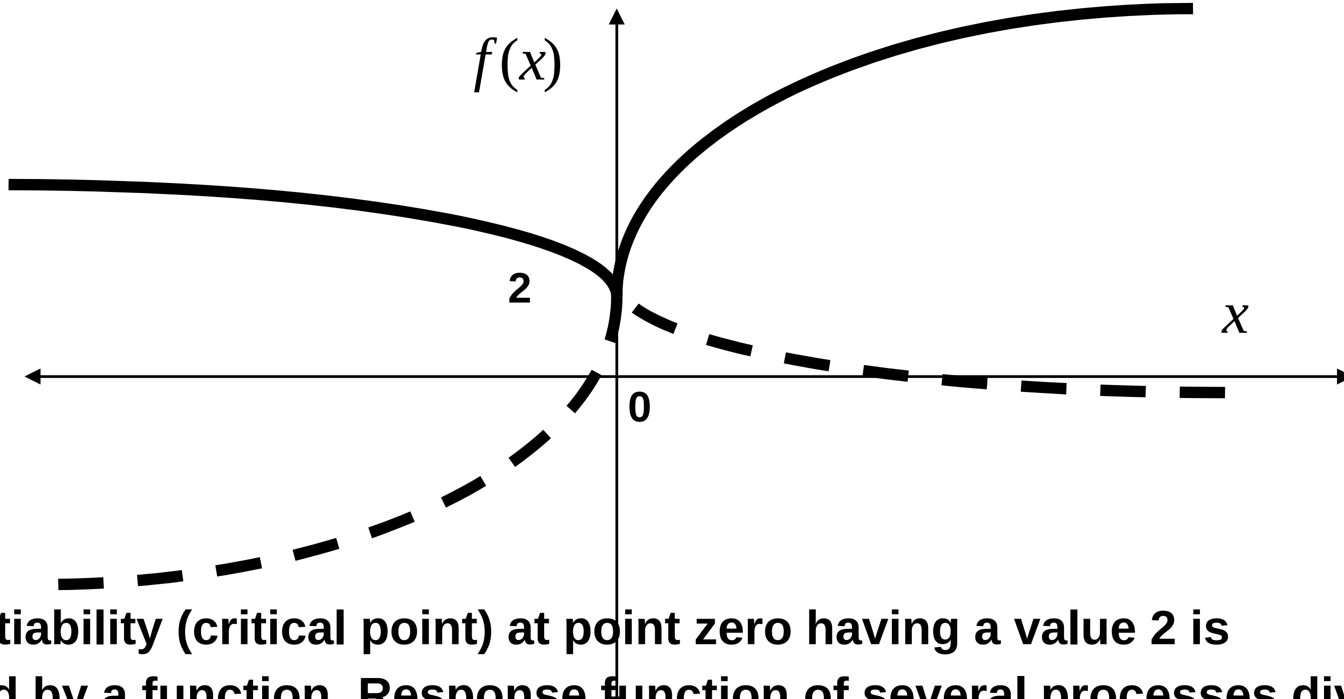
Phase Transition (Non-Differentiability) at critical point:

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^h$$

$$x_0 = 0; P(x) = 2 + 3x$$

Critical Point at zero and the polynomial is linear.

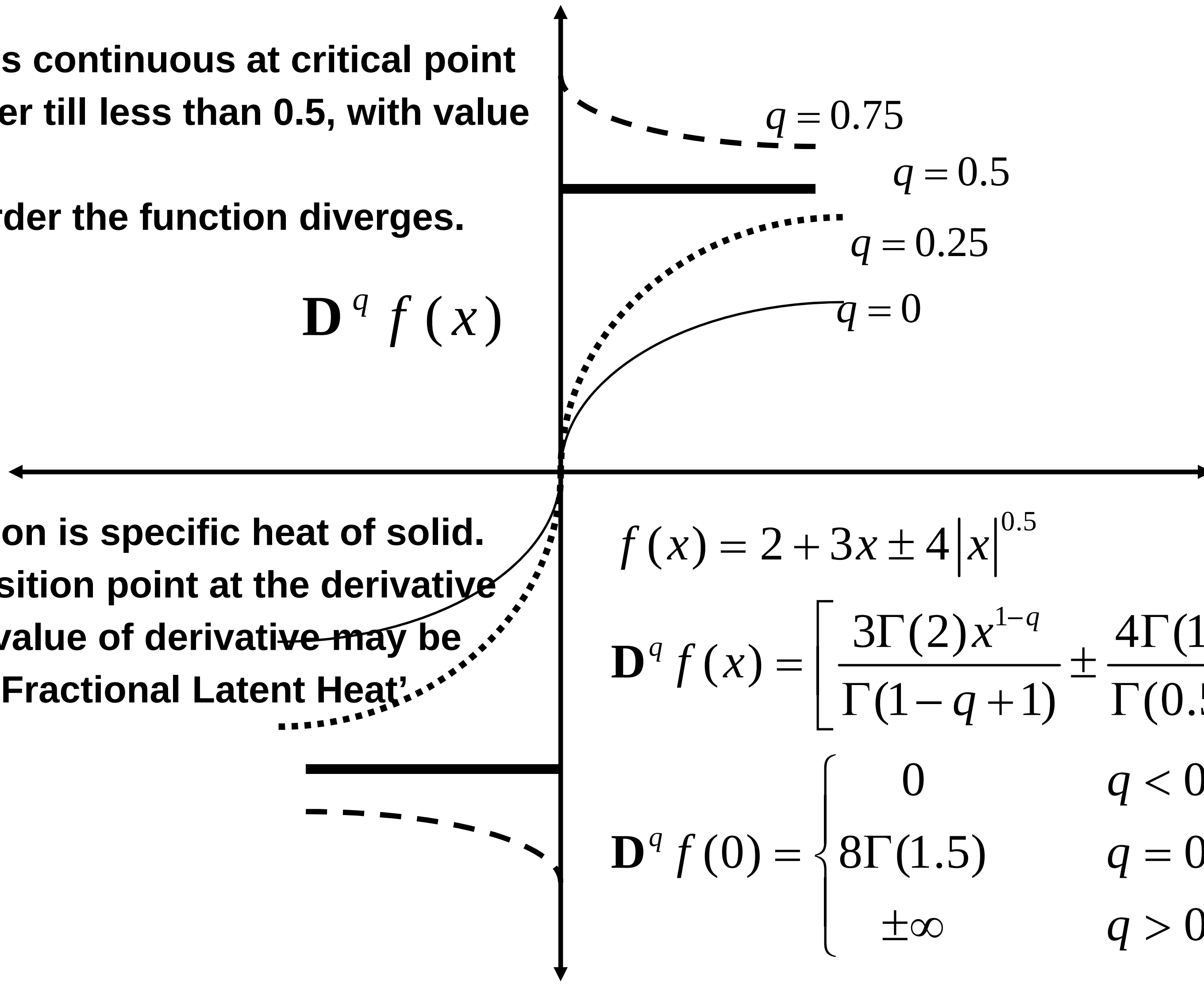
$$f(x) = (2 + 3x) \pm 4 \left| x \right|^{1/2}$$



Non-Differentiability (critical point) at point zero having a value 2 is approximated by a function. Response function of several processes diverge algebraically near critical point. Example Vander wall's equation at critical point.

Fractional Differentiability at Critical Point:

The function is continuous at critical point from zero order till less than 0.5, with value zero.
Beyond 0.5 order the function diverges.



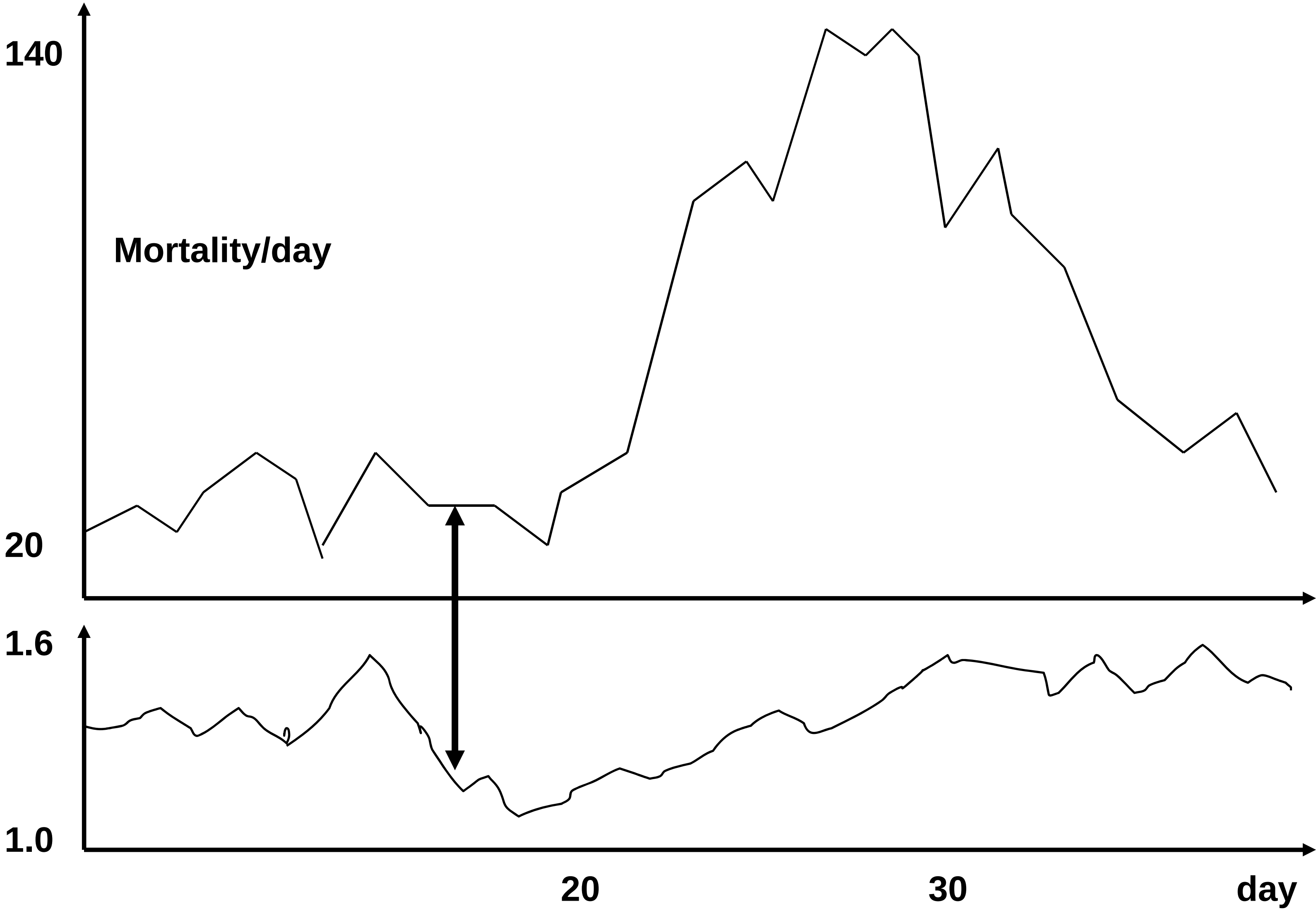
Say the function is specific heat of solid.
At phase transition point at the derivative order 0.5 the value of derivative may be Regarded as 'Fractional Latent Heat'.

$$f(x) = 2 + 3x \pm 4|x|^{0.5}$$

$$\mathbf{D}^q f(x) = \left[\frac{3\Gamma(2)x^{1-q}}{\Gamma(1-q+1)} \pm \frac{4\Gamma(1.5)x^{0.5-q}}{\Gamma(0.5-q+1)} \right]$$

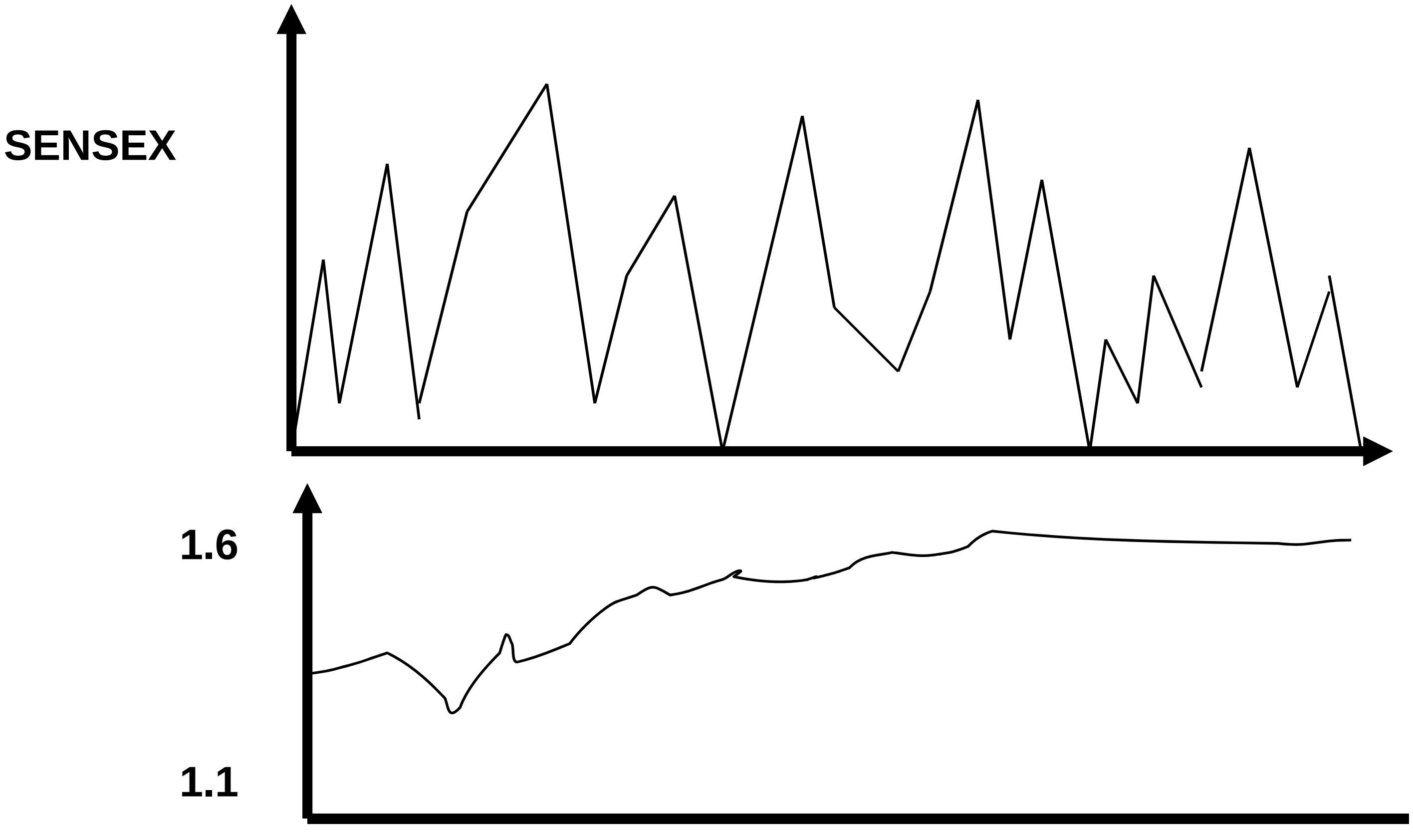
$$\mathbf{D}^q f(0) = \begin{Bmatrix} 0 & q < 0.5 \\ 8\Gamma(1.5) & q = 0.5 \\ \pm\infty & q > 0.5 \end{Bmatrix}$$

Fractal Dimension indicating on-set of Epidemic:



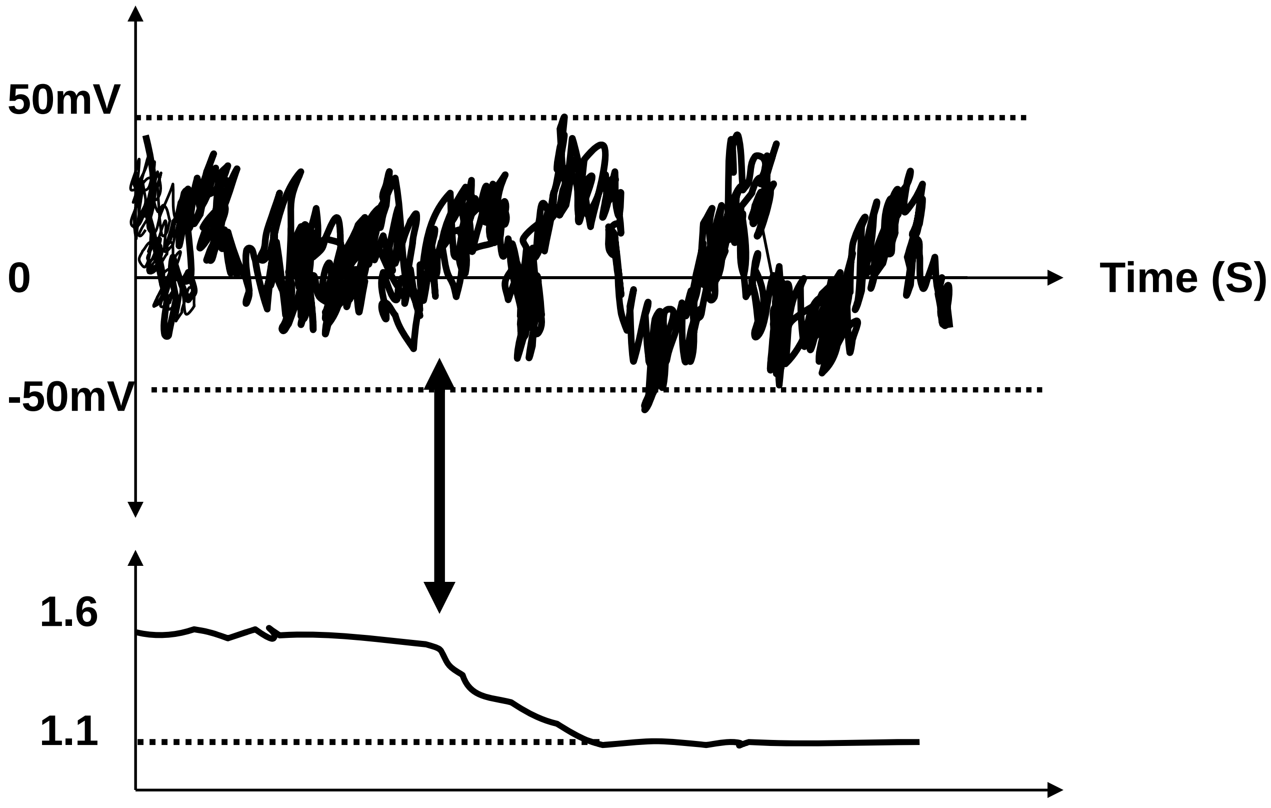
The cause of epidemics exhibited significant change in fractal dimension. Initially behaved as Brownian Motion 1.4-1.6, then dropped to 1.3-1.1 indicating on set of burst between 0-16 day (became regular from irregular) and again raised to 1.4 behaves variably.

A normal sensex of stock market



The dimension shows normal irregular 'Brownian' trading, with dimension slewing towards 1.5-1.6 indicating no bull bear or crash or financial irregularity! Trading is regular with normal irregularity as expected like White-Noise.

Exactly where the signal starts in high noise background:



Signal buried in 85% white noise, the change in dimension indicates the first arrival time of signal.

Identification of singularity by LFD:

1. Single singularity:

$$f(x) = a x^{\alpha} \qquad 0 < \alpha < 1$$

At 'zero' critical order gives the 'order of singularity and LFD $\mathbf{D}^{\alpha} f(0) = a\Gamma(\alpha+1)$ **gives strength of singularity.**

2. Multiple singularity:

$$f(x) = a x^{\alpha} + b x^{\beta} \qquad 0 < \alpha < \beta < 1$$

$$\mathbf{D}^{\alpha} f(0) = a\Gamma(\alpha+1) + b\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} = a\Gamma(\alpha+1)$$

Write:

$$G(x,\alpha) = f(x) - \left[f(0) + \frac{\mathbf{D}^{\alpha} f(0)}{\Gamma(\alpha+1)} x^{\alpha} \right] = b x^{\beta}$$

$$\frac{d^q G(x,\alpha)}{dx^q} = b\frac{\Gamma(\beta+1)}{\Gamma(\beta-q+1)} x^{\beta-q}$$

$q = \beta$ **is critical order**

This way one extracts secondary singularity hidden by primary singularity

Fractional Taylor's series by LFD:

Let $$F(x_0, x-x_0; q) = \frac{d^q\left(f(x)-f(x_0)\right)}{\left[d(x-x_0)\right]^q}$$ **it is clear that** $$\mathbf{D}^q f(x_0) = F(x_0, 0, q)$$

Using RL Integration, and by integration by parts we get

$$f(x) - f(x_0) = \frac{1}{\Gamma(q)} \int_0^{x-x_0} \frac{F(x_0, \xi; q)}{(x-x_0-\xi)^{-q+1}} d\xi$$

$$= \frac{1}{\Gamma(q)} \left[F(x_0, \xi; q) \int (x-x_0-\xi)^{q-1} d\xi \right]_0^{x-x_0} + \frac{1}{\Gamma(q)} \int_0^{x-x_0} \frac{dF(x_0, \xi; q)}{d\xi} \frac{(x-x_0-\xi)^q}{q}$$

Provided the II term exists!

$$f(x) - f(x_0) = \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} (x-x_0)^q + \frac{1}{\Gamma(q+1)} \int_0^{x-x_0} \frac{dF(x_0, \xi; q)}{d\xi} (x-x_0-\xi)^q d\xi$$

$$f(x) = f(x_0) + \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} (x-x_0)^q + R_q(x, x_0)$$ **for** $0 < q < 1$

More general:

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{\Gamma(n+1)} \Delta^n + \frac{\mathbf{D}^q f(x_0)}{\Gamma(q+1)} \Delta^q + R_q(x, \Delta)$$

$$\Delta = x - x_0 > 0$$

Usage:

1. LFD provides the coefficient A **in approximating function** $f(x)$

By $f(x_0) + \frac{\mathrm{A}}{\Gamma(q+1)}(x - x_0)^q$ **in vicinity of** x_0 **.The terms are non trivial**

for $q = \alpha$

For $q = 1$ **we get equation for tangent** $f(x) = f(x_0) + \mathbf{D}^1(x_0)[(x - x_0)^q]$
This forms an equivalence class modeled by linear behavior. All curves passing through a point x_0 **having same tangent.**

2. Analogously all the functions (curves) with same 'critical order' α **and same** $\mathbf{D}^q$ **will form an equivalence class modeled by power law** x^α
This generalizes definition of tangents.

3 Useful to approximate irregular (non-differentiable) functions by piece-wise smooth (scaling) function; and survey of singularities.

4. Useful as Fractional curve fitting, start point of 'Fractional Differential Geometry'.

Non-Differentiabilty is actuality

A fractal space time was suggested in 1983 by G. Ord and L. Nottale but till today there is no mathematical model of such space.

Einstein suggested that true understanding of quantum physics could imply to give up differentiability. . Therefore, 'non-differentiability', to describe the space-time in quantum mechanics was suggested early by Einstein without pursuit, and suggested in 1993 by Nottale. , which leads to scale relativity.

Giving up the differentiability means that we have to deal with 'nowhere differentiable functions'; are thus described as monsters. There are no tools nor analytic characterization for nowhere differentiable functions other than notion of fractal dimensions. We thus approximate these nowhere differentiable functions by a 'mean function', which follows the geometry of the approximated function.

Apparently the framework of the formalism that provides analysis and accurate description of nowhere differentiable functions differs radically from that one provides analysis and description of the differentiable functions in a deep sense, other than differentiability and non differentiability.

Unreachable and Uncertain Graphs and Systems

This ensures that there is uncertainty; also we cannot for sure make the judgment about the next position after an instant. With view of our discussions about differentiability we give an interpretation of uncertainty principle mathematically

Let $f \in C^0([a,b])$ **and let** $\Gamma_f = \{(t, f(t)) \in \mathbb{R}^2 / t \in [a,b]\}$ **be the graph of** $f(t)$

We say that Γ_f **is observable if** $f(t)$ **is differentiable on** $]a,b[$ **and we say that** Γ_f **is non observable if** $f(t)$ **is nowhere differentiable on** $]a,b[$

The observable curve, graph or trajectory Γ_f **is the one that can be drawn with our tools, that is to say its location in a given reference frame and its instantaneous rate of change can be determined for all** $t \in]a,b[$ **which leads to obtain an exact shape of the curve graph or trajectory** Γ_f **the precise location of its key features everywhere That is reachable curve or trajectory or graph. The non observable (unreachable) graph curve or trajectory is the one that cannot be drawn in a given reference frame, and an exact shape as well as a precise location of its key features cannot be obtained anywhere, that is to say it is impossible to determine its instantaneous rate of change for all** $t \in]a,b[$ **The non observable graph is unreachable in a given reference frame, yet its mean is always reachable**

Scale resolution & approximation of non-differentiable function

Let $[a, b] \in \mathbb{R}$ **be an interval, with length** $l = b - a$ **it is possible to construct a family of subintervals of length** $l = (b-a)/n$ **with** $n \geq 1$ **This kind of subdivision is countable.**

For non countable subdivisions of an unit interval, let us consider a length $l = 1/\beta, \beta > 0$ **and we call 'resolution' the quotient of the unit by a real number** $\beta > 0$ **and denote** $\tilde{\delta} = 1/\beta$ **This resolution** $\tilde{\delta}$ **takes infinite values between** 0 **and** $+\infty$

Resolution $\tilde{\delta} = 0$ **as** $\beta \to \infty$ **and thus we have** $\tilde{\delta} \in [0, \infty[$ **. We can restrict to any value other than infinity, say** $\delta \in [0, \varepsilon]$ $0 < \varepsilon \ll 1$ **in order to do neighborhood approx. of the no-where differentiable functions**

Call it reduced resolution set $\mathcal{R}_f$

We represent a continuous $f(x) \in C^0([a,b])$ **non differentiable function by a mean function which is continuous-is approximation in its neighborhood, by a differentiable function**

$$f(x, \delta) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(\xi) \mathrm{d}\xi \qquad \forall x \in [a, b] \qquad \delta \in [0, 1]$$

A single variable non-differentiable function is approximated by a two variable function Therefore a rough two-D graph is approximated by a 'surface'!!

Approximation domain by mean function

The mean function $f(x,\delta)=\dfrac{1}{2\delta}\int\limits_{x-\delta}^{x+\delta} f(\xi)\mathrm{d}\xi \qquad \forall x\in[a,b] \qquad \delta\in[0,1]$

The derivative of the mean function is given as $f'(x,\delta)=\dfrac{f(x+\delta)-f(x-\delta)}{2\delta}$

For $\delta=0$ **we have** $f(x,0)=\lim_{\delta\to 0} f(x,\delta)$ **which is nowhere differentiable and** $f(x,\delta)$ **is always differentiable** $\forall\delta\in\left]0,\varepsilon\right] \qquad 0<\varepsilon\ll 1$

Therefore rules for the domain of approximation we write as

$$\mathcal{R}_f=\left\{\delta\in\mathbb{R}^+ / f(x,\delta)\,\text{is differentiable on}\ \left]a,b\right[\right\} \qquad \delta\in\left]0,\varepsilon\right] \qquad 0<\varepsilon\ll 1$$

$$\mathcal{R}_f=\left]0,\varepsilon\right]\text{for}\quad f-\text{differentiable on}\ \left]a,b\right[\in\mathbb{R}$$

$$\mathcal{R}_f=\left[0,\varepsilon\right]\text{for}\quad f-\text{nowhere differentiable on}\ \left]a,b\right[\in\mathbb{R}$$

The forward and backward mean are defined as:

$$f^+(x,\delta)=\frac{1}{\delta}\int\limits_{x}^{x+\delta} f(\xi)\mathrm{d}\xi \qquad f^-(x,\delta)=\frac{1}{\delta}\int\limits_{x-\delta}^{x} f(\xi)\mathrm{d}\xi$$

$$f^+(x,0)=\lim_{\delta\to 0} f^+(x,\delta)=f(x) \qquad f^-(x,0)=\lim_{\delta\to 0} f^-(x,\delta)=f(x)$$

Smaller the δ **closer the** $f^+(x,\delta)$ **and** $f^-(x,\delta)$ **curves are to the original** $f(x)$

Then $\forall(x,\delta)\in[a,b]\times\mathcal{R}_f$ **we have**

$$f(x)=f^+(x,\delta)+\delta\left(\frac{\partial f^+(x,\delta)}{\partial\delta}-\frac{\partial f^+(x,\delta)}{\partial x}\right) \qquad f(x)=f^-(x,\delta)+\delta\left(\frac{\partial f^-(x,\delta)}{\partial\delta}+\frac{\partial f^-(x,\delta)}{\partial x}\right)$$

Making uncertain approximately certain in neighborhood

Unreachable can be reached by derivative of the surface $z = f^{+}(x,\delta)$

$$z = f^{+}(x,\delta) = \frac{1}{\delta}\int_{x}^{x+\delta} f(\xi)\,\mathrm{d}\xi \qquad f(x) = f^{+}(x,\delta) + \delta\left(\frac{\partial f^{+}(x,\delta)}{\partial\delta} - \frac{\partial f^{+}(x,\delta)}{\partial x}\right)$$

Therefore the simultaneous determination of the deviation of the function and the normal average rate of change of the function to accuracy is thus subject to ‘regularity’ of the main function. More the roughness more is the deviation and its uncertainty.

The proof is as:

$$f^{+}(x,\delta) = \frac{1}{\delta}\int_{x}^{x+\delta} f(\xi)\,\mathrm{d}\xi$$

$$\frac{\partial}{\partial\delta} f^{+}(x,\delta) = -\frac{1}{\delta^{2}}\int_{x}^{x+\delta} f(\xi)\,\mathrm{d}\xi + \frac{1}{\delta} f(x+\delta)$$

$$\frac{\partial}{\partial\delta} f^{+}(x,\delta) = -\frac{1}{\delta} f^{+}(x,\delta) + \frac{1}{\delta} f(x+\delta)$$

Moreover we also have $$\frac{\partial}{\partial x} f^{+}(x,\delta) = \frac{f(x+\delta) - f(x)}{\delta}$$

From above two that is by subtraction, we get the desired result

$$\delta\left(\frac{\partial}{\partial\delta} f^{+}(x,\delta) - \frac{\partial}{\partial x} f^{+}(x,\delta)\right) = f(x) - f^{+}(x,\delta)$$

Deviation and rate of deviation of uncertain function

$\forall \delta \in \mathcal{R}_f$, that in reduced set of scale, we define deviation of the function as

$$\Delta_\delta f(t) = f^+(t,0) - f^+(t,\delta) \equiv f(t) - f^+(t,\delta)$$

For a nowhere differentiable function we have

$$\frac{\Delta_\delta f(t)}{\delta} = \left(\frac{\partial}{\partial \delta} f^+(t,\delta) - \frac{\partial}{\partial t} f^+(t,\delta) \right) \qquad \forall t \in \left]a,b\right[\qquad \mathcal{R}_f = \left]0,\varepsilon\right]$$

Since $\delta \neq 0$ we could divide by it; above we got from our previous derivations.

We define the average rate of change of the deviation w.r.t δ

$$\Delta_\delta m(t) = -\frac{\Delta_\delta f(t)}{\delta} = \frac{f^+(t,\delta) - f^+(t,0)}{\delta}$$

Under the condition $\Delta_\delta m(t) \neq 0; \quad \forall \delta \neq 0; \quad \forall t \in \left]a,b\right[$ we define normal rate of deviation

as $$\Delta_\delta n(t) = -\frac{1}{\Delta_\delta m(t)} = \frac{\delta}{\Delta_\delta f(t)}$$

Geometrically Deviation and rate of deviation of uncertain function

Since $f(t) = f^{+}(t,0)$ **then** $\forall \delta \in \mathcal{R}_f$

$$\Delta_\delta m(t_0) = \frac{-(\Delta_\delta f(t_0))}{\delta} = \frac{[f^{+}(t_0,\delta) - f^{+}(t_0,0)]}{\delta}$$ **thus** $$\Delta_\delta m(t_0) = \frac{-\Delta_\delta f(t_0)}{\delta}$$

is the average rate of change of $f^{+}(t_0,\delta)$ **with respect to** δ **in the interval** $[0,\delta]$

at a fixed t_0 **Meaning that** $\Delta_\delta m(t_0)$ **is slope of the line through the points** $\{t_0, \delta, f^{+}(t_0,\delta)\}$

and point $\{t_0, 0, f^{+}(t_0,0)\}$

Therefore the quantity $\Delta_\delta n(t_0) = \dfrac{-1}{\Delta m_\delta(t_0)} = \dfrac{\delta}{\Delta_\delta f(t_0)}$ **represents the average rate of**

change of $f^{+}(t,\delta)$ **along a line perpendicular to line defined by slope of above**

$$\Delta m_\delta(t_0)\Delta_\delta n(t_0) = -1$$

Charterising non differentiable function with deviation functions

The $f(t)$ **is differentiable on** $]a,b[$ **implies that** $\exists \delta \in \mathcal{R}_f \, / \, \forall t \in]a,b[$ **and** $\Delta_\delta f(t) = 0$

Simple arguments explains this; if $f(t)$ **is differentiable on** $]a,b[$ **then** $\mathcal{R}_f = [0, \varepsilon]$

by the rule what we wrote about using reduced scaling and domain approximation.

For $\delta_0 = 0 \in \mathcal{R}_f$ **we have** $f(t) = f^+(t,0) = \lim_{\delta \to 0} f^+(t,\delta)$ **such that** $\Delta f_\delta(t) = 0$

and if $\delta_0 \neq 0$ **then by the relation** $\Delta_\delta f(t) = f^+(t,0) - f^+(t,\delta) \equiv f(t) - f^+(t,\delta_0)$ **we have**

$f(t) \cong f^+(t,\delta_0)$ **implying that** $f'(t) = \dfrac{[f(t+\delta_0) - f(t)]}{\delta_0}$ **Thus** $f(t)$

is differentiable for $\Delta_\delta f(t) = 0$

Conversely for $f(t)$ **which is non differentiable on we have** $\Delta_\delta f(t) \neq 0$

May be uncertainty principle

Thus we have from $\dfrac{\Delta_\delta f(t)}{\delta} = \left(\dfrac{\partial}{\partial \delta} f^+(t,\delta) - \dfrac{\partial}{\partial t} f^+(t,\delta) \right)$ **and** $\Delta_\delta m(t) = -\dfrac{\Delta_\delta f(t)}{\delta} = \dfrac{f^+(t,\delta) - f^+(t,0)}{\delta}$

and $\Delta_\delta n(t) = -\dfrac{1}{\Delta_\delta m(t)} = \dfrac{\delta}{\Delta_\delta f(t)}$

For $f(t) \in C^0([a,b])$ **a nowhere differentiable one.**

$$\Delta_\delta f(t) \Delta_\delta n(t) = \delta$$

Say we have position graph $x(t)$ **which is nowhere differentiable and then if**

$$\Delta x(t) \Delta p(t) \sim h \qquad \Delta x : \text{deviation}; \qquad \Delta p : \text{normal rate of change}$$

Cannot measure deviation in position and deviation in momentum to a accuracy given by above

Here we hypothyse a different approach, with out quantum mechanical frame work, we perhaps can describe the uncertainty principle via precise mathematical description of mean errors of Δx **and** Δp **using the properties of continuous but nowhere differentiable functions. The relation of the Planks constant to our deviation variable** δ **should depend on the physical system.**

Give up differentiability thus:

The standard uncertainty of 1927 by W. Heisenberg is $\Delta x \Delta p \sim h$

In similar time E. H. Kennard gave the theoretical framework that presented

$$\sigma_x \sigma_p \geq \hbar / 2 \qquad \hbar = h / 2\pi$$

Where σ_x **and** σ_p **represents standard deviation of position function and momentum function**

While viewing the uncertainty principle in the way as classical one has to 'hunt' for probability density function for the position variable; whereas the non-differentiabilty class of function of position may be other way of looking at this

It is true to state that the uncertainty principle was a surprising discovery of the twentieth century that reinforced the 'divorce' between classical point of view and quantum point of view regarding the possibility to associate 'simultaneous exact' values of the position and momentum to any physical system-and was perhaps the beginning of start of 'giving up the differentiabilty'!!

Line/surface/volume integrals of Fractal Distribution:

Fractal Distribution represented by Fractional Continuous Medium and then we perform the integration. The fractional Integrals are considered as an approximate integrals on fractals. This type of new approach is applicable in processes where fractal features of the process or the medium impose the necessity of using non traditional tools in regular smooth physical equations. Smoothening the microscopic characteristics over physically infinitesimal Volume/surface/line transforms the initial fractal distribution into fractional continuum model. The order of fractional integration is of fractal dimension.

$$ {}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)du $$

$$ {}_0D^{-d} f(r) = \int_V f(r)dV_d \approx \int_V \frac{r^{d-3}}{\Gamma^3(d/3)} dV_3 \qquad dV_d = K_3(r,d)dV_3 \quad K_3(r,d) = \frac{r^{d-3}}{\Gamma^3(d/3)} $$

$$ 2 < d < 3 $$

$$ dS_d = K_2(r,d)dS_2 \qquad 1 < d < 2 \qquad K_2(r,d) = \frac{r^{d-2}}{\Gamma^2(d/2)} $$

$$ dL_d = K_1(r,d)dL_1 \qquad 0 < d < 1 \qquad K_1(r,d) = \frac{r^{d-1}}{\Gamma(d)} $$

Some laws on Fractal Geometries:

Flux through a fractal surface:

A flowing quantity trough a fractal surface be represented as:

$$\phi_{S_d} = \int_S \left(J(r,t) \bullet dS_d \right) \qquad dS_d \equiv K_2(r,d) dS_2 \qquad dS_d = \frac{r^{d-2}}{\Gamma^2(d/2)} dS_2$$

Gauss's law on Fractal:

$$\int_{\partial W} \left(J(r,t) \bullet dS_2 \right) = \int_W \mathbf{div}[J(r,t)] dV_3$$

$$\int_{\partial W} \left(J(r,t) \bullet dS_d \right) = \int_W (K_3(r,d_3))^{-1} \mathbf{div}[K_2(r,d_2) J(r,t)] dV_d$$

Stroke's law on Fractal:

$$\int_L \left(E \bullet dL_1 \right) = \int_S [\mathbf{curl} E] dS_2$$

$$\int_L \left(E \bullet dL_d \right) = \int_S \left(K_2(r,d_2) \right)^{-1} [\mathbf{curl} K_1(r,d_1) E] dS_d$$

Existence of Magnetic charges?

In normal cases of smooth geometry $\mathbf{div} B = 0$ indicating no magnetic charges at point exists . Magnetic mono-pole not possible.

Fractional generalization however gives: $\mathbf{div}[K_2(r, d_2)B] \neq 0$

$$\mathbf{div} B = B.\mathbf{grad} K_2(r, d_2)$$

For $d_2 \neq 2$ $\mathbf{grad} K_2(r, d_2) \neq 0$ indicating $\mathbf{div} B \neq 0$
Existence of 'magnetic monopole charges' with magnitude of

$$e_m \approx B.\nabla K_2(r, d_2)$$

For fractal distribution we have thus all sets of conservation laws and set of Maxwell equations and electrodynamics do get modified. This method perhaps is suitable for dusty plasma cases.

Several dimensions an overview:

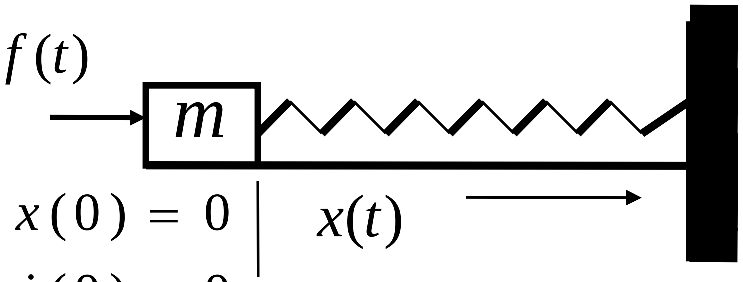
1. **Embedding Euclidian dimension: The regular dimension which holds the structure.** d

1. **Fractal dimension : Describes irregularity and roughness, locally or globally. For self similar the dilation symmetry is uniform, but the self affine has anisotropic dilation symmetry.** d_F

2. **Spectral dimension: Described by density of states or the relaxation of stimulus and order of it at asymptotic late times.** d_S

$$d \geq d_F \geq d_S$$

Generalization of Newtonian mechanics and differential equations



$$m\ddot{x}(t) + b_0\dot{x}(t) + kx(t) = f(t)$$

$$x(0) = 0$$
$$\dot{x}(0) = 0$$

Mass concentrated at point
Mass less spring
Frictionless spring
Infinite wall

$$(ms^2 + b_0 s + k)X(s) = F(s)$$

Spring with friction →

$$k_q s^q X(s) = F_{sp}(s)$$
$$0 \le q \le 1$$

$$(ms^2 + b_0 s + k_q s^q + k)X(s) = F(s)$$

$$(ms^2 + b_0 s + k_{q_n} s^{q_n} + k_{q_{n-1}} s^{q_{n-1}} + \ldots + k_{q_1} s^{q_1} + k_{q_0})X(s) = F(s)$$

$$\sum_{n=0}^{2}\left[k_n s^{q_n}\right]X(s) = F(s)$$

$$\left(\int_0^2 k(q)s^q dq\right)X(s) = F(s)$$

Distributed mass
Spring with mass
Spring with friction
Damping with spring action
Non conservation system
Leaky wall/termination

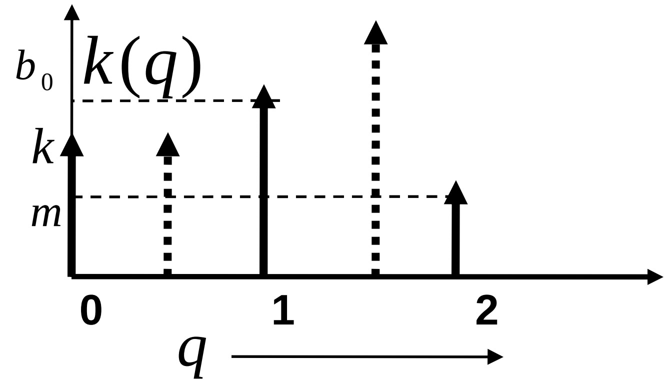
System Identification & order distribution

Integer Order:

$$m\ddot{x}(t)+b_0\dot{x}(t)+kx(t)=f(t)$$

$$(ms^2+b_0s+k)X(s)=F(s)$$

$$\left\{\int_0^\infty [m\delta(q-2)+b_0\delta(q-1)+k\delta(q)]s^q dq\right\}X(s)=F(s)$$



Fractional Order

$$(ms^2+b_1s^{3/2}+b_0s+k_1s^{1/2}+k_0)X(s)=F(s)$$

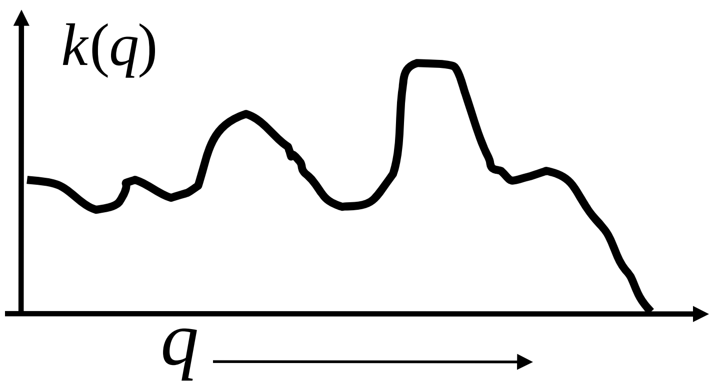
$$\{[m\delta(q-2)+b_1\delta(q-1.5)+b_0\delta(q-1)+k_1\delta(q-0.5)+k_0\delta(q)]s^q dq\}X(s)=F(s)$$

$$m\frac{d^2x(t)}{dt^2}+b_1\frac{d^{3/2}x(t)}{dt^{3/2}}+b_0\frac{dx(t)}{dt}+k_1\frac{d^{1/2}x(t)}{dt^{1/2}}+k_0=f(t)$$

Continuous Order

$$\left(\int_0^\infty k(q)s^q dq\right)X(s)=F(s)$$

$$\left\{\mathcal{L}^{-1}\left(\int_0^\infty k(q)s^q dq\right)\right\}*x(t)=f(t)$$



A 'first order' system with half order term can oscillate

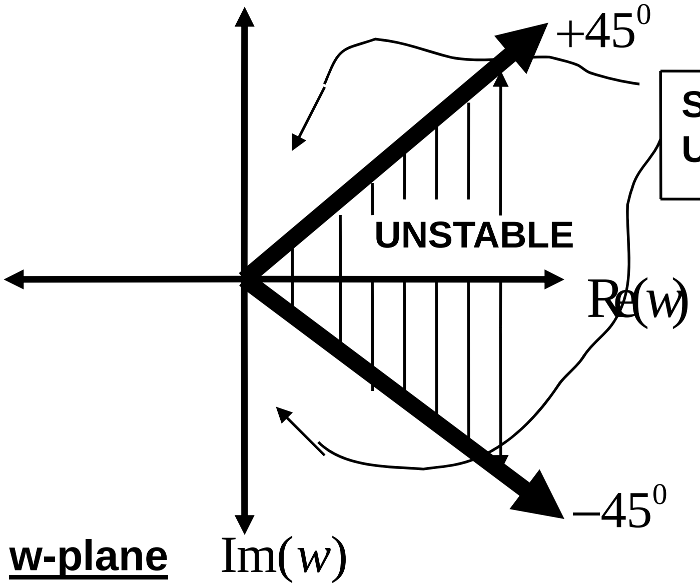
Concept of w-plane conformal mapping

$$\frac{d}{dt}x(t) + a\frac{d^{1/2}}{dt^{1/2}}x(t) + x(t) = f(t)$$

$$sX(s) + as^{1/2}X(s) + X(s) = F(s)$$

Characteristic equation is: $s + a\sqrt{s} + 1$ **in s-plane**

let $s^{1/2} = w$ **then** $w^2 + aw + 1$ **Is characteristic equation in w-plane.** $\arg w = \frac{1}{2}\arg s, \mathrm{mod}(w) = \sqrt{\mathrm{mod}(s)}$



$a < -\sqrt{2}$	$\arg(w) < \pm 45^0$	$\arg(s) < \pm 90^0$	**Unstable**
$0 > a > -\sqrt{2}$	$\pm 45^0 - \pm 90^0$	$\pm 90^0 - \pm 180^0$	**Stable**
$0 < a < 2$	$\pm 90^0 - \pm 180^0$	$\pm 180^0 - \pm 360^0$	**Hyper damped**
$a > 2$	$\arg(w) > \pm 180^0$	$\arg(s) > \pm 360^0$	**Ultra damped**

A first order system with fractional term may become unstable can have oscillatory behaviour and can behave as stable second order stable under damped systems
Classical order definition with number of energy storage element and or number of initial condition can give misleading information about the response in presence of fractional order terms.

Study of Chaos in Chua's system:

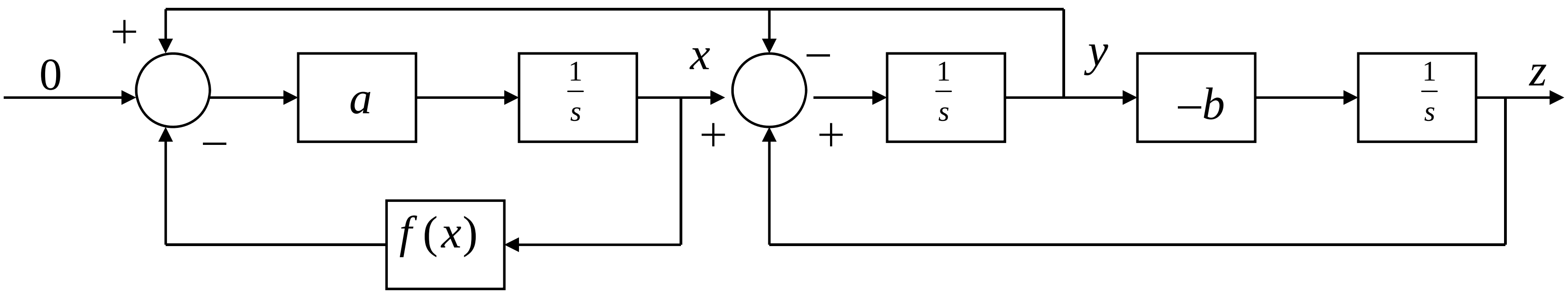
State space representation by replacing piece-wise non linearly by a cubic NL.

$$\frac{dx}{dt} = a\left[y + f(x)\right]$$

$$\frac{dy}{dt} = x - y + z$$

$$\frac{dz}{dt} = -by$$

$$f(x) = \frac{x - 2x^3}{7}, \quad b = \frac{100}{7}$$



Chaos in Fractional Order Chua's system:

$$s \to s^q$$

$$\frac{d^q x}{dt^q} = a\left[y + \frac{x - 2x^3}{7} \right]$$

$$\frac{d^q y}{dt^q} = x - y + z$$

$$\frac{d^q z}{dt^q} = -\frac{100}{7} y$$

q	Mathematical Order	Approximations	a	λ_1	λ_2	λ_3
0.9	2.7	9.0	12.75	>0	<0	<0
1.0	3.0	3.0	9.5	>0	<0	<0
1.1	3.3	18.0	7.0	>0	<0	<0

Generalized-Relaxation in complex process some comments:

. Sufficiently high micro structural disorder can lead statistically to macroscopic behavior well approximated by Fractional Calculus.

. Damping (relaxation) behavior of materials if modeled by Linear Differential Equations (LDE); with constant coefficient cannot include 'long-memory, that fractional order derivatives require. (Long range correlations)

. Rubber molecules (presumably) cannot remember past here (perhaps) LDE with constant coefficient can be involved. Such systems have 'exponential-decay'-system without memory. For large times the value goes to zero-(quickly). (Short range correlation)

. Many materials with 'complex' microscopic dissipative mechanisms may macroscopically show Fractional Order Differential Equation behavior. Damping (relaxation) models may involve relatively fewer fitted parameters compared to integer order complex models.

. Fractional Order behavior may be an artifact of a dissipative mechanism in Fractal Landscape, involving different scales as compared to conventional Euclidean scales. In those systems the flow of time may also be non-uniform with fractal time series.

Epilogue to Generalized Relaxation & Fractional Calculus:

Fractional Calculus methods have been invoked (recently) to model relaxation processes in complex systems. This has lead to interesting discussions into nature of transport coefficients appropriately/equations to describe these complex materials. This observation is leading to thought to have 'Universality' of Disordered Material Relaxation" !!

Relaxation integral I_t^{ϕ} where ϕ characterizes 'degree of intermittency' in the relaxation process-and exact solutions of these integrals (may) describe relaxation in condense matter-'intermittency in relaxation'

Extension in this above model of relaxation , to include β 'dynamic heterogeneity ' arising out from particle clustering (too may be included) which is ubiquitous in condense matter. With $I_t^{\phi,\beta}$ intermittency plus heterogeneity.

$I_t^{\phi} f(t)$ **Is Fractional Integration of arbitrary order.**

$D_t^{\phi} f(t)$ **Is Fractional Differentiation of arbitrary order**

Defines the processes involving Fractal Landscape-and gives wonderful world

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Fractional Calculus
may have a different expression and is generalization of classical calculus

Perhaps can be utilized to express nature in exact way!!
to express happenings of a real system which is non-linear irregular chaotic

Remember that ‘Mathematical tools’ go far beyond
our physical understanding

yet we have several miles to go!!

to appreciate wonderful world of mathematics that lays
in between one complete integration and one complete differentiation.
and to answer the question “which mathematics” nature follows
well also is it non-differentiability all the way in future!!