Physics Colloquium
25-02-2010

Half and One-half derivatives in Physics

-a reality-

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Prelude:

I (by profession an Electrical and Electronics Engineer) can state that **"Mathematics goes far beyond our physical understanding"**.

Mathematics is what nature understands, but which one? We Scientists & Engineers try to find and keep on finding**- "that particular mathematics".**

As an engineer my past one and half decade of work is mainly to ascribe **physical/engineering/geometrical** sense to wonderful **three hundred years old topic** of **fractional calculus**-make **product and science out of this subject**.

Why? Because it was for search to make **"Fuel Efficient Control System"** for Nuclear Reactors!!

My small contribution is also to relate real life processes and explanation with physical behavior to the wonderful tool of mathematics that is fractional calculus; also to give physical sense to solution of solvable extraordinary differential equations, a way gives merger of two definitions of Fractional Derivative , Riemann-Liouvelii and Caputo, and this way only integer order states are required to solve FDE.

Still continuing……… and is unfinished.

Still learning Fractional Calculus
by fractions!!

Book references:

Functional Fractional Calculus for System Identifications & Controls (2007). Springer Verlag Germany.

BARC library Book Section No.204472-(517.97:621.039.56-B08)

Cited by several researches across the globe in pure and applied science & engineering

Mathematico-Physics of Generalized Calculus

(a course work book for PG PhD students in limited numbers)

Department of Applied Mathematics Calcutta University library, Calcutta Mathematical Society, Jadavpur University Library , Department of Physics Jadavpur Univ. BARC library, Power Engineering Department Jadavpur Univ. IRPE Library Calcutta University,

BARC Library Book Section No. 510:56:53:B10-207150

Edition-II of Functional Fractional Calculus

for print (60% more than Edition-I)with publisher since **February2011**, may come by this year middle.

Few journal papers references (2006-2011):

1. International Journal of Applied Mathematics & Statistics. Vol. 17; No. J10, June 2010, pp. 44-76***"Generalized Dynamic System Solution by Decomposed Physical Reactions"***.
2. International Journal Modeling and Simulation in Engineering. Volume 2010, ID739675, pp. 1-19. ***"Solution of Extraordinary Differential Equations with Physical Reasoning by Obtaining Modal Reaction Series".***
3. International Journal of Applied Mathematics & Statistics. Vol. 21; No. J11, 2011, pp131-140. ***"Fractional Stochastic Modeling for Random Dynamic Delays in Computer Control System".***
4. Elsevier ISA-Transactions 49(2010) pp.196-206 December 2009, "***Fractional Order Phase Shaper Design with Bode Integral for Iso-damped control system".***
5. Int. J. of Nuclear Energy Science & Technology, Vol.5, No.2, pp105-113***"Solution of Coupled Fractional Neutron Diffusion Equation with Delayed Neutron".***
6. Geophysical Journal No.2 T 31,2009 pp147-159. ***"Fractional Calculus to describe Half-Space Geophysical Analysis for Transient Electro-Magnetic Method".***
7. Int. J. Nuclear Energy Science & Technology Vol.3 No.2, 2006, pp139-159, ***"Fractional divergence for neutron flux profile in nuclear reactor".***
8. IEEE Transactions on Nuclear Science, Vol.57, No.3, pp1602-1612, June 2010, ***"Design of Fractional Order Phase Shaper for Iso-Damped Control of PHWR under Step-Back".***
9. Int. J of Applied Mathematics & Statistics, Vol. 23, D11, pp64-74, 2011 ***"Convergence of Riemann-Liouvelli and Caputo Derivative for Practical Solution of Fractional Order Differential Equations."***
10. Condensed Matter Archives arxiv:1012.08V2 dated 10-12-2010, "***Oscillatory Spreading and Surface Instability of a Non-Newtonian Fluid under compression."*** (Also being critically revised by Physics Review Letters)
11. Journal of Process Control Elsevier (ISA-Transactions)-in press, ***"On Selection of Tuning Rule for Higher FOPID Controllers to Control Higher Order Processes"***

And several more .

(1-10) Are present in BARC library

Generalization of theory of numbers and calculations

$2^3 = 2 \times 2 \times 2 = 8$ Can be visualized

$2^{0.5} = \exp\{(0.5)\ln 2\} = 1.414$ Number exists but hard to visualize how.

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ Is a visualized quantity, but what about $(5.5)!$

Generalized factorial as GAMMA FUNCTION $(5.5)! = \Gamma(1+5.5) = \Gamma(6.5) = 287.88$

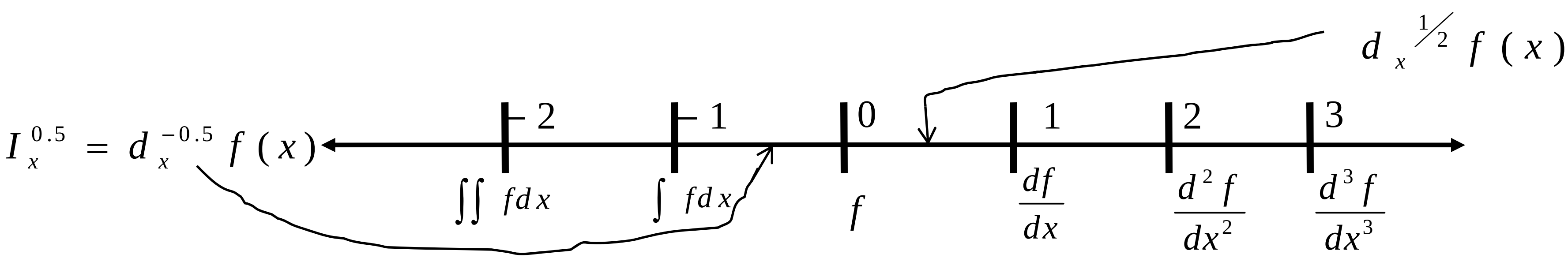
$$x^r = e^{r \ln x}, r \in \Re e$$

$$x! = \Gamma(x+1) = x\Gamma(x)$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

$$\Gamma(x) = \lim_{n \to \infty} \frac{(n!)n^x}{x(x+1)(x+2)\ldots(x+n)}$$

Wonderful universe of mathematics lays in between one full integration and one full differentiation. Fractional Calculus is Generalization of Newtonian-Leibeniz's integer-order calculus, not a paradox what Leibniz in 30th September, 1695 wrote to L Hopital's query. Well a three hundred plus year old topic.



Fractional integration-antiderivative

Repeated *n*-fold integration generalization to arbitrary order

$$I_t^1 f(t) = d_t^{-1} f(t) = \int_0^t f(\tau)d\tau$$

$$I_t^2 f(t) = d_t^{-2} f(t) = \int_0^t \int_0^t f(\tau)d\tau d\tau = \int_0^t (t-\tau) f(\tau)d\tau$$

$$I_t^3 f(t) = d_t^{-3} f(t) = \int_0^t \int_0^t \int_0^t f(\tau)d\tau d\tau d\tau = \frac{1}{2}\int_0^t (t-\tau)^2 f(\tau)d\tau$$

$$I_t^n f(t) = d_t^{-n} f(t) = \int_0^t \int_0^t \underbrace{\cdots\cdots\cdots\cdots\int_0^t f(\tau)d\tau}_{n} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau$$

$$n \in Z^+$$

$$I_t^\phi f(t) = d_t^{-\phi} f(t) = \frac{1}{\Gamma(\phi)}\int_0^t (t-\tau)^{\phi-1} f(\tau)d\tau$$

$$\phi \in R^+$$

$$I_t^{0.5} f(t) = d_t^{-0.5} f(t) = \frac{1}{\Gamma(0.5)}\int_0^t (t-\tau)^{-0.5} f(\tau)d\tau$$

$$I_t^{1.5} f(t) = d_t^{-1.5} f(t) = \frac{1}{\Gamma(1.5)}\int_0^t (t-\tau)^{0.5} f(\tau)d\tau$$

Fractional Integration of exponential function & origin of Miller-Ross function & higher trigonometric functions:

$$f(t) = e^{at}$$

$$d^{-v} e^{at} = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} e^{a\xi} d\xi$$

$$x = t - \xi \qquad \text{change of variable}$$

$$d^{-v} e^{at} = \frac{e^{at}}{\Gamma(v)} \int_0^t x^{v-1} e^{-ax} dx$$

Above is not elementary function, but is closely related to incomplete Gamma function

$$\gamma^*(v,t) = \frac{1}{t^v \Gamma(v)} \int_0^t \xi^{v-1} e^{-\xi} d\xi$$

$$d^{-v} e^{at} = t^v e^{at} \gamma^*(v, at) = E_t(v, a)$$

similarly

$$d^{-v} \cos at = \frac{1}{\Gamma(v)} \int_0^t \xi^{v-1} \cos a(t-\xi) d\xi = C_t(v,a)$$

$$d^{-v} \sin at = \frac{1}{\Gamma(v)} \int_0^t \xi^{v-1} \sin a(t-\xi) d\xi = S_t(v,a)$$

Miller-Ross function-and higher trigonometric functions with properties

$$E_t(w,c) = t^w \sum_{n=0}^{\infty} \frac{(ct)^n}{\Gamma(1+n+w)}$$

$$E_t(v,ia) = t^v \left[\sum_{k(even)}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma(v+k+1)} + i \sum_{k(odd)}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma(v+k+1)} \right]$$

$$C_t(v,a) = t^v \sum_{j=0}^{\infty} \frac{(-1)^j (at)^{2j}}{\Gamma(v+2j+1)}; S_t(v,a) = t^v \sum_{k(odd)}^{\infty} \frac{(-1)^{\frac{k-1}{2}} (at)^k}{\Gamma(v+k+1)}$$

$$E_t(v,ia) = C_t(v,a) + iS_t(v,a)$$

$$C_t(0,a) = \cos at, S_t(0,a) = \sin at, E_t(0,a) = e^{at}$$

$$d_t^1 E_t(v,a) = E_t(v-1,a) \qquad d_t^1 C_t(v,a) = C_t(v-1,a) \qquad d_t^1 S_t(v,a) = S_t(v-1,a)$$

$$\int_0^t E_\xi(v,a)d\xi = E_t(v+1,a) \qquad \int_0^t C_\xi(v,a)d\xi = C_t(v+1,a) \qquad \int_0^t S_\xi(v,a)d\xi = S_t(v+1,a)$$

…………………..and several more interesting relations

Well the basic functions are from Higher Transcendental Functions **Mittag-Leffler** and its Variants like **Agarwal, Srivastava, Robotnov-Hartley, Erderyl, Generalized R functions** all have base from **Hyper geometric series** even called **Higher Transcendental functions**, **Fox functions. Miller-Ross function** is one of them (1903 ML-1999 RH, MR, functions)

Fractional derivative

$$\frac{d^n f(x)}{dx^n} = D_x^n f(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\cdots\cdots\frac{d}{dx}}_{n} f(x)$$

$$\frac{d^n}{dx^n}\{x^m\} = m(m-1)(m-2)\cdots\cdots(m-n+1)x^{m-n}$$

$$\Gamma(m+1) = m(m-1)(m-2)\cdots\cdots(m-n+1)\Gamma(m-n+1)$$

$$\frac{d^n}{dx^n}\{x^m\} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$$

$$\frac{d^{0.5}}{dx^{0.5}}\{x\} = \frac{\Gamma(1+1)}{\Gamma(1-0.5+1)}x^{1-0.5} = \frac{\sqrt{x}}{\Gamma(1+0.5)} = \frac{\sqrt{x}}{0.5\Gamma(0.5)} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

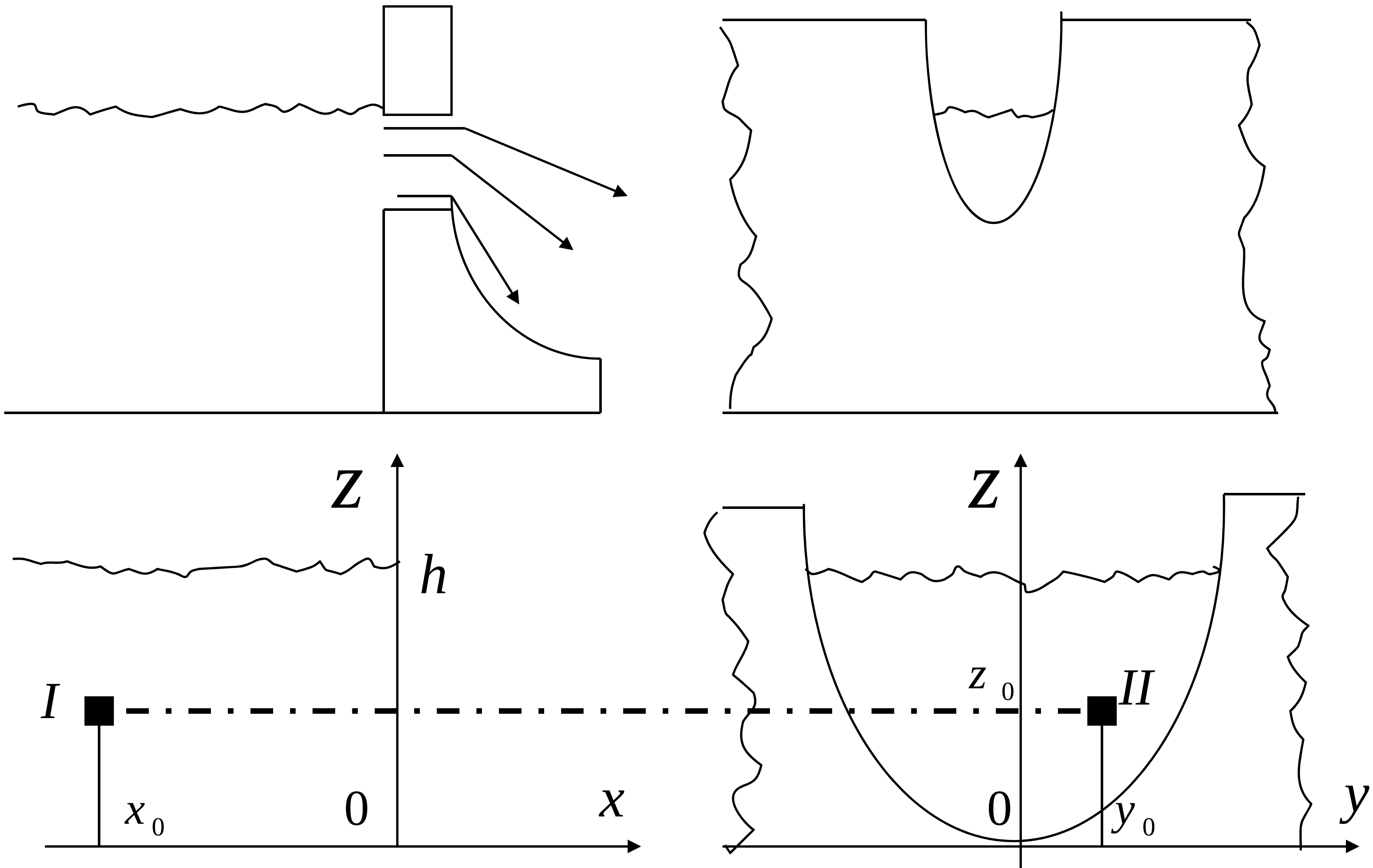
$$\frac{d^{0.5}}{dx^{0.5}}\{C\} = \frac{C}{\sqrt{x\pi}} \neq 0$$

For positive index the process is differentiation

For negative index the process is integration

Fractional derivative of a constant is not zero

Flow rate vis-à-vis notch shape for flow of water through dam weir.



I and *II* are same fluid element, as it moves from point-*I* (x_0, y_0, z_0) to point-*II* $(0, y_0, z_0)$ along the same “tube of flow”.

Bernoulli's equation for hydrodynamics for point *I* and *II* by head balancing gives:

$$\frac{P_I}{\rho} + g z_0 + \frac{1}{2} V_I^2 = \frac{P_{II}}{\rho} + g z_0 + \frac{1}{2} V_{II}^2$$

Assuming point-*I* is far enough upstream $V_I << V_{II}; V_I \approx 0$

Point-*I* pressure is atmospheric pressure plus the pressure exerted by the column of water of height $(h - z_0)$.Since point-*II* is at the same plane, that of notch, the pressure at point-*II* is atmospheric pressure.

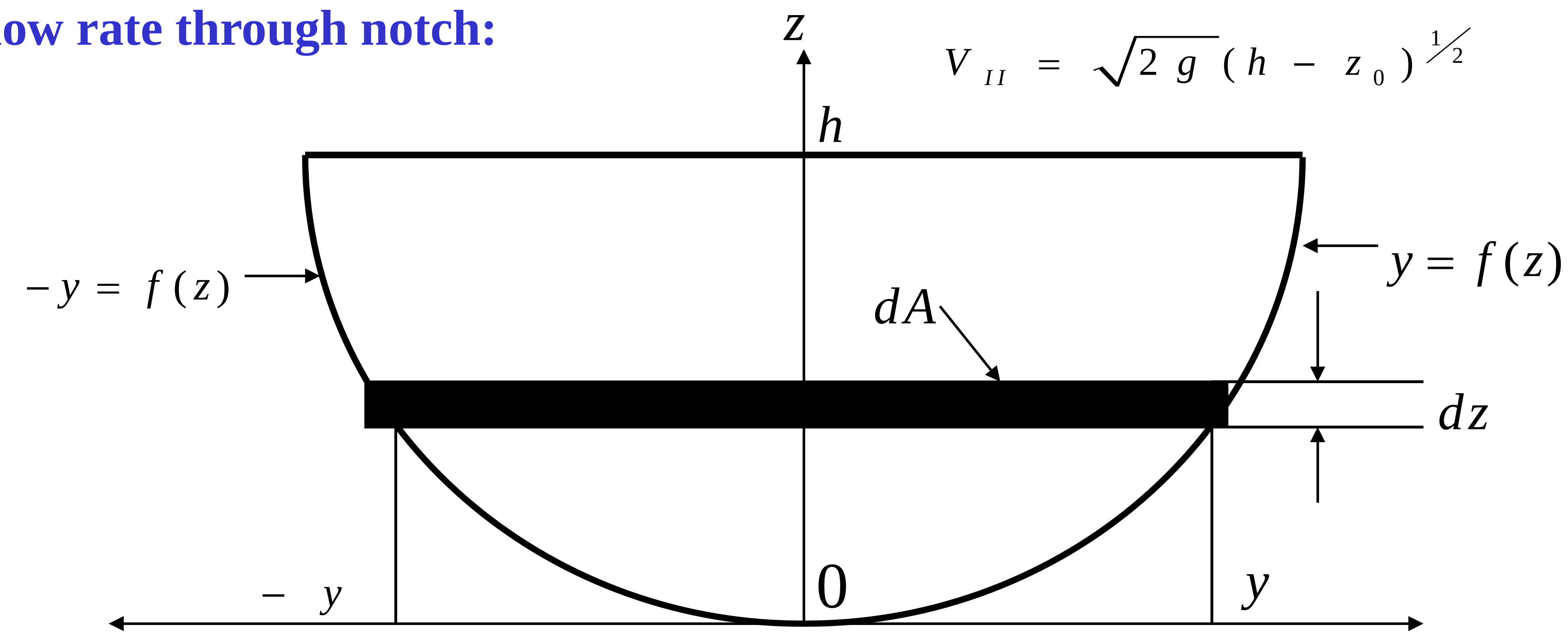
Therefore: $$P_I - P_{II} \equiv (\rho g) \times (h - z_0)$$

And

$$V_{II} = \sqrt{2 g} (h - z_0)^{1/2}$$

Flow rate through notch:

$$V_{II} = \sqrt{2g}(h - z_0)^{1/2}$$



$$dA = 2y\,dz$$

$$dA = 2f(z)\,dz$$

The incremental rate of flow of flow of water through '*dA*' is $dQ = V.dA$

$$dQ = 2\sqrt{2g}(h - z)^{1/2} f(z)dz$$

Total flow of water through the notch:

$$Q = \int_0^h dQ = 2\sqrt{2g}\int_0^h (h - z)^{1/2} f(z)dz$$

Flow rate-vis-à-vis notch profile:

$$Q = \int_0^h dQ = 2\sqrt{2g}\int_0^h (h-z)^{1/2} f(z)\,dz$$

The flow rate (total) through weir notch: $Q(h) = \int_0^h dQ = 2\sqrt{2g}\int_0^h (h-z)^{1/2} f(z)\,dz$

When written in notation of FC the formulation is**:** $Q(h) = \sqrt{2g\pi}\, D^{-3/2} f(h)$

This can be modified as: $D^{3/2}\left[Q(h)\right] = \sqrt{2g\pi}\, D^{3/2}\left[D^{-3/2} f(h)\right]$

Giving: $f(h) = \dfrac{1}{\sqrt{2g\pi}} D^{3/2} Q(h)$

Let us suppose that flow rate profile we want is: $Q(z) = kz^{\lambda}$

Then the notch profile is:

$$f(z) = \frac{1}{\sqrt{2g\pi}} D^{3/2} Q(z) = \frac{k}{\sqrt{2g\pi}} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-3/2)} z^{\lambda-3/2} = \frac{k}{\sqrt{2g\pi}} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-1/2)} z^{\lambda-3/2}$$

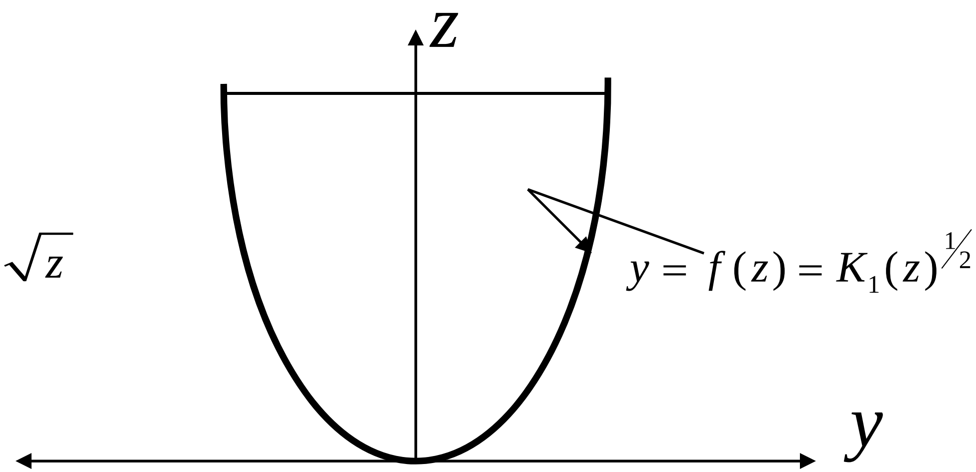
The expression of 1.5 folds integration is used here is

$$I_t^{1.5} f(t) = d_t^{-1.5} f(t) = \frac{1}{\Gamma(1.5)}\int_0^t (t-\tau)^{0.5} f(\tau)\,d\tau = \frac{2}{\sqrt{\pi}}\int_0^t (t-\tau)^{1/2} f(\tau)\,d\tau$$

Particular cases for notch profile for particular flow rates function:

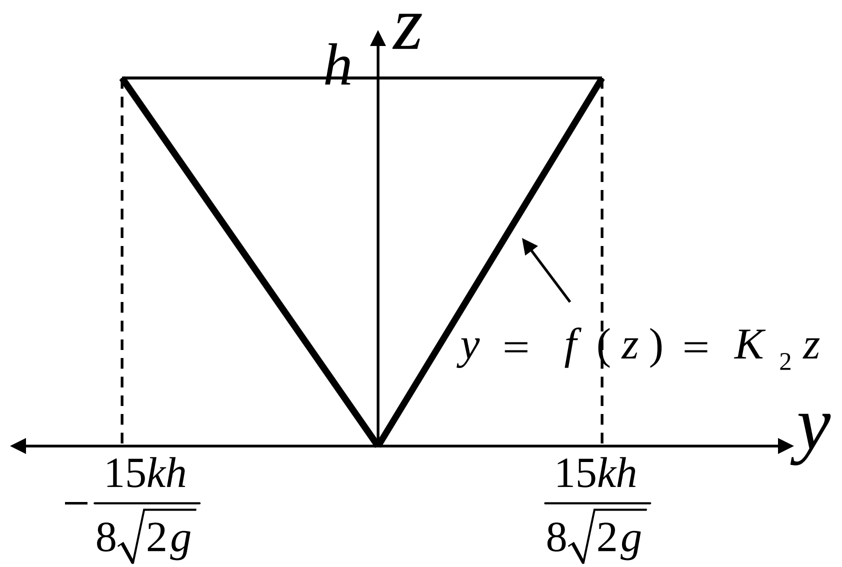
If the flow rate is: $Q(z) = kz^2; \lambda = 2$

Then notch profile is: $y = f(z) = \dfrac{4k}{\pi\sqrt{2g}}\sqrt{z}$



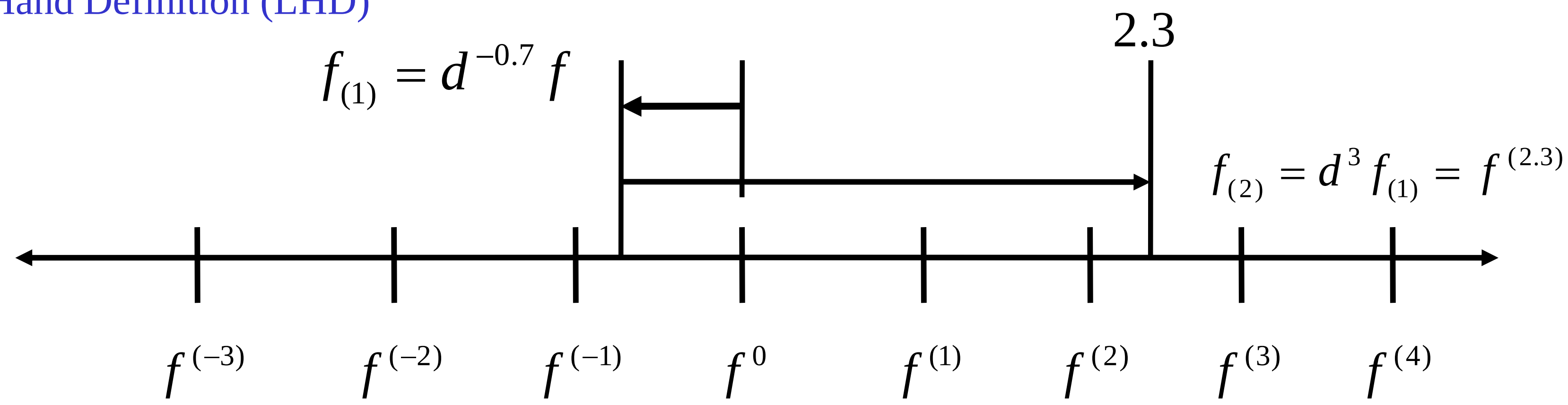
If the flow rate is $Q(z) = kz^{5/2}; \lambda = 5/2$

The notch profile is: $y = f(z) = \dfrac{15k}{8\sqrt{2g}} z$

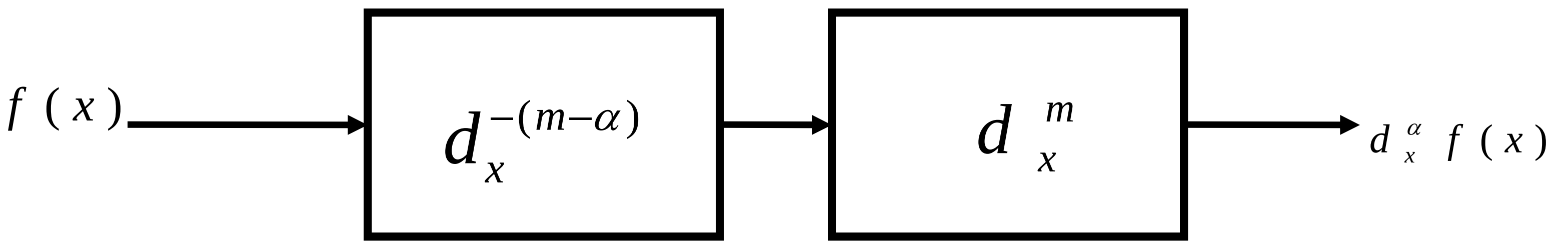


Riemann Liouvelli (RL) Fractional derivative

Left Hand Definition (LHD)



Here '*m*' is the integer just greater than fractional order of derivative



$$d_x^{\alpha} f(x) = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^x (x-\tau)^{-\alpha-1+m} f(\tau)\,d\tau\right]$$

Fractional differentiation of exponential Miller-Ross & trigonometric functions

$$f(t) = e^{at}$$

$$d_t^{\mu} e^{at} = d_t^{m}[d_t^{-v} e^{at}]; \qquad \mu = m - v$$

$$d_t^{-v} e^{at} = E_t(v, a)$$

$$d_t^{m} E_t(v, a) = E_t(v - m, a) = E_t(-\mu, a)$$

$$d_t^{\mu} e^{at} = E_t(-\mu, a)$$

Similarly, one can have

$$d_t^{\mu} \cos at = C_t(-\mu, a)$$

$$d_t^{\mu} \sin at = S_t(-\mu, a)$$

$$f(t) = E_t(\lambda, a)$$

$$d_t^{\mu} E_t(\lambda, a) = d_t^{m}[d_t^{-v} E_t(\lambda, a)]; \qquad \mu = m - v$$

$$d_t^{-v} E_t(\lambda, a) = E_t(\lambda + v, a)$$

$$d_t^{\mu} E_t(\lambda, a) = E_t(\lambda - \mu, a)$$

Similarly, one can have

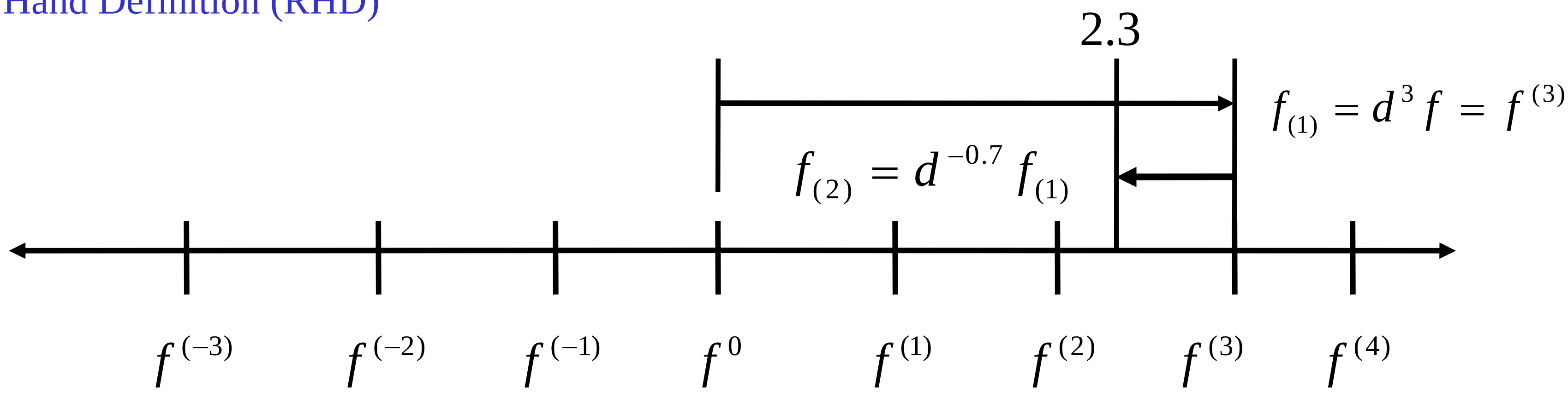
$$d_t^{\mu} C_t(\lambda, a) = C_t(\lambda - \mu, a)$$

$$d_t^{\mu} S_t(\lambda, a) = S_t(\lambda - \mu, a)$$

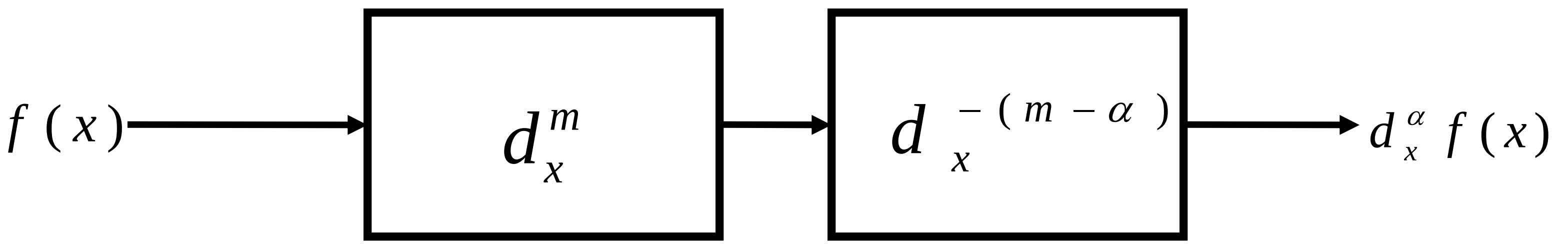
…………………for several more cases of Functions for Fractional Differ-integrations consult books FFC MPGC, elaborately solved there.

Caputo (1967) Fractional derivative

Right Hand Definition (RHD)



Here '*m*' is the integer just greater than the fractional order derivative



$$d_x^\alpha f(x) = \left[\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{-\alpha-1+m} \frac{d^m f(\tau)}{d\tau^m} d\tau \right]$$

Michelle Caputo in March 2010 sent me email appreciating and congratulating on the modern treatments contained in I-edition of FFC book, which he read with interest; greatest honor PRIS award of my life to get recognition from founder of fractional derivative 85years old living legend Italian mathematician

Caputo-Riemann Liouvelli Fractional Derivative Equivalence

Only if the initial conditions are static to zero should these be equal.

$$\frac{1}{\Gamma(m-q)}\int_a^x \frac{f^{(m)}}{[x-y]^{q-n+1}}dy + \sum_{k=0}^{m-1}\frac{[x-a]^{k-q} f^{(k)}(a)}{\Gamma(k-q+1)} = \frac{d^m}{dx^m}\left[\frac{1}{\Gamma(m-q)}\int_a^x \frac{f(y)}{[x-y]^{q-m+1}}dy\right]$$

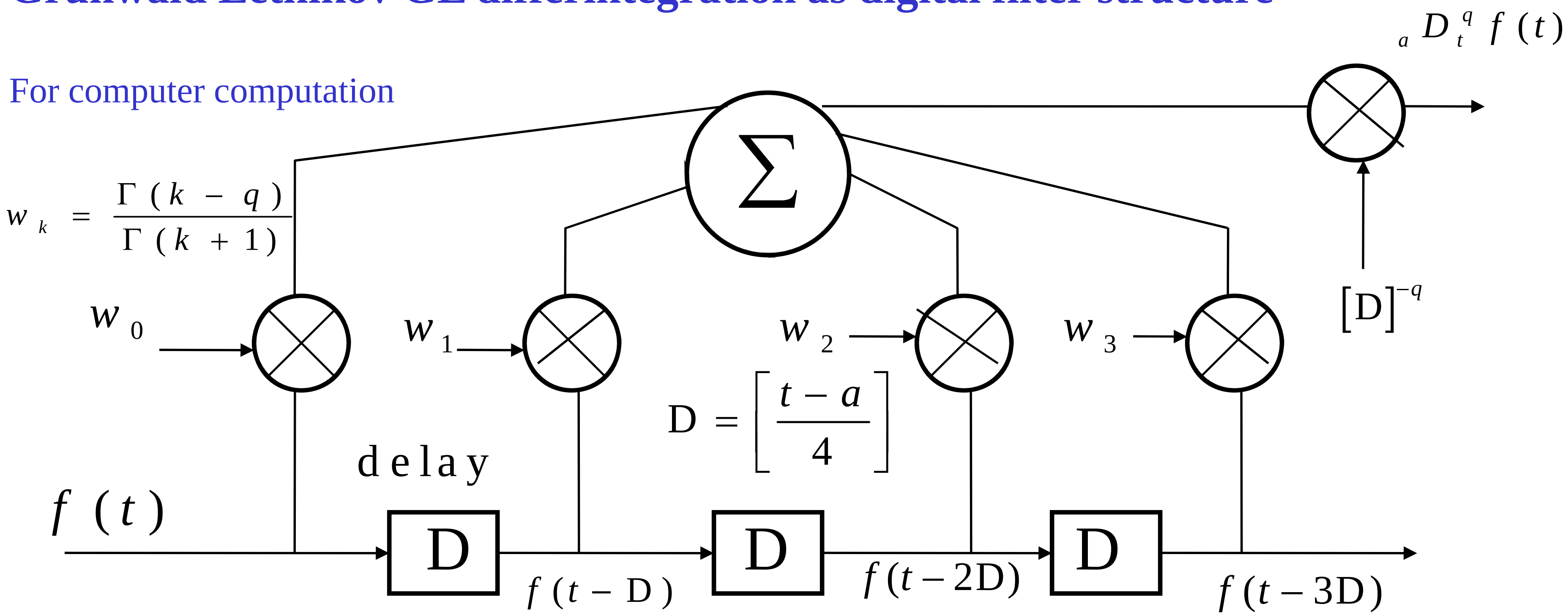
$$m > q$$

$$[\,{}_a d_t^q\, f(t)]_{RL} = [\,{}_a^C d_t^q\, f(t)] + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k-q+1)}(t-a)^{k-q}; n > q$$

This is generalization of fundamental theorem if calculus, i.e. differentiation of integration commutes and integration of differentiation is separated by initial values of function at start.

Grunwald Letnikov GL differintegration as digital filter structure

For computer computation

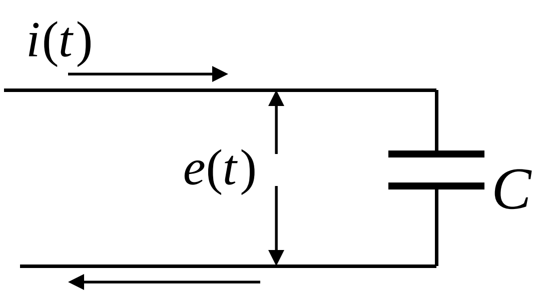


$${}_aD_t^q f(t) = \lim_{\Delta T \to 0} \frac{(\Delta T)^{-q}}{\Gamma(-q)} \sum_{k=0}^{N-1} \frac{\Gamma(k-q)}{\Gamma(k+1)} f(t-k\Delta T) = \lim_{\Delta \to 0} [\Delta]^{-q} \sum_{k=0}^{N-1} w_k f(t-k\Delta)$$

This is obtained by generalizing finite differences to *n* folds and Riemann Sum formula generalization of *n* fold Trapezoidal rule: The fractional differentiation and integration same but weights differ.

For detail derivation refer the books

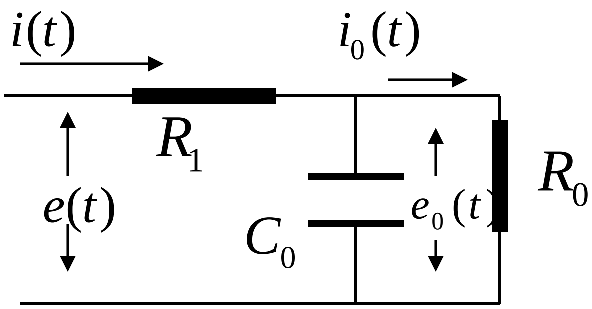
Self similar repeated prolong structure terminal relation and semi-differentiation



$$e(t) = \frac{1}{C}\int_0^t i(\xi)\,d\xi = \frac{1}{C}\frac{d^{-1}}{dt^{-1}}i(t)$$

$$i(t \le 0) = 0 = e(t \le 0)$$

$$s \leftrightarrow \frac{d}{dt}; \qquad s^{-1} \leftrightarrow \frac{d^{-1}}{dt^{-1}}; \qquad s^{\nu} \leftrightarrow \frac{d^{\nu}}{dt^{\nu}}$$



$$i_0(t) = \frac{e_0(t)}{R_0} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$i(t) - i_0(t) = C_0\frac{de_0(t)}{dt} \ldots\ldots\ldots\ldots\ldots (2)$$

$$i(t) = \frac{e(t) - e_0(t)}{R_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eliminating $e_0(t)$ & $i_0(t)$ from (1), (2) & (3)

$$[R_0 + R_1]i(t) + R_0R_1C_0\frac{di(t)}{dt} = e(t) + R_0C_0\frac{de(t)}{dt}$$

Continued Fraction Expansion form:

$i(0) = 0 = e(0)$

Given initial values

$$[R_0 + R_1]i(t) + R_0 R_1 C_0 \frac{di(t)}{dt} = e(t) + R_0 C_0 \frac{de(t)}{dt}$$

This above relation was obtained from the previous circuit

$$I(s)[R_0 + R_1 + R_0 R_1 C_0 s] - R_0 R_1 C_0 i(0) = E(s)[1 + R_0 C_0 s] - R_0 C_0 e(0)$$

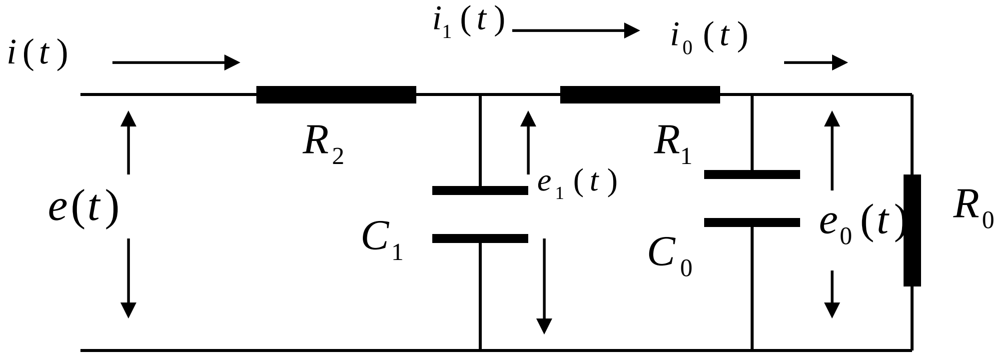
We get the above byLaplace Tranforming the time domain equation

$$\frac{E(s)}{I(s)} = \frac{R_0 + R_1 + R_0 R_1 C_0 s}{1 + R_0 C_0 s}$$

This is well known transfer function representation

$$\frac{E(s)}{R_1 I(s)} = 1 + \frac{\frac{1}{R_1 C_0}}{s + \frac{1}{R_0 C_0}}$$

Expand the circuit further:



$$i(t) = \frac{e(t) - e_1(t)}{R_2}$$

$$i(t) - i_1(t) = C_1 \frac{d e_1(t)}{dt}$$

$$\frac{E(s)}{R_2 I(s)} = 1 + \frac{\frac{1}{R_2 C_1}}{s + \frac{I_1(s)}{C_1 E_1(s)}}$$

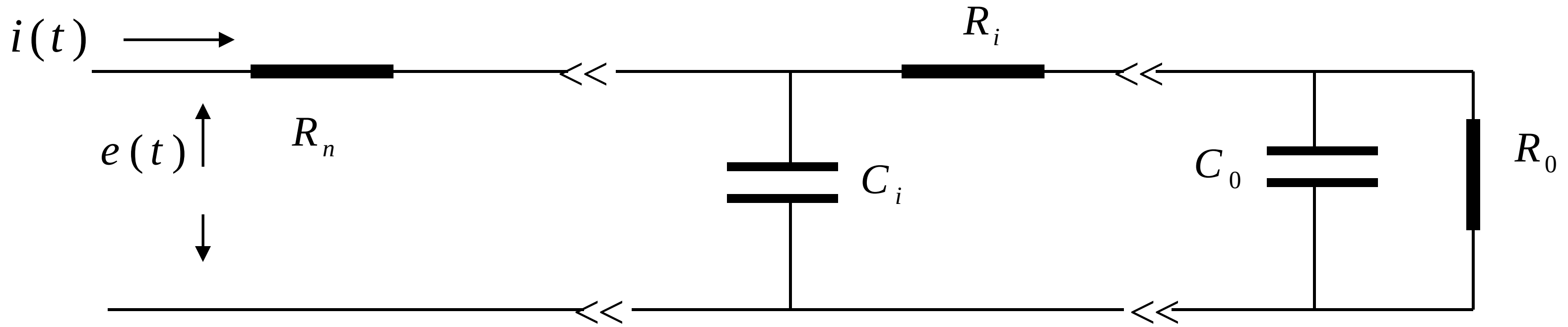
$$\frac{E_1(s)}{R_1 I_1(s)} = 1 + \frac{\frac{1}{R_1 C_0}}{s + \frac{1}{R_0 C_0}}$$

$$\omega_0 \equiv (\tau_0)^{-1} = (R_0 C_0)^{-1}, \omega_1 \equiv (R_1 C_0)^{-1}$$

$$\omega_2 \equiv (R_1 C_1)^{-1}, \omega_3 \equiv (R_2 C_1)^{-1}$$

$$\frac{E(s)}{R_2 I(s)} = 1 + \frac{\omega_3}{s + \frac{\omega_2}{1 + \frac{\omega_1}{s + \omega_0}}}$$

Generalizing and expanding to infinity



$$\frac{E(s)}{R_n I(s)} = 1 + \cfrac{\omega_{2n-1}}{s + \cfrac{\omega_{2n-2}}{1 + \cfrac{\omega_{2n-3}}{s + \ldots \cfrac{\omega_1}{s + \omega_0}}}} = 1 + \frac{\omega_{2n-1}}{s +}\frac{\omega_{2n-3}}{1 +}\frac{\omega_{2n-3}}{s +}\ldots\frac{\omega_2}{1 +}\frac{\omega_1}{s +}\frac{\omega_0}{1}$$

$$\omega_{2j} = (R_j C_j)^{-1}; \qquad \omega_{2j+1} = (R_{j+1} C_j)^{-1}$$

Simplifying CFE

$$\frac{E(s)}{R_n I(s)} = 1 + \frac{\omega_{2n-1}}{s+} \frac{\omega_{2n-2}}{1+} \frac{\omega_{2n-3}}{s+} \ldots\ldots \frac{\omega_2}{1+} \frac{\omega_1}{s+} \frac{\omega_0}{1}$$

$$v_j = \frac{\omega_j}{s}$$

$$\frac{E(s)}{R_n I(s)} 1 + \frac{v_{2n-1}}{1+} \frac{v_{2n-2}}{1+} \frac{v_{2n-3}}{1+} \ldots\ldots\ldots\ldots \frac{v_2}{1+} \frac{v_1}{1+} \frac{v_0}{1}$$

$$C_0 = C_1 = C_2 = \ldots\ldots\ldots\ldots\ldots = C_{n-1} = C$$

$$R_0 = R_1 = R_2 = \ldots\ldots\ldots\ldots\ldots = R_{n-1} = R; R_n = \frac{1}{2} R$$

$$v_{2n-1} = \frac{2}{RCs} = 2v$$

$$\frac{2E(s)}{RI(s)} = 1 + 2 \frac{v}{1+} \frac{v}{1+} \frac{v}{1+} \ldots\ldots\ldots\ldots \frac{v}{1+} \frac{v}{1}$$

CFE in limit of very large number of stages:

By induction $$\frac{2E(s)}{RI(s)} = 1 + 2\frac{v}{1+}\frac{v}{1+}\cdots\cdots\cdots\frac{v}{1+}\frac{v}{1} \quad \cdots\cdots\cdots(1)$$

$$\frac{v}{1+}\frac{v}{1+}\cdots\cdots\frac{v}{1+}\frac{v}{1} = \frac{\sqrt{4v+1}}{1+\left[\frac{\sqrt{4v+1}-1}{\sqrt{4v+1}+1}\right]^{2n+1}} - \frac{\sqrt{4v+1}}{2} - \frac{1}{2} \quad \cdots\cdots\cdots(2)$$

From (1) and (2) and dividing by $2\sqrt{v}$ we obtain:

$$\frac{E(s)}{I(s)}\sqrt{\frac{Cs}{R}} = \sqrt{\frac{4v+1}{4v}}\left[\frac{[\sqrt{4v+1}+1]^{2n+1} - [\sqrt{4v+1}-1]^{2n+1}}{[\sqrt{4v+1}+1]^{2n+1} + [\sqrt{4v+1}-1]^{2n+1}}\right] \cdots\cdots\cdots(3)$$

Graphically one estimate RHS of (3) to unity as for large ‘*n*’ and seemingly wide spread of “*v*”; implying, RHS is within 2% of unity for wide frequency/time range

$$\frac{E(s)}{I(s)}\sqrt{\frac{Cs}{R}} \approx 1 \qquad 6 \le v \le \frac{1}{6}n^2$$

$$E(s) \approx \sqrt{\frac{R}{Cs}}I(s) \qquad 6RC \le \frac{1}{s} \le \frac{1}{6}n^2RC$$

$$e(t) \approx \sqrt{\frac{R}{C}}\frac{d^{-1/2}}{dt^{-1/2}}i(t) \qquad 6RC \le t \le \frac{1}{6}n^2RC$$

A normal diffusion equation

Start with continuity equation $\frac{d}{dt}M(r,t) = -j(r,t)$,

where

$$M(r,t) = \int_0^r dr P(r,t)$$

and $j(r,t)$ is the total probability current at *r* from origin.

The above equation must be supplemented by constitutive equation relating the current *j* (*r*, *t*) to the probability density function p.d.f *P* (*r*, *t*) i.e.

$$j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r}$$

From these we get diffusion equation in differential form from the above Fick's law.

$$\frac{\partial}{\partial t}P(r,t) = \mathbb{D}_0 \frac{\partial^2}{\partial r^2}P(r,t)$$

The plume is Gaussian and MSD is linear with time, for delta function at origin

$$P(r,t) = \frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}$$

A normal diffusion equation & its fractional calculus version

Take Laplace Transform of the constitutive equation

$$j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r} \text{ when LaplaceTranformed we get: } \quad j(r,s) = -\mathbb{D}_0 \frac{\partial}{\partial r} P(r,s)$$

The plume is Gaussian and MSD is linear with time, for delta function at origin

$$P(r,t) = \frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}$$

Using Laplace of Gaussian, we get (Refer Laplace Table FFC book)

$$P(r,s) = \mathcal{L}\{P(r,t)\} = \mathcal{L}\left\{\frac{1}{\sqrt{\pi \mathbb{D}_0 t}} e^{\left(-\frac{r^2}{4\mathbb{D}_0 t}\right)}\right\} = \left(\frac{1}{\sqrt{\mathbb{D}_0 s}}\right) e^{\left(-r\sqrt{\frac{s}{\mathbb{D}_0}}\right)}$$

Taking first derivative of above we get

$$\frac{d}{dr} P(r,s) = \frac{-1}{\sqrt{\mathbb{D}_0 s}} \sqrt{\frac{s}{\mathbb{D}_0}} e^{-r\sqrt{s/\mathbb{D}_0}} = -\frac{1}{\mathbb{D}_0} e^{-r\sqrt{s/\mathbb{D}_0}}$$

and then from Laplace transformed constitutive equation with slight manipulation, we get

$$j(r,s) = \sqrt{s\mathbb{D}_0}\, P(r,s)$$

Use $\sqrt{s} \leftrightarrow \mathcal{L}\left\{d^{1/2} / dt^{1/2}\right\}$ and we get time domain expression of current as

$$j(r,t) = \sqrt{\mathbb{D}_0} \frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}}$$

, put this in $j(r,t) = -\mathbb{D}_0 \frac{\partial P(r,t)}{\partial r}$ constitutive equation

we get **fractional calculus version** of standard diffusion equation

$$\frac{\partial^{1/2} P(r,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0} \frac{\partial P(r,t)}{\partial r}$$

This is standard Brownian process BM with linear MSD

A Fractional Brownian Motion Process (FBM) simplest way-a passing remark

Fractional Brownian Motion (FBM) is simplest mathematical model extension of Gaussian stochastic process (random walk) whose variance (MSD) does not scale linearly with time its p.d.f. is:

FBM is natural generalization of BM, here with a stretched exponential

$$P_{\mathrm{FBM}}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t^{2/d_w}}} e^{\left(-\frac{x^2}{4\mathbb{D}_0 t^{2/d_w}}\right)}$$

$$\left\langle x^2(t) \right\rangle \equiv 2\mathbb{D}_0 t^{2/d_w} \qquad 2 \leq d_w < \infty$$

d_w Anomalous diffusion exponent

Brownian Case is with anomalous diffusion exponent as 2 is, MSD is $\left\langle x_{\mathrm{BM}}^2(t) \right\rangle \equiv 2\mathbb{D}_0 t$ plume following:

$$P_{\mathrm{BM}}(x,t) = \frac{1}{\sqrt{4\pi \mathbb{D}_0 t}} e^{\left(-\frac{x^2}{4\mathbb{D}_0 t}\right)} \qquad d_w = 2 \qquad \frac{\partial^{1/2} P_{\mathrm{BM}}(x,t)}{\partial t^{1/2}} = -\sqrt{\mathbb{D}_0} \frac{\partial P_{\mathrm{BM}}(x,t)}{\partial |x|}$$

For non-Brownian case:

$$\frac{\partial^{1/d_w} P_{\mathrm{FBM}}(x,t)}{\partial t^{1/d_w}} = -\mathrm{A} \frac{\partial P_{\mathrm{FBM}}(x,t)}{\partial |x|}$$

Transport phenomena in complex systems such as random fractal structures exhibit anomalous features which are qualitatively different from the standard regular systems. In the case of fractals such anomalies are due to constraint on the transport process on all lengths scales. These constraints may be seen as temporal correlations existing on time scales.

A Fractional Brownian Motion Process (FBM), represented with memory

A Fractional Brownian Motion Process (FBM), represented with memory, described as integral transform of Brownian Motion (BM). The convolution with memory kernel that is $K_M(t)$.

$$x_{\mathrm{FBM}}(t) = \int_{-\infty}^{t} K_M(t-\tau)\,dx_{\mathrm{BM}}(\tau)$$

Where $x_{\mathrm{FBM}}(t)$ and $x_{\mathrm{BM}}(t)$ are the position of the particle undergoing the FBM and BM process respectively.

$$K_M(t-\tau) = \begin{cases} (t-\tau)^{(1/d_w)-(1/2)} - (-\tau)^{(1/d_w)-(1/2)} & ;\tau < 0 \\ (t-\tau)^{(1/d_w)-(1/2)} & ;0 < \tau < t \end{cases}$$

The kernel resembles the singular memory kernel associated with fractional integral (derivatives).

$$I_t^{\phi} f(t) = d_t^{-\phi} f(t) = \frac{1}{\Gamma(\phi)} \int_0^t (t-\tau)^{\phi-1} f(\tau)\,d\tau$$

With anomalous exponent equal to 2 we get a no-memory case with $K_M(t-\tau) = 1$ in this case

$$x_{\mathrm{FBM}}(t) = x_{\mathrm{BM}}(t)$$

A walker undergoing FBM remembers his past, while a walker undergoing BM does not remember its past. Well a walker can remember its past and have preferences in same direction giving persistence walk, or a walker remembering its past can change its direction giving anti-persistence walks, are the cases of anomalous transport. This gives concept of sub or super diffusion, in FBM context.

Note, the variable $x_{\mathrm{FBM}}(t)$ may not be physical distance. It could be

1. Computer net work delay.
2. Could be price of stock market.
3. Could be random returns of insurance system.
4. Could be infected population by swine flu

In general could be variable for physical systems represented by random stochastic process.

Non-Linear MSD cases-Anomalous diffusion

$$\left\langle X^2(t) \right\rangle \approx t^{2H}$$

$H = \frac{1}{2}$ Normal Random walk and normal diffusion
Brownian Motion, with no memory

$$\left\langle X^2(t) \right\rangle \approx t$$

$0 \leq H < \frac{1}{2}$ Anti Persistent Random walk and sub (slow) diffusion
With short term memory
The process decays monotonically to zero
hyperbolically

$$\left\langle X^2(t) \right\rangle \approx t^{1/2}$$

$$H = 1/4$$

$1 > H > \frac{1}{2}$ Persistent Random walk and super (fast) diffusion
Long-term- ‘lingering' memory (Long Ranged
Dependence) LRD, Autocorrelation
decays as power law, Fractional Brownian Motion

$$\left\langle X^2(t) \right\rangle \approx t^{3/2}$$

$$H = 3/4$$

H **Hurst exponent**

The Fickian diffusion generalized with Euclidian dimension and geometrical parameter:

The Fick's diffusion equation for vector form is thus:

$$\frac{\partial}{\partial t} u(\overline{X}, t) = \mathbb{D} \nabla^2 u(\overline{X}, t)$$

For isotropic case

$$\frac{\partial}{\partial t} u(r, t) = \mathbb{D}\, r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} u(r, t)$$

d -is Euclidian dimension of 1, 2, 3. The Laplacian operator is generalized 1- is planar 2- cylindrical, and 3- spherical coordinates

One may too re-write the Laplacian operator with geometrical parameter g with 0- for planar, ½ for cylindrical and 1-for spherical coordinate (geometry), as:

$$\frac{\partial}{\partial t} u(x,t) = \mathbb{D} \frac{\partial^2}{\partial r^2} u(x,t) + \frac{2g\mathbb{D}}{r} \frac{\partial}{\partial r} u(x,t)$$

These generalization of Laplacian operator give integer order diffusion equation. Can these number be arbitrary. Well yes they can and then the Laplacian becomes fractional differential equation-giving fractional order diffusion equation. Well, if the diffusing species is moving in a matrix of well distributed obstacles or traps or attractors can they rate at fast or slow? There is thus possibility of this not following Gaussian or integer order law.

Phase Table for the Fractional Diffusion Equation

$$\frac{\partial}{\partial t}\phi(x,t) = {}_0D_t^{1-\alpha}(\mathbb{D}_{\alpha,\mu})\frac{\partial^\mu}{\partial x^\mu}\phi(x,t)$$

We are used to $\alpha = 1, \mu = 2$ The fractional order comes as observation of asymptotic behavior in space time relaxation.

Temporal Fractional Order α	Spatial Fractional Order μ	Type of Walk	Average Waiting Time T	Jump-Length Variance σ^2	Nature of Diffusion
$0<\alpha<1$	$0<\mu<2$	Long-Jump	∞	∞	Non-Markovian
$\alpha \geq 1$	$0<\mu<2$	Long-Jump	$<\infty$	∞	Markovian
$0<\alpha<1$	$\mu \geq 2$	Sub-diffusion	∞	$<\infty$	Non-Markovian
$\alpha \geq 1$	$\mu \geq 2$	Brownian	$<\infty$	$<\infty$	Markovian

Clearly we can have argument that massive particle as neutron cannot jump infinitely far. For such massive particles finite velocity of propagation exists making instantaneous long jumps impossible. But in reality we have nuclear reactors where the dimensions are large especially for high power reactors. The neutrons do have 'coupling' between spatially distributed 'point' reactors. These power reactors are having dimensions much larger than the average diffusion lengths of neutron-and are called coupled core reactors. The spatial heterogeneity in small scales do manifest as fractal dimensions in space which makes the anomalous transport of neutrons contrary to belief that it can only reside and 'walk' as local Brownian motion. Long-jumps can therefore take place for these 'massive' neutrons-in a 'fractal' heterogeneous spatial backdrop, along with long wait times, in heterogeneous lattice. The 'Fractal background' helps to have walk-through, making long jumps possible!

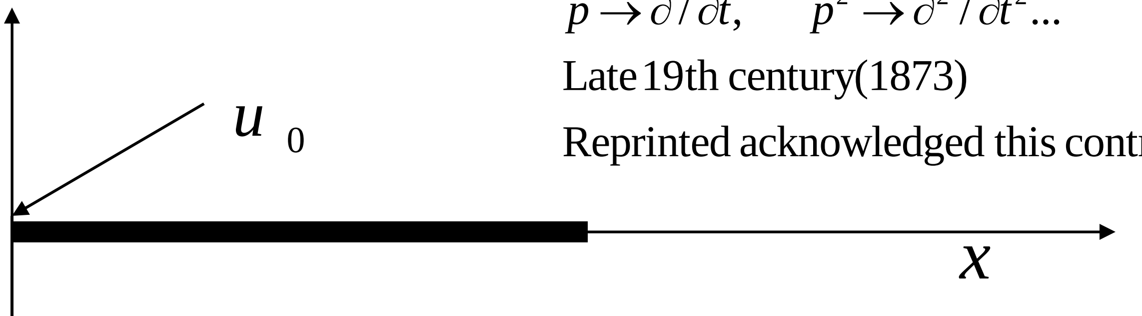
Partial Differential Equation & Operational calculus Heat flow in semi-infinite System

Oliver Heaviside

$p \to \partial/\partial t, \quad p^2 \to \partial^2/\partial t^2 \ldots$

Late 19th century(1873)

Reprinted acknowledged this contribution (1922)



Semi-infinite system diffusion equation PDE

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

$$a^2 = \frac{c_p \rho}{k}, \quad a^2 = RC$$

Initial condition $\quad u(x,0) = 0, \; x > 0$

Boundary condition $\quad u(0,t) = u_0$

Operator $\quad s \equiv \partial/\partial t$

$$\frac{\partial^2 u}{\partial x^2} - a^2 s u = 0$$

Solution

$$\frac{\partial^2 u}{\partial x^2} - a^2 s u = 0$$

$$m = \pm a\sqrt{s}$$

$$u(x,s) = A\exp(-a\sqrt{s}\,x) + B\exp(+a\sqrt{s}\,x)$$

$$x \to \infty, u(x,0) = 0, B = 0$$

$$x \to 0, u(0,t) = u_0 = A$$

$$u(x,s) = u_0\exp(-a\sqrt{s}\,x)$$

Now we will use FC concepts from the above response function of Oliver Heaviside's steps of operational calculus. Note he used the operational calculus with *p* variable, and later formalized as Laplace Operator

Expanding exponential as power series:

$$u(x,s) = u_0 + u_0\sum_{n=1}^{\infty}\frac{(-ax\sqrt{s})^n}{n!} = u_0 + \sum_{n=1}^{\infty}\frac{(-ax)^n (s)^{\frac{n}{2}}}{n!}u_0$$

segregating odd & even terms and then rearranging:

$$u(x,s) = u_0 - \sum_{m=0}^{\infty}\frac{(ax)^{2m+1}}{(2m+1)!}s^m\left(s^{1/2}u_0\right) + \sum_{n=0}^{\infty}\frac{(ax)^{2n}}{(2n)!}s^n u_0$$

putting

$$d^{1/2}u_0 \to s^{1/2}u_0 \equiv \frac{u_0}{\sqrt{\pi t}}; d^n u_0 \to s^n u_0 = 0$$

$$s^m \to d^m / dt^m$$

Putting this we get the even terms as *n*-th integer order derivative components of a constant a zero.

Solution (contd.)

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \left[\frac{d^m}{dt^m} t^{-1/2} \right] = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma(-\frac{1}{2}+1)}{\Gamma(-\frac{1}{2}-m+1)} t^{-\frac{1}{2}-m}$$

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(ax)^{2m+1}}{(2m+1)!} \frac{\Gamma\left(1/2\right)}{\Gamma\left(\frac{1}{2}-m\right)} \frac{1}{t^{m+1/2}}$$

using $\Gamma(-m + 1/2) = \dfrac{[-4]^m m!\sqrt{\pi}}{(2m)!}$ We obtain the following:

$$u(x,t) = u_0 - \frac{u_0}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+\frac{1}{2}}}$$

writing $\dfrac{(ax)^{2m+1}}{(2m+1)2^{2m} t^{m+\frac{1}{2}}} \equiv 2 \displaystyle\int_0^{y=\frac{ax}{2\sqrt{t}}} y^{2m} dy$ and using $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (x^2)^m \equiv \exp(-x^2)$

$$u(x,t) = u_0 - \frac{2u_0}{\sqrt{\pi}} \int_0^{ax/2\sqrt{t}} \exp(-y^2)\,dy$$

Semi-Infinite Lossy Transmission line

$$\frac{\partial v(x,t)}{\partial x} = -i(x,t)R$$

$$\frac{\partial i(x,t)}{\partial x} = -C\frac{\partial v(x,t)}{\partial t}$$

Differentiating first one w.r.t. *x* and then putting in the second one

$$\frac{\partial^2 v}{\partial x^2} = -R\frac{\partial i}{\partial x} = RC\frac{\partial v}{\partial t}$$ Choose $\alpha = \frac{1}{RC}$ to get diffusion equation

$$\frac{\partial v(x,t)}{\partial t} = \alpha\frac{\partial^2 v(x,t)}{\partial x^2}$$

Condition $v(0,t) = v_I(t),\ v(\infty,t) = 0$

$v(x,0)$, given with $i(x,t) = -\frac{1}{R}\frac{\partial v(x,t)}{\partial x}$

In this formulation (v) is the voltage (i) is the current (v_I) is time dependent input variable. A classical method of using iterated Laplace is used to solve this problem: (Refer FFC)

Driving point impedance is:

$$Z(0,s) = \frac{V(0,s)}{I(0,s)} = \sqrt{\frac{R}{C}}\frac{1}{\sqrt{s}}$$

Semi-Derivative $$i(x,t) = \frac{1}{R\sqrt{\alpha}}\frac{d^{1/2}v(x,t)}{dt^{1/2}}$$

Semi Derivative (or Integral) is natural part of “Normal-Diffusion” equation!!!

Circuit Synthesis

Synthesis of fractional order immittances
Newton method of root evaluation

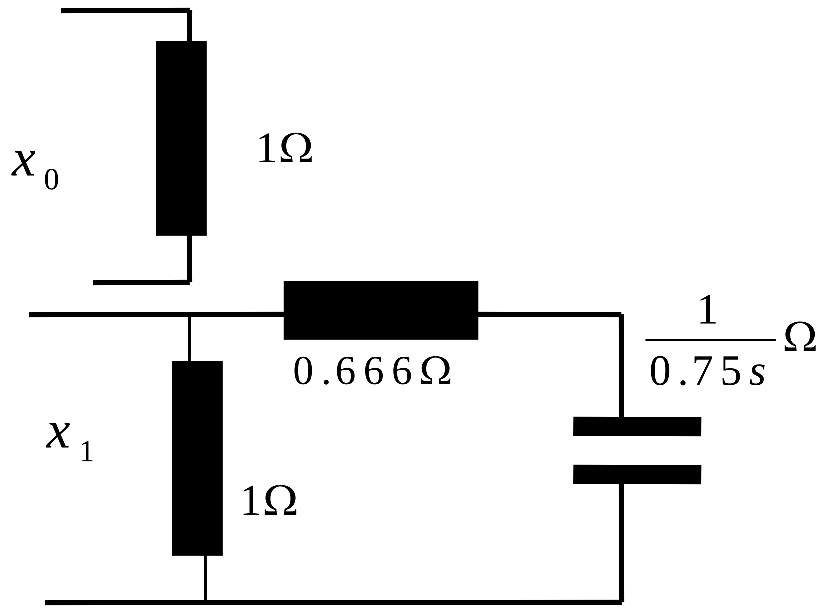
$$a = x^n,\ x = (a)^{1/n},\ x_0 = 1$$

$$x_k = x_{k-1}\frac{(n-1)(x_{k-1})^n + (n+1)a}{(n+1)(x_{k-1})^n + (n-1)a}$$

$$n = 3, \qquad a = \frac{1}{s}, \qquad x_0 = 1,$$

$$x_1 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s+2}{2s+1}$$

$$x_2 = \left(\frac{1}{s}\right)^{1/3} = \frac{1}{\sqrt[3]{s}} = \frac{s^5 + 24s^4 + 80s^3 + 92s^2 + 42s + 4}{4s^5 + 42s^4 + 92s^3 + 80s^2 + 24s + 1}$$

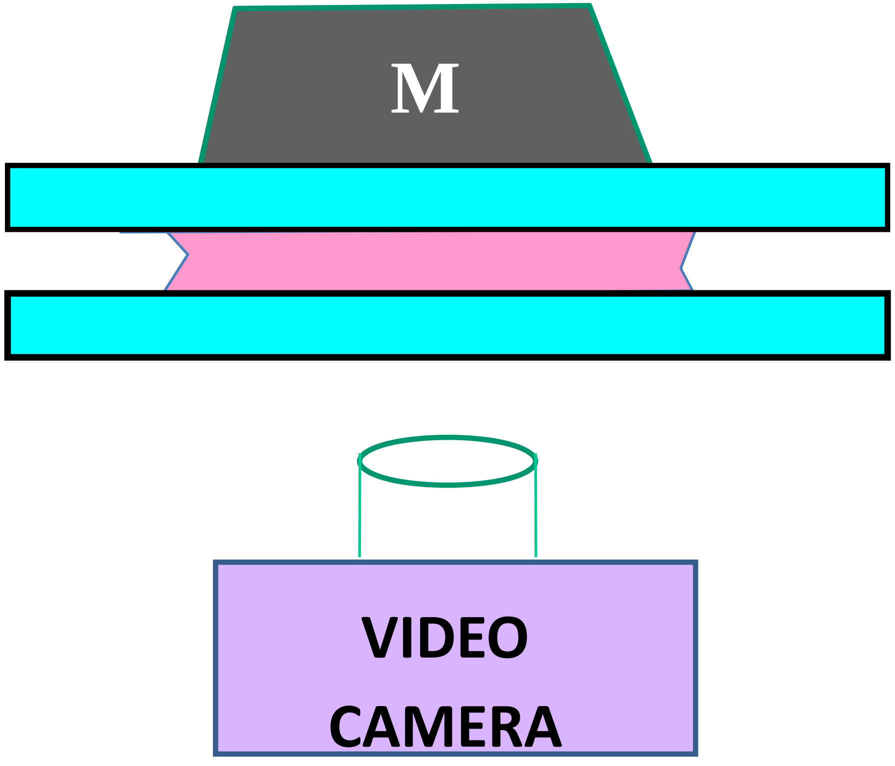


This way one can synthesize approximately the fractional Laplace operator thus Fractional Differ-integral circuit of any order!!

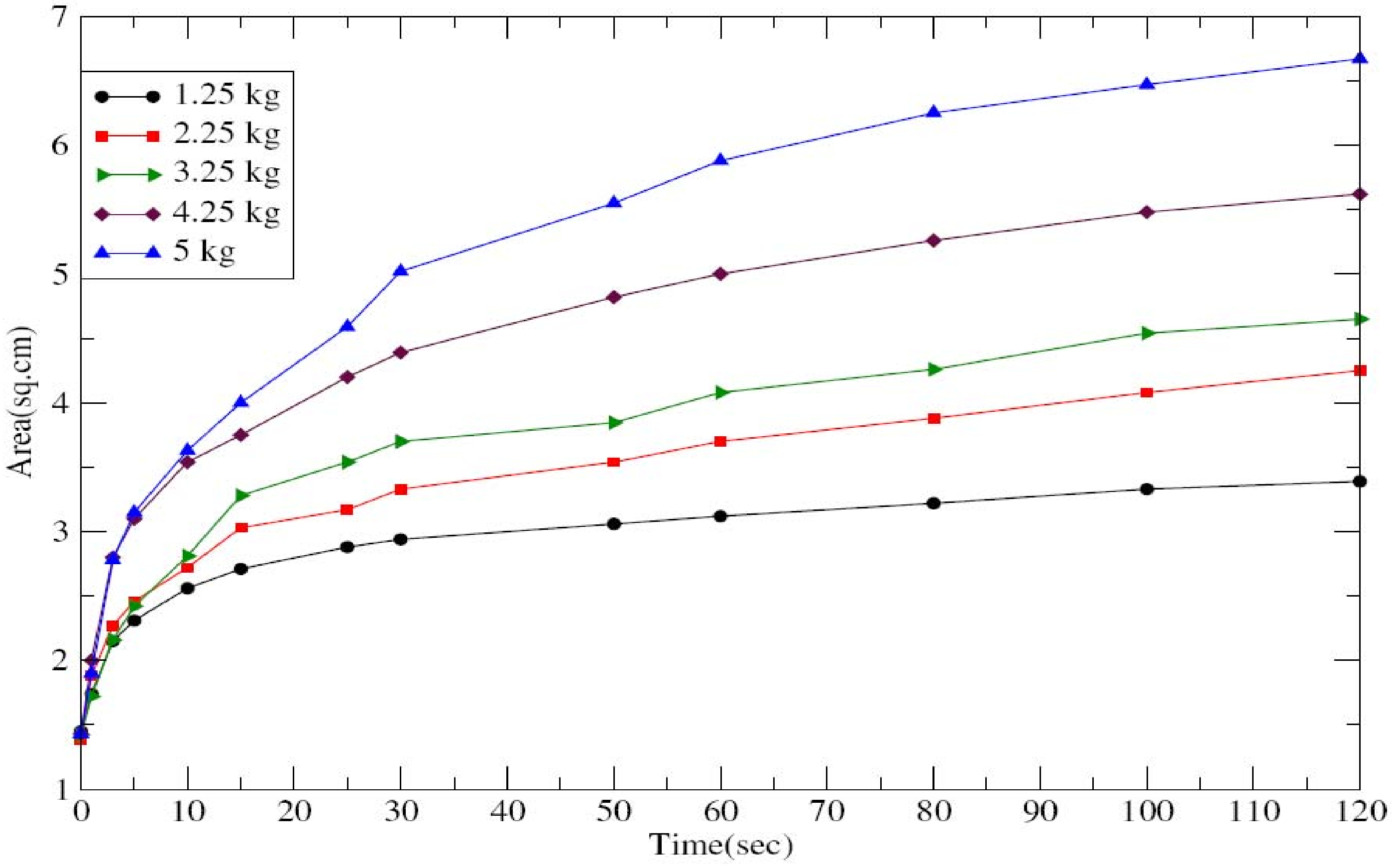
$$s^{0.5} = \frac{s+3}{3s+1}, \qquad \left(\frac{1}{s^{0.5}}\right) = \frac{3s+1}{s+3}, \qquad \frac{1}{s^{0.25}} = \frac{3s+5}{5s+3}, \qquad \frac{1}{s^{0.15}} = \frac{1+1.35s}{1.35s+s}$$

Oscillatory Spreading of Non Newtonian fluid Under Compression

Studies at Condense Mater Research Centre Dept. of Physics JU under scholarship scheme for M.Sc. PG project, and extra course MPGC

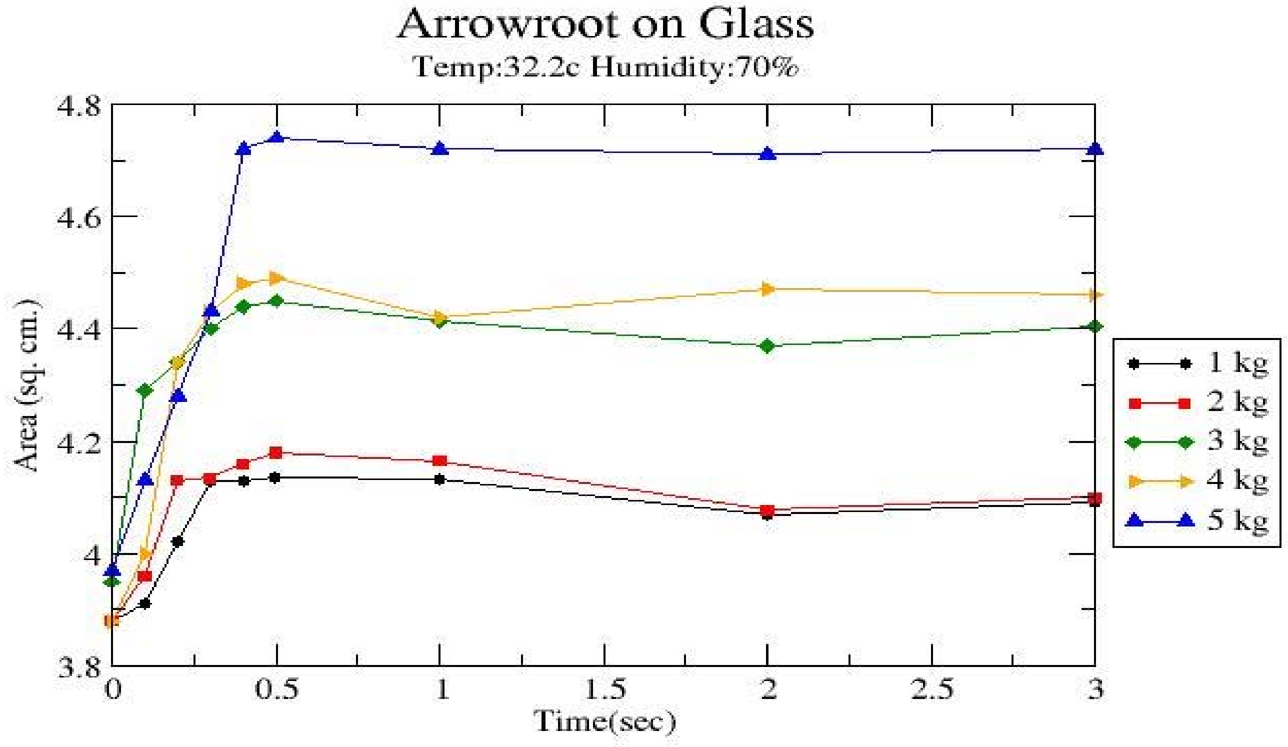


Case of Newtonian Fluid



Non-Newtonian case

Area-Time plot



Viscoelasticity with variable fractional order value

$$\beta \frac{d^q}{dt^q}\varepsilon(t) + Y\varepsilon(t) = \sigma(t)$$

$$\sigma(t) = \sigma_0 H(t)$$

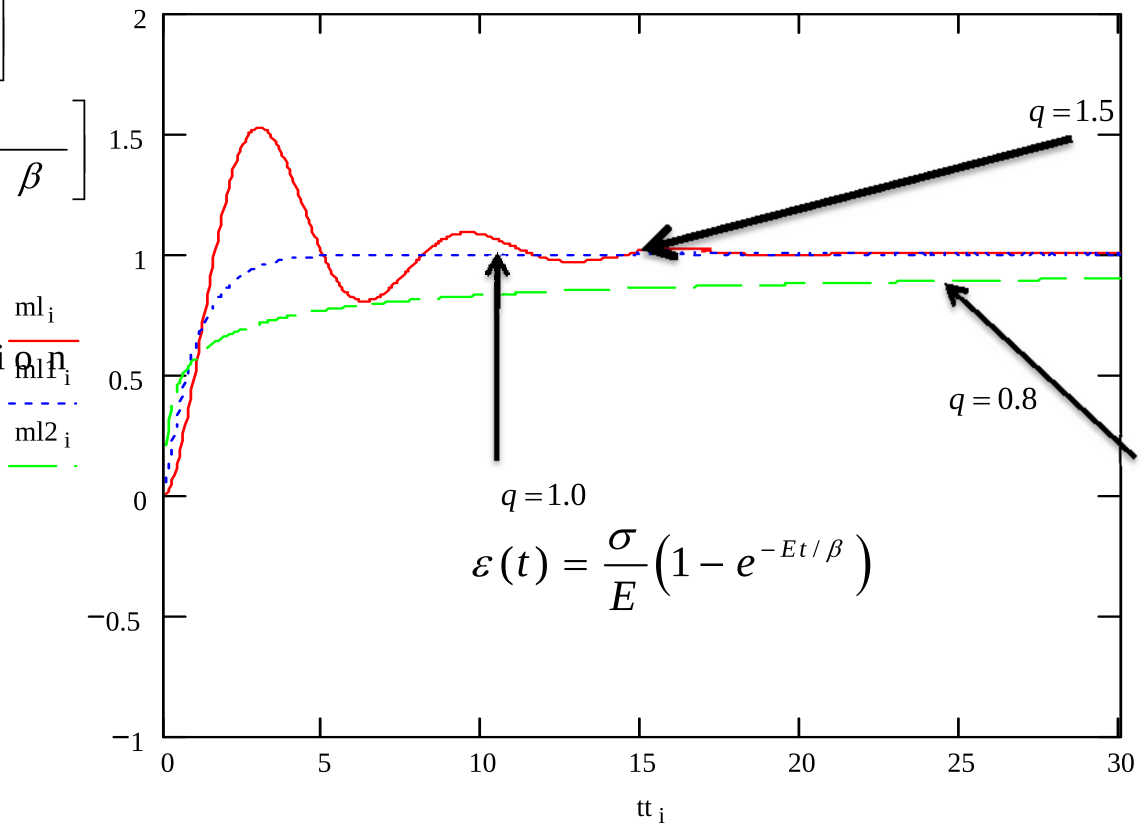
$$\sigma < \sigma_c$$

$$\varepsilon(s) = \frac{\sigma_0}{\beta}\left[\frac{1}{s(s^q + Y/\beta)}\right]$$

$$= \frac{\sigma_0}{E}\left[\frac{1}{s} - \frac{s^{q-1}}{s^q + Y/\beta}\right]$$

$$\varepsilon(t) = \frac{\sigma}{E}\left[1 - E_q\left(-\frac{Yt}{\beta}\right)\right]$$

E_q : Mittag Leffler function



Memory Integrals:

$$\frac{d^q}{dt^q}\varepsilon(t) + B\varepsilon(t) = \frac{\sigma(t)}{\beta} \qquad B = Y/\beta$$

$$\frac{d\varepsilon(t)}{dt} = -\int_0^t K(t-\tau)\varepsilon(\tau)d\tau = -K(t) * \varepsilon(t)$$

Represents Memory Integral i.e. all instances for $\tau = 0$ to $\tau = t$ contribute to situation at $\tau = t$

1. Memory breaks down i.e. Markovian Case:

$$K(t) = K_0\delta(t)$$

$$\frac{d}{dt}\varepsilon(t) = -\int_0^t K_0\delta(t-\tau)\varepsilon(\tau)d\tau = -K_0\varepsilon(t)$$

$$\varepsilon(t) = \varepsilon_0 \exp\{-K_0 t\}$$

$$\frac{d\varepsilon(t)}{dt} = -\frac{\varepsilon(t)}{\tau} = -K(t) * \varepsilon(t)$$

<u>How the process time constant is related to Kernel of Memory integral, good research case?</u>

2. The opposite case Constant Memory i.e. leading to oscillatory case

$$K(t) = K_0$$

$$\frac{d^2}{dt^2}\varepsilon(t) = -K_0\varepsilon(t)$$

$$\varepsilon(t) = \varepsilon_0 \cos(\sqrt{K_0}\, t)$$

Memory Integral with power law Kernel

3. Relaxation for Fractional Differential/Integral equation & its Memory Kernel

$$K(t) = K_0 t^{q-2}; \; 0 < q \le 2$$

$$\frac{d}{dt}\varepsilon(t) = -\frac{1}{\tau^q}\left[{}_0D_t^{1-q}\varepsilon(t)\right]$$

$$\tau^q = \left[K_0 \Gamma(q-1)\right]^{-1}$$

Apply ${}_0D_t^{-1}$ on both sides to get: $\varepsilon(t) - \varepsilon_0 = -\tau^{-q}\, {}_0D_t^{-q}\varepsilon(t)$

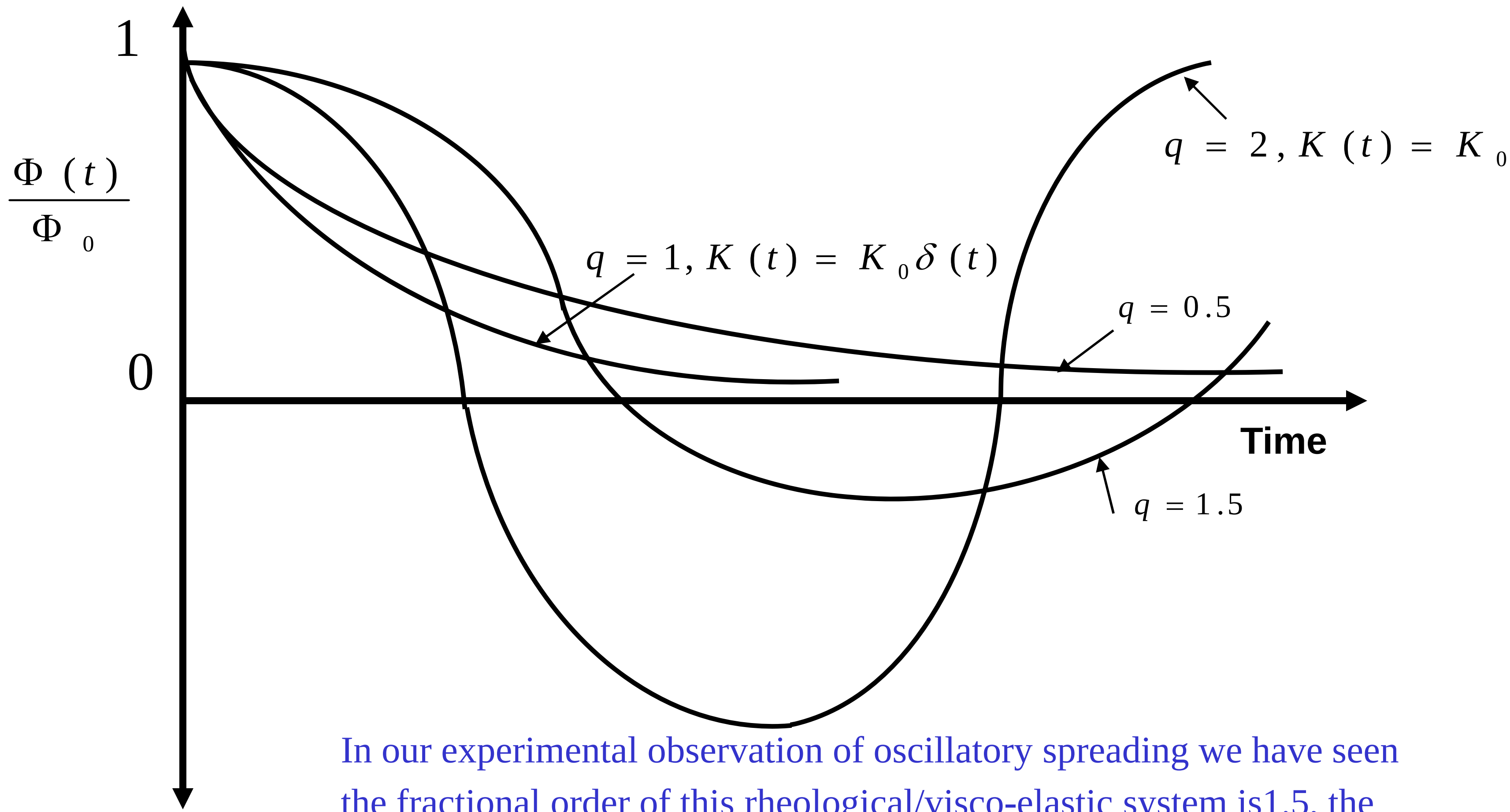
Apply ${}_0D_t^{q}$ on both sides to get FDE

$${}_0D_t^{q}\varepsilon(t) - \varepsilon_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\varepsilon(t)$$

Using Fractional Derivative of constant **C** as , non zero, that is $C\, t^{-q} / \Gamma(1-q)$

Memory Kernel & Fractional Differential Equation for Relaxation kinetics

$$ {}_0D_t^q\,\Phi(t) - \Phi_0 \frac{t^{-q}}{\Gamma(1-q)} = -\tau^{-q}\,\Phi(t) $$



In our experimental observation of oscillatory spreading we have seen the fractional order of this rheological/visco-elastic system is1.5, the system relaxation is memorized. Only pure solid system and fluid will relax without memory, else the relaxation is in between Newtonian and pure fluid. Well no pure resistance, capacitance and inductance, exists hence the volt-current relations are too governed by fractional differential equations

Hardwire set up to control DC Motor servo position system with Fractional Order -PID circuits under three patents

$$u(t) = K_P e(t) + K_I \{ {}_0D_t^{-\alpha} e(t) \} + K_D \{ {}_0D_t^{\beta} e(t) \}$$

$$e(t) = r(t) - c(t)$$

$$\mathrm{PI}^{\alpha}\mathrm{D}^{\beta} \qquad \alpha, \beta \in \Re(0,1)$$

Fractional Order Differential Equation solved in circuits

No computers!!

Line/surface/volume integrals of Fractal Distribution:

Fractal Distribution represented by Fractional Continuous Medium and then we perform the integration.
The fractional Integrals are considered as an approximate integrals on fractals. This type of new approach is applicable in processes where fractal features of the process or the medium impose the necessity of using non traditional tools in regular smooth physical equations.
Smoothening the microscopic characteristics over physically infinitesimal Volume/surface/line transforms the initial fractal distribution into fractional continuum model. The order of fractional integration is of fractal dimension.

$${}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} f(u)\,du$$

$${}_0D^{-d} f(r) = \int_V f(r)\,dV_d \approx \int_V \frac{r^{d-3}}{\Gamma^3(d/3)} dV_3 \qquad dV_d = K_3(r,d)\,dV_3 \quad K_3(r,d) = \frac{r^{d-3}}{\Gamma^3(d/3)}$$

$$2 < d < 3$$

$$dS_d = K_2(r,d)\,dS_2 \qquad 1 < d < 2 \qquad K_2(r,d) = \frac{r^{d-2}}{\Gamma^2(d/2)}$$

$$dL_d = K_1(r,d)\,dL_1 \qquad 0 < d < 1 \qquad K_1(r,d) = \frac{r^{d-1}}{\Gamma(d)}$$

Some laws on Fractal Geometries:

Flux through a fractal surface:

A flowing quantity trough a fractal surface be represented as:

$$\phi_{S_d} = \int_S \left(J(r,t) \bullet dS_d \right) \qquad dS_d \equiv K_2(r,d)\,dS_2 \qquad dS_d = \frac{r^{d-2}}{\Gamma^2(d/2)} dS_2$$

$$1 < d < 2$$

Gauss's law on Fractal:

$$\int_{\partial W} \left(J(r,t) \bullet dS_2 \right) = \int_W \mathbf{div}[J(r,t)]\,dV_3$$

$$dS_d = K_2(r,d)\,dS_2 \qquad dV_d = K_3(r,d)\,dV_3$$

$$\int_{\partial W} \left(J(r,t) \bullet dS_d \right) = \int_W (K_3(r,d_3))^{-1} \mathbf{div}[K_2(r,d_2) J(r,t)]\,dV_d$$

Stroke's law on Fractal:

$$\int_L \left(E \bullet dL_1 \right) = \int_S [\mathbf{curl} E]\,dS_2$$

$$dL_d = K_1(r,d)\,dL_1 \qquad dS_d = K_2(r,d)\,dS_2$$

$$\int_L \left(E \bullet dL_d \right) = \int_S \left(K_2(r,d_2) \right)^{-1} [\mathbf{curl} K_1(r,d_1) E]\,dS_d$$

Existence of Magnetic charges?

In normal cases of smooth geometry $\mathbf{div}\, B = 0$ indicating no magnetic charges at point exists . Magnetic mono-pole not possible.

Fractional generalization however gives: $\mathbf{div}[K_2(r, d_2) B] \neq 0$

$$\mathbf{div}\, B = B.\mathbf{grad}\, K_2(r, d_2)$$

For $d_2 \neq 2$; $\mathbf{grad}\, K_2(r, d_2) \neq 0$ indicating $\mathbf{div}\, B \neq 0$

Existence of 'magnetic monopole charges' with magnitude of

$$e_m \approx B.\nabla K_2(r, d_2)$$

For fractal distribution we have thus all sets of conservation laws and set of Maxwell equations and electrodynamics do get modified.

This method perhaps is suitable for dusty plasma cases.

Local Fractional Derivative (LFD) Kolwankar-Gangal (KG)

For a function $f : [0,1] \to \mathbf{R}$, the limit

$$\mathbf{D}^q f(x_0) = \lim_{x \to x_0} \frac{d^q \big(f(x) - f(x_0) \big)}{\left[d(x - x_0) \right]^q}$$

exists and is finite; then we say LFD of order q where $0 < q < 1$ at $x = x_0$ exits

In this definition the lower limit x_0 is treated as a constant. The subtraction of $f(x_0)$ corrects for the fact that fractional derivative of constant (in RL) is not zero. Where the limit $x \to x_0$ is taken to remove non-local contents. This LFD (removing the non-local contents) allows the study of point wise behavior of $f(x)$.

$$\mathbf{D}^1 f(0) = \lim_{x \to 0} \frac{d}{dx} f(x)$$ Slope at origin!

Critical Order LFD and Fractal (Box) dimension

$f : [0,1] \to \mathbf{R}$, be a continuous (real) function

If $\lim_{x \to x_0} \frac{d^q \left(f(x) - f(x_0) \right)}{\left[d(x - x_0) \right]^q} = 0$ for $q < \alpha$

Then $\dim_B f(x_0) \leq 2 - \alpha$

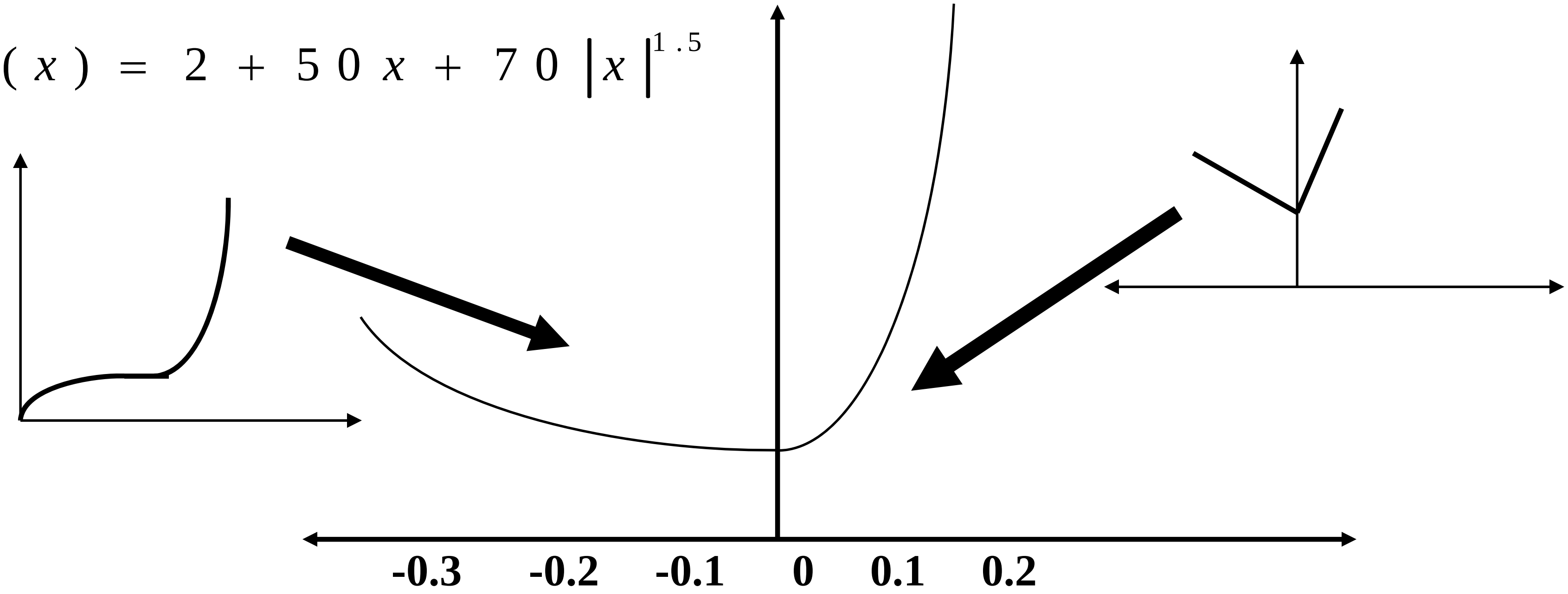
Holder Exponent $\alpha(x_0)$ of a function $f(x)$ defined by this is the largest exponent such that there exists a polynomial $P_n(x)$ that satisfies

$$\left| f(x) - P_n(x - x_0) \right| = C \left| x - x_0 \right|^{\alpha}$$

There is clear change in behavior when q crosses the Critical Order. $\alpha(x_0)$

Abrupt phase transition to continuous phase transition:

$$f(x) = 2 + 50x + 70|x|^{1.5}$$

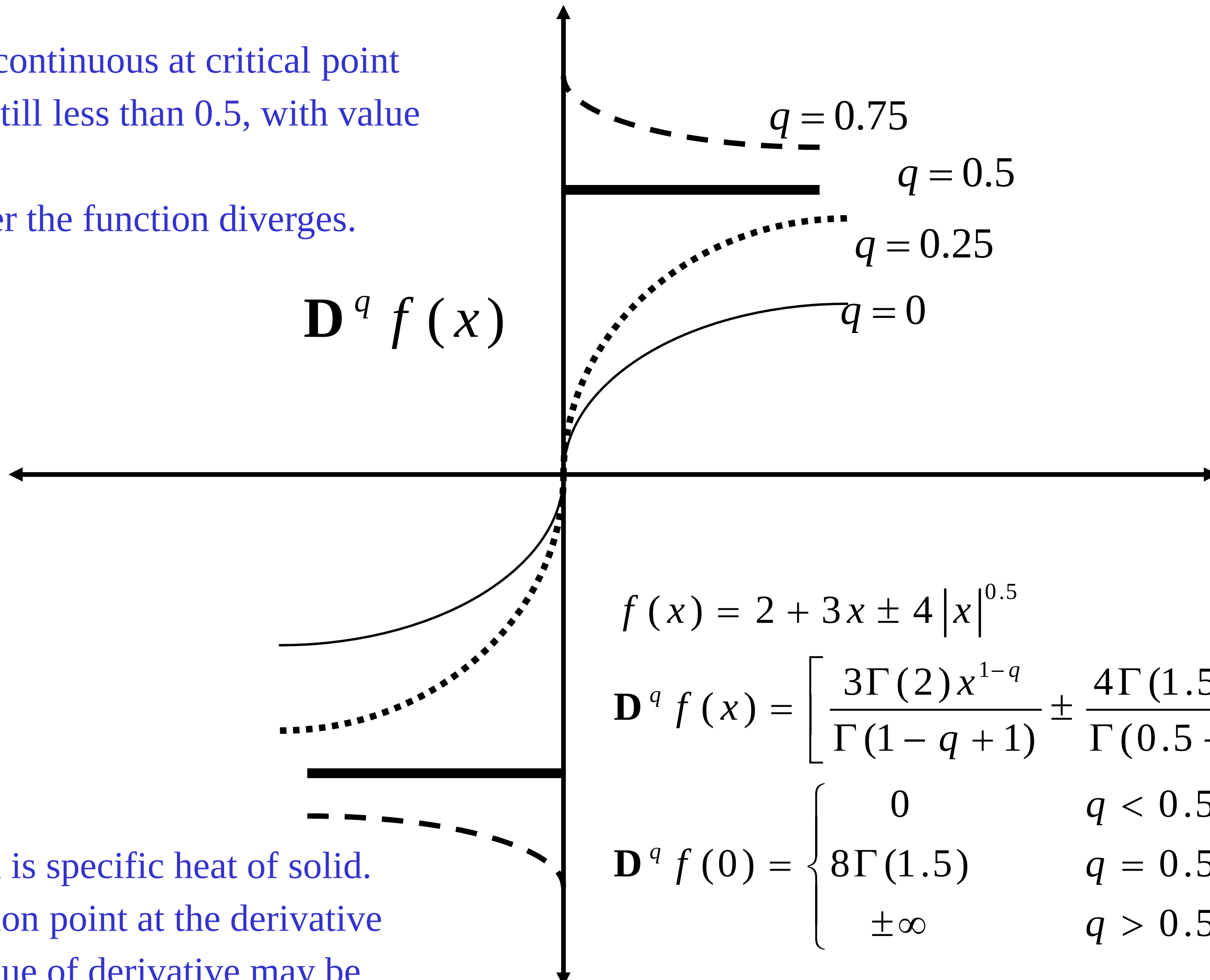


This is notion to extend Ehrenfest's classification of thermodynamic phase transition, magnetic property at critical point, or yield point (stain) beyond critical stress to continuous transition.

In simplified terms magnify the critical point which takes place abruptly and approximate by polynomial to get Fractional Differentiability at critical point. Non-differentiability can be magnified and studied near critical points.

Fractional Differentiability at Critical Point:

The function is continuous at critical point from zero order till less than 0.5, with value zero.
Beyond 0.5 order the function diverges.



$$f(x) = 2 + 3x \pm 4|x|^{0.5}$$

$$\mathbf{D}^q f(x) = \left[\frac{3\Gamma(2)x^{1-q}}{\Gamma(1-q+1)} \pm \frac{4\Gamma(1.5)x^{0.5-q}}{\Gamma(0.5-q+1)} \right]$$

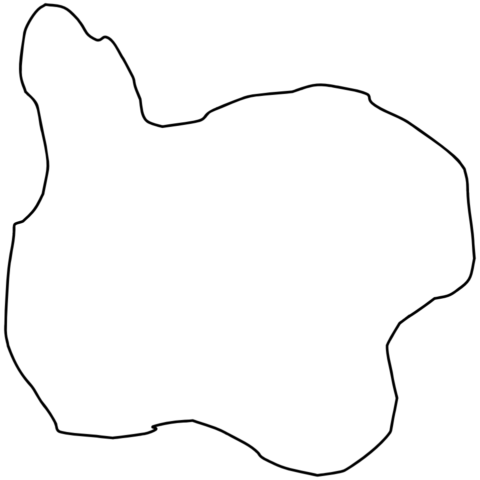
$$\mathbf{D}^q f(0) = \begin{Bmatrix} 0 & q < 0.5 \\ 8\Gamma(1.5) & q = 0.5 \\ \pm\infty & q > 0.5 \end{Bmatrix}$$

Say the function is specific heat of solid.
At phase transition point at the derivative order 0.5 the value of derivative may be regarded as ‘Fractional Latent Heat’

Does '*d* / *dt*' represent accumulation or loss always

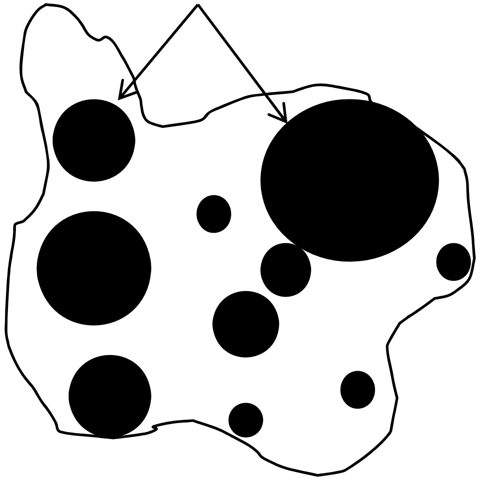
Well if there are temporary traps then?

Well if the elementary element (area, volume etc) be not a point quantity?



$$\frac{d}{dt}\phi = \text{GAIN} - \text{LOSS?}$$

Traps or Island (Forbidden zone)

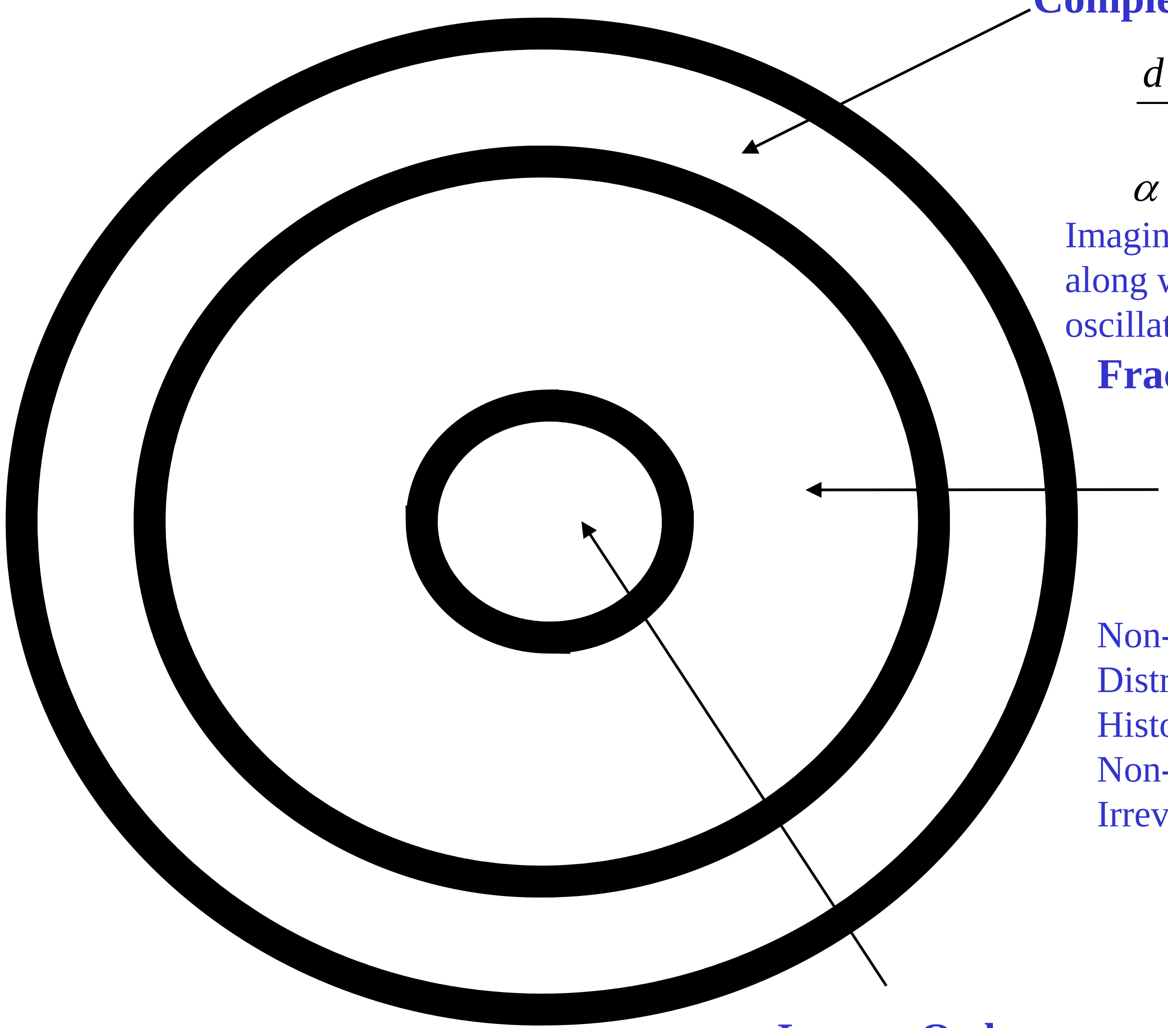


$$\frac{d^{\alpha}}{dt^{\alpha}}\{\phi\} = \text{GAIN} - \text{LOSS?}$$

Some are entraps temporarily indicating slow rate of change than *d* / *dt*

The particles cannot have the island paths indicating fast rate of change than *d* / *dt*

The Generalized Calculus



Complex-Order

$$\frac{d^{\alpha+i\beta} f(z)}{dz^{\alpha+i\beta}}$$

$$\alpha, \beta \in \Re e$$

Imaginary time response
along with real time, fractal
oscillatory frequency response

Fractional order

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}}$$

$$\alpha \in \Re e$$

Non-local
Distributed
History/heredity
Non-Markovian
Irreversibly

Integer Order

Newtonian
Point property
Symmetry

$$\frac{d^{n} f(x)}{dx^{n}}$$

$$n \in Z$$

Thanking You All

………………………………..it was a fraction of
Fractional Calculus

and I have to travel several light years, to know, as at present know nothing……what is Fractional Calculus, and

to see know
'What Mathematics Nature Should Follow'
"That is the Greatest Discovery by Newton"